LOCAL LIPSCHITZ CONTINUITY FOR ENERGY INTEGRALS WITH SLOW GROWTH

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ABSTRACT. We consider some energy integrals under slow growth and we prove that the local minimizers are locally Lipschitz continuous. Many examples are given, either with subquadratic $p,q-$growth and/or anisotropic growth.

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1. PROLOGUE

When concerned with the $W^{1,\infty}$ or $C^{1,\alpha}$ regularity of local minimizers of energy integrals of the calculus of variations of the type

$$F(u)=\int_{\Omega} f\left(Du\left(x\right)\right)\,dx \tag{1.1}$$

we are naturally led to require a *qualified convexity condition* on the energy integrand $f:\mathbb{R}^n\to\mathbb{R}$; more precisely, on the quadratic form of the $n\times n$ matrix of the second derivatives $D^2f=\left(f_{\xi_i\xi_j}\right)$ of f

$$g_1\left(|\xi|\right)|\lambda|^2\leq\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(\xi\right)\lambda_i\lambda_j\leq g_2\left(|\xi|\right)|\lambda|^2\,,\quad\forall\;\lambda,\xi\in\mathbb{R}^n, \tag{1.2}$$

where $g_1,g_2:[0,+\infty)\to[0,+\infty)$ are given nonnegative real functions which allow us to control the *ellipticity* in the minimization problem. Of course $g_1\left(t\right)\leq g_2\left(t\right)$ for all $t\in[0,+\infty)$; if g_1 is positive and there exists a constant $M\geq 1$ such that $g_2\left(t\right)\leq Mg_1\left(t\right)$ for all $t\in[0,+\infty)$ then we say that the we are dealing with an *uniformly elliptic* problem. This is the case when the quadratic form $\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(\xi\right)\lambda_i\lambda_j$ has a simpler equivalent behavior as $g_1\left(|\xi|\right)|\lambda|^2$ and $g_2\left(|\xi|\right)|\lambda|^2$ and the regularity process can work easier. However the assumption $g_2\leq Mg_1$ rules out many interesting energy integrals; in this paper we do not assume this uniformly elliptic condition.

For instance, in the special case $f\left(\xi\right)=g\left(|\xi|\right)$ with $g:[0,+\infty)\to\mathbb{R}$, a direct computation (see for instance (6.3) in [31] and [34]) shows that

$$g_1\left(t\right)=\min\left\{g''\left(t\right),\frac{g'\left(t\right)}{t}\right\},\quad g_2\left(t\right)=\max\left\{g''\left(t\right),\frac{g'\left(t\right)}{t}\right\},\quad\forall\;t\in[0,+\infty)\,,$$

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where g', g'' are the first and the second derivatives of g. In this context the *uniformly elliptic* case corresponds to compare $g''(t)$ and $\frac{g'(t)}{t}$; i.e. to ask for the existence of two positive constants m, M such that $m\frac{g'(t)}{t} \leq g''(t) \leq M\frac{g'(t)}{t}$ for all $t \in (0, +\infty)$. The $p-$Laplacian $f(\xi) = |\xi|^p$ with $p > 1$ is a main example of *uniformly elliptic energy integrand*, with $g(t) = t^p$ and $\frac{g'(t)}{t} = \frac{1}{p-1}g''(t) = pt^{p-2}$. Within this uniformly elliptic context - however nonlinearities of possibly non-polynomial type are allowed - we quote the global (i.e., up to the boundary) Lipschitz regularity results by Cianchi-Maz'ya for a class of quasilinear elliptic equations [9] and for a class of nonlinear elliptic systems [10].

Also some energy integrands of $p, q-$*growth* can be uniformly elliptic; for instance an integrand, which does not behave like a power, but which however is an uniformly elliptic energy integrand, is $f(\xi) = |\xi|^{a+b\sin(\log\log|\xi|)}$; in this case $f(\xi)$ is a convex function for $|\xi| \geq e$ if $a, b > 0$ and $a > 1 + b\sqrt{2}$. This function f satisfies the $p, q-$*growth conditions* with $p = a - b$ and $q = a + b$. It can be shown (see [4],[5]) that $f(\xi)$ satisfies the Δ_2-condition. To notice however that some convex functions $f(\xi) = g(|\xi|)$ of $p, q-$growth with $p > 1$ and $q > p$ arbitrarily close to p exist, they do not satisfy the Δ_2-condition and the corresponding variational problem are not uniformly elliptic; see Krasnosel'skij-Rutickii [27, p. 28–29], Focardi-Mascolo [22, p. 342–343], Chlebicka [8, Section 2.4] and Bögelein-Duzaar-Marcellini-Scheven [5, Remark 3.3].

In this research we are concerned with the $W^{1,\infty}$ regularity of the local minimizers of energy integrals of the calculus of variations of the type (1.1), when the quadratic form of the second derivatives $D^2 f = \left(f_{\xi_i\xi_j}\right)$ of f is governed by (1.2) where $g_1, g_2 : [0, +\infty) \to [0, +\infty)$ are given nonnegative real functions, not only of polynomial type. The first local Lipschitz-continuity results under this general context has been proposed in the '90s in [31],[32] by assuming, among other conditions, that $g_1(t), g_2(t)$ are increasing functions in $[0, +\infty)$. This, when typified by the model case $g_1(t) = t^{p-2}, g_2(t) = t^{q-2}$, gives $q \geq p \geq 2$. The approach to regularity, governed by (1.2) with general g_1, g_2 functions not necessarily monotone functions, was given by Marcellini-Papi [34]. Related regularity results, with energy-integrands $f(x, \xi) = g(x, |\xi|)$ depending on x too, are due to Mascolo-Migliorini [35], Beck-Mingione [2], Di Marco-Marcellini [18], De Filippis-Mingione [16]. See also Apushkinskaya-Bildhauer-Fuchs [1] for a local gradient bound of a-priori bounded minimizers.

More precisely, in [2] Beck-Mingione consider the vector-valued case of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f(x, \xi) = g(|\xi|) + h(x)u$ with the main part $g(|\xi|)$ modulus dependent, as well as in [35],[18],[16]; they also study the scalar case $m = 1$ with the more general integrand not modulus dependent, i.e. of the form $f(x, \xi) = g(\xi) + h(x)u$, however with a growth assumption from below of power type for some fixed exponent greater than 1 (see (1.33) in [2]).

Here we focus our attention to *slow growth* integrands, for which the state-of-art is not so established. We give a general local $W^{1,\infty}$ regularity result for the minimizers of energy integrals of the type (1.1),(1.2) with an energy integrand $f = f(\xi)$ not necessarily depending on the modulus of ξ and with $g_1, g_2 : [0, +\infty) \to [0, +\infty)$ nonnegative decreasing real functions (more precisely we require that only g_2 is a decreasing function), not only of polynomial type. Precise statements can be found in the next section. We treat *general slow growth conditions* under the ellipticity condition (1.2), where $g_2(t)$ is a *decreasing* (not necessarily strictly decreasing) function with respect to t; of course in the model case $g_2(t) = M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}$ this corresponds to $q \leq 2$. As already said, in this article we do not assume uniformly elliptic conditions, nor the modulus dependence as $f(\xi) = g(|\xi|)$.

In this regularity field specific references for *slow growth* are Fuchs-Mingione [23], who concentrated on the *nearly-linear growth*, such as for instance the *logarithmic case* $f(\xi) = g(|\xi|) = |\xi|\log(1+|\xi|)$; also Bildhauer, in his book [3], considered *nearly-linear growth.* Leonetti-Mascolo-Siepe [28] considered the *subquadratic* $p,q-$*growth* with $1 < p < q < 2$, with energy densities for instance the type $f(\xi) = g(|\xi|) = |\xi|^p \log^{\alpha}(1+|\xi|)$.

Here we emphasize some examples which enter in our regularity theory and which seem not to be considered in the mathematical literature on this subject. The first one is complementary to the case considered by Bousquet-Brasco [6] for exponents $p_i \geq 2$ for all $i = 1, 2, \ldots, n$; in fact here we can treat the model energy-integral (see Example 3.3)

$$F_1(u) = \int_{\Omega} \sum_{i=1}^{n} \left(1 + u_{x_i}^2\right)^{\frac{p_i}{2}} \, dx \tag{1.3}$$

when $1 < p_i \leq 2$ for all $i = 1, 2, \ldots, n$. In Section 4 we propose some further examples of anisotropic energy integrands which seem to be new in the mathematical literature on this subject.

The mail regularity result that we propose in this manuscript is Theorem 2.1 stated in the next section. It gives a more general regularity result than similar results that can be found in the recent mathematical literature on $p,q-$growth; see in particular the Remark 4.3 for details. Also the integral $\int_{\Omega} |Du| \log^a(1+|Du|)\, dx$, for every $a > 0$, enters in the regularity result of Theorem 2.1. Of course a by-product of our general Theorem 2.1 is also the $p,q-$growth case, when the ellipticity conditions (1.2) are satisfied with $g_1(|\xi|) = m\, |\xi|^{p-2}$, $g_2(|\xi|) = M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}$, for some positive constants m, M and exponents $1 < p \leq q \leq 2$ such that $\frac{q}{p} < 1 + \frac{2}{n}$. As well known, this condition guarantees the Lipschitz continuity of the solutions also when $q \geq p > 1$ and classically this is nowadays a well known constraint for the $p,q-$growth (see [30],[32],[33]).

The regularity results are stated in the next section, while in Sections 3 and 4 some examples are considered in more details. The other sections are devoted to the proofs.

2. Introduction and statement of the main results

We assume that $f : \mathbb{R}^n \to [0, +\infty)$ is a convex function in $\mathcal{C}(\mathbb{R}^n) \cap \mathcal{C}^2(\mathbb{R}^n \backslash B_{t_0}(0))$ for some $t_0 \geq 0$, satisfying the following growth condition: there exist two continuous functions

$g_1, g_2 : [t_0\,, +\infty) \to (0\,, +\infty)$ and positive constants C_1, C_2, α, β and $\mu \in [0\,, 1]$ such that

$$\begin{cases} g_1\left(|\xi|\right)|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(\xi\right)\lambda_i\lambda_j \leq g_2\left(|\xi|\right)|\lambda|^2\ , \quad \forall\,\lambda,\xi\in\mathbb{R}^n,\ |\xi|\geq t_0 \\ t\mapsto t^{\mu}g_2(t) \text{ is decreasing and } t\mapsto tg_2(t) \text{ is increasing} \\ \left(g_2(t)\right)^{\frac{n-2}{n}} \leq C_1 t^{2\beta}g_1(t), \quad \frac{1}{n}<\beta<\frac{2}{n},\ \forall\,t\geq t_0 \\ g_2(|\xi|)|\xi|^2 \leq C_2\left[1+f(\xi)\right]^{\alpha}, \quad \alpha>1,\ \forall\,\xi\in\mathbb{R}^n,\ |\xi|\geq t_0 \\ f(\xi)/\,|\xi|\to+\infty \quad \text{as } |\xi|\to\infty \end{cases} \tag{2.1}$$

where $\frac{n-2}{n}$ in $(2.1)_3$, in the case $n = 2$, must be replaced with any fixed positive number less than $1 - \beta$.

