Hamiltonian/Stroh formalism for reversible poroelasticity (and thermoelasticity)

Andrea Nobili[1]

Department of Engineering Enzo Ferrari, University of Modena and Reggio Emilia, via Vivarelli 10, 41125 Modena, Italy

Abstract

Stroh's sextic formalism represents the equilibrium equations of anisotropic elasticity in a particularly attractive form, that is most suitable for studying layered and composite materials and time harmonic problems. Taking advantage of the fact that the Stroh formalism really amounts to the canonical form of the equations in the Hamiltonian sense, the case of Biot's reversible (i.e. no fluid dissipation) poroelasticity is here addressed, in the absence of a fluid pressure gradient. This framework is the same as thermoelasticity of perfect conductors. Two Hamiltonian formulations are developed: the first describes both the solid and the fluid phases and it exhibits, besides energy conservation, momentum conservation, as a result of pressure uniformity. The second is restricted to the solid skeleton and parallels anisotropic elasticity, although with Stroh matrices that account for fluid coupling. The case of weak fluid-solid coupling is also considered and it produces a perturbation from anisotropic elasticity with the same structure as incompressibility, although in an "opposing" manner. This comparison suggests that the incompressibility limit introduced by Biot should be revised. The energy conservation integral and the edge impedance matrix are also illustrated.

Keywords: Stroh formalism, reversible poroelasticity, thermoelasticity, Hamiltonian form

Email address: andrea.nobili@unimore.it (Andrea Nobili)

1. Introduction

The foundation of Stroh sextic formalism is laid out in a pair of celebrated papers concerning dislocations (Stroh, 1958) and harmonic motion (Stroh, 1962) in generally anisotropic materials (in plane strain). This framework provides a substantial improvement over the already established Eshelby-Reid-Shockley form of the equations of elastostatics (Ting, 1996). Indeed, although both methods share the fact that mechanical features are interpreted and described under the unifying lens of linear algebra, Stroh's approach exhibits very distinctive features for the involved matrices. In fact, solutions are built in terms of eigenvalues and eigenvectors of a block matrix, the so-called Stroh fundamental elastic matrix, endowed with many striking properties (Barnett, 2000). Since then, the method has been extensively applied to composite materials, harmonic wave propagation, crack and dislocations, instability and many more topics (Ting, 1996). Given its success, it is little wonder that extensions of the method have been proposed outside its original domain, to address, for example, constrained materials (Chadwick and Smith, 1977), anisotropic plates (Fu, 2007) and internally constrained micro-polar solids (Nobili and Radi, 2022). In general, the success of the procedure hangs on the careful choice of the unknown variables, which can be rather tricky unless somehow guided. In fact, many contributions exist in the literature where a trial-and-error approach was used (see Fu (2007)). As an illustration, Hwu (2003) analyses coupled stretching-bending modes in anisotropic laminates through a modification of the Lekhnitskii formalism (for details on which see Ting (1996)) in an attempt to recover the properties specific to the Stroh form. Recently, Fu (2003) developed a Stroh-like formulation for determining the dispersion relation of edge waves in generally anisotropic plates under the sole restriction that the mid-plane is a plane of material symmetry. In that work, Fu capitalized on the observation that the Stroh formalism is really an Hamiltonian formulation where a space variable is treated in time-like fashion, already available in the literature (Barnett, 2000), to develop a guiding principle for the right choice of the unknown pairs, namely the principle of energy conjugation. Successively, this approach was used in (Fu and Brookes, 2006) to study edge waves in asymmetrically laminated plates, for which in-plane and out-of-plane deformations are coupled by anisotropy. By the same method, Fu (2007) studies incompressible anisotropic materials and anisotropic plates, and results are later extended by Edmondson and Fu (2009) to generally constrained and pre-stressed anisotropic materials. The procedure paves the way for the application of the surface-impedance matrix for studying localized waves (Fu, 2005).

