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Indeterminacy and well-posedness of the non-local theory of Rayleigh waves

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ABSTRACT

The latest literature stance holds that, in a 2D framework, the non-local theory of elasticity, as developed by Eringen, is fundamentally inconsistent because "it does not satisfy the equations of motion for [the] non-local stresses". In fact, it is believed that the differential form of this theory, that is accessible when the attenuation function is a Green function and that is well-posed, gives different results from the integral formulation. We show that these ideas are ill-conceived, provided that we adopt the *kernel modification approach*, by which the constitutive boundary conditions (CBCs) embedded in the integral formulation are reconciled with the natural boundary conditions of the problem at hand. Indeed, this kernel modification strategy, which was first introduced by the authors for 1D non-local models, is necessary to avoid that the problem becomes over-constrained through (possibly conflicting) natural and constitutive boundary conditions, and consequently ill-posed. Once the problem is made well-posed, we show that (1) failure to satisfy the equations of motion is not only expected, but it is in fact necessary, (2) for the example case of surface waves propagating in a stress-free half-plane, the integral and the differential formulations coincide, (3) for a force problem, the non-local theory is generally *indeterminate* because it lacks compatibility: consequently, for a unique solution, an extra boundary condition is needed, and (4) multiple Rayleigh wave branches appear as a consequence of non-locality.

1. Introduction

One of the most noteworthy defect of the classical theory of elasticity is its lack of an internal length-scale that incorporates the experimental finding that matter is fundamentally granular, which fact plays a very important role for extremely localized stress/deformation fields. The infinite stress field that appears at the tip of a crack is often taken as a case in point of this deficiency. This theory prediction possesses strong technical implications, for it prevents the use of a maximum stress criterion to guide crack propagation. Instead, an energy criterion must be reverted to, through the concept of stress intensity factor (SIF). The need to overcome this limitation is also especially felt when dealing with nano-structures, whose physical dimensions are comparable with that of the underlying micro-structure (Karlicic et al., 2015), which fact is known to produce significant deviations from the predictions obtained at large scales. The theory of non-local elasticity, often credited to Eringen and Edelen (1972) but really originated by Kröner (1967), has attracted considerable attention in the literature, precisely to address such issues. For the largest part, this theory has been adopted to study 1D micro- and nano-devices, in view of their great practical importance (Benvenuti &

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Simone, 2013; Mikhasev & Nobili, 2020; Peddieson et al., 2003; Radi et al., 2021; Wang & Liew, 2007). For numerical methods within non-local elasticity in connection to layered media, one may refer to Chakraborty (2007). This theory belongs to the wide area of generalized continua (Altenbach et al., 2011) and accounts for long-range interactions, in contrast to strain-gradient theories (Nobili, 2021) which account for short-range effects. The underlying principle of this theory is to express the material response, the so-called non-local stress, through an integral, over the entire domain, of the familiar local stress. Most importantly, this integral formulation makes use of an *attenuation function* (kernel) which may be coming from some micro-mechanical insight, and, for this reason, it is considered a constitutive assumption.

The non-local theory of elasticity, however, suffers from the major defect of implicitly introducing extra boundary conditions, named *constitutive*, through the choice of the attenuation functions. The presence of these constitutive boundary conditions (CBCs), in addition to the natural boundary conditions (BCs), leads to the problem being over-constrained and, consequently, ill-posed (Kaplunov et al., 2023). Interestingly, this shortcoming seems absent in the differential formulation of the problem, which is accessible when the attenuation function (kernel) is the Green's function of some differential operator $\mathfrak{L}$. Well-posedness of the differential formulation, however, is only apparent, for it emerges as a result of the improper dealing with the integral formulation, namely from the fact that the CBCs are simply ignored. As a consequence, the differential and the integral formulation of the problem yield different results.

Understandably, such features of the non-local theory have caused much confusion in the literature, which, for the sake of simplicity, we may divide into two groups. On the one hand, an overwhelming number of contributions simply ignore the issue and solve the differential problem with no concern whether this also solves the integral problem. As a matter of fact, this approach amounts to solving the classical problem with some elaborate inertia forces, see, among many examples, Bhat and Manna (2024), Biswas (2020), Kaur et al. (2019), Lu et al. (2007), Pramanik and Manna (2024) and references therein. On the other hand, some authors acknowledge the problem with the integral formulation and revert to the differential formulation instead, often adding extra assumptions, claiming that this is the only correct pathway for the solution of the non-local problem. As a case in point, one may refer to the concept of *weakly non-local* theory introduced by Anh et al. (2023) and Vinh et al. (2024) or to the two-phase non-local model (TPNM), see Mikhasev and Nobili (2020) and Mikhasev et al. (2024), first introduced by Eringen (1972). Although TPNM remediates ill-posedness, it suffers from the major shortcoming, that it shares with any strain-gradient model, of necessitating extra boundary conditions of difficult physical interpretation and determination, not to mention the need to somehow define the phase parameters (Nobili & Pramanik, 2025).

Matters are further complicated by the very recent realization that the solution of the differential problem does not satisfy the equations of motion. This result was first observed by Kaplunov et al. (2022) through an asymptotic procedure in the context of Rayleigh wave propagation, and it was interpreted in support of the idea that the integral problem is ill-posed. The point is further pressed in Kaplunov et al. (2023) and motivates the adoption of a fully differential approach in Pham and Vu (2024), in an attempt to depart from the "defective" integral model. In this last contribution, the solution of the differential model is used to determine the local stresses and, from these, the non-local stresses are calculated by solving the PDE of which the integral formulation is the Green's function, i.e. avoiding the integral formulation. Crucially, this PDE is supplemented by the problem BCs, as opposed to the CBCs. We shall show that this approach is equivalent to solving the integral formulation provided that the attenuation functions are modified to agree with the natural boundary conditions.

In this paper, we investigate the propagation of Rayleigh waves within the integral non-local model, by adopting the *kernel modification approach*. The idea of kernel modification first appeared in Borino et al. (2003), in the context of damage theory, and was later borrowed by Eptaimeros et al. (2016) to warrant the unit property of the kernel over a finite (as opposed to infinite) domain. This concept of kernel modification was embraced and extended in Nobili and Pramanik (2025) and Pramanik and Nobili (2025) to reconcile the CBCs with the BCs and thus guarantee that the non-local problem becomes well-posed. This result is achieved while keeping the symmetric character of the kernel, which is indispensable to warrant that the strain energy is a quadratic functional of the strain. As a consequence of kernel modification, the attenuation functions can no longer be completely and freely defined, as in a constitutive property. Instead, they are subjected to the effect of the domain boundaries, from which they borrow the relevant boundary conditions. It is observed that this boundary effect only constrains the arbitrariness of the CBC, while the choice of the operator $\mathfrak{L}$ remains free. Indeed, kernel modification really acts only in the close proximity of the boundaries, the bulk of the material remaining largely unaffected, see Nobili and Pramanik (2025) and Pramanik and Nobili (2025). Hence, in this restricted sense, the constitutive nature of the attenuation function remains intact. That boundaries affect the constitutive response may not appear so strange, especially in connection with nano-structures, as the development of surface elasticity shows (Radi et al., 2021). It is emphasized that, in a 2D problem, three attenuation functions must be defined and only two of these necessitate kernel modification, because only two BCs are given. It is also observed that, as it appears in Pham and Vu (2024), this modification is explicitly linked to the success of the differential model, which makes use of the natural BCs. Finally, we point out that the same attenuation function is adopted for all stresses in Chakraborty (2007), which approach fails for general BCs.

Once well-posedness is established, in Section 2.1 we develop the solution to the differential problem and in Section 2.2 we show that its failure to comply with the equations of motion is expected, because it provides the very conditions to the full determination of the solution. Besides, a closed-form proof that the differential and the integral solutions indeed correspond is given, that hinges on an integral representation of the Bessel function developed in Appendix A. Interestingly, Section 3 shows that the non-local problem of elasticity remains *indetermined*, because an extra CBC is required which may be given arbitrarily. This is a consequence of the lack of a unique compatibility condition for the non-local stresses. As a result, one attenuation function can be prescribed freely and it does not require modification. Finally, Section 4 develops the solution for SH waves, and conclusions are drawn in Section 5. For the reader's convenience, Appendix C collects a table of symbols used throughout the paper.