It is worth to highlight that we require uniform convexity and growth assumptions on $f = f\,(\xi)$ only for large value of $|\xi|$ ([7],[19],[20],[21]). We say that $u \in W^{1,1}_{loc}(\Omega)$ is a *local minimizer* of the integral functional F in (1.1) if $f\,(Du) \in L^1_{loc}(\Omega)$ and

$$\int_{\Omega'} f(Du)\,dx \leq \int_{\Omega'} f(Du + D\varphi)\,dx$$

for every open set Ω', $\overline{\Omega'} \subset \Omega$ and for every $\varphi \in W^{1,1}_0(\Omega')$. The result for *slow growth conditions*, under the ellipticity condition (1.2) with $g_1\,(t)$ and $g_2\,(t)$ general functions, can be stated as follows.

Theorem 2.1 (general growth)**.** *Let us assume that f satisfies the growth assumptions in (2.1), with the parameters α, β, μ related by the condition*

$$2 - \mu - \alpha(n\beta - \mu) > 0\,. \tag{2.2}$$

Then any minimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ of (1.1) is locally Lipschitz continuous in Ω and, for every $0 < \rho < R$, $\bar{B}_R \subset \Omega$, there exists a positive constant C depending on ρ, R, C_1, C_2, α, β, μ, $g_2(t_0)$, such that

$$\|Du\|_{L^\infty(B_\rho\,;\mathbb{R}^n)} \leq C\left\{\frac{1}{(R-\rho)^n}\int_{B_R}\left\{1+f(Du)\right\}dx\right\}^{\theta} \tag{2.3}$$

where $\theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}$.

When we specialize Theorem 2.1 to the *subquadratic $p, q-$growth* we obtain:

Corollary 2.2 ($p, q-$growth)**.** *Let $f = f\,(\xi)$ be a convex function in $\mathcal{C}(\mathbb{R}^n) \cap \mathcal{C}^2(\mathbb{R}^n \backslash B_{t_0}(0))$ for some $t_0 \geq 0$, satisfying the ellipticity conditions*

$$m\,|\xi|^{p-2}|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j \leq M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\lambda|^2, \quad \forall\,\lambda,\xi\in\mathbb{R}^n : |\xi|\geq t_0\,, \tag{2.4}$$

for some positive constants m, M and exponents p, q, $1 \leq p \leq q \leq 2$, such that

$$\frac{q}{p} < 1 + \frac{2}{n}. \tag{2.5}$$

Then every local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *to the energy integral in* (1.1) *is of class* $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *and there exists a constant* $C > 0$*, depending only on* p, q, n, m, M*, such that, for all* ρ, R *with* $0 < \rho < R \leq \rho + 1$*,*

$$\|Du\|_{L^{\infty}(B_{\rho};\mathbb{R}^n)} \leq C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} \{1 + f(Du)\}\, dx \right\}^{\frac{2}{(n+2)p-nq}}. \tag{2.6}$$

Let us briefly sketch the tools and the techniques to prove the above regularity results. A first step is an *a priori estimate* for smooth minimizers through the interpolation result stated in Lemma 5.1. The second step is an approximation procedure: we construct a sequence of smooth strictly convex functions f_k, each of them being equal to f for large $|\xi|$, in the same outlook in [36],[32]. More in details, if u is a local minimizer of (1.1), we consider the sequence of variational problems in a ball B_R, $\overline{B}_R \subset \Omega$, with as integrand a suitable perturbation of f_k and boundary value data $u_\epsilon = u * \varphi_\varepsilon$, where φ_ε are smooth mollifiers. Each u_ϵ satisfies the *bounded slope condition*; then, by the well know existence and Lipschitz regularity theorem by Hartman-Stampacchia [26] each problem has a unique Lipschitz continuous solution v_ϵ. By applying the a-priori estimate to the sequence of the solutions we get an uniform control in L^∞ of the gradient of v_ϵ, which allows us to transfer the Lipschitz continuity property to the original minimizer u.

The plan of the paper is the following: in Sections 3, 4 we present some examples, some of them being new in this context of general growth conditions. In Section 5 we give the *interpolation lemma*. In Section 6 we prove the a priori estimate for functionals with *general slow growth* by means of the interpolation lemma. In the last section we prove the regularity results. As we show in the next section, the class of energy integrals that we consider is quite large, not only polynomial unbalanced $p, q-$*subquadratic growth* as in the Corollary 2.2, but also *logaritmic growth* (as in Examples 3.1 and 3.2) and *anisotropic behaviour* (Example 3.3).

3. Examples

In this section we present some examples of density function f for which the above assumptions hold.

Example 3.1. $f(\xi) = |\xi|(\log|\xi|)^a$, $a > 0$, $|\xi| \geq t_0 \geq 1$. For large t $(2.1)_1$ holds for $g_1(t) = \frac{a}{2}\frac{(\log t)^{a-1}}{t}$ and $g_2(t) = (1+a)\frac{(\log t)^a}{t}$. It is easy to check that $(2.1)_2$ and $(2.1)_3$ hold for every $\beta > \frac{1}{n}$. Since $g_2(|\xi|)|\xi|^2 = (1+a)f(\xi)$, $(2.1)_4$ holds for every $\alpha > 1$. Moreover, for every $\mu < 1$, $t^\mu g_2(t)$ is decreasing in $[t_0\,, +\infty)$, choosing $\alpha > \frac{1}{n\beta - 1}$, (2.2) follows. Therefore Theorem 2.1 applies for every $a > 0$.

Example 3.2. $f(\xi) = (|\xi|+1)L_k(|\xi|)$, $g(t) = (1+t)L_k(t)$, $k \in \mathbb{N}$, L_k defined as:

$$L_1(t) = \log(1+t)\,, \qquad L_{k+1}(t) = \log(1 + L_k(t))\,;$$

therefore

$$L_1'(t) = \frac{1}{1+t}\,, \quad L_{k+1}'(t) = \frac{L_k'(t)}{1+L_k(t)} = \frac{1}{(1+t)(1+L_1(t))\cdots(1+L_{k-1}(t))}\,.$$

Then we get

$$g'(t) = L_k(t) + \frac{1}{(1+L_1(t))\cdots(1+L_{k-1}(t))} \quad \Longrightarrow \quad g_2(t) = \frac{2}{1+t} L_k(t)\,;$$

$$g''(t) = \frac{1}{(1+t)(1+L_1(t))\cdots(1+L_{k-1}(t))}\left[1 - \sum_{i=1}^{k-1} \frac{1}{(1+L_1(t))\cdots(1+L_i(t))}\right]$$

$$\Longrightarrow \quad g_1(t) = \frac{1}{2(1+t)(1+L_1(t))\cdots(1+L_{k-1}(t))}\,.$$

Similarly to the Example 3.1, for t large enough, $\mu = 1$ and $\beta > \frac{1}{n}$, $(2.1)_2$ and $(2.1)_3$ hold. Moreover, $g_2(|\xi|)|\xi|^2 \leq 2f(\xi)$; therefore $(2.1)_4$ holds for every $\alpha > 1$. Since we can choose α and β such that (2.2) holds, every local minimizer of the corresponding integral is locally Lipschitz continuous (see [23] for related results).

Example 3.3. Next we consider the *anisotropic case* of the energy integral in (1.1) with

$$f(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}\,, \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \tag{3.1}$$

where the exponents p_i are greater than or equal to 2 for all $i = 1, 2, \ldots, n$. Of course $f(\xi)$ in (3.1) is a convex function in $\mathbb{R}^n$. Note that the $n \times n$ matrix of the second derivatives $D^2 f = \left(f_{\xi_i \xi_j}\right)$ of f is diagonal and the corresponding quadratic form is given by

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\,\lambda_i \lambda_j = \sum_{i=1}^{n} p_i\,(p_i - 1)\,|\xi_i|^{p_i - 2}\,|\lambda_i|^2\,, \quad \forall\,\lambda, \xi \in \mathbb{R}^n. \tag{3.2}$$

This quadratic form is positive semidefinite but is not definite if (at least) one of the exponents p_i is greater than 2; in fact, if for instance $p_1 > 2$, then $\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\,\lambda_i \lambda_j = 0$ when $\xi = (\xi_1, 0, \ldots, 0) \neq 0$ and $\lambda = (0, \lambda_2, \ldots, \lambda_n) \neq 0$. Nevertheless, in spite of this lack of uniform convexity, without using the quadratic form in (3.2), the local $L^\infty-$bound of the minimizers has been established in [11]-[15],[17],[24] under some optimal conditions on the exponents $p_i > 1$. More recently Bousquet-Brasco [6] proved that *bounded* minimizers of the energy integral (1.1), with f as in (3.1), are locally Lipschitz continuous in Ω under the condition $p_i \geq 2$ for all $i = 1, 2, \ldots, n$.

In our context of slow growth we emphasize the locally Lipschitz regularity that we deduce by Theorem 2.1 when $1 < p_i \leq 2$ for all $i = 1, 2, \ldots, n$, which should make more complete the case considered by Bousquet-Brasco [6]. More precisely, we have to change the model example $f(\xi)$ in (3.1) since $f : \mathbb{R}^n \to \mathbb{R}$ there is not a function of class C^2 around $\xi = 0$ when $p_i < 2$ for some $i \in \{1, 2, \ldots, n\}$. The corresponding not-singular model is

$$f(\xi) = \sum_{i=1}^{n} \left(1 + \xi_i^2\right)^{\frac{p_i}{2}}\,, \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n. \tag{3.3}$$

Similarly to (3.2) we obtain the quadratic form of the $n\times n$ matrix of the second derivatives $D^2 f=\left(f_{\xi_i\xi_j}\right)$ of f in (3.3)

$$\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j=\sum_{i=1}^{n} p_i\left(1+(p_i-1)\,\xi_i^2\right)\left(1+\xi_i^2\right)^{\frac{p_i}{2}-2}|\lambda_i|^2\,,\quad \forall\,\lambda,\xi\in\mathbb{R}^n. \tag{3.4}$$

Since $p_i-2\le 0$ for every $i\in\{1,2,\ldots,n\}$, then

$$\begin{aligned}\left(1+(p_i-1)\,\xi_i^2\right)\left(1+\xi_i^2\right)^{\frac{p_i}{2}-2} &\ge (p_i-1)\left(1+\xi_i^2\right)^{\frac{p_i-2}{2}}\\ &\ge (p_i-1)\left(1+|\xi|^2\right)^{\frac{p_i-2}{2}}\ge (p-1)\left(1+|\xi|^2\right)^{\frac{p-2}{2}}\end{aligned}$$

where $p=:\min\{p_i : i=1,2,\ldots,n\}$ .We obtain

$$\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j\ge p\,(p-1)\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2\,,\quad \forall\,\lambda,\xi\in\mathbb{R}^n. \tag{3.5}$$

Again for every $i\in\{1,2,\ldots,n\}$, since $p_i-2\le 0$ we also have

$$p_i\left(1+(p_i-1)\,\xi_i^2\right)\left(1+\xi_i^2\right)^{\frac{p_i}{2}-2}\le p_i\left(1+\xi_i^2\right)^{\frac{p_i-2}{2}}\le p_i$$

and thus from (3.4) we deduce

$$\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j\le 2\,|\lambda|^2\,,\quad \forall\,\lambda,\xi\in\mathbb{R}^n. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we have

$$g_1(|\xi|)\,|\lambda|^2\le\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(\xi)\,\lambda_i\lambda_j\le g_2(|\xi|)\,|\lambda|^2\,,\quad \forall\,\lambda,\xi\in\mathbb{R}^n, \tag{3.7}$$

where $g_1,g_2:[0,+\infty)\to(0,+\infty)$ are the nonnegative real functions defined by $g_1(t)=p\,(p-1)\,(1+t^2)^{\frac{p-2}{2}}$ and $g_2(t)=g_2$ constantly equal to 2. By Corollary 2.2 with $q=2$ we obtain the further regularity result too.