Thus far, a classical Stroh formalism could be retrieved, by which a right eigenvalue problem is finally obtained (as presently explained). Yet, the Hamiltonization of any mechanical model may be carried out by the same principles and the outcome, in general, may not correspond to a classical Stroh-like structure. As a case in point, Fu and Kaplunov (2012) study waves localized at the edge of isotropic thin cylindrical shells and find that the fundamental elastic matrix is in fact wavenumber dependent. This result, which is typical of dispersive systems, is also retrieved by Nobili and Radi (2022) in the context of the indeterminate couple-stress theory of elasticity. The structure of the Stroh formalism is now supplemented by a right hand side that is proportional to the unknowns (i.e. the problem is still linear). Therefore, the very form of the Stroh-like canonical system already reveals important informations on the problem under scrutiny.

Biot's poroelasticity is a very successful phenomenological theory with enormous practical implications in the fields of seismology and seismic exploration, geology and geotechnical structures, soil testing and characterization, to name only a few Dullien (2012). The literature on this topic is very extensive and moves in many directions, for example, concerning wave propagation in porous media, see the review paper by Corapcioglu and Tuncay (1996). Efforts in the direction of connecting this theory to the theory of mixtures or to micromechanical theories have been long going, with mixed success, see, among many, Lopatnikov and Cheng (2004). Extensions of the theory have been proposed in the many directions, for example introducing double (Berryman and Wang,

2000) or multi- porosities (Pramanik et al., 2024), finite elastic deformations (Norris and Grinfeld, 1995) or even piezoelectric effects (Sharma, 2010). Still, no Stroh-like formalism may be traced in the literature, possibly on the grounds that multi-field theories may prove impervious to this framework.

In this paper, we Hamiltonize the equations of Biot's poroelasticity in the absence of dissipation, i.e. in the context of reversible processes (thermostatics) and in the absence of a fluid pressure gradient (Biot, 1955, 1956a). This same framework may be applied to thermoelastostatics of perfect conductors, where in fact temperature plays the role of the fluid pressure (Biot, 1956b). Inertia effects are only considered inasmuch as they may be incorporated into the material properties in the form of time-harmonic contributions. Focus is set on the determination of the canonical formalism and on the properties it reveals.

2. Reversible poroelasticity

Let $\boldsymbol{u}$ and $\boldsymbol{U}$ denote the displacement in the solid and in the fluid phase, respectively. Besides, the fluid-to-solid displacement per unit volume of the poroelastic medium reads

$$\boldsymbol{w} = f(\boldsymbol{U} - \boldsymbol{u}), \tag{1}$$

where f is the *effective porosity*, generally not uniform, that represents the interconnected pore space. In particular, the porosity is defined as the ratio between the volume of interconnected pores, V_p, over the bulk volume V_b, the latter being obtained by $V_b = V_p + V_s$, i.e. summing the pore volume to the volume occupied by the solid skeleton, see (Biot, 1955). Following Biot (1962), in this theory closed porosity is assumed to be part of the solid skeleton. Also, we let

$$e = \operatorname{div} \boldsymbol{u}, \quad \zeta = -\operatorname{div} \boldsymbol{w}, \tag{2}$$

that provide the volume *increment* for the solid and the fluid phase, respectively (indeed the fluid increment is obtained by the inflow of $\boldsymbol{w}$). In particular, we have the connection

$$-\zeta = (\boldsymbol{U} - \boldsymbol{u}) \cdot \operatorname{grad} f + f(\epsilon - e), \tag{3}$$

where we have let $\epsilon = \operatorname{div} \boldsymbol{U}$. In the case of uniform porosity, we retrieve the result given in Biot (1962)