2. On the well-posedness of Eringen's non-local theory of elasticity

We begin by outlining the approach originally followed by Eringen and Kim (1974) to address propagation of travelling harmonic waves. This approach was later revised by Kaplunov et al. (2022) to conclude that "the integral nonlocal model for an elastic half-space is inconsistent, since the nonlocal stresses provided by [this] theory do not satisfy the equations of motion". We shall indeed show that, apart from the usual problem attached with the presence of constitutive boundary conditions (CBCs) embedded in the kernel and extensively discussed for 1D models, failure to satisfy the motion equations is in fact expected and not a problem in itself.

Let us consider the elastic half-plane $\mathbb{V} = \{(x_1, x_3) : 0 \le x_3 < \infty\}$ in plane strain. Accordingly, the nonzero displacement components are

$$u_1 = u_1(x_1, x_3, t), \quad u_3 = u_3(x_1, x_3, t), \tag{1}$$

while the governing equations of motion read

$$t_{11,1} + t_{31,3} = \rho \ddot{u}_1, \quad t_{13,1} + t_{33,3} = \rho \ddot{u}_3, \tag{2}$$

where a subscript comma denotes differentiation with respect to the relevant space coordinate, i.e. $t_{11,1} = \partial t_{11}/\partial x_1$, and a superposed dot indicates time differentiation, i.e. $\ddot{u}_3 = \partial^2 u_3/\partial t^2$. Besides, ρ denotes the mass density of the material. Here, following Eringen, t_{ij}, $i, j \in \{1, 3\}$, express the so-called *non-local stress components*, that are obtained averaging over the whole domain through the *attenuation functions* $\alpha_{ij}(\boldsymbol{x}, \boldsymbol{\xi})$

$$t_{ij}(\boldsymbol{x}, t) = \int_{\mathbb{V}} \alpha_{ij}(\boldsymbol{x}, \boldsymbol{\xi})\, \sigma_{ij}(\boldsymbol{\xi}, t) \mathrm{d}\boldsymbol{\xi}, \quad \boldsymbol{x}, \boldsymbol{\xi} \in \mathbb{V}. \quad \text{(no summation)} \tag{3}$$

In general, different attenuation functions (or kernels) may be adopted for different non-local stress components, provided they satisfy the fundamental properties of unity, impulsivity and symmetry, see Nobili and Pramanik (2025) and Pramanik and Nobili (2025) for further details. In particular, the kernel functions depend on the internal length parameter, $\epsilon > 0$, that expresses the stress diffusion length scale; by the impulsivity requirement, we demand that $\lim_{\epsilon \to 0} \alpha_{ij} = \delta$, where $\delta = \delta(\boldsymbol{x})$ denotes Dirac's delta (generalized) function.

In contrast, σ_{ij} represents the *local stress components*, that are obtained from the linear deformation components

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{4}$$

through the usual constitutive relations

$$\sigma_{ij} = C_{ijkl}\, e_{kl}, \tag{5}$$

where C_{ijkl} are the elastic constants and summation over twice-repeated subscripts is assumed. Here, for the sake of simplicity, we assume a homogeneous isotropic material, for which

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk}, \tag{6}$$

where μ and λ are the usual Lamé parameters and δ_{ij} is the Kronecker's symbol (the danger of confusion with the delta function should not be great). The problem formulation is completed by the traction-free boundary conditions

$$t_{13} = t_{33} = 0, \quad \text{at} \quad x_3 = 0, \tag{7}$$

which are properly formulated in terms of the non-local stress components (see Pham & Vu, 2024 for reference to the several literature contributions which incorrectly employ the local stresses instead).

Together, Eqs. ((2),(3),(4),(6)), supplemented by the BCs (7), describe the nonlocal problem of plane elasticity, in the unknowns (1). For the problem to be fully described, we also need to provide the attenuation functions α_{ij}. In this form, the problem is governed by a complex system of integro-partial differential equations subject to a set of boundary conditions, some of which are explicit, as in Eqs. (7), while some others are implicit, for they are embedded in the choice of the attenuation functions. Matters are greatly simplified, at least in terms of the solution process, if, following Eringen (1983), the attenuation functions are chosen in the form of the Green's function attached to some differential operator, the usual choice being the Helmholtz operator

$$\mathfrak{L} = 1 - \epsilon^2\, \triangle, \tag{8}$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$ represents the 2D Laplace operator. With this move, the integral nature of the problem may be eliminated by application of the operator $\mathfrak{L}$, and this leads to a purely differential formulation that is easier to deal with. Indeed, upon application of the Helmholtz operator to Eqs. (3), one obtains the differential formulation for the non-local stresses

$$\mathfrak{L} t_{ij} = \sigma_{ij}, \tag{9}$$

which is oblivious of the CBCs. Similarly, applying the operator $\mathfrak{L}$ to the equations of motion (2) and using Eqs. (9), we have

$$\left.\begin{aligned} \sigma_{11,1} + \sigma_{13,3} &= \rho \mathfrak{L} \ddot{u}_1, \\ \sigma_{13,1} + \sigma_{33,3} &= \rho \mathfrak{L} \ddot{u}_3, \end{aligned}\right\} \tag{10}$$

which, using the stress–displacement relations (6) and expressing the operator $\mathfrak{L}$ as in Eq. (8), yields

$$\left.\begin{aligned}(\lambda+2\mu)u_{1,11}+(\lambda+\mu)u_{3,13}+\mu u_{1,33}&=\rho\ddot{u}_1-\epsilon^2\rho\left(\ddot{u}_{1,11}+\ddot{u}_{1,33}\right),\\(\lambda+\mu)u_{1,13}+\mu u_{3,11}+(\lambda+2\mu)u_{3,33}&=\rho\ddot{u}_3-\epsilon^2\rho\left(\ddot{u}_{3,11}+\ddot{u}_{3,33}\right),\end{aligned}\right\} \tag{11}$$

that is a special case of Eqs. (15, 16) in Chakraborty (2007). Indeed, Eqs. (11) correspond[1] to Eq. (6.9.15) in Eringen (2002) and "replace the Navier equations of classical elasticity". Indeed, they provide a fully differential formulation of the problem, which stands regardless of the CBCs embedded in the integral formulation and, in this sense, it is oblivious of these. As a result, much confusion has appeared in the literature owing to the fact that, while the differential formulation (11) can always be solved, subjected to the natural BCs, the relevant solution may not correspond to that of the integral problem, from which it originated. Indeed, it is important to note that the equivalence between the integral (3) and the differential formulation (9) holds strictly for an infinite domain, i.e., $\mathbb{V}=\mathbb{R}^3$. If the domain $\mathbb{V}$ is a proper subset of $\mathbb{R}^3$ with a boundary $\mathbb{S}$, then the non-local stress defined in Eq. (3) naturally satisfies a constitutive boundary condition (CBC) of the form

$$\frac{\partial t_{ij}(\boldsymbol{x},t)}{\partial \boldsymbol{n}}+\beta(\boldsymbol{x},t)t_{ij}(\boldsymbol{x},t)=0,\quad \boldsymbol{x}\in\mathbb{S}, \tag{12}$$

where $\boldsymbol{n}$ denotes the outward unit normal to $\mathbb{S}$ at $\boldsymbol{x}$ and the function $\beta(\boldsymbol{x},t)$ is determined by the properties of the kernel function $\alpha_{ij}(\boldsymbol{x},\boldsymbol{\xi})$. Thus, for a domain $\mathbb{V}\subset\mathbb{R}^3$ (i.e., a domain with a non-empty boundary), the integral Eq. (3) is equivalently represented by the differential Eq. (9), provided that the latter is supplemented by the constitutive boundary condition (12).

As we shall presently see, and as it is the case for 1D nonlocal elasticity (Nobili & Pramanik, 2025), the fact that the integral and the differential formulation differ emerges whenever the CBCs embedded in the attenuation functions fail to comply with the natural BCs (7), whence the integral problem is ill-posed. If, on the other hand, the attenuation functions are chosen in such a way so as to satisfy the natural boundary conditions, the problem may be solved in both the integro-differential as well as in the purely differential forms, and the corresponding solutions coincide. It is remarked that, as it will be explained, considerations of well-posedness have no connection with the fact that the solution of the differential problem fails to satisfy the motion Eqs. (2). Indeed, as illustrated by Nobili and Pramanik (2025), the well-posed character of the problem is a matter of *kernel modification*, according to which the kernel function may be modified so as to replace the CBCs attached to the original kernel with another set that complies with the natural BCs. Accordingly, the attenuation functions cannot be decided at the onset, irrespectively of the problem at hand, and in this sense they become part of the unknowns. In fact, the only thing that can be successfully established at the onset is the differential operator $\mathfrak{L}$, of which the attenuation functions are the Green's functions. This feature of the problem is especially relevant for restricted 1D model, such it is the case of the Euler–Bernoulli beam theory, for which multiple natural boundary conditions appear against only one available attenuation function, see Pramanik and Nobili (2025). In contrast, as we shall presently see, the plain strain problem, and in general any 2D and 3D problem, suffers from the opposite defect, by which multiple attenuation functions are available and only a few natural BCs exist which enable to define the appropriate CBCs. In this spirit, provided the CBCs are made to comply with the natural BCs, the nonlocal problem of elasticity is indetermined.