Corollary 3.4 (anisotropic energy integrals with slow growth). *Let $f=f(\xi)$ be the model convex function in* (3.3)*, with $1<p_i\le 2$ for all $i=1,2,\ldots,n$. If*

$$\frac{2}{p}<1+\frac{2}{n}\quad\Leftrightarrow\quad p>\frac{2n}{n+2},\quad \textit{where}\quad p=:\min_{i\in\{1,2,\ldots,n\}}\{p_i\}\,, \tag{3.8}$$

then every local minimizer $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ to the energy integral (1.1)*, with $f(\xi)$ in* (3.3)*, is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and there exists a constant $C>0$ depending only on p,n,m,M, such that, for all ρ,R with $0<\rho<R\le\rho+1$,*

$$\|Du(x)\|_{L^\infty(B_\rho;\mathbb{R}^n)}\le\left(\frac{C}{(R-\rho)^n}\int_{B_R}\{1+f(Du)\}\,dx\right)^{\frac{2}{(n+2)p-2n}}.$$

Note that when $n=2$ the bound in (3.8) simply reduces to $1<p_i\leq 2$ for all $i=1,2,\ldots,n$. More generally we can consider energy integrands of the form

$$f(\xi)=\sum_{i=1}^{n}g(\xi_i)\,, \quad \text{or} \quad f(\xi)=\sum_{i=1}^{n}g_i(\xi_i)\,, \tag{3.9}$$

where, for instance, $g(t)$ or $g_i(t)$ are one of the functions considered above in Examples 3.1 and 3.2.

4. New examples of anisotropic energy functions

We provide some applications of our Theorem 2.1 and we infer the Lipschitz continuity of the local minimizers to some class of functionals with anisotropic behaviour.

Example 4.1. Consider

$$f(\xi)=\sqrt{\sum_{i=1}^{n}\left(1+|\xi_i|^2\right)^{p_i}}, \qquad p_i>1, \quad \forall\, i=1,\ldots,n. \tag{4.1}$$

With the same argument of Example 3.3 we have

$$\frac{1}{\sqrt{n^p}}\left(1+|\xi|^2\right)^{\frac{p}{2}}\leq f(\xi)\leq\sqrt{n}\left(1+|\xi|^2\right)^{\frac{q}{2}}$$

where $p=\min_i p_i$ and $q=\max_i p_i$. Let us denote by $Q(\xi,\lambda)$ the quadratic form

$$Q(\xi,\lambda)=\sum_{i,j=1}^{n}f_{\xi_i\xi_i}(\xi)\,\lambda_i\lambda_j\,, \quad \forall\,\lambda,\xi\in\mathbb{R}^n. \tag{4.2}$$

We have

$$\begin{aligned}
f_{\xi_i\xi_i}&=-\frac{p_i^2\xi_i^2(1+\xi_i^2)^{2p_i-2}}{(\sum_{k=1}^n(1+\xi_k^2)^{p_k})^{\frac{3}{2}}}+\frac{p_i(1+\xi_i^2)^{p_i-2}\left(1+(2p_i-1)\xi_i^2\right)}{(\sum_{i=1}^n(1+\xi_i^2)^{p_i})^{\frac{1}{2}}}, \qquad i=1,\ldots n,\\
f_{\xi_i\xi_j}&=-\frac{p_ip_j\xi_i\xi_j(1+\xi_i^2)^{p_i-1}(1+\xi_j^2)^{p_j-1}}{(\sum_{k=1}^n(1+\xi_k^2)^{p_k})^{\frac{3}{2}}}, \qquad i,j=1,\ldots n, \ \ i\neq j,
\end{aligned}$$

and then

$$\begin{aligned}
Q(\xi,\lambda)\left(\sum_{k=1}^{n}(1+\xi_k^2)^{p_k}\right)^{\frac{3}{2}}&=-\left(v\cdot w\right)^2\\
&\quad+\left(\sum_{k=1}^{n}(1+\xi_k^2)^{p_k}\right)\sum_{i=1}^{n}p_i(1+\xi_i^2)^{p_i-2}\left(1+(2p_i-1)\xi_i^2\right)\lambda_i^2,
\end{aligned}$$

where $v_i = p_i \xi_i (1+\xi_i^2)^{\frac{p_i}{2}-1}\lambda_i$ and $w_i = (1+\xi_i^2)^{\frac{p_i}{2}}$. Therefore

$$Q(\xi,\lambda)\left(\sum_{k=1}^n (1+\xi_k^2)^{p_k}\right)^{\frac{1}{2}} \leq \sum_{i=1}^n p_i (1+\xi_i^2)^{p_i-2}\left(1+(2p_i-1)\xi_i^2\right)\lambda_i^2$$
$$\leq (2q^2-q)\sum_{i=1}^n \left[(1+\xi_i^2)^{p_i}\right]^{1-\frac{1}{p_i}}\lambda_i^2 \leq 2q^2\left[\sum_{k=1}^n (1+\xi_k^2)^{p_k}\right]^{1-\frac{1}{q}}|\lambda|^2.$$

For $|\xi|\geq 1$, if $q\leq 2$ we have

$$\begin{aligned} Q(\xi,\lambda) &\leq 2q^2\left(\sum_{k=1}^n (1+\xi_k^2)^{p_k}\right)^{\frac{q-2}{2q}}|\lambda|^2 \leq 2q^2\left(1+\frac{1}{n}|\xi|^2\right)^{p\frac{q-2}{2q}}|\lambda|^2 \\ &\leq C|\xi|^{\frac{p}{q}(q-2)}|\lambda|^2, \end{aligned} \tag{4.3}$$

instead, if $q\geq 2$ we obtain

$$Q(\xi,\lambda)\leq 2q^2\left(\sum_{k=1}^n (1+\xi_k^2)^{p_k}\right)^{\frac{q-2}{2q}}|\lambda|^2 \leq 2q^2\left(n\left(1+|\xi|^2\right)^q\right)^{\frac{q-2}{2q}}|\lambda|^2\leq C|\xi|^{q-2}|\lambda|^2.$$

Moreover, since

$$(v\cdot w)^2\leq |v|^2|w|^2=\sum_{i=1}^n p_i^2\xi_i^2(1+\xi_i^2)^{p_i-2}\lambda_i^2\sum_{k=1}^n(1+\xi_k^2)^{p_k}$$

we have

$$Q(\xi,\lambda)\left(\sum_{k=1}^n (1+\xi_k^2)^{p_k}\right)^{\frac{1}{2}} \geq -\sum_{i=1}^n p_i^2\xi_i^2(1+\xi_i^2)^{p_i-2}\lambda_i^2$$
$$+\sum_{i=1}^n p_i(1+\xi_i^2)^{p_i-2}\left(1+(2p_i-1)\xi_i^2\right)\lambda_i^2=\sum_{i=1}^n p_i(1+\xi_i^2)^{p_i-2}\left(1+(p_i-1)\xi_i^2\right)\lambda_i^2.$$

For every $q>1$ and $|\xi|\geq 1$ we deduce

$$\begin{aligned} Q(\xi,\lambda) &\geq \left(\sum_{k=1}^n (1+\xi_k^2)^{p_k}\right)^{-\frac{1}{2}}(p^2-p)\sum_{i=1}^n(1+\xi_i^2)^{p_i-1}\lambda_i^2 \\ &\geq \frac{p^2-p}{\sqrt{n}}(1+\max_i\{|\xi_i|\}^2)^{p-1-\frac{q}{2}}|\lambda|^2\geq c|\xi|^{2p-2-q}|\lambda|^2. \end{aligned} \tag{4.4}$$

We note explicitly that if $1<p<q$ then $2p-2-q<p-2$. Therefore, by denoting

$$r=2p-q \qquad \text{and} \qquad s=\frac{p}{q}(q-2)+2 \tag{4.5}$$

with $r\leq p\leq q\leq s\leq 2$, by (4.3) and (4.4) we obtain that $f(\xi)$ in (4.1) satisfies the assumptions $(2.1)_1$ and $(2.1)_2$ with

$$g_1(t)=c\,t^{r-2} \qquad \text{and} \qquad g_2(t)=C\,t^{s-2}. \tag{4.6}$$

Therefore the function $f(\xi)$ in (4.1) satisfies all assumptions in (2.1) with

$$\mu = 2 - s, \qquad \beta = \frac{n-2}{2n}\, s - \frac{r}{2} + \frac{2}{n} \qquad \text{and} \qquad \alpha = \frac{s}{p},$$

when we impose the bounds

$$\alpha < \frac{2-\mu}{n\beta - \mu} \qquad \Longleftrightarrow \qquad s < \frac{2}{n}\, p + r.$$

We are in the conditions to apply Theorem 2.1. In the next Corollary 4.2 we state what we have proved by the computations above for this example, about the energy integral

$$F_2(u) = \int_\Omega \left(\sum_{i=1}^{n} \left(1 + |u_{x_i}|^2\right)^{p_i} \right)^{\frac{1}{2}} dx\,. \tag{4.7}$$

Corollary 4.2. *Let* $1 < p = \min_i p_i \le q = \max_i p_i \le 2$ *and* r, s *as in (4.5). If*

$$s < \frac{2}{n}\, p + r \qquad \Longleftrightarrow \qquad \frac{q}{p} < 1 + \frac{2}{n} - 2\left(\frac{1}{p} - \frac{1}{q}\right), \tag{4.8}$$

then the local minimizers of the energy integral F_2 *in* (4.7) *are locally Lipschitz continuous in* Ω.

Remark 4.3. At a first glance we may think that the assumption (4.8) of Corollary 4.2 is more restrictive that the similar assumption $\frac{q}{p} < 1 + \frac{2}{n}$ in Corollary 2.2, valid under the general $p, q-$growth. But, if we apply correctly Corollary 2.2 to F_2, on the contrary we had a more restrictive assumption than the above condition (4.8). In fact by (4.6) we have here $g_1(t) = c\, t^{r-2}$, $g_2(t) = C\, t^{s-2}$ and the estimate of the quadratic form $(2.1)_1$ becomes

$$c\, |\xi|^{r-2} |\lambda|^2 \le \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi)\, \lambda_i \lambda_j \le C\, |\xi|^{s-2} |\lambda|^2, \quad \forall\, \lambda, \xi \in \mathbb{R}^n : |\xi| \ge 1\,. \tag{4.9}$$

Then Corollary 2.2 applied to F_2 gives the regularity of minimizers under the bound $s < \left(1 + \frac{2}{n}\right) r = \frac{2}{n}\, r + r$, which is a more restrictive condition than the above assumption (4.8) $s < \frac{2}{n}\, p + r$, since $r = 2p - q = p + (p - q) < p$ when $p < q$.

Therefore the general bound of Corollary 2.2 gives a less precise result than Theorem 2.1 when applies to the energy integral (4.7). This fact also shows that Theorem 2.1 gives a more general regularity result than similar results that can be found in the recent mathematical literature on $p, q-$growth.