$$\epsilon = e - f^{-1}\zeta. \tag{4}$$

Let the rank-2 tensor $\boldsymbol{T}$ denote the *total stress*, that is obtained summing the stress in the solid phase $\boldsymbol{\sigma}$ with the stress in the fluid phase $\boldsymbol{\sigma}_f = -fp_f\mathbf{1}$, where, here and after, $\mathbf{1}$ is the rank-3 identity tensor and p_f is the fluid pressure (positive when compressive) per unit area of the fluid phase. Sometimes, to refer pressure to the unit bulk area, the shorthand $\sigma_f = -fp_f$ is introduced. Let (O, x_1, x_2, x_3) denote an orthogonal reference frame and $\boldsymbol{n}$ be the unit vector normal to any relevant directed surface S. Alongside the axis (x_1, x_2, x_3), we introduce an orthonormal set of basis vectors, $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$, such that $\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \delta_{ij}$, with the usual understanding that twice repeated subscripts are summed over in the set $\{1, 2, 3\}$. Here, δ_{ij} is zero for $i \neq j$ and 1 for $i = j$. We define the fundamental force vectors in a generally anisotropic medium with elastic constants c_{ijkl}

$$\boldsymbol{t_1} = \boldsymbol{T}\boldsymbol{e_1} = \mathrm{Q}\boldsymbol{u}_{,1} + \mathrm{R}\boldsymbol{u}_{,2} - r\zeta\boldsymbol{e_1}, \tag{5a}$$

$$\boldsymbol{t_2} = \boldsymbol{T}\boldsymbol{e_2} = \mathrm{R}^T\boldsymbol{u}_{,1} + \mathrm{T}\boldsymbol{u}_{,2} - r\zeta\boldsymbol{e_2}, \tag{5b}$$

where $Q_{ij} = c_{i1j1}$, $R_{ij} = c_{i1j2}$, $T_{ij} = c_{i2j2}$ are the usual Stroh matrices. In particular, Q and T are symmetric, i.e. $\mathrm{Q} = \mathrm{Q}^T$ and $\mathrm{T} = \mathrm{T}^T$, and positive definite, provided the strain energy is a positive function (Ting, 1996, §6.1). Here, r denotes the cross coupling term between volume changes in the solid and in the fluid (denoted by C in (Biot, 1962, Eq.(3.5)), and by Q/f in (Corapcioglu and Tuncay, 1996, Eq.(2.16))). In this paper, we assume that cross coupling occurs in isotropic fashion, for transverse anisotropy see, for example, Biot (1955, Eq.(3.2)). Besides, it is assumed that dependent variables are independent from x_3, i.e. $\partial/\partial x_3() = 0$. In a steady-state motion with velocity v in the x_1-direction, the matrix Q is simply replaced by $\mathrm{Q} - \rho v^2\mathrm{I}$, where ρ is the density of the solid skeleton and I is the identity matrix. Besides, for an isotropic solid, it is $c_{ijkl} = 2\mu\delta_{ik}\delta_{jl} + \lambda_c\delta_{kl}\delta ij$, where μ and λ_c are the Lamé moduli (Nobili and Radi, 2022).

For a compressible fluid, it is

$$p_f = -re + m\zeta, \tag{6}$$

where m is the compressibility modulus for the fluid, defined as the fluid pressure required to force a unit volume of fluid into the pore structure while keeping the solid volume unchanged, i.e. $e = 0$. As pointed out by Biot and Willis (1957), the reversibility assumption, by which a stored elastic potential is admitted, identifies the coupling coefficient in the last term in (5) with that in (6). Also, following Biot and Willis (1957), it is

$$f \leq \alpha = r/m < 1. \tag{7}$$

Biot (1956b) showed that this framework parallels that of thermo-elasticity, with the pressure p_f playing the same role as temperature.