2.1. Travelling wave solution

Let us develop the theory for the special case of harmonic disturbances, such that the displacement and the stress components move as harmonic plane waves in the x_1 direction

$$\left\{u_i,\sigma_{ij},t_{ij}\right\}(x_1,x_3,t)=\left\{U_i,S_{ij},T_{ij}\right\}(x_3)\,e^{\imath k(x_1-ct)},\quad i,j=1,3, \tag{13}$$

where k is the wavenumber, c is the phase velocity of the wave and $\imath^2=-1$. Accordingly, the differential formulation (11) may be rewritten as

$$\left.\begin{aligned}\left(c_2^2-k^2\epsilon^2c^2\right)U_1''+k^2\left\{c^2(1+k^2\epsilon^2)-c_1^2\right\}U_1+\imath k\left(c_1^2-c_2^2\right)U_3'=0,\\\left(c_1^2-k^2\epsilon^2c^2\right)U_3''+k^2\left\{c^2(1+k^2\epsilon^2)-c_2^2\right\}U_3+\imath k\left(c_1^2-c_2^2\right)U_1'=0,\end{aligned}\right\} \tag{14}$$

where $c_1=\sqrt{(\lambda+2\mu)/\rho}$ and $c_2=\sqrt{\mu/\rho}$ are the longitudinal and the shear wave velocity, respectively, and prime denotes differentiation with respect to the only remaining space variable x_3. We let the shorthands $\epsilon_1=k\epsilon$ and $\kappa=c_1/c_2>1$. The system (14) leads to the single fourth order ODE

$$\begin{aligned}&\left(\epsilon_1^2v^2-1\right)\left(\kappa^2-\epsilon_1^2v^2\right)U_1''''+k^2\left(v^2\left(2\epsilon_1^4v^2+2\epsilon_1^2\left(v^2-1\right)-1\right)-\kappa^2\left(v^2\left(2\epsilon_1^2+1\right)-2\right)\right)U_1''\\&+k^4\left(v^2\left(\epsilon_1^2+1\right)-1\right)\left(\kappa^2-v^2\left(\epsilon_1^2+1\right)\right)U_1=0,\end{aligned} \tag{15}$$

whose solutions decaying along the x_3 direction amount to

$$\left.\begin{aligned}U_1(x_3)&=k^{-1}\left[A_1e^{-kb_1x_3}+A_2e^{-kb_2x_3}\right],\\U_3(x_3)&=\imath k^{-1}\left[b_1A_1e^{-kb_1x_3}+\frac{A_2}{b_2}e^{-kb_2x_3}\right],\end{aligned}\right\} \tag{16}$$

[1] Letting $\gamma=0$ to account for the Helmholtz operator only.

where A_1, A_2 are dimensionless arbitrary constants, and we have let the dimensionless wavenumbers for bulk waves

$$b_1 = \sqrt{1 - \frac{v^2}{\kappa^2 - \epsilon_1^2 v^2}}, \quad b_2 = \sqrt{1 - \frac{v^2}{1 - \epsilon_1^2 v^2}}, \tag{17}$$

together with the dimensionless wave speed $v = c/c_2$. In Eqs. (17) the branch of the square root is chosen such that $\Re(b_{1,2}) > 0$, where, for any complex number z, it is $z = \Re(z) + \imath\Im(z)$. A pair of real bulk wavenumbers is obtained provided that $v < 1/b_3$ with $b_3 = \sqrt{1+\epsilon_1^2}$. Plugging Eqs. (16) into Eqs. (6), and in view of Eq. (4), the amplitude of the local stress components σ_{ij} is immediately obtained

$$S_{ij}(x_3) = \mu \left[P_{(i+j)/2} A_1 e^{-kb_1x_3} + Q_{(i+j)/2} A_2 e^{-kb_2x_3}\right], \quad i, j = 1, 3, \tag{18}$$

where

$$\left.\begin{array}{l} P_1 = \imath\left\{\kappa^2 - \left(\kappa^2 - 2\right) b_1^2\right\}, \quad P_2 = -2b_1, \quad P_3 = \imath\left\{\kappa^2 - 2 - \kappa^2 b_1^2\right\}, \\ Q_1 = -Q_3 = 2\imath, \quad Q_2 = -\frac{1+b_2^2}{b_2}. \end{array}\right\} \tag{19}$$

In order to proceed to the determination of the non-local stress components, we need to define the attenuation functions. In particular, we shall choose all of them as the Green's functions attached to the Helmholtz operator (8). Furthermore, we specify the CBCs of t_{13} and t_{33} according to the natural BCs (7) coupled with the decay condition at infinity. Consequently, we set

$$\alpha_{13}(x_1, x_3, \xi_1, \xi_3) = \alpha_{33}(x_1, x_3, \xi_1, \xi_3) = G_1(x_1, x_3; \xi_1, \xi_3) - G_1(x_1, x_3; \xi_1, -\xi_3), \tag{20}$$

where

$$G_1(x_1, x_3; \xi_1, \xi_3) = \frac{1}{2\pi\epsilon^2} K_0\left(\frac{\sqrt{(x_1-\xi_1)^2 + (x_3-\xi_3)^2}}{\epsilon}\right), \tag{21}$$

and $K_0(r)$ is the zero-th order modified Bessel function of the second kind (Abramowitz & Stegun, 1948, §9.6). The appearance in (20) of the difference of two Bessel functions is motivated by the method of images and warrants that

$$\alpha_{13}(x_1, 0, \xi_1, \xi_3) = \alpha_{33}(x_1, 0, \xi_1, \xi_3) \equiv 0, \tag{22}$$

so that, as a consequence of such property, the non-local stress components (3) feature a pair of CBCs corresponding to the natural BCs (7), alongside decay at infinity. It is interesting to observe that in Kaplunov et al. (2022, Eq.(2.2)), only one Bessel function is used as the attenuation function, and consequently, the integral formulation is doomed to fail. This motivates the appearance of a boundary layer, which attempts to accommodate for conflicting boundary conditions. In contrast, the same combination of two Bessel functions as in (20) appears in Pham and Vu (2024) where, however, it is never really used because the (equivalent) differential formulation (9) is employed instead. It is important to emphasize that the two Bessel function attenuation kernel (20) is no longer of the difference type, i.e. it is no longer a difference kernel, and this fact has been already observed in Nobili and Pramanik (2025) and Pramanik and Nobili (2025), with reference to 1D problems, in connection with the idea that accounting for the presence of the domain boundary $\mathbb{S}$ inevitably leads to an inhomogeneous kernel.

Remarkably, we have no BCs at our disposal to uniquely define the CBCs for t_{11}, which may be thus arbitrarily chosen. To fix ideas, we define α_{11} as a single Bessel function, namely

$$\alpha_{11}(x_1, x_3, \xi_1, \xi_3) = G_1(x_1, x_3; \xi_1, \xi_3), \tag{23}$$

and this choice should be contrasted with that in Pham and Vu (2024), where the combination (20) is adopted instead. Consequently, as a result of the latter choice and upon accounting for Eringen's representation for the non-local stress (3), the equally arbitrary CBC $t_{11} = 0$ at $x_3 = 0$ is introduced by Pham and Vu (2024). As it can be deduced by direct inspection (see the Appendix A), our choice of a single Bessel function leads, instead, to the CBC

$$t_{11,3} - \epsilon^{-1} b_3 t_{11} = 0, \quad \text{at } x_3 = 0, \tag{24}$$

that is at variance with the second option for the CBC for t_{11} suggested in Pham and Vu (2024, Eq.(4.30)), the latter being motivated by an analogy with the 1D problem. We thus see that the non-local problem is non uniquely defined, inasmuch as some CBCs are concerned. Indeed, a definite choice must be made for *some* attenuation functions only, to comply with the natural BCs of the problem and avoid ill-posedness, while the others remain arbitrary (at least in terms of CBCs). This matter is further elucidated in Section 3.