Example 4.4. Let

$$h(\xi) = \sqrt{\sum_{i=1}^{n} |\xi_i|^{2p_i}}, \qquad p_i \ge 1, \tag{4.10}$$

$p = \min_i p_i$ and $q = \max_i p_i \le 2$, and

$$\overline{Q}(\xi, \lambda) = \sum_{i,j=1}^{n} h_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j\,, \quad \forall\, \lambda, \xi \in \mathbb{R}^n. \tag{4.11}$$

We prove that the associated quadratic form to h is semidefinite, i.e. $\overline{Q}(\xi,\lambda)\geq 0$. In fact

$$\overline{Q}(\xi,\lambda)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{3}{2}} = -\sum_{i,j=1}^{n}\operatorname{sign}(\xi_i\xi_j)p_ip_j|\xi_i|^{2p_i-1}|\xi_j|^{2p_j-1}\lambda_i\lambda_j + \sum_{k=1}^{n}|\xi_k|^{2p_k}\sum_{i=1}^{n}(2p_i^2-p_i)|\xi_i|^{2p_i-2}\lambda_i^2. \tag{4.12}$$

By proceeding as above we have

$$\overline{Q}(\xi,\lambda)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{3}{2}} = -(v\cdot w)^2+\sum_{k=1}^{n}|\xi_k|^{2p_k}\sum_{i=1}^{n}(2p_i^2-p_i)|\xi_i|^{2p_i-2}\lambda_i^2,$$

$v_i = p_i|\xi_i|^{p_i-1}\lambda_i$ and $w_i=\operatorname{sign}(\xi_i)|\xi_i|^{p_i}$. In this case the quadratic form $\overline{Q}$ is degenerate ($\overline{Q}(\xi,\lambda)=0$ if $(\xi\cdot\lambda)=0$) but positive semidefinite:

$$\overline{Q}(\xi,\lambda)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{3}{2}} \geq \sum_{k=1}^{n}|\xi_k|^{2p_k}\sum_{i=1}^{n}(p_i^2-p_i)|\xi_i|^{2p_i-2}\lambda_i^2\geq 0.$$

On the other hand, if $\max_i\{|\xi_i|\}\geq 1$,

$$\begin{aligned}\overline{Q}(\xi\,,\lambda)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{1}{2}} &\leq \sum_{i=1}^{n}(2p_i^2-p_i)|\xi_i|^{2p_i-2}\lambda_i^2 = \sum_{i=1}^{n}(2p_i^2-p_i)(|\xi_i|^{2p_i})^{1-\frac{1}{p_i}}\lambda_i^2\\ &\leq (2q^2-q)\sum_{i=1}^{n}\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{1-\frac{1}{p_i}}\lambda_i^2\end{aligned}$$

since $\sum_{k=1}^{n}|\xi_k|^{2p_k}\geq(\max_i\{|\xi_i|\})^{2p}\geq 1$,

$$\overline{Q}(\xi\,,\lambda)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{1}{2}}\leq(2q^2-q)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{1-\frac{1}{q}}|\lambda|^2.$$

Now, again using $\max_i\{|\xi_i|\}\geq 1$,

$$\sum_{k=1}^{n}|\xi_k|^{2p_k}\geq(\max_i\{|\xi_i|\})^{2p}\quad\Longrightarrow\quad\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{1}{2}-\frac{1}{q}}\leq\left(\max_i\{|\xi_i|\}^{2p}\right)^{\frac{1}{2}-\frac{1}{q}},$$

therefore

$$\begin{aligned}\overline{Q}(\xi\,,\lambda)&\leq(2q^2-q)\left(\sum_{k=1}^{n}|\xi_k|^{2p_k}\right)^{\frac{1}{2}-\frac{1}{q}}|\lambda|^2\leq(2q^2-q)\left(\max_i\{|\xi_i|\}\right)^{p\frac{q-2}{q}}|\lambda|^2\\&\leq C|\xi|^{p\frac{q-2}{q}}|\lambda|^2.\end{aligned}$$

In this case $\overline{Q}(\xi,\lambda)\leq C|\xi|^{q-2}|\lambda|^2$ when $q\geq 2$. We denote by

$$s=\frac{p}{q}(q-2)+2 \tag{4.13}$$

with $1 < p = \min_i p_i \le q = \max_i p_i \le 2$. We consider the ellipticity conditions (2.4), with s replaced by q, for the function

$$f(\xi) = |\xi|^p + h(\xi). \tag{4.14}$$

Since

$$\frac{s}{p} < 1 + \frac{2}{n} \quad \Longleftrightarrow \quad \frac{q}{p} < 1 + \frac{q}{n}\,,$$

from Corollary 2.2 we obtain the proof of a further regularity result for the following energy integral

$$F_3(u) = \int_\Omega |Du|^p + \left(\sum_{i=1}^{n} |u_{x_i}|^{2p_i} \right)^{\frac{1}{2}} dx. \tag{4.15}$$

Corollary 4.5. *If* $1 < p = \min_i p_i \le q = \max_i p_i \le 2$ *satisfy*

$$\frac{q}{p} < 1 + \frac{q}{n} \quad \Longleftrightarrow \quad q < p^* =: \frac{np}{n-p}\,, \tag{4.16}$$

then any local minimizers of F_3 *in* (4.15) *is locally Lipschitz continuous in* Ω.

We can consider also different integrands related with h in (4.10). By taking in account Example 3.1 and Example 3.2, we can consider, for $1 < q \le 2$, $s = 3 - \frac{2}{q} \ge q$,

$$f(\xi) = |\xi|(\log|\xi|)^a + \sqrt{\sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^{2q}}, \qquad a > 0, \tag{4.17}$$

or

$$f(\xi) = |\xi| L_k(|\xi|) + \sqrt{\sum_{i=1}^{n-1} |\xi_i|^2 + |\xi_n|^{2q}}, \tag{4.18}$$

Assumption (2.1) holds with $g_2(t) = Ct^{s-2}$ and respectively

$$g_1(t) = c\,\frac{(\log t)^{a-1}}{t} \qquad \text{or} \qquad g_1(t) = \frac{c}{(1+t)(1+L_1(t))\cdots(1+L_{k-1}(t))}\,,$$

$$\mu = 2 - s, \quad \beta > \frac{n-2}{2n}s - \frac{1}{2} + \frac{2}{n}, \quad \alpha > s\,.$$

Therefore, if $s < 1 + \frac{2}{n}$, by Theorem 2.1 the corresponding local minimizers are locally Lipschitz continuous.

5. Interpolation lemma

As usual we denote by B_R a generic ball of radius R compactly contained in Ω and by B_ϱ a ball of radius $\varrho < R$ concentric with B_R.

Lemma 5.1 (interpolation)**.** *Let* $v \in L^\infty_{\mathrm{loc}}(\Omega)$ *and let us assume that for some* $\vartheta \ge 1$, $c > 0$ *and for every* ϱ *and* R *such that* $0 < \rho < R$

$$\|v\|_{L^\infty(B_\varrho)}^{\frac{1}{\vartheta}} \le \frac{c}{(R-\varrho)^n} \int_{B_R} |v|\; dx\,. \tag{5.1}$$

Then, for every $\lambda \in \left(\frac{\vartheta-1}{\vartheta}, 1\right)$ (i.e., in particular with $\vartheta(1-\lambda) < 1$) there exists a constant c_λ such that, for every $\varrho < R$,

$$\|v\|_{L^\infty(B_\varrho)}^{\frac{1-\vartheta(1-\lambda)}{\vartheta}} \leq \frac{c_\lambda}{(R-\varrho)^n} \int_{B_R} |v|^\lambda \, dx. \tag{5.2}$$

Proof. Fixed $\lambda \in \left(\frac{\vartheta-1}{\vartheta}, 1\right)$, we make use of the interpolation inequality

$$\int_{B_\varrho} |v| \, dx = \int_{B_\varrho} |v|^{1-\lambda} |v|^\lambda \, dx \leq \|v\|_{L^\infty(B_\varrho)}^{1-\lambda} \int_{B_\varrho} |v|^\lambda \, dx \, .$$

By the assumption (5.1) we obtain

$$\int_{B_\varrho} |v| \, dx \leq \left(\frac{c}{(R-\varrho)^n} \int_{B_R} |v| \, dx \right)^{\vartheta(1-\lambda)} \int_{B_\varrho} |v|^\lambda \, dx \, .$$

We denote by $\gamma := \vartheta(1-\lambda)$ and we observe that $0 < \gamma < 1$ since $\lambda > \frac{\vartheta-1}{\vartheta}$. Thus the previous estimate has the equivalent form

$$\int_{B_\varrho} |v| \, dx \leq c^\gamma \int_{B_\varrho} |v|^\lambda \, dx \cdot \left(\frac{1}{(R-\varrho)^n} \int_{B_R} |v| \, dx \right)^\gamma . \tag{5.3}$$

Given ϱ_0 and R_0, with $0 < \varrho_0 < R_0 \leq \varrho_0 + 1$, we define a decreasing sequence ϱ_k by $\varrho_k = R_0 - \frac{R_0 - \varrho_0}{2^k}$, $k = 0, 1, 2, \ldots$ In (5.3) we pose $\varrho = \varrho_k$ and $R = \varrho_{k+1}$. Since $R - \varrho = \varrho_{k+1} - \varrho_k = \frac{R_0-\varrho_0}{2^{k+1}}$, we obtain

$$\int_{B_{\varrho_k}} |v| \, dx \leq c^\gamma \int_{B_{R_0}} |v|^\lambda \, dx \cdot \left(\frac{2^{n(k+1)}}{(R_0-\varrho_0)^n} \int_{B_{\varrho_{k+1}}} |v| \, dx \right)^\gamma ,$$

Denote $B_k = \int_{B_{\varrho_k}} |v| \, dx$ for $k = 0, 1, 2, \ldots$. The last inequality becomes

$$B_k \leq c^\gamma \int_{B_{R_0}} |v|^\lambda \, dx \cdot \frac{2^{n\gamma(k+1)}}{(R_0-\varrho_0)^{n\gamma}} B_{k+1}^\gamma \, .$$

We start to iterate with $k = 0, 1, 2, \ldots$

$$B_0 \leq c^\gamma \int_{B_{R_0}} |v|^\lambda \, dx \cdot \frac{2^{n\gamma}}{(R_0-\varrho_0)^{n\gamma}} B_1^\gamma$$

$$\leq c^\gamma \int_{B_{R_0}} |v|^\lambda \, dx \cdot \frac{2^{n\gamma}}{(R_0-\varrho_0)^{n\gamma}} \left(c^\gamma \int_{B_{R_0}} |v|^\lambda \, dx \cdot \frac{2^{n\gamma\cdot 2}}{(R_0-\varrho_0)^{n\gamma}} B_2^\gamma \right)^\gamma$$

and for general $k = 1, 2, 3, \ldots$ we have

$$B_0 \leq \left(\frac{c^\gamma \int_{B_{R_0}} |v|^\lambda \, dx}{(R_0-\varrho_0)^{n\gamma}} \right)^{\sum_{i=0}^{k-1} \gamma^i} (2^n)^{\sum_{i=1}^{k} i\gamma^i} (B_k)^{\gamma^k} \, .$$

Since $0 < \gamma < 1$, passing to the limit as $k \to \infty$, $\sum_{i=0}^\infty i\gamma^i < \infty$ and $\sum_{i=0}^\infty \gamma^i = \frac{1}{1-\gamma}$. Moreover the increasing sequence $B_k = \int_{B_{\varrho_k}} |v| \, dx$ is bounded by $\int_{B_{R_0}} |v| \, dx$ for $k = 0, 1, 2, \ldots$ Thus

$(B_k)^{\gamma^k} = \left(\int_{B_{\varrho_k}} |v|\, dx\right)^{\gamma^k} \le \left(\int_{B_{R_0}} |v|\, dx\right)^{\gamma^k}$ and the right hand side converges to 1 as $k \to \infty$. Therefore, in the limit as $k \to \infty$, there exists a constant c_1 such that

$$B_0 = \int_{B_{\varrho_0}} |v|\, dx \le c_1 \left(\frac{1}{(R_0 - \varrho_0)^{n\gamma}} \int_{B_{R_0}} |v|^\lambda\, dx \right)^{\frac{1}{1-\gamma}} . \tag{5.4}$$