We are now in the position to write the potential elastic energy minus the work done by the applied external forces over the body B (i.e. the total energy in the sense of Eshelby)

$$\mathcal{L} = \int_B W \mathrm{d}V - \int_{\partial B} \left(\boldsymbol{t_0} \cdot \boldsymbol{u} - p_{f0} \boldsymbol{n} \cdot \boldsymbol{w} \right) \mathrm{d}S, \tag{8}$$

where we have let the stored potential energy density

$$W = \tfrac{1}{2} \left(\boldsymbol{T} \cdot \operatorname{grad} \boldsymbol{u} + p_f \zeta \right). \tag{9}$$

Here, $\boldsymbol{t_0}$ and p_{f0} are the prescribed surface force and fluid pressure over the body boundary ∂B with unit normal $\boldsymbol{n}$. For the sake of simplicity, no body force is considered. With a little abuse of notation yet in favour of tidiness, an interposed dot denotes the scalar product between both tensor and vector pairs, i.e. in components $\boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{ij}$ and $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$, respectively. By strong ellipticity, Q and T are positive definite and $m > 0$.

In a reversible process (i.e. thermostatics), the imposed boundary pressure should not trigger movement of the fluid phase and, therefore, p_{f0} is constant on the surface ∂B and p_f is equally constant *throughout the body* (Biot, 1962). This pressure distribution holds also in steady state motion of the solid provided that we take no account of dissipation. The mins sign associated with the fluid pressure p_{f0} is a consequence of it being positive in compression. In Eq.(8), we have let the shorthand

$$\boldsymbol{T} \cdot \operatorname{grad} \boldsymbol{u} = \boldsymbol{t_1} \cdot \boldsymbol{u}_{,1} + \boldsymbol{t_2} \cdot \boldsymbol{u}_{,2},$$

where it is understood that $\operatorname{grad} \boldsymbol{u} = \boldsymbol{u}_{,1} \otimes \boldsymbol{e_1} + \boldsymbol{u}_{,2} \otimes \boldsymbol{e_2}$ and a subscript comma denotes differentiation, e.g. $\boldsymbol{u}_{,1} = \partial \boldsymbol{u}/\partial x_1$. Here, we have used the vector dyadic that, for any pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, yields the rank-2 tensor $\boldsymbol{a} \otimes \boldsymbol{b}$ such that, for any vector $\boldsymbol{c}$, $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$.

The equilibrium equations read

$$\boldsymbol{t_1}_{,1} + \boldsymbol{t_2}_{,2} = \boldsymbol{o}, \tag{10a}$$

$$\operatorname{grad} p_f = \boldsymbol{o}, \tag{10b}$$

where the last is the equilibrium version of Darcy's law[1], see Biot (1962, Eq.(7.2)). Without loss of generality, we assume that the boundary conditions are only expressed in terms of forces

$$\boldsymbol{T}\boldsymbol{n} = \boldsymbol{t_0} \quad \text{and} \quad p_f = p_{f0}, \qquad \text{for } \boldsymbol{x} \in \partial B, \tag{11}$$

where ∂B is the frontier of the body B. By the divergence theorem , one can rewrite the total energy in terms of a single volume integral

$$\mathcal{L} = -\int_B L \mathrm{d}V,$$

where we have introduced the Lagrangian density L

$$L(\boldsymbol{u}_{,1}, \boldsymbol{u}_{,2}, \operatorname{div} \boldsymbol{w}) = \tfrac{1}{2}\boldsymbol{T} \cdot \operatorname{grad} \boldsymbol{u} + \tfrac{1}{2} p_f \zeta. \tag{12}$$

In light of Eqs.(5) and (6), this may be rewritten as

$$\begin{aligned} L(\boldsymbol{u}_{,1}, \boldsymbol{u}_{,2}, \boldsymbol{w}_{,1}, \boldsymbol{w}_{,2}) = {} & \tfrac{1}{2}\boldsymbol{u}_{,1} \cdot \mathrm{Q}\boldsymbol{u}_{,1} + \boldsymbol{u}_{,1} \cdot \mathrm{R}\boldsymbol{u}_{,2} + \tfrac{1}{2}\boldsymbol{u}_{,2} \cdot \mathrm{T}\boldsymbol{u}_{,2} \\ & - \tfrac{1}{2} r\zeta(\boldsymbol{e_1} \cdot \boldsymbol{u}_{,1} + \boldsymbol{e_2} \cdot \boldsymbol{u}_{,2}) + \tfrac{1}{2}(-re + m\zeta)\zeta, \end{aligned} \tag{13}$$