2.2. Solution of the integral formulation

The non-local stress components $t_{ij} (i, j = 1, 3)$ are obtained from

$$t_{ij}(x_1, x_3, t) = \int_0^\infty \mathrm{d}\xi_3 \int_{-\infty}^\infty \alpha_{ij}\left(x_1, x_3; \xi_1, \xi_3\right) \sigma_{ij}(\xi_1, \xi_3, t)\, \mathrm{d}\xi_1, \quad i, j = 1, 3, \tag{25}$$

and, by virtue of the integral representation formula (A.6) for the Bessel function, we are in a position to actually compute the integrals in closed form, without recourse to any approximations. Indeed, plugging Eqs. (13) and (18) into the integrals (25) and choosing a favourable order of integration, the amplitude of the non-local stress components follow (for the details of the derivations see the Appendix B)

$$T_{11}(x_3) = \mu\left[\frac{P_1 A_1}{1+\epsilon_1^2\left(1-b_1^2\right)}\left(e^{-kb_1x_3} - \frac{b_3+\epsilon_1 b_1}{2b_3}e^{-k\frac{b_3}{\epsilon_1}x_3}\right) + \frac{Q_1 A_2}{1+\epsilon_1^2\left(1-b_2^2\right)}\left(e^{-kb_2x_3} - \frac{b_3+\epsilon_1 b_2}{2b_3}e^{-k\frac{b_3}{\epsilon_1}x_3}\right)\right], \tag{26a}$$

$$T_{13}(x_3) = \mu\left[\frac{P_2 A_1}{1+\epsilon_1^2\left(1-b_1^2\right)}\left(e^{-kb_1x_3} - e^{-k\frac{b_3}{\epsilon_1}x_3}\right) + \frac{Q_2 A_2}{1+\epsilon_1^2\left(1-b_2^2\right)}\left(e^{-kb_2x_3} - e^{-k\frac{b_3}{\epsilon_1}x_3}\right)\right], \tag{26b}$$

$$T_{33}(x_3) = \mu\left[\frac{P_3 A_1}{1+\epsilon_1^2\left(1-b_1^2\right)}\left(e^{-kb_1x_3} - e^{-k\frac{b_3}{\epsilon_1}x_3}\right) + \frac{Q_3 A_2}{1+\epsilon_1^2\left(1-b_2^2\right)}\left(e^{-kb_2x_3} - e^{-k\frac{b_3}{\epsilon_1}x_3}\right)\right]. \tag{26c}$$

It can be easily verified that the non-local stress components (26) satisfy the natural boundary conditions (7), as well as the extra CBC (24) and the decay property at infinity. Indeed, they may be equally obtained through solving the differential problem (9) subject to the boundary conditions ((7), (24)) and to the decay condition at infinity, where σ_{ij} are given by (18) and the travelling wave assumption (13). In fact, this very approach is followed in Pham and Vu (2024), although assuming the CBC $t_{11} = 0$, in the belief that the integral approach cannot be pursued because it is ill-posed.

Most notably, the boundary conditions ((7), (24)) hold regardless of the arbitrary constants A_1, A_2. It is therefore no surprise that the motion equations are not satisfied, because indeed they provide the very conditions for the determination of such constants. The following argument motivates this result. It should be emphasized that the assumptions which were introduced for the attenuation functions α_{13} and α_{33}, and more specifically for the CBCs attached to them, namely Eqs. (22), are not enough no guarantee that the natural BCs (7) are indeed satisfied. This is because, appealing to the theory of Green's functions (Lanczos, 1996, §5.6), it is, in general,

$$\tilde{t}_{ij}(\boldsymbol{x}) = \int_{\mathbb{V}} \alpha_{ij}(\boldsymbol{x},\xi)\sigma_{ij}(\xi)\mathrm{d}\xi - \epsilon^2\int_{\partial\mathbb{V}}\left(\frac{\mathrm{d}\alpha_{ij}}{\mathrm{d}\boldsymbol{n}}\tilde{t}_{ij} - \frac{\mathrm{d}\tilde{t}_{ij}}{\mathrm{d}\boldsymbol{n}}\alpha_{ij}\right)\mathrm{d}\boldsymbol{\xi}, \tag{27}$$

where α_{ij} is the Green's function of the Helmholtz operator $\mathfrak{L}$ and $\boldsymbol{n}$ is the outward unit normal to the volume $\mathbb{V}$. Clearly, the first integral gives the volume contribution and amounts to Eringen's definition (25) of the local stresses t_{ij}. Obviously, the second term, which is a surface contribution, needs to disappear for the Eringen approach to be valid, i.e. for having $\tilde{t}_{ij} = t_{ij}$. This outcome only occurs when $\alpha_{ij} \equiv 0$ *and when* $\tilde{t}_{ij} \equiv 0$. Indeed, if this is not the case, Eringen's expression for the non-local stresses is incomplete and does not amount to the Green's function of the differential problem. In fact, this double requirement guarantees that $\tilde{t}_{ij} = t_{ij}$ and that the strain energy $\int_{\mathbb{V}} t_{ij}\epsilon_{ij}\mathrm{d}\boldsymbol{x}$ is a quadratic functional of strain. Yet, so far, we have only demanded that $\alpha_{ij}(x_1, 0, \xi_1, \xi_3) \equiv 0$, i.e. *a restriction has been cast upon the attenuation function*, but nothing has been required for $\tilde{t}_{ij}$ on the surface, and precisely this last constraint amounts to the motion equations being satisfied. In other words, the requirement that the motion equations are satisfied is a further condition, which needs to be enforced to obtain self-adjointness of the problem and, ultimately, to obtain the enforcement of the natural boundary conditions (7). It is therefore clear that the fact that the motion Eqs. (2) are not satisfied when plugging in the non-local stresses t_{ij} that were obtained from the differential formulation is in no way connected to the integral theory being ill-posed (which fact rests entirely on the requirement that the CBCs be a subset of the natural BCs). Instead, failure to satisfy the motion equation is a natural feature which emerges from the fact that the boundary conditions (7) have not yet been enforced properly on the system.

In a similar fashion, one may equally argue that the fact that the motion equations are not immediately satisfied amounts to the fact that $\tilde{t}_{ij}$ and t_{ij} are not equal, which means that the differential and the integral problem do not correspond. In this alternative interpretation, enforcing the fulfilment of the motion equations amounts to requiring that the integral and the differential formulations are equivalent.

A third argument may be given that explains why it is natural to expect that the solution (16) of the differential problem (11) fails to comply with the motion Eqs. (2). Indeed, the Navier Eqs. (11) enforce a stronger (in the sense of differentiation) form of equilibrium, namely Eqs. (10) as opposed to (2), which fact introduces a spurious part of the solution. In fact, the solution of the stronger form (10) differs from the solution of the motion Eqs. (2) by any homogeneous solution of the Helmholtz operator $\mathfrak{L}$, i.e. by its kernel set. Such spurious solution provides precisely the non-vanishing terms in the motion Eqs. (2) and one may easily see, either by direct application of the Helmholtz operator or by appealing to the theory of the Green's function (Lanczos, 1996), that this spurious solution may be written in the form given by the last integral in (27), whence we are back to the previous argument.

At this stage, some comments are in order. First, it is remarkable that, in the non-local theory, the boundary conditions take the form of (i.e. are enforced through) the motion equations: a situation that, to the best of our knowledge, has no counterparts in other frameworks. Second, so far only homogeneous conditions have been imposed on t_{13} and t_{33} (and equally on α_{13} and α_{33}), and yet more general conditions may be applied, in the form (12), without significantly affecting to the procedure. In other words, the appearance of homogeneous conditions is not so essential.