Fixed $\varrho < R$ we consider $\overline{\varrho} = \frac{R+\varrho}{2}$ and, by combining the assumption (5.1) and (5.4), since $R - \overline{\varrho} = \overline{\varrho} - \varrho$ and $\gamma = \vartheta\,(1-\lambda)$,

$$\|v\|_{L^\infty(B_\varrho)} \le \left(\frac{c}{(\overline{\varrho} - \varrho)^n} \int_{B_{\overline{\varrho}}} |v|\, dx \right)^{\vartheta}$$

$$\le \left(\frac{c \cdot c_1}{(\overline{\varrho} - \varrho)^n} \left(\frac{1}{(R - \overline{\varrho})^{n\gamma}} \int_{B_R} |v|^\lambda\, dx \right)^{\frac{1}{1-\gamma}} \right)^{\vartheta}$$

$$\le c_2 \left(\frac{1}{(R - \overline{\varrho})^{n(1-\gamma)+n\gamma}} \int_{B_R} |v|^\lambda\, dx \right)^{\frac{\vartheta}{1-\gamma}} = c_3 \left(\frac{1}{(R - \varrho)^{n}} \int_{B_R} |v|^\lambda\, dx \right)^{\frac{\vartheta}{1-\vartheta(1-\lambda)}},$$

which gives (5.2) and the proof of the Lemma is concluded. □

6. A priori estimates

In order to simplify the notations, without loss of generality in this section we assume that $t_0 = 1$. First of all we give a technical result.

Lemma 6.1. *Let us assume that* $(2.1)_2$ *and* $(2.1)_3$ *hold. Then for every* $\gamma \ge 0$ *there exists a constant* $C_3 = C_3(C_1\,, g_2(1)) > 0$ *independent of* γ, *such that*

$$C_3 \left[1 + g_2(1+t)^{\frac{1}{2^*}} \frac{(1+t)^{\frac{\gamma}{2}+1-\beta}}{\left(\frac{\gamma}{2}+1-\beta\right)^2} \right] \le 1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \sqrt{g_1(1+s)}\, ds \tag{6.1}$$

for every $t \ge 0$, *where , for* $n > 2$, $2^* = \frac{2n}{n-2}$, *while, for* $n = 2$, 2^* *can be any number greater than* $\frac{2}{1-\beta}$.

Proof. If $t \ge 0$ then, by assumption $(2.1)_3$

$$1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \sqrt{g_1(1+s)}\, ds \ge 1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \frac{1}{\sqrt{C_1}} (1+s)^{-\beta} g_2(1+s)^{\frac{1}{2^*}}\, ds.$$

On the other hand, since g_2 is decreasing, $g_2(1+s)^{\frac{1}{2^*}} \ge g_2(1+t)^{\frac{1}{2^*}}$ and therefore

$$1 + \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \int_0^t (1+s)^{\frac{\gamma-2}{2}-\beta} s\, ds \le 1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \frac{1}{\sqrt{C_1}} (1+s)^{-\beta} g_2(1+s)^{\frac{1}{2^*}}\, ds.$$

By Lemma 2.2 in [21], we have that (see (2.6) here): let $\alpha_0 > 0$ there exists a constant c depending on α_0, but independent of $\alpha \geq \alpha_0$, such that

$$(1+t)^\alpha \leq c\,\alpha^2 \left(1 + \int_0^t (1+s)^{\alpha-2} s\, ds\right). \tag{6.2}$$

In our case $\alpha := \frac{\gamma-2}{2} - \beta + 2 = \frac{\gamma}{2} + 1 - \beta$ and $\alpha \geq \alpha_0 := \frac{2}{2^*}$. Inequality (6.2) is valid for all $t \geq 0$ so in particular for $t \geq 1$ and it entails

$$\int_0^t (1+s)^{\alpha-2} s\, ds \geq \frac{(1+t)^\alpha}{c\,\alpha^2} - 1.$$

The last inequality implies

$$\begin{aligned} & 1 + \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \int_0^t (1+s)^{\frac{\gamma-2}{2}-\beta} s\, ds \\ \geq\ & 1 + \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \left[\frac{(1+t)^{\frac{\gamma}{2}+1-\beta}}{c\left(\frac{\gamma}{2}+1-\beta\right)^2} - 1\right] \\ =\ & 1 + \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \frac{(1+t)^{\frac{\gamma}{2}+1-\beta}}{c\left(\frac{\gamma}{2}+1-\beta\right)^2} - \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}}. \end{aligned}$$

Now we observe that, for every $t \geq 0$, since g_2 is decreasing,

$$\frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \leq \frac{1}{\sqrt{C_1}} g_2(1)^{\frac{1}{2^*}} =: \tilde{C}_1.$$

Thus summing up

$$1 + \tilde{C}_1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \sqrt{g_1(1+s)}\, ds \geq 1 + \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \frac{(1+t)^{\frac{\gamma}{2}+1-\beta}}{c\left(\frac{\gamma}{2}+1-\beta\right)^2}$$

which in turn implies

$$\begin{aligned} & (1+\tilde{C}_1)\left[1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \sqrt{g_1(1+s)}\, ds\right] \\ \geq\ & 1 + \tilde{C}_1 + \int_0^t (1+s)^{\frac{\gamma-2}{2}} s \sqrt{g_1(1+s)}\, ds \\ \geq\ & 1 + \frac{1}{\sqrt{C_1}} g_2(1+t)^{\frac{1}{2^*}} \frac{(1+t)^{\frac{\gamma}{2}+1-\beta}}{c\left(\frac{\gamma}{2}+1-\beta\right)^2} \end{aligned}$$

therefore, by setting $\tilde{C}_2 := (1+\tilde{C}_1)\sqrt{C_1}\, c$, we get (6.1) for $C_3 = \frac{1}{\tilde{C}_2}$. We note explicitly that $\tilde{C}_2$ may depend on n but it is independent of γ. □

Lemma 6.2. *Assume that f satisfies the growth assumptions* $(2.1)_1$, $(2.1)_2$, $(2.1)_3$. *In addiction, assume that $f(\xi)$ is of class $\mathcal{C}^2(\mathbb{R}^n)$ and for every $M > 0$ there exists a positive constant $\ell = \ell(M)$ such that*

$$\ell\,|\lambda|^2 \leq \sum_{i,j=1}^n f_{\xi_i \xi_j}(\xi)\,\lambda_i\,\lambda_j \qquad \forall \lambda, \xi \in \mathbb{R}^n,\ |\xi| \geq M. \tag{6.3}$$

If $u \in W^{1,\infty}_{\rm loc}(\Omega)$ is a a local minimizer of (1.1), then for every $0 < \rho < R$, $\bar{B}_R \subset \Omega$ there exists a positive constants c_4 depending only on C_1, β, $g_2(1)$, such that

$$\begin{aligned}\left(\|1 + (|Du| - 1)_+\|_{L^\infty(B_\rho)}\right)^{2-n\beta} \\ \leq \frac{c_4}{(R-\rho)^n} \int_{B_R} (1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+)\, dx.\end{aligned} \tag{6.4}$$

Proof. Since the local minimizer u is in $W^{1,\infty}_{\rm loc}(\Omega)$, it satisfies the Euler equation: for every open set Ω' compactly contained in Ω we have

$$\int_\Omega \sum_{i=1}^n f_{\xi_i}(Du)\, \varphi_{x_i}\, dx = 0 \qquad \forall \varphi \in W^{1,2}_0(\Omega').$$

Moreover, by the techniques of the difference quotient (see for example [25, Ch. 8, Sect. 8.1]), $u \in W^{2,2}_{\rm loc}(\Omega)$, then the second variation holds:

$$\int_\Omega \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du) u_{x_j x_k} \varphi_{x_i}\, dx = 0, \qquad \forall k = 1, \dots, n, \;\; \forall \varphi \in W^{1,2}_0(\Omega').$$

For fixed $k = 1, \dots, n$ let $\eta \in \mathcal{C}^1_0(\Omega')$ be equal to 1 in B_ρ, with support contained in B_R, such that $|D\eta| \leq \frac{2}{(R-\rho)}$, and consider $\varphi = \eta^2\, u_{x_k}\, \Phi((|Du| - 1)_+)$ with Φ non negative, increasing, locally Lipschitz continuous on $[0, +\infty)$, such that $\Phi(0) = 0$. Here $(a)_+$ denotes the positive part of $a \in \mathbb{R}$; in the following we denote $\Phi((|Du| - 1)_+) = \Phi(|Du| - 1)_+$. Then a.e. in Ω

$$\varphi_{x_i} = 2\eta\, \eta_{x_i} u_{x_k} \Phi(|Du| - 1)_+ + \eta^2 u_{x_i x_k} \Phi(|Du| - 1)_+ + \eta^2 u_{x_k} \Phi'(|Du| - 1)_+ [(|Du| - 1)_+]_{x_i}.$$

Proceeding along the lines of [32], we therefore deduce that

$$\begin{aligned}&\int_\Omega 2\eta \Phi(|Du| - 1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du) u_{x_j x_k} \eta_{x_i} u_{x_k}\, dx \\ &+ \int_\Omega \eta^2 \Phi(|Du| - 1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du) u_{x_j x_k} u_{x_i x_k}\, dx \\ &+ \int_\Omega \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du) u_{x_j x_k} u_{x_k} [(|Du - 1|)_+]_{x_i}\, dx = 0.\end{aligned}$$

We estimate the first integral in the previous equation by using the Cauchy-Schwarz inequality and the Young inequality so that

$$
\begin{aligned}
&\left| \int_\Omega 2\eta\Phi(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_jx_k}\eta_{x_i}u_{x_k}\,dx \right| \\
&\quad \le \int_\Omega 2\Phi(|Du|-1)_+ \left(\eta^2 \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_ix_k}u_{x_jx_k}\right)^{\frac{1}{2}} \left(\sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\right)^{\frac{1}{2}} dx \\
&\quad \le \frac{1}{2}\int_\Omega \eta^2\Phi((|Du|-1)_+) \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_ix_k}u_{x_jx_k}\,dx \\
&\qquad + 2\int_\Omega \Phi((|Du|-1)_+) \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\,dx.
\end{aligned}
$$

Therefore we deduce

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \eta^2\Phi(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_ix_k}u_{x_jx_k}\,dx \\
&\qquad + \int_\Omega \eta^2\Phi'(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_k}[(|Du-1|)_+]_{x_i}\,dx \\
&\qquad \le 2\int_\Omega \Phi(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\,dx.
\end{aligned}
$$