[1] Darcy's law emerges from considering an irreversible process and the attached *dissipation function*, that is a quadratic form in $\partial \boldsymbol{w}/\partial t$

where $\boldsymbol{u}_{,1} \cdot \boldsymbol{e_1} + \boldsymbol{u}_{,2} \cdot \boldsymbol{e_2} = \operatorname{div} \boldsymbol{u} = e$. Clearly, this formulation admits the Stroh formalism because, unlike internally constraints solids (Nobili and Radi, 2022), both displacement vectors, $\boldsymbol{u}$ and $\boldsymbol{w}$, appear only in differentiated form. The Lagrangian density becomes

$$
\begin{aligned}
L(\boldsymbol{u}_{,1}, \boldsymbol{u}_{,2}, \boldsymbol{w}_{,1}, \boldsymbol{w}_{,2}) &= \tfrac{1}{2}\boldsymbol{u}_{,1} \cdot \mathrm{Q}\boldsymbol{u}_{,1} + \boldsymbol{u}_{,1} \cdot \mathrm{R}\boldsymbol{u}_{,2} + \tfrac{1}{2}\boldsymbol{u}_{,2} \cdot \mathrm{T}\boldsymbol{u}_{,2} \\
&+ r(\boldsymbol{e_1} \cdot \boldsymbol{w}_{,1} + \boldsymbol{e_2} \cdot \boldsymbol{w}_{,2})(\boldsymbol{e_1} \cdot \boldsymbol{u}_{,1} + \boldsymbol{e_2} \cdot \boldsymbol{u}_{,2}) + \tfrac{1}{2} m(\boldsymbol{e_1} \cdot \boldsymbol{w}_{,1} + \boldsymbol{e_2} \cdot \boldsymbol{w}_{,2})^2, \quad (14)
\end{aligned}
$$

whence the Euler-Lagrange equations read

$$
\frac{d}{dx_1}\frac{\partial L}{\partial \boldsymbol{u}_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial \boldsymbol{u}_{,2}} = 0, \tag{15a}
$$

$$
\frac{d}{dx_1}\frac{\partial L}{\partial \boldsymbol{w}_{,1}} + \frac{d}{dx_2}\frac{\partial L}{\partial \boldsymbol{w}_{,2}} = 0, \tag{15b}
$$

which is clearly in the Stroh form once we settle for either coordinate to act as a time-like variable, say x_2 as in Fu (2007). Eq.(15a) gives

$$
(\mathrm{Q}\boldsymbol{u}_{,1} + \mathrm{R}\boldsymbol{u}_{,2} - r\zeta\boldsymbol{e_1})_{,1} + (\mathrm{R}^T\boldsymbol{u}_{,1} + \mathrm{T}\boldsymbol{u}_{,2} - r\zeta\boldsymbol{e_2})_{,2} = \boldsymbol{o}, \tag{16}
$$

that corresponds to the equilibrium equation (10a), provided that we account for (5). Similarly, Eq.(15b) lends

$$
(re - m\zeta)_{,1}\boldsymbol{e_1} + (re - m\zeta)_{,2}\boldsymbol{e_2} = \boldsymbol{o}, \tag{17}
$$

that indeed amounts to Eq.(10b), once acknowledging for (6).