The non-local stress components (26) satisfy the governing equations of motion (2) if

$$\frac{2b_3^2 P_2 - \iota\epsilon_1\left(b_3+\epsilon_1 b_1\right)P_1}{1+\epsilon_1^2\left(1-b_1^2\right)}A_1 + \frac{2b_3^2 Q_2 - \iota\epsilon_1\left(b_3+\epsilon_1 b_2\right)Q_1}{1+\epsilon_1^2\left(1-b_2^2\right)}A_2 = 0, \tag{28a}$$

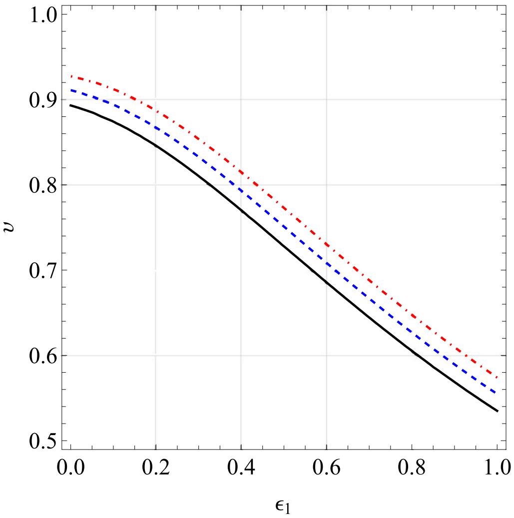

Fig. 1. Dispersion diagram of Rayleigh waves: Dimensionless phase velocity υ against the dimensionless non-local parameter $\epsilon_1 = \epsilon k$, for different values of the Poisson's ratio, namely $\nu = 0.1$ (black, solid), $\nu = 0.2$ (blue, dashed) and $\nu = 0.3$ (red, dot-dashed). It is emphasized that only the fundamental branch is shown, the presence of multiple branches being illustrated in Fig. 2.

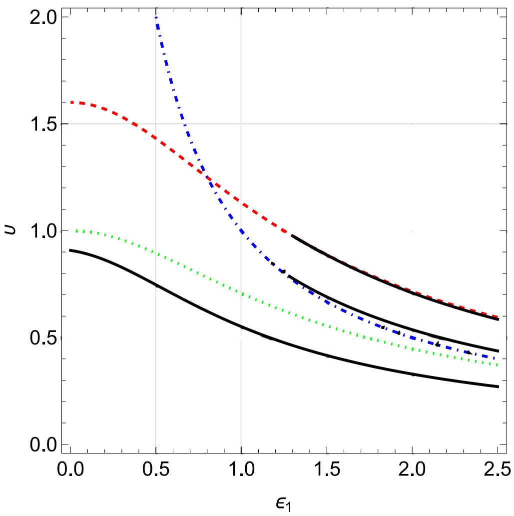

Fig. 2. Dispersion diagram of Rayleigh waves: Dimensionless phase velocity υ against the dimensionless non-local parameter $\epsilon_1 = \epsilon k$, for $\kappa = 1.6$. Multiple branches (black, solid) are shown against the bulk waves $b_1 = 0$ (red, dashed), $b_2 = 0$ (green, dotted) and $\upsilon = \epsilon_1^{-1}$ (blue, dot-dashed).

$$\frac{b_3 P_3 - \iota\epsilon_1 P_2}{1+\epsilon_1^2\left(1-b_1^2\right)} A_1 + \frac{b_3 Q_3 - \iota\epsilon_1 Q_2}{1+\epsilon_1^2\left(1-b_2^2\right)} A_2 = 0, \tag{28b}$$

which is a homogeneous algebraic system in the unknowns A_1 and A_2. To obtain non-trivial solutions of this system, the determinant of the coefficient matrix must be zero and this condition yields the dispersion relation for the propagation of elastic waves

$$2b_3^3\left(4b_1 + \iota P_3 Q_2\right) + 2\iota\epsilon_1 b_3^2\left(P_1 + P_3\right) + \iota\epsilon_1^2 b_3\left(P_1 Q_2 + 2b_1 P_1 + 2b_2 P_3 + 4\iota b_1\right) - \epsilon_1^3 b_1\left(4b_2 - \iota P_1 Q_2\right) = 0. \tag{29}$$

It is a straightforward matter to verify that, in the limit of the purely local elasticity, i.e. as $\epsilon \to 0$, the dispersion Eq. (29) reduces to

$$4\sqrt{1-\upsilon^2}\sqrt{1-\frac{\upsilon^2}{\kappa^2}} - \left(2-\upsilon^2\right)^2 = 0, \tag{30}$$

thus recovering the classical secular equation for Rayleigh waves. Here, we have used the connection $\kappa = \sqrt{2(1-\nu)/(1-2\nu)}$, where ν denotes Poisson's ratio.

Fig. 1 plots the (first) real solution of the dispersion relation (29) in terms of the phase velocity v as a function of the non-local parameter ϵ_1. It appears that the phase velocity v decreases with increasing ϵ_1, while it exhibits an increasing trend with the Poisson's ratio ν. Fig. 2 reveals that, as a result of non-locality, three dispersion branches are present for $\epsilon_1 > \epsilon_{crit}$ and that the two highest modes sit very close to the bulk modes $b_1 = 0$ and $b_2 = 0$. Indeed, we can say that, just like Rayleigh waves are perturbations of shear waves, similarly, the second and third branches are perturbations of the corresponding bulk waves. Interestingly, all branches are decaying depthwise and, unlike for the couple stress theory (Nobili et al., 2020), no depth-travelling modes are present. Indeed, it seems that this wave pattern is here observed for the first time.

3. Indeterminacy of the non-local theory

So far we have seen that, provided that the CBCs embedded in the attenuation functions α_{13} and α_{33} are reconciled with the natural boundary conditions (7), the non-local problem becomes well-posed and both the integral and the differential formulations lead to the same solution. We also saw that the extra CBC (24) is required to fix the attenuation function α_{11} for the non-local stress t_{11}, which does not emerge from standard mechanical considerations. We now discuss this deficiency in some more depth and show that it is a manifestation of the problem indeterminacy.

Let us consider a stress problem, wherein *all* boundary conditions are expressed in terms of the non-local stresses. For the sake of clarity, it is remarked that this is not the case of this wave propagation problem, for which only the BCs at $x_3 = 0$ qualify, while the decay condition at infinity is enforced on displacement. Clearly, in a plane problem, three non-local stresses are present, namely t_{11}, t_{13} and t_{33}, and for their determination we need three PDEs and three boundary conditions. However, as it is the case for the classical theory, only two PDEs and two boundary conditions stand, respectively the motion Eqs. (10) and the natural BCs, say (7). In the classical theory, this deficiency is remediated by the *compatibility condition*

$$(1-\nu)\left(\sigma_{11,33}+\sigma_{33,11}\right)-\nu\left(\sigma_{11,11}+\sigma_{33,33}\right)-2\sigma_{13,13}=0, \tag{31}$$

that, using the classical motion equations, takes the usual form

$$\triangle\left(\sigma_{11}+\sigma_{33}\right)=\frac{\rho}{1-\nu}\left(\ddot{u}_{1,1}+\ddot{u}_{3,3}\right). \tag{32}$$

In the non-local problem, compatibility still holds in the form (31) for the local stresses, whence, making use of (9), we get

$$\mathfrak{L}\left[(1-\nu)\left(t_{11,33}+t_{33,11}\right)-\nu\left(t_{11,11}+t_{33,33}\right)-2t_{13,13}\right]=0,$$

that is

$$(1-\nu)\left(t_{11,33}+t_{33,11}\right)-\nu\left(t_{11,11}+t_{33,33}\right)-2t_{13,13}=\bar{h}, \tag{33}$$

where $\bar{h}=\bar{h}(x_1,x_3,t)$ is *any* homogeneous solution of the operator $\mathfrak{L}$. Then, using the motion Eqs. (10), one arrives at

$$\triangle\left(t_{11}+t_{33}\right)-\frac{\rho}{1-\nu}\left(\ddot{u}_{1,1}+\ddot{u}_{3,3}\right)=h, \tag{34}$$

which replaces (33) of classical elasticity. In particular, for the Helmholtz operator, we can eliminate the Laplacian operator and obtain

$$t_{11}+t_{33}-\sigma_{11}-\sigma_{33}-\epsilon^2\frac{\rho}{1-\nu}\left(\ddot{u}_{1,1}+\ddot{u}_{3,3}\right)=\epsilon^2 h. \tag{35}$$

It is emphasized that the only condition standing on h is the requirement that it decays at infinity, because this emerges from the same requirement holding on t_{11}, t_{33}, u_1 and u_3 (note that this is an extra condition anyway). Other than this, h remains undetermined, hence the need for the extra condition (24). In the classical limit $\epsilon = 0$, h drops out from Eq. (35) that provides a connection for the determination of t_{11} (which we know to be equal to σ_{11}) once t_{33} is fixed, for example by the natural boundary conditions of the problem. This is no longer the case for the non-local theory, because, in general, t_{11} remains undefined, through the homogeneous solution h of the Helmholtz operator, once t_{33}, u_1 and u_3 are defined. As it was already observed, adding a solution of the homogeneous equation $\mathfrak{L}h = 0$ amounts to the modification of the CBC for t_{11}. In other words, for Eringen's non-local elasticity to be *determinate*, in either integral or differential form, an extra boundary condition needs to be (freely) prescribed on the non-local stresses. Conversely, to achieve well-posedness, the attenuation functions draw their CBCs from the natural BCs of the problem.