Since a.e. in Ω

$$
[(|Du|-1)_+]_{x_i} = \begin{cases} (|Du|)_{x_i} = \frac{1}{|Du|}\sum_k u_{x_ix_k}u_{x_k} & \text{if } |Du|>1, \\ 0 & \text{if } |Du|\le 1, \end{cases}
$$

by summing up in the previous chain of inequalities with respect to $k=1,\dots,n$ we obtain

$$
\sum_{k=1}^n\sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_k}[(|Du|-1)_+]_{x_i} = |Du|\sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)[(|Du-1|)_+]_{x_j}[(|Du|-1)_+]_{x_i}
$$

therefore we deduce the estimate

$$
\begin{aligned}
&\int_\Omega \eta^2\Phi(|Du|-1)_+ \sum_{k,i,j=1}^n f_{\xi_i\xi_j}(Du)u_{x_jx_k}u_{x_ix_k}\,dx \\
&\quad + \int_\Omega \eta^2|Du|\Phi'(|Du|-1)_+ \sum_{i,j=1}^n f_{\xi_i\xi_j}(Du)[(|Du-1|)_+]_{x_j}[(|Du-1|)_+]_{x_i}\,dx \\
&\quad \le 4\int_\Omega \Phi(|Du|-1)_+ \sum_{k,i,j=1}^n f_{\xi_i\xi_j}(Du)\eta_{x_i}u_{x_k}\eta_{x_j}u_{x_k}\,dx.
\end{aligned}
$$

Using the inequality $|D(|Du|-1)_+|^2 \le |D^2u|^2$ and the ellipticity condition in $(2.1)_1$ we obtain

$$\begin{aligned}
&\int_\Omega \eta^2[\Phi(|Du|-1)_+ + |Du|\Phi'(|Du|-1)_+]g_1(1+(|Du|-1)_+))\,|D(|Du|-1)_+|^2\,dx\\
&\quad= \int_\Omega \eta^2[\Phi(|Du|-1)_+ + |Du|\Phi'(|Du|-1)_+]g_1(|Du|)\,|D(|Du|-1)_+|^2\,dx\\
&\quad\le 4\int_\Omega |D\eta|^2\,\Phi(|Du|-1)_+\,g_2(|Du|)\,|Du|^2\,dx\\
&\quad= 4\int_\Omega |D\eta|^2\,\Phi(|Du|-1)_+\,g_2(1+(|Du|-1)_+))\,|Du|^2\,dx.
\end{aligned} \tag{6.5}$$

Let us define

$$G(t) = 1+\int_0^t \sqrt{\Phi(s)\,g_1(1+s)}\,ds \qquad \forall t\ge 0. \tag{6.6}$$

By Jensen's inequality and the monotonicity of Φ, since $t\mapsto tg_2(t)$ is increasing,

$$\begin{aligned}
G(t) &= 1+\int_0^t \sqrt{\Phi(s)(1+s)g_1(1+s)\frac{1}{1+s}}\,ds \le 1+\int_0^t \sqrt{\Phi(s)(1+s)g_2(1+s)\frac{1}{1+s}}\,ds\\
&\le 1+\sqrt{\Phi(t)(1+t)g_2(1+t)}\int_0^t \frac{1}{\sqrt{1+s}}\,ds \le 1+2\sqrt{\Phi(t)(1+t)g_2(1+t)}\,\sqrt{1+t},
\end{aligned}$$

hence $[G(t)]^2 \le 8\,[1+\Phi(t)(1+t)^2g_2(1+t)]$. On the other hand

$$\begin{aligned}
&|D(\eta(G((|Du|-1)_+)|^2\\
&\quad\le 2\,|D\eta|^2[G((|Du|-1)_+)]^2 + 2\eta^2[G'((|Du|-1)_+)]^2\,|D((|Du|-1)_+)|^2\\
&\quad\le 16\,|D\eta|^2\,(1+\Phi(|Du|-1)_+\,g_2(|Du|)|Du|^2) + 2\,\eta^2\,\Phi(|Du|-1)_+ g_1(|Du|)\,|D(|Du|)|^2.
\end{aligned}$$

Since $\Phi(|Du\,(x)\,|-1)_+ = 0$ when $|Du\,(x)\,|\le 1$, by (6.5) we get

$$\begin{aligned}
&\int_\Omega |D(\eta\,G((|Du|-1)_+)|^2\,dx\\
&\le 24\int_\Omega |D\eta|^2\,(1+\Phi(|Du|-1)_+ g_2(|Du|)|Du|^2)\,dx\\
&= 24\int_\Omega |D\eta|^2\,(1+\Phi(|Du|-1)_+\,g_2(1+(|Du|-1)_+)\,(1+(|Du|-1)_+)^2\,dx.
\end{aligned} \tag{6.7}$$

Let us assume

$$\Phi(t) = (1+t)^{\gamma-2}t^2 \qquad \gamma\ge 0. \tag{6.8}$$

By the Sobolev inequality, there exists a constant c_S such that

$$\left\{\int_\Omega [\eta\,G((|Du|-1)_+)]^{2^*}\,dx\right\}^{2/2^*} \le c_S\int_\Omega |D(\eta(G(|Du|-1)_+))|^2\,dx \tag{6.9}$$

where $2^* = \frac{2n}{n-2}$ if $n>2$ and a number greater than $\frac{2}{1-\beta}$ if $n=2$. We apply (6.1) with the choice $t = (|Du|-1)_+$

$$G((|Du|-1)_+) \;=\; 1+\int_0^{(|Du|-1)_+} (1+s)^{\frac{\gamma-2}{2}}\,s\sqrt{g_1(1+s)}\,ds$$

$$\geq \; C_3 \left[1 + g_2(1+(|Du|-1)_+)^{\frac{1}{2^*}} \frac{(1+(|Du|-1)_+)^{\frac{\gamma}{2}+1-\beta}}{\left(\frac{\gamma}{2}+1-\beta\right)^2} \right]$$

thus by (6.7) we obtain that there exists $c = c(C_3) > 0$ such that, for all $\gamma \geq 0$,

$$\begin{aligned} &\left\{ \int_\Omega \eta^{2^*} (1 + (1+(|Du|-1)_+)^{(\gamma+2-2\beta)\frac{2^*}{2}} \, g_2(1+(|Du|-1)_+)) \, dx \right\}^{\frac{2}{2^*}} \\ &\quad \leq \; 16\, c \left(\frac{\gamma}{2}+1-\beta\right)^4 \int_\Omega |D\eta|^2 \, (1 + (1+(|Du|-1)_+)^{\gamma+2} \, g_2(1+(|Du|-1)_+) \, dx \\ &\quad \leq \; c \; (\gamma+2)^4 \int_\Omega |D\eta|^2 \, \left[1 + (1+(|Du|-1)_+)^{\gamma+2} \, g_2(1+(|Du|-1)_+ \right] \, dx \end{aligned} \tag{6.10}$$

where we used once more (6.7) and (6.9). The iteration process follows now the arguments contained in [34]; for the sake of clarity we focus on the main steps. From now on, we label the constants; this will be useful in the sequel. We set $\delta := (\gamma + 2)$ and we notice that, since $\gamma \geq 0$, then $\delta \geq 2$. Then

$$\begin{aligned} &\left\{ \int_{B_\rho} \left[1 + (1+(|Du|-1)_+)^{(\delta-2\beta)\frac{2^*}{2}} \; g_2(1+(|Du|-1)_+ \right] \, dx \right\}^{\frac{2}{2^*}} \\ &\qquad \leq \; c_1 \left(\frac{\delta^2}{R-\rho} \right)^2 \int_{B_R} \left[1 + (1+(|Du|-1))_+^{\delta} g_2(1+(|Du|-1)_+) \right] \, dx, \end{aligned} \tag{6.11}$$

$c_1 = c_1(C_3) > 0$, for all $\delta \geq 2$. We fix $\bar\rho$ and $\bar R$ such that $\bar\rho < \bar R$ and we introduce the decreasing sequence of radii $\{\rho_i\}_{i\geq 0}$

$$\rho_i = \bar\rho + \frac{\bar R - \bar\rho}{2^i}, \qquad \forall i \geq 0,$$

observing that $\bar\rho < \rho_{i+1} < \rho_i < \bar R = \rho_0$. Correspondingly we define as well the increasing sequence of exponents $\{\delta_i\}_{i\geq 0}$ such that

$$\delta_0 = 2 \qquad \text{and} \qquad \delta_{i+1} = (\delta_i - 2\beta)\,\frac{2^*}{2}, \qquad \forall i \geq 0.$$

First of all we check that $\delta_i \geq 2$ for all $i \geq 0$. By induction this is equivalent to require $\beta < 1 - \frac{2}{2^*}$ that is $2^* > \frac{2}{1-\beta}$ and this is always satisfied.

We can rewrite (6.11) with $\rho = \rho_{i+1}$, $R = \rho_i$, $\delta = \delta_i$. For every $i \geq 0$ we then obtain

$$\begin{aligned} &\left\{ \int_{B_{\rho_{i+1}}} \left[1 + (1+(|Du|-1)_+)^{\delta_{i+1}} g_2((1+|Du|-1)_+) \right] \, dx \right\}^{\frac{2}{2^*}} \\ &\qquad \leq \; c_1 \left(\frac{\delta_i^2 \, 2^{i+1}}{\bar R - \bar\rho} \right)^2 \int_{B_{\rho_i}} \left[1 + (1+(|Du|-1)_+)^{\delta_i} g_2(1+(|Du|-1)_+) \right] \, dx. \end{aligned}$$

By iterating the previous inequality, we are able to deduce

$$\left\{ \int_{B_{\rho_{i+1}}} \left[1 + (1 + (|Du| - 1)_+)^{\delta_{i+1}} g_2(1 + (|Du| - 1)_+)\right] \, dx \right\}^{\left(\frac{2}{2^*}\right)^{i+1}} \leq c_2 \int_{B_{\bar{R}}} \left[1 + (1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+)\right] \, dx,$$

where, by induction we computed

$$\begin{aligned} \delta_{i+1} &= 2\left(\frac{2^*}{2}\right)^{i+1} - 2\beta \sum_{k=1}^{i+1} \left(\frac{2^*}{2}\right)^k = 2\left(\frac{2^*}{2}\right)^{i+1} \left[1 - \beta \sum_{k=0}^{i} \left(\frac{2}{2^*}\right)^k\right] \\ &= 2\left(\frac{2^*}{2}\right)^{i+1} \left[1 - \beta \frac{1 - \left(\frac{2}{2^*}\right)^{i+1}}{1 - \frac{2}{2^*}}\right] = 2\left(\frac{2^*}{2}\right)^{i+1} \left[1 - \beta \frac{2^*}{2^* - 2}\right] + 2\beta \frac{2^*}{2^* - 2} \end{aligned} \tag{6.12}$$

and where

$$\begin{aligned} c_2 &= \prod_{k=0}^{+\infty} \left[\frac{c_1}{(\bar{R} - \bar{\rho})^2} \delta_k^2 \, 2^{k+1}\right]^{\left(\frac{2}{2^*}\right)^k} \leq \prod_{k=0}^{+\infty} \left[\frac{c_1}{(\bar{R} - \bar{\rho})^2} 4\left(\frac{2^*}{2}\right)^{2k} 2^{k+1}\right]^{\left(\frac{2}{2^*}\right)^k} \\ &\leq \left(\frac{8c_1}{(\bar{R} - \bar{\rho})^2}\right)^{\sum_{k=0}^{+\infty}\left(\frac{2}{2^*}\right)^k} (2^*)^{\sum_{k=0}^{\infty} 2k\left(\frac{2}{2^*}\right)^k} =: \frac{c_3}{(\bar{R} - \bar{\rho})^{\frac{2\,2^*}{2^*-2}}}. \end{aligned}$$

Now, by $(2.1)_2$ we have that,

$$1 + t^2 g_2(t) \leq \frac{t}{g_2(1)} g_2(1) + t^2 g_2(t) \leq \left(\frac{1}{g_2(1)} + 1\right) t^2 g_2(t). \tag{6.13}$$