3. Hamiltonian formalism

We now introduce the Hamiltonian formalism by treating x_2 as a time-like variable (Fu, 2007). Consequently, differentiation with respect to x_2 will be denoted by a superscript dot. For reasons that shall be presently apparent, we let

$$
\bar{\mathrm{Q}} = \mathrm{Q} - \frac{r^2}{m}\boldsymbol{e_1} \otimes \boldsymbol{e_1}, \quad \bar{\mathrm{R}} = \mathrm{R} - \frac{r^2}{m}\boldsymbol{e_1} \otimes \boldsymbol{e_2}, \quad \bar{\mathrm{T}} = \mathrm{T} - \frac{r^2}{m}\boldsymbol{e_2} \otimes \boldsymbol{e_2}, \tag{18}
$$

whence we may rewrite (5) as

$$
\boldsymbol{t_1} = \bar{\mathrm{Q}}\boldsymbol{u}_{,1} + \bar{\mathrm{R}}\dot{\boldsymbol{u}} - r\frac{p_f}{m}\boldsymbol{e_1}, \tag{19a}
$$

$$
\boldsymbol{t_2} = \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \bar{\mathrm{T}}\dot{\boldsymbol{u}} - r\frac{p_f}{m}\boldsymbol{e_2}. \tag{19b}
$$

Eq.(14) becomes

$$L(\boldsymbol{u}_{,1},\dot{\boldsymbol{u}},\boldsymbol{w}_{,1},\dot{\boldsymbol{w}}) = \tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\bar{\mathrm{Q}}\boldsymbol{u}_{,1} + \boldsymbol{u}_{,1}\cdot\bar{\mathrm{R}}\dot{\boldsymbol{u}} + \tfrac{1}{2}\dot{\boldsymbol{u}}\cdot\bar{\mathrm{T}}\dot{\boldsymbol{u}} \\ + \tfrac{1}{2}m^{-1}\left[r\left(\boldsymbol{e_1}\cdot\boldsymbol{u}_{,1} + \boldsymbol{e_2}\cdot\dot{\boldsymbol{u}}\right) + m(\boldsymbol{e_1}\cdot\boldsymbol{w}_{,1} + \boldsymbol{e_2}\cdot\dot{\boldsymbol{w}})\right]^2. \quad (20)$$

from which conjugate momenta are immediately obtained

$$\boldsymbol{p_1} = \frac{\partial L}{\partial \dot{\boldsymbol{u}}} = \boldsymbol{t_2}, \quad (21a)$$

$$\boldsymbol{p_2} = \frac{\partial L}{\partial \dot{\boldsymbol{w}}} = (re - m\zeta)\boldsymbol{e_2} = -p_f\boldsymbol{e_2}. \quad (21b)$$

Solving Eq.(21b) for ζ gives

$$\zeta = m^{-1}\left(p_f + re\right), \quad (22)$$

while solving Eq.(19b) for $\dot{\boldsymbol{u}}$ gives

$$\dot{\boldsymbol{u}} = \bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right). \quad (23)$$

Scalar multiplication of (23) throughout by $\boldsymbol{e_2}$ lends

$$\dot{\boldsymbol{u}}\cdot\boldsymbol{e_2} = \zeta_1\bar{\mathrm{T}}^{-1}\bar{\boldsymbol{t}}_{\boldsymbol{2}}\cdot\boldsymbol{e_2}, \quad (24)$$

where we have let the shorthand

$$\bar{\boldsymbol{t}}_{\boldsymbol{2}} = \boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + r\left(\zeta - \frac{r}{m}\boldsymbol{u}_{,1}\cdot\boldsymbol{e_1}\right)\boldsymbol{e_2},$$

and, as in Fu (2007, Eq.(3.12)), it is

$$\zeta_1^{-1} = 1 + \frac{r^2}{m}\boldsymbol{e_2}\cdot\bar{\mathrm{T}}^{-1}\boldsymbol{e_2} > 1, \quad (25)$$

whose last term is always positive by virtue of strong ellipticity (see the Appendix). Hence, plugging (24) into (23), it is finally

$$\dot{\boldsymbol{u}} = \bar{\mathrm{T}}^{-1}\mathrm{P}\bar{\boldsymbol{t}}_{\boldsymbol{2}}, \quad (26)$$

having let the projector (we have used the symmetry of $\bar{\mathrm{T}}$)

$$\mathrm{P} = \boldsymbol{1} - \frac{r^2}{m}\zeta_1\boldsymbol{e_2}\otimes\bar{\mathrm{T}}^{-1}\boldsymbol{e_2}. \quad (27)$$