4. SH-waves in a non-local half-space

Let us consider SH-waves that propagate in the x_1 direction and decay in the x_3 direction of the elastic half-plane $\mathbb{V} = \left\{(x_1,x_3) : x_3 \geq 0\right\}$. For out-of-plane motions, the only nonzero displacement component is

$$u_2 = u_2(x_1,x_3,t), \tag{36}$$

and the local stress components $\sigma_{2j}(j=1,3)$ follow in the usual way

$$\sigma_{2j} = \mu u_{2,j}, \quad j = 1,3. \tag{37}$$

The equation of motion for SH-wave propagation reads

$$t_{21,1}+t_{23,3}=\rho\ddot{u}_2, \tag{38}$$

subjected to a single traction-free boundary condition

$$t_{23} = 0, \text{ at } x_3 = 0. \tag{39}$$

The nonzero non-local stress components $t_{2j} (j = 1, 3)$ are obtained through the usual integral relation (3) from the relevant local stress σ_{2j}. In particular, we set

$$t_{21}(x_1, x_3, t) = \int_0^\infty \mathrm{d}\xi_3 \int_{-\infty}^\infty G_1\left(x_1, x_3; \xi_1, \xi_3\right) \sigma_{21}(\xi_1, \xi_3, t)\, \mathrm{d}\xi_1, \tag{40a}$$

$$t_{23}(x_1, x_3, t) = \int_0^\infty \mathrm{d}\xi_3 \int_{-\infty}^\infty G_2\left(x_1, x_3; \xi_1, \xi_3\right) \sigma_{23}(\xi_1, \xi_3, t)\, \mathrm{d}\xi_1, \tag{40b}$$

where the attenuation function $\alpha_{23} = G_2$ is chosen as the difference of two Bessel functions so that the associated CBC corresponds to the equilibrium boundary condition (39). In contrast, we are free to choose the attenuation function α_{21} and we decide, as before, to stick with the single Bessel function G_1. The integral definition (40) admits the equivalent differential formulation

$$\mathfrak{L} t_{2j} = \sigma_{2j}, \quad j = 1, 3, \tag{41}$$

provided that this is accompanied by the boundary condition (39) and by the constitutive boundary condition (cf. Eq. (A.16))

$$t_{21,3} - \epsilon^{-1} b_3\, t_{21} = 0, \text{ at } x_3 = 0. \tag{42}$$

Upon applying the operator $\mathfrak{L} = 1 - \epsilon^2 \triangle$ to the equation of motion (38) and making use of Eqs. (41), we obtain the equation of motion in terms of the local stresses

$$\sigma_{21,1} + \sigma_{23,3} = \rho \ddot{u}_2 - \epsilon^2 \rho \left(\ddot{u}_{2,11} + \ddot{u}_{2,33}\right). \tag{43}$$

As it was already observed, this step is bound to introduce spurious solutions corresponding to the kernel of the operator $\mathfrak{L}$. Consequently, appealing to the stress–displacement relation (37), we retrieve the fully differential formulation of the problem

$$\mu \left(u_{2,11} + u_{2,33}\right) = \rho \ddot{u}_2 - \epsilon^2 \rho \left(\ddot{u}_{2,11} + \ddot{u}_{2,33}\right), \tag{44}$$

whose solutions, however, need to be confronted with the original equation of motion (38) to rule out the spurious contributions.

Consideration of harmonic waves travelling along the x_1 direction yields

$$\left\{u_2, \sigma_{2j}, t_{2j}\right\}(x_1, x_3, t) = \left\{U_2, S_{2j}, T_{2j}\right\}(x_3)\, e^{\imath k (x_1 - ct)}, \quad j = 1, 3, \tag{45}$$

which, plugged into (44) and accounting for decay along x_3, provides

$$U_2(x_3) = k^{-1} A_3 e^{-k b_2 x_3}, \tag{46}$$

where A_3 is a dimensionless arbitrary constant to be determined by enforcing the boundary conditions. Eq. (37), together with Eqs. ((45),(46)), lends the amplitude of the local stress components

$$S_{21}(x_3) = \imath \mu A_3 e^{-k b_2 x_3}, \quad S_{23}(x_3) = -\mu b_2 A_3 e^{-k b_2 x_3}. \tag{47}$$

from which the amplitude of the non-local stress components can be immediately obtained by integration according to the definition (40), i.e.

$$T_{21}(x_3) = \frac{\imath \mu A_3}{1 + \epsilon_1^2 \left(1 - b_2^2\right)} \left(e^{-k b_2 x_3} - \frac{b_3 + \epsilon_1 b_2}{2 b_3} e^{-k \frac{b_3}{\epsilon_1} x_3}\right), \tag{48a}$$

$$T_{23}(x_3) = -\frac{\mu b_2 A_3}{1 + \epsilon_1^2 \left(1 - b_2^2\right)} \left(e^{-k b_2 x_3} - e^{-k \frac{b_3}{\epsilon_1} x_3}\right). \tag{48b}$$

It can be easily verified that the non-local stress components defined by Eqs. (48) satisfy the BC (39) as well as the CBC (42). In fact, they may be equally obtained from the differential formulation (41) supplemented by the CBC (42) and the BC (39). Notably, such boundary conditions hold regardless of the constant A_3, which can only be determined from the original equation of motion (38) that gives the algebraic relation

$$\frac{b_2 b_3 - \epsilon_1 \frac{b_3 + \epsilon_1 b_2}{2 b_3}}{1 + \epsilon_1^2 \left(1 - b_2^2\right)} A_3 = 0. \tag{49}$$

Thus, looking for non-trivial solutions of this equation, we obtain the dispersion relation of SH-waves, namely

$$2 b_2 (1 + \epsilon_1^2) - \epsilon_1 \left(b_3 + \epsilon_1 b_2\right) = 0, \tag{50}$$

according to which the dimensionless phase velocity ($v = c/c_2$) immediately follows

$$v^2 = \frac{1 + \frac{3}{4}\epsilon_1^2}{\left(1 + \epsilon_1^2\right)^2}. \tag{51}$$

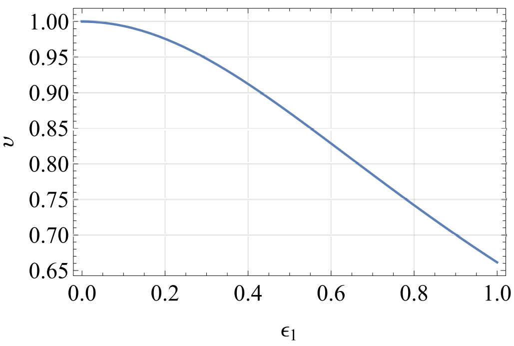

Fig. 3. Dispersion diagram of SH localized waves: Dimensionless phase velocity v against the dimensionless non-local parameter $\epsilon_1 = \epsilon k$, where k is the wavenumber.

It is straightforward to see that, in the limit of purely local elasticity, i.e., as $\epsilon \to 0$ (equivalently $\epsilon_1 \to 0$), the dispersion Eq. (51) simplifies to $c = c_2$, indicating the absence of SH localized waves propagating in a traction-free local elastic half-space. However, in the presence of non-local effects, the surface SH-wave becomes dispersive, with its phase velocity decreasing while the non-local parameter ϵ_1 increases, as illustrated in Fig. 3.