So we can write

$$\left[\int_{B_{\bar{\rho}}} (1 + (|Du| - 1)_+)^{\delta_{i+1}} g_2(1 + (|Du| - 1)_+) \, dx\right]^{\left(\frac{2}{2^*}\right)^{i+1}} \leq \frac{c_4}{(\bar{R} - \bar{\rho})^{\frac{2\,2^*}{2^*-2}}} \int_{B_{\bar{R}}} \left[(1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+)\right] \, dx,$$

$c_4 = c_3 \left(\frac{1}{g_2(1)} + 1\right)$. Finally, by (6.12), $\delta_{i+1} \left(\frac{2}{2^*}\right)^{i+1} \to \left[2 - \beta \frac{2\,2^*}{2^*-2}\right]$ as $i \to +\infty$, so passing to the limit we obtain

$$\begin{aligned} &\left(\|1 + (|Du| - 1)_+\|_{L^\infty(B_\rho)}\right)^{2 - \frac{22^*}{2^*-2}\beta} \\ &\qquad = \lim_{i \to +\infty} \left[\int_{B_{\bar{\rho}}} (1 + (|Du| - 1)_+)^{\delta_{i+1}} g_2(1 + (|Du| - 1)_+) \, dx\right]^{\left(\frac{2}{2^*}\right)^{i+1}} \\ &\qquad \leq \frac{c_5}{(\bar{R} - \bar{\rho})^{\frac{2\,2^*}{2^*-2}}} \int_{B_{\bar{R}}} (1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+) \, dx. \end{aligned}$$

Therefore (6.4) is proved, in fact $n = \frac{2\,2^*}{2^*-2}$ if $n > 2$ and $2 < \frac{2\,2^*}{2^*-2}$ if $n = 2$. Notice that $0 < 2 - n\beta < 1$ since $\frac{1}{n} < \beta < \frac{2}{n}$. □

Lemma 6.3. *Assume that f satisfies the assumptions of previous lemma and (2.2) and $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a local minimizer of (1.1). Then for every $0 < \rho < R$, $\bar{B}_R \subset \Omega$, there exists a positive constant $C = C(\rho, R, C_1, C_2, \alpha, \beta, \mu, g_2(t_0))$, such that*

$$\|Du\|_{L^\infty(B_\rho)} \le C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} (1 + f(Du))\, dx \right\}^\theta \tag{6.14}$$

with $\theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}$.

Proof. Set

$$V = V(x) = (1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+).$$

By $(2.1)_2$ we have

$$\|V\|_{L^\infty(B_\rho)} \le C_\mu \left(\|1 + (|Du| - 1)_+\|_{L^\infty(B_\rho)}\right)^{2-\mu}$$

so inequality (6.4) becomes

$$\left(\|V\|_{L^\infty(B_\rho)}\right)^{\frac{2-n\beta}{2-\mu}} \le \frac{c_5}{(R-\rho)^n} \int_{B_R} V(x)\, dx.$$

Let $\alpha > 1$ satisfies (2.2) we can apply Lemma 5.1 with $v = V$, $\vartheta = \frac{2-\mu}{2-n\beta}$ and $\lambda = \frac{1}{\alpha}$. In fact we have

$$\alpha(n\beta - \mu) < 2 - \mu \quad \Longleftrightarrow \quad \frac{2-\mu}{2-n\beta}\left(1 - \frac{1}{\alpha}\right) < 1 \quad \Longleftrightarrow \quad \vartheta(1-\lambda) < 1. \tag{6.15}$$

Therefore we deduce the existence of a constant c_6 such that, for every $\varrho < R$, the following estimate holds

$$\|V\|^{\frac{1-\vartheta(1-\lambda)}{\vartheta}}_{L^\infty(B_\rho)} = \|V\|^{\frac{2-\mu-\alpha(n\beta-\mu)}{(2-\mu)\alpha}}_{L^\infty(B_\rho)} \le \frac{c_6}{(R-\rho)^n} \int_{B_R} |V|^{\frac{1}{\alpha}}\, dx.$$

Now, by $(2.1)_4$, if $|Du| \ge 1$,

$$V = (1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+) = |Du|^2 g_2(|Du|) \le C_2\,(1 + f(Du))^\alpha,$$

otherwise

$$V = (1 + (|Du| - 1)_+)^2 g_2(1 + (|Du| - 1)_+) = g_2(1) \le g_2(1)\,(1 + f(Du))^\alpha.$$

Therefore

$$\|V\|_{L^\infty(B_\rho)} \le \left[\frac{c_7}{(R-\rho)^n} \int_{B_{R_0}} (1 + f(Du))\, dx \right]^{\frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}} \tag{6.16}$$

holds for $c_7 := \max\{C_2\,, g_2(1)\}^{\frac{1}{\alpha}} c_6$. Finally, since by $(2.1)_2$ $V \ge g_2(1)|Du|$, (6.14) holds for $C = c_7^{\frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}}/g_2(1)$ and $\theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}$. □

7. Proofs of the results of Section 2

We use the following approximation Lemma.

Lemma 7.1. *Assume that* $f : \mathbb{R}^n \to [0\,,+\infty)$ *be a convex function and* $v \in W^{1,1}_{\rm loc}(\Omega)$ *such that* $f(Dv) \in L^1_{\rm loc}(\Omega)$. *For* Ω' *open set compactly contained in* Ω *and* φ_ε *be smooth mollifiers with support in* $B_\varepsilon(0)$ *we define* $v_\varepsilon = v * \varphi_\varepsilon \in \mathcal{C}^\infty(\Omega')$ *i.e.*

$$v_\varepsilon(x) = \int_{B_\varepsilon(0)} \varphi_\varepsilon(y) v(x-y)\,dy, \qquad x \in \Omega'. \tag{7.1}$$

Then, for every open ball B_ρ *compactly contained in* Ω',

$$\lim_{\varepsilon\to 0} \int_{B_\rho} f(Dv_\varepsilon)\,dx = \int_{B_\rho} f(Dv)\,dx. \tag{7.2}$$

Proof. By Jensen's inequality

$$f(Dv_\varepsilon(x)) \le \int_{B_\varepsilon(0)} \rho_\varepsilon(y)\, f(Dv(x-y))\,dy.$$

By integrating over B_ρ for ε sufficiently small we obtain

$$\begin{aligned}\int_{B_\rho} f(Dv_\varepsilon(x))\,dx &\le \int_{B_\varepsilon(0)} \rho_\varepsilon(y) \int_{B_\rho} f(Dv(x-y))\,dx\,dy \\ &\le \int_{B_\varepsilon(0)} \rho_\varepsilon(y)\,dy \int_{B_{\rho+\varepsilon}} f(Dv(x))\,dx \le \int_{B_{\rho+\varepsilon}} f(Dv(x))\,dx\end{aligned}$$

and then

$$\limsup_{\varepsilon\to 0} \int_{B_\rho} f(Dv_\varepsilon(x))\,dx \le \int_{B_\rho} f(Dv(x))\,dx.$$

By the other hand, Dv_ε converges to Dv in $L^1(B_\rho\,,\mathbb{R}^n)$, then the lower semicontinuity of the integral yields

$$\liminf_{\varepsilon\to 0} \int_{B_\rho} f(Dv_\varepsilon(x))\,dx \ge \int_{B_\rho} f(Dv(x))\,dx$$

proving (7.2). □

Proof of Theorem 2.1. Consider the functional (1.1) with f satisfying (2.1). For every $k \in \mathbb{N}$, let us consider the sequence f_k defined as follows (see [36]):

$$f_k(\xi) = f(\xi)(1-\phi(\xi)) + (f\phi) * \eta_k(\xi), \tag{7.3}$$

where η_k are standard mollifiers and $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \le \phi(\xi) \le 1$ for every $\xi \in \mathbb{R}^n$, $\phi(\xi) = 1$ if $|\xi| \le t_0 + 1$ and $\phi(\xi) = 0$ if $|\xi| \ge t_0 + 2$. $f_k \in \mathcal{C}^2(\mathbb{R}^n)$ and

$$f_k(\xi) = \begin{cases} f(\xi) & |\xi| \ge t_0 + 2 \\ (f\phi) * \eta_k(\xi) & |\xi| \le t_0 + 1. \end{cases}$$

Therefore the sequence $\{f_k\}_k$ converges to f uniformly and we can suppose $|f(\xi) - f_k(\xi)| \le 1$ for every $\xi \in \mathbb{R}^N$ and $k \in \mathbb{N}$. Moreover, for sufficiently large k, f_k is a convex function and $D^2 f_k(\xi)$ is positive defined for $|\xi| > t_0 + 1$. Since $f_k(\xi) = f(\xi)$ for $|\xi| > t_0 + 2$, then (2.1) holds with $t_0 + 2$ instead of t_0.

Let $h \in \mathcal{C}^2([0,+\infty))$ be the positive, increasing, convex function defined by

$$h(t) = \begin{cases} \frac{1}{8}(6t^2 - t^4 + 3) & t \in [0\,,1) \\ t & t \in [1\,,+\infty). \end{cases}$$

Observe that $h \in \mathcal{C}^2(\mathbb{R}^2)$, h'' is non negative and $h'' > 0$ in $[0\,,1)$. For $k \in \mathbb{N}$ denote

$$\tilde{f}_k(\xi) = f_k(\xi) + \frac{1}{k}\, h\left(\frac{|\xi|}{t_0+2}\right)$$

and define the integral functional

$$F_k(v) = \int_{B_R} \tilde{f}_k(Dv)\, dx.$$

Notice that $\tilde{f}_k \in \mathcal{C}^2(\mathbb{R}^n)$ and that it is uniformly convex on compact subsets of $\mathbb{R}^n$. Let B_R be a ball compactly contained in Ω and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional (1.1). Since $u_\varepsilon \in \mathcal{C}^2(\overline{B}_R)$, then it verifies the bounded slope condition (see for example [26] and [25, Theorem 1.1 and Theorem 1.2]) and then F_k has unique minimizer $v_{\varepsilon,k}$ among Lipschitz continuous functions in B_R with boundary value u_ε on ∂B_R. By $(2.1)_1$ and [34, (3.3)],

$$g_1(|\xi|)|\lambda|^2 \le \sum_{i,j} (\tilde{f}_k)_{\xi_i\xi_j}(\xi)\lambda_i\lambda_j \le \left[g_2(|\xi|)| + \frac{1}{k(t_0+2)}\frac{1}{|\xi|}\right]|\lambda|^2,$$

for every $\lambda, \xi \in \mathbb{R}^n$, $|\xi| \ge t_0+2$. On the other hand, since by $(2.1)_2$ $t\, g_2(t) \ge (t_0+2)g_2(t_0+2)$ for $t \ge t_0+2$,

$$g_2(t) + \frac{1}{k(t_0+2)}\frac{1}{t} \le g_2(t) + \frac{g_2(t)}{k(t_0+2)^2 g_2(t_0+2)} \le 2g_2(t)$$

for k sufficiently large. Therefore $\tilde{f}_k$ satisfies (2.1) with t_0+2 instead of t_0, $2g_2$ instead of g_2 and constants C_1 and C_2 independent from k. Moreover, as $k \to \infty$, $\tilde{f}_k \to f$ uniformly on compact subsets of $\mathbb{R}^n$ and then

$$\lim_{k\to\infty} \int_{B_R} \tilde{f}_k(Dv)\, dx = \int_{B_R} f(Dv)\, dx \qquad \text{for every } v \in W^{1,\infty}(B_R).$$