We note that

$$\mathrm{P}\boldsymbol{e_2} = \zeta_1 \boldsymbol{e_2}, \quad \text{and} \quad \bar{\mathrm{T}}^{-1}\mathrm{P} \in \mathrm{Sym}\,, \tag{28}$$

whence Eq.(26) may be rewritten as

$$\dot{\boldsymbol{u}} = \bar{\mathrm{T}}^{-1}\mathrm{P}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T \boldsymbol{u}_{,1}\right) + r\zeta_1 \left(\zeta - \frac{r}{m}\boldsymbol{u}_{,1}\cdot\boldsymbol{e_1}\right)\bar{\mathrm{T}}^{-1}\boldsymbol{e_2}. \tag{29}$$

Indeed, scalar multiplication by $\boldsymbol{e_2}$, in view of the properties (28), immediately lends (24).

In similar fashion, in light of Eqs.(2,24), Eq.(22) yields

$$-\dot{\boldsymbol{w}}\cdot\boldsymbol{e_2} = \boldsymbol{w}_{,1}\cdot\boldsymbol{e_1} + \frac{p_f}{m} + \frac{r}{m}\boldsymbol{u}_{,1}\cdot\boldsymbol{e_1} + \frac{r}{m}\bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right)\cdot\boldsymbol{e_2}. \tag{30}$$

We introduce the Hamiltonian density

$$\begin{aligned} H &= \boldsymbol{t_2}\cdot\dot{\boldsymbol{u}} + \boldsymbol{p_2}\cdot\dot{\boldsymbol{w}} - L \\ &= \boldsymbol{t_2}\cdot\dot{\boldsymbol{u}} - p_f\boldsymbol{e_2}\cdot\dot{\boldsymbol{w}} - \tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\bar{\mathrm{Q}}\boldsymbol{u}_{,1} - \boldsymbol{u}_{,1}\cdot\bar{\mathrm{R}}\dot{\boldsymbol{u}} - \tfrac{1}{2}\dot{\boldsymbol{u}}\cdot\bar{\mathrm{T}}\dot{\boldsymbol{u}} - \tfrac{1}{2}m^{-1}p_f^2, \end{aligned}$$

whence

$$\begin{aligned} H = \tfrac{1}{2}&\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right)\cdot\bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right) \\ &+ p_f\left(\boldsymbol{w}_{,1}\cdot\boldsymbol{e_1} + \frac{p_f}{m} + \frac{r}{m}\boldsymbol{u}_{,1}\cdot\boldsymbol{e_1}\right) - \tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\bar{\mathrm{Q}}\boldsymbol{u}_{,1} - \tfrac{1}{2}m^{-1}p_f^2, \end{aligned}$$

and finally

$$\begin{aligned} H = \tfrac{1}{2}&\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right)\cdot\bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right) \\ &+ p_f\left(\boldsymbol{w}_{,1} + \frac{r}{m}\boldsymbol{u}_{,1}\right)\cdot\boldsymbol{e_1} - \tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\bar{\mathrm{Q}}\boldsymbol{u}_{,1} + \tfrac{1}{2}\frac{p_f^2}{m}. \end{aligned} \tag{31}$$

As well known, the canonical equations may be grouped in two sets, described by the vector canonical equations

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \text{and} \quad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}. \tag{32}$$