5. Conclusions

We study Rayleigh and shear horizontal (SH) localized waves in an elastic half-plane within the integral form of the theory of non-local elasticity. Since this theory is generally over-determined, as a result of the presence of implicit boundary conditions, named constitutive, which are embedded in the attenuation functions, a kernel modification procedure is adopted. By this procedure, already introduced by the authors for 1D non-local problems in Nobili and Pramanik (2025), constitutive boundary conditions (CBCs) are made to coincide with the natural boundary conditions (BCs) of the problem, while the symmetric nature of the kernel is preserved to guarantee a quadratic strain energy density. Once the problem is thus made well-posed, it may be solved by adopting the differential formulation and then plugging the result into the integral expressions to determine the non-local stresses. This last step is possible, in closed form, in light of an integral representation for the Bessel function, that is presented in Appendix A. The thus obtained expressions for the non-local stresses fail to comply with the motion equations, a result already highlighted by Kaplunov et al. (2022) in asymptotic form for the single Bessel function kernel and there interpreted as a further proof that the integral formulation "appears to be ill-posed" (Kaplunov et al., 2023). Instead, we show that this outcome is not only expected but indeed necessary to the full determination of the solution. Besides, the solution corresponds to what can be obtained following a completely differential approach, as in Pham and Vu (2024), thus demonstrating that the integral and the differential formulations are indeed equivalent. Finally, we show that the non-local theory, provided that it is made consistent by kernel modification, is also *indeterminate*, in the sense that one CBC can be set arbitrarily. This result comes as a consequence of the lack of a unique set of compatibility conditions for the non-local stresses (because compatibility only stands for *local* stresses). This last feature has no counterpart in reduced 1D models, such as beams. Discussion of the dispersion relations for Rayleigh and SH waves completes the paper and highlights several solution branches which seem to have gone unnoticed.

CRediT authorship contribution statement

Andrea Nobili: Writing – review & editing, Supervision, Software, Investigation, Funding acquisition, Conceptualization. **Dipendu Pramanik:** Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Green functions and corresponding CBCs

In this section, we present the construction of the Green's function for the differential operator $\mathfrak{L} \equiv 1 - \epsilon^2 \Delta$ and derive the corresponding constitutive boundary conditions (CBCs) for harmonic plane waves in the $x_1 - x_3$ plane.

To get the Green's function of the Helmholtz operator in the plane we look for solutions of

$$\mathfrak{L}G(x_1, x_3) = \delta(x_1)\delta(x_3), \tag{A.1}$$

that decay at infinity. Since the problem is even, we may apply the cosine Fourier transform in both spatial directions, for example

$$\bar{G}(s, x_3) = \frac{2}{\pi} \int_0^\infty G(x_1, x_3) \cos(s x_1) \mathrm{d}x_1, \tag{A.2}$$

whose inverse transform is

$$G(x_1, x_3) = \int_0^\infty \bar{G}(s, x_3) \cos(s x_1) \mathrm{d}s. \tag{A.3}$$

A simple calculation shows that

$$G(x_1, x_3) = \frac{1}{\pi^2 \epsilon^2} \iint_0^\infty \frac{\cos(s\epsilon^{-1} x_1)\cos(p\epsilon^{-1} x_3)}{1 + s^2 + p^2} \mathrm{d}s\mathrm{d}p, \tag{A.4}$$

which is even and symmetric in x_1 and x_3. It is possible to perform one integration, although this inevitably destroys the symmetry of the expression

$$G(x_1, x_3) = \frac{1}{2\pi\epsilon^2} \int_0^\infty \frac{\cos(s\epsilon^{-1} x_1)\exp(-\sqrt{1+s^2}\epsilon^{-1}|x_3|)}{\sqrt{1+s^2}} \mathrm{d}s, \tag{A.5}$$

where the absolute value has been introduced to preserve the even character of the function. In particular, letting $x_3 = 0$, one retrieves the formula (Abramowitz & Stegun, 1948, §9.6.21). Equivalently, one may write

$$G_1(x_1, x_3; \xi_1, \xi_3) = \frac{1}{4\pi^2\epsilon^2} \iint_{-\infty}^\infty \frac{\exp\left[-\imath\left\{\eta_1\epsilon^{-1}(x_1 - \xi_1) + \eta_3\epsilon^{-1}(x_3 - \xi_3)\right\}\right]}{1 + \eta_1^2 + \eta_3^2} \mathrm{d}\eta_1\mathrm{d}\eta_3, \tag{A.6}$$

which can be written after performing the integration with respect to η_1 as

$$G_1(x_1, x_3; \xi_1, \xi_3) = \frac{1}{4\pi\epsilon^2} \int_{-\infty}^\infty \exp\left[-\frac{|x_1 - \xi_1|}{\epsilon}\sqrt{1+\eta_3^2}\right] \frac{\exp\left[-\imath\eta_3\epsilon^{-1}(x_3 - \xi_3)\right]}{\sqrt{1+\eta_3^2}} \mathrm{d}\eta_3. \tag{A.7}$$

Thus, after some manipulations

$$G_1(x_1, x_3; \xi_1, \xi_3) = \frac{1}{2\pi\epsilon^2} \int_0^\infty \exp\left[-\frac{\sqrt{(x_1 - \xi_1)^2 + (x_3 - \xi_3)^2}}{\epsilon}\cosh(z)\right] \mathrm{d}z, \tag{A.8}$$

that corresponds to the closed analytical expression given in Olver et al. (2010, §10.32) as

$$G_1(x_1, x_3; \xi_1, \xi_3) = \frac{1}{2\pi\epsilon^2} K_0\left(\frac{\sqrt{(x_1 - \xi_1)^2 + (x_3 - \xi_3)^2}}{\epsilon}\right). \tag{A.9}$$

Now, we proceed to determine the CBC for the Green's function $G_1(x_1, x_3; \xi_1, \xi_3)$ in the semi-infinite domain $x_3 \geq 0$ for the problem

$$T(x_1, x_3) = \int_0^\infty \mathrm{d}\xi_3 \int_{-\infty}^\infty G_1(x_1, x_3; \xi_1, \xi_3)\, \sigma(\xi_1, \xi_3)\, \mathrm{d}\xi_1. \tag{A.10}$$

Using the integral representation of the Green's function defined in Eq. (A.6), we have

$$T(x_1, x_3) = \frac{1}{4\pi^2\epsilon^2} \int_0^\infty \mathrm{d}\xi_3 \int_{-\infty}^\infty \mathrm{d}\xi_1 \int_{-\infty}^\infty \mathrm{d}\eta_3 \int_{-\infty}^\infty \frac{e^{-\imath\left\{\eta_1\epsilon^{-1}(x_1-\xi_1)+\eta_3\epsilon^{-1}(x_3-\xi_3)\right\}}}{1 + \eta_1^2 + \eta_3^2} \sigma(\xi_1, \xi_3)\mathrm{d}\eta_1. \tag{A.11}$$

To obtain a CBC of the form defined in Eq. (12), we set up the integral and perform the integration with respect to η_1 as follows

$$\frac{\partial T(x_1, x_3)}{\partial x_3} + \alpha\, T(x_1, x_3) = \frac{1}{4\pi\epsilon^2} \int_0^\infty \mathrm{d}\xi_3 \int_{-\infty}^\infty \mathrm{d}\eta_3 \int_{-\infty}^\infty \frac{\alpha - \imath\epsilon^{-1}\eta_3}{\sqrt{1+\eta_3^2}} e^{-\frac{|x_1-\xi_1|}{\epsilon}\sqrt{1+\eta_3^2} - \imath\eta_3\epsilon^{-1}(x_3-\xi_3)} \sigma(\xi_1, \xi_3)\mathrm{d}\xi_1, \tag{A.12}$$

where α is a constant to be determined. Consideration of harmonic waves propagating in the x_1 direction yields

$$\sigma(x_1, x_3) = S(x_3)\exp(\imath k x_1), \tag{A.13}$$

whence, after performing the integration with respect to ξ_1, Eq. (A.12) becomes

$$\frac{\partial T(x_1, x_3)}{\partial x_3} + \alpha\, T(x_1, x_3)$$

$$= \frac{1}{2\pi\epsilon}\exp(\imath k x_1)\int_0^{\infty} d\xi_3 \int_{-\infty}^{\infty} \frac{\alpha - \imath\epsilon^{-1}\eta_3}{\eta_3^2 + b_3^2}\exp\left\{-\imath\eta_3\epsilon^{-1}(x_3-\xi_3)\right\} S(\xi_3) d\eta_3. \tag{A.14}$$