Since $\tilde{f}_k$ is uniformly convex on compact sets and $v_{\varepsilon,k}$ is Lipschitz continuous in B_R, all the assumptions of Lemma 6.3 hold and, by the a priori estimate (6.14), for every $\rho < R$ there exists a positive constant C depending on ρ, R, α, β, μ, C_1, C_2, $g_2(t_0)$, but independent on k such that

$$\begin{aligned} \|Dv_{\varepsilon,k}\|_{L^\infty(B_\rho)} &\le C\left\{\frac{1}{(R-\rho)^n}\int_{B_R} 1 + \tilde{f}_k(Dv_{\varepsilon,k})\, dx\right\}^\theta \\ &\le C\left\{\frac{1}{(R-\rho)^n}\int_{B_R} 1 + \tilde{f}_k(Du_\varepsilon)\, dx\right\}^\theta, \end{aligned}$$

$\theta = \frac{(2-\mu)\alpha}{2-\mu-\alpha(n\beta-\mu)}$, where the last inequality depends by the minimality of $v_{\varepsilon,k}$. Therefore, for every $\varepsilon > 0$ we have

$$\limsup_{k\to\infty} \|Dv_{\varepsilon,k}\|_{L^\infty(B_\rho)} \le C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} (1 + f(Du_\varepsilon)\, dx \right\}^\theta = M_\varepsilon.$$

The sequence $v_{\varepsilon,k}$ is bounded in $W^{1,\infty}(B_\rho)$ with respect to k, then there exists a subsequence $k_j \to 0$, such that $\{v_{\varepsilon,k_j}\}$ is weakly* convergent to $\bar{v}_\varepsilon$ in $W^{1,\infty}(B_\rho)$ and for every $\rho < R$

$$\|D\bar{v}_\varepsilon\|_{L^\infty(B_\rho)} \le C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} (1 + f(Du_\varepsilon)\, dx \right\}^\theta . \tag{7.4}$$

We prove that v_{ε,k_j} converges to $\bar{v}_\varepsilon$ in $W^{1,1}(B_R)$ and then $\bar{v}_\varepsilon \in u + W_0^{1,1}(B_R)$. Indeed by $(2.1)_5$ and the minimality of v_{ε,k_j}, as $j \to \infty$ we have

$$\int_{B_R} f(Dv_{\varepsilon,k_j})\, dx \le 1 + \int_{B_R} \tilde{f}_{k_j}(Dv_{\varepsilon,k_j})\, dx \le 1 + \int_{B_R} \tilde{f}_{k_j}(Du_\varepsilon)\, dx \to 1 + \int_{B_R} f(Du_\varepsilon)\, dx.$$

By de la Vallée-Poussin Theorem we can choose the sequence k_j such that $Dv_{\varepsilon,k_j} \rightharpoonup D\bar{v}_\varepsilon$ in $L^1(B_R)$ and then $v_{\varepsilon,k_j} - u_\varepsilon \rightharpoonup (\bar{v}_\varepsilon - u_\varepsilon) \in W_0^{1,1}(B_R)$. On the other hand, for every $\delta > 0$ and for every k sufficiently large, $|\tilde{f}_k(\xi) - f(\xi)| \le \delta$, for every $|\xi| \le M + 1$, therefore by the minimality of v_{ε,k_j}

$$\begin{aligned} \int_{B_\rho} f(Dv_{\varepsilon,k_j})\, dx &= \int_{B_\rho} (f(Dv_{\varepsilon,k_j}) - \tilde{f}_{k_j}(Dv_{\varepsilon,k_j})) + \tilde{f}_{k_j}(Dv_{\varepsilon,k_j})\, dx \\ &\le \int_{B_R} \tilde{f}_{k_j}(Du_\varepsilon)\, dx + \delta|B_R|. \end{aligned}$$

By lower semicontinuity in $W^{1,1}(B_R)$, passing to the limit for $j \to \infty$, we get

$$\begin{aligned} \int_{B_\rho} f(D\bar{v}_\varepsilon)\, dx \le \liminf_{j\to\infty} \int_{B_\rho} f(Dv_{\varepsilon,k_j})\, dx &\le \lim_{j\to\infty} \int_{B_R} \tilde{f}_{k_j}(Du_\varepsilon)\, dx + \delta|B_R| \\ &= \int_{B_R} f(Du_\varepsilon)\, dx + \delta|B_R| \end{aligned}$$

for every $\delta > 0$ and $\rho < R$, and then for $\rho \to R$ and $\delta \to 0$

$$\int_{B_R} f(D\bar{v}_\varepsilon)\, dx \le \int_{B_R} f(Du_\varepsilon)\, dx$$

Again by de la Vallée-Poussin Theorem and (7.4), we have that there exists a sequence $\varepsilon_j \to 0$ such that $\bar{v}_{\varepsilon_j} - u_{\varepsilon_j} \rightharpoonup \bar{v} - u$ in $W_0^{1,1}(B_R)$ and $\{\bar{v}_{\varepsilon_j}\}_j$ is weakly * convergent to $\bar{v}$ in $W^{1,\infty}(B_\rho)$ for every $0 < \rho < R$. By the lower semicontinuity of the functional

$$\int_{B_R} f(D\bar{v})\, dx \le \liminf_{j\to\infty} \int_{B_R} f(D\bar{v}_{\varepsilon_j})\, dx \le \lim_{j\to\infty} \int_{B_R} f(Du_{\epsilon_j})\, dx = \int_{B_R} f(Du)\, dx. \tag{7.5}$$

Then $\bar{v}$ is a minimizer for (1.1) with $\Omega = B_R$. Moreover from (7.2) and (7.4) we have that $\bar{v}_\varepsilon$ converges to $\bar{v}$ in $W^{1,\infty}_{\text{loc}}$ and

$$\begin{aligned} \|D\bar{v}\|_{L^\infty(B_\rho)} \leq \liminf_{j\to\infty} \|D\bar{v}_{\varepsilon_j}\|_{L^\infty(B_\rho)} &\leq \lim_{j\to\infty} C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} (1+f(Du_{\varepsilon_j})\, dx \right\}^\theta \\ &= C \left\{ \frac{1}{(R-\rho)^n} \int_{B_R} (1+f(Du)\, dx \right\}^\theta . \end{aligned} \tag{7.6}$$

Therefore $\bar{v}$ and u are two different minimizers of F in B_R. Since $f(\xi)$ is strictly convex for $|\xi| > t_0$, by proceeding as in [19] it is possible to prove that the set

$$E_0 := \left\{ x \in B_R : \left| \frac{Du(x) + D\bar{v}(x)}{2} \right| > t_0 \right\}.$$

has zero measure. Therefore

$$\|Du\|_{L^\infty(B_\rho)} \leq \|Du + D\bar{v}\|_{L^\infty(B_\rho)} + \|D\bar{v}\|_{L^\infty(B_\rho)} \leq 2t_o + \|D\bar{v}\|_{L^\infty(B_\rho)}.$$

□

Lemma 7.2. *Let f be a positive function in $\mathcal{C}^2(\{\xi \in \mathbb{R}^n : |\xi| \geq t_0\})$ satisfying ellipticity conditions* (2.4). *Then there exists $\bar{t} \geq t_0$ such that*

$$\frac{m}{2(p-1)}|\xi|^p \leq f(\xi) \leq \frac{2M}{q-1}|\xi|^q, \qquad |\xi| \geq \bar{t} \tag{7.7}$$

if $1 < p \leq q$, $f(\xi) \geq \frac{m}{2}|\xi|\log|\xi|$ or $f(\xi) \leq 2M|\xi|\log|\xi|$ if $p = 1$ or $q = 1$ respectively.

Proof. Without loss of generality we can suppose $t_0 = 1$. We will prove the left hand side in (7.7), the right hand side being analogous. We consider the real function $\varphi : [1, +\infty) \to \mathbb{R}$ defined by $\varphi(t) = f\left(t\frac{\xi}{|\xi|}\right)$. Then $\varphi \in \mathcal{C}^2[1, +\infty)$ and its first and second derivatives hold

$$\varphi'(t) = \left(D_\xi f\left(t\frac{\xi}{|\xi|}\right), \frac{\xi}{|\xi|} \right); \qquad \varphi''(t) = \sum_{i,j=1}^n f_{\xi_i\xi_j}\left(t\frac{\xi}{|\xi|}\right)\frac{\xi_i\xi_j}{|\xi|^2}$$

and, by (2.4), $\varphi''(t) \geq mt^{p-2}$. Therefore, integrating from 1 to t we obtain

$$\varphi'(t) - \varphi'(1) \geq \begin{cases} \frac{m}{p-1}(t^{p-1}-1) & \text{if } p > 1 \\ m\log t & \text{if } p = 1 \end{cases}$$

and again

$$\varphi(t) - \varphi(1) \geq Df\left(\frac{\xi}{|\xi|}\right)(t-1) + \begin{cases} \frac{m}{p-1}(t^p - t) & \text{if } p > 1 \\ m(t\log t - t) & \text{if } p = 1. \end{cases}$$

Therefore, for $t = |\xi|$ we have

$$f(\xi) \geq -L(|\xi|-1) + \begin{cases} \frac{m}{p-1}(|\xi|^p - |\xi|) & \text{if } p > 1 \\ m(|\xi|\log|\xi| - |\xi|) & \text{if } p = 1 \end{cases}$$

where $L = \max\{|Df(v)| : |v| = 1\}$. □

Proof of Corollary 2.2. We have to prove that the assumptions of Theorem 2.1 are satisfied. First of all, notice that assumptions $(2.1)_1$ and$(2.1)_2$ hold for $g_1(t) = mt^{p-2}$ and $g_2(t) = Mt^{q-2}$. Moreover, for every $t \geq 1$,

$$(g_2(t))^{\frac{2}{2^*}} = M^{2/2^*} t^{(q-2)\frac{2}{2^*}} = mt^{p-2} \frac{M^{2/2^*}}{m} t^{(q-2)\frac{2}{2^*}-(p-2)},$$

then $(2.1)_3$ holds for $\beta \geq \bar{\beta} = \frac{n-2}{2n} q - \frac{p}{2} + \frac{2}{n}$. By Lemma 7.2, for $\alpha \geq \bar{\alpha} = q/p$ and $|\xi|$ large enough,

$$g_2(|\xi|)|\xi|^2 = M|\xi|^q \leq M \left(\frac{2p-2}{m} \right)^\alpha \left(\frac{m}{2p-2} |\xi|^p \right)^\alpha \leq M \left(\frac{4}{m} \right)^\alpha [1 + f(\xi)]^\alpha$$

and $(2.1)_4$ holds. Moreover, by Lemma 7.2 also $(2.1)_5$ holds. Therefore, all the assumptions of Theorem 2.1 are satisfied if (2.2) is satisfied, that is

$$\alpha < \frac{q}{n\beta + q - 2} \quad \Longleftrightarrow \quad \frac{q}{p} < \frac{q}{\frac{n}{2}q - \frac{n}{2}p} \quad \Longleftrightarrow \quad \frac{q}{p} < \frac{n+2}{n}$$

since $\mu = 2 - q$. In order to obtain the correct exponent in (2.6), notice that in this case by (6.16) we can conclude

$$\|Du\|^q_{L^\infty(B_\rho)} \leq \left[\frac{c_7}{(R-\rho)^n} \int_{B_{R_0}} (1 + f(Du))\, dx \right]^{\frac{q\alpha}{q-\alpha(n\beta+q-2)}}.$$

Since $\frac{q\alpha}{q-\alpha(n\beta+q-2)} = \frac{2}{(n+2)p-nq}$, we get (2.6). □

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