In the first group we have

$$\dot{\boldsymbol{u}} = \frac{\partial H}{\partial \boldsymbol{t_2}} = \bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right), \tag{33}$$

and

$$\dot{\boldsymbol{w}}\cdot\boldsymbol{e_2} = -\frac{\partial H}{\partial p_f} = -\boldsymbol{w}_{,1}\cdot\boldsymbol{e_1} - \frac{p_f}{m} - \frac{r}{m}\boldsymbol{u}_{,1}\cdot\boldsymbol{e_1} - \frac{r}{m}\boldsymbol{e_2}\cdot\bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right), \tag{34}$$

that correspond to Eq.(23) and to (30), respectively. The second group provides the equilibrium equations. Indeed, one gets

$$\dot{\boldsymbol{t_2}} = -\frac{\partial H}{\partial \boldsymbol{u}} = -\left[\bar{\mathrm{R}}\bar{\mathrm{T}}^{-1}\left(\boldsymbol{t_2} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \frac{r}{m}p_f\boldsymbol{e_2}\right) + \bar{\mathrm{Q}}\boldsymbol{u}_{,1} - \frac{r}{m}p_f\boldsymbol{e_1}\right]_{,1} \tag{35}$$

that, accounting for (23), whereby $\bar{\mathrm{T}}^{-1}$ times the term in round brackets gives $\dot{\boldsymbol{u}}$, and in light of the first of (5), amounts to (10a). By the same token,

$$-\dot{p_f}\boldsymbol{e_2} = -\frac{\partial H}{\partial \boldsymbol{w}} = (p_f\boldsymbol{e_1})_{,1} \tag{36}$$

that is immediately (10b). Incorporating the dissipation function into this formulation, may provide the starting point for addressing the general case of irreversible poroelasticity.

3.1. Reduced Hamiltonian

Looking at Eq.(21b) and recalling that p_f is constant throughout the body, as a result of the equilibrium equation (10b), one realises that, besides energy conservation, another motion invariant is available. Indeed, this formulation possesses translational invariance with respect to $\dot{\boldsymbol{w}}$. This is an outcome of the fact that, unlike $\boldsymbol{u}$, $\boldsymbol{w}$ appears in the Lagrangian only through its divergence ζ, and therefore one may assume $\boldsymbol{w} = \operatorname{grad}\varphi$ without loss of generality, the solenoidal contribution to $\boldsymbol{w}$ being irrelevant to the present purposes, see (Biot, 1962, Eq.(7.13)). This feature is specific to reversible poroelasticity and it is lost when encompassing for dissipation. Consequently, $\boldsymbol{w}_{,1}\cdot\boldsymbol{e_1}$ and $\dot{\boldsymbol{w}}\cdot\boldsymbol{e_2}$ are not (globally) independent from one another. To avoid dealing with this constraint, a more convenient approach consists of replacing ζ in (20) through the connection (22) to get

$$\hat{L}(\boldsymbol{u}_{,1},\dot{\boldsymbol{u}}) = \tfrac{1}{2}\boldsymbol{u}_{,1}\cdot\bar{\mathrm{Q}}\boldsymbol{u}_{,1} + \boldsymbol{u}_{,1}\cdot\bar{\mathrm{R}}\dot{\boldsymbol{u}} + \tfrac{1}{2}\dot{\boldsymbol{u}}\cdot\bar{\mathrm{T}}\dot{\boldsymbol{u}}, \tag{37}$$

having dispensed with the irrelevant constant term $\frac{1}{2}p_f^2/m$. In this form, the system matches anisotropic elasticity, provided that the Stroh matrices (18) are used. It is also emphasized that, in this reduced formulation (37), only the solid skeleton is represented. The Euler-Lagrange equation reads

$$\hat{\boldsymbol{t}_{\mathbf{1}}}_{,1} + \dot{\hat{\boldsymbol{t}_{\mathbf{2}}}} = 0, \tag{38}$$

having let the force vectors

$$\hat{\boldsymbol{