Finally, integrating with respect to η_3, we have

$$\frac{\partial T(x_1,x_3)}{\partial x_3} + \alpha T(x_1,x_3) = \frac{1}{2\epsilon}\exp(\imath k x_1)\int_0^{\infty} \frac{\alpha - \mathrm{sgn}(x_3-\xi_3)\epsilon^{-1}b_3}{b_3}\exp\left\{-\frac{|x_3-\xi_3|}{\epsilon}b_3\right\} d\xi_3, \tag{A.15}$$

where $sgn(x)$ is the *sign function*. Thus, it is straightforward to see that, at $x_3 = 0$, the above integral evaluates to zero for $\alpha = -\epsilon^{-1}b_3$. This implies that the CBC for the case of harmonic plane waves and for the Green's function G_1, is given by

$$\frac{\partial T(x_1,x_3)}{\partial x_3} - \epsilon^{-1}b_3 T(x_1,x_3) = 0, \text{ at } x_3 = 0. \tag{A.16}$$

An alternative approach to derive the constitutive boundary conditions (CBCs) is to analyse the boundary term arising in the integral representation of the non-local stress based on Green's function theory. From Eq. (27), the boundary term is given by

$$\mathfrak{BT} = \int_{\partial\mathbb{V}}\left(\frac{d\alpha_{ij}}{d\boldsymbol{n}}\tilde{t}_{ij} - \frac{d\tilde{t}_{ij}}{d\boldsymbol{n}}\alpha_{ij}\right) d\boldsymbol{\xi}, \tag{A.17}$$

where the $\tilde{t}_{ij}$ must satisfy appropriate boundary conditions to vanish the boundary term for a self-adjoint boundary value problem and ensure the symmetry of the associated Green's function.

For a half-space domain $x_3 \geq 0$, subject to decay conditions at infinity, the boundary term in Eq. (A.17) reduces to

$$\mathfrak{BT} = \int_{-\infty}^{\infty}\left(\frac{d\alpha_{ij}}{d\xi_3}(x_1,x_3;\xi_1,0)\,\tilde{t}_{ij}(\xi_1,0) - \frac{d\tilde{t}_{ij}}{d\xi_3}(\xi_1,0)\,\alpha_{ij}(x_1,x_3;\xi_1,0)\right) d\xi_1. \tag{A.18}$$

Now, for a specific case where $\alpha_{ij} = G_1(x_1,x_3;\xi_1,\xi_3)$ (cf. Eq. (A.7)), the boundary term can be written as

$$\mathfrak{BT} = \frac{1}{4\pi\epsilon^2}\iint_{-\infty}^{\infty} e^{-\frac{|x_1-\xi_1|}{\epsilon}\sqrt{1+\eta_3^2}} e^{-\imath\eta_3\epsilon^{-1}x_3}\frac{\imath\epsilon^{-1}\eta_3\tilde{t}_{ij}(\xi_1,0) - \frac{\partial\tilde{t}_{ij}}{\partial\xi_3}(\xi_1,0)}{\sqrt{1+\eta_3^2}} d\xi_1 d\eta_3. \tag{A.19}$$

For a time-harmonic formulation, assuming $\tilde{t}_{ij}(x_1,x_3) = \tilde{T}(x_3)e^{\imath k x_1}$, the boundary term simplifies to

$$\mathfrak{BT} = \frac{1}{4\pi\epsilon^2}\int_{-\infty}^{\infty} e^{-\imath\eta_3\epsilon^{-1}x_3}\frac{\imath\epsilon^{-1}\eta_3\tilde{T}(0) - \tilde{T}'(0)}{\sqrt{1+\eta_3^2}}\left[\int_{-\infty}^{\infty} e^{-\frac{|x_1-\xi_1|}{\epsilon}\sqrt{1+\eta_3^2}} e^{\imath k\xi_1} d\xi_1\right] d\eta_3. \tag{A.20}$$

After performing the integrals over ξ_1 and η_3, we obtain

$$\mathfrak{BT} = \frac{1}{2\epsilon}\frac{\epsilon^{-1}b_3\tilde{T}(0) - \tilde{T}'(0)}{b_3} e^{-\epsilon^{-1}b_3x_3} e^{\imath k x_1}. \tag{A.21}$$

For the boundary term $\mathfrak{BT}$ to vanish, the function $\tilde{T}(x_3)$, representing the amplitude of the harmonic non-local stress, must satisfy the boundary condition

$$\tilde{T}'(0) - \epsilon^{-1}b_3\tilde{T}(0) = 0. \tag{A.22}$$

This provides the CBC for the Green's function $\alpha_{ij} = G_1(x_1,x_3;\xi_1,\xi_3)$ for the non-local representation (27) with no boundary integral.

Appendix B. Integration of the non-local stress components

In this Section, we carry out the integration of the local stress components to get the non-local stress components according to the integral representation (25). For the sake of definiteness, we provide the derivation of $t_{11}(x_1,x_3)$ by making use of the attenuation function $\alpha_{11} = G_1$ and accounting for the travelling wave assumption (13). Consequently, we need to calculate

$$T_{11}(x_3)e^{\imath k x_1} = \int_0^{\infty} d\xi_3 \int_{-\infty}^{\infty} G_1\left(x_1,x_3;\xi_1,\xi_3\right) S_{11}(\xi_3)e^{\imath k\xi_1} d\xi_1, \tag{B.1}$$

that, by the integral representation (A.7) for $G_1(x_1,x_3;\xi_1,\xi_3)$ and in view of the exponential form of the local stress $S_{11} = Pe^{-kb_1x_3}$, becomes

$$T_{11}(x_3)e^{\imath k x_1} = \frac{P}{4\pi\epsilon^2}\int_0^{\infty} d\xi_3\int_{-\infty}^{\infty} d\eta_3\int_{-\infty}^{\infty}\frac{e^{-\frac{|x_1-\xi_1|}{\epsilon}\sqrt{1+\eta_3^2}}}{\sqrt{1+\eta_3^2}} e^{-\imath\eta_3\epsilon^{-1}(x_3-\xi_3)} e^{-kb_1\xi_3} e^{\imath k\xi_1} d\xi_1. \tag{B.2}$$

Upon performing the integration in the specific order ξ_1, η_3 and ξ_3, one gets, successively,

$$T_{11}(x_3) = \frac{P}{2\pi\epsilon}\int_0^{\infty} d\xi_3\int_{-\infty}^{\infty}\frac{e^{-\imath\eta_3\epsilon^{-1}(x_3-\xi_3)}}{\eta_3^2+b_3^2} e^{-kb_1\xi_3} d\eta_3, \tag{B.3}$$

$$T_{11}(x_3) = \frac{P}{2\epsilon}\int_0^{\infty}\frac{e^{-\frac{|x_3-\xi_3|}{\epsilon}b_3}}{b_3} e^{-kb_1\xi_3} d\xi_3, \tag{B.4}$$

and finally

$$T_{11}(x_3) = \frac{P}{1+\epsilon_1^2\left(1-b_1^2\right)}\left(e^{-kb_1x_3} - \frac{b_3+\epsilon_1 b_1}{2b_3}e^{-k\frac{b_3}{\epsilon_1}x_3}\right).$$

Appendix C. Nomenclature

A_1, A_2, A_3	Arbitrary integrating constants	$\boldsymbol{x}$, x_1, x_3	Field point vector and its coordinates
b_1, b_2	Wavenumber for bulk waves	$\mathbb{R}^3$	3D Euclidean space
c	Phase velocity	$\mathbb{V}$, $\mathbb{S}$	Elastic half-plane and its boundary
c_1, c_2	Longitudinal and shear wave velocity	$\mathfrak{L}$	Helmholtz operator
C_{ijkl}	Elastic constants	α_{ij}	Attenuation functions
e_{ij}	Linear deformation components	δ	Dirac's delta
G, G_1	Kernel functions	δ_{ij}	Kronecker's symbol
h, $\bar{h}$	Homogeneous solutions of the operator $\mathfrak{L}$	$\triangle$	2D Laplace operator
$\imath$	Imaginary unit	ϵ	Non-local internal length parameter
k	Wavenumber	κ	Velocity ratio
K_0	Zero-th order modified Bessel function of the second kind	λ, μ	Lamé parameters
$\boldsymbol{n}$	Unit normal vector	ν	Poisson's ratio
t_{ij}, $\tilde{t}_{ij}$	Non-local stress components	$\boldsymbol{\xi}$, ξ_1, ξ_3	Source point vector and its components
t	time variable	ρ	mass density
u_1, u_2, u_3	Particle displacement components	σ_{ij}	Local stress components
U_i, S_{ij}, T_{ij}	Amplitude of the harmonic wave displacement, local and non-local stress components	υ	Dimensionless phase velocity

Data availability

No data was used for the research described in the article.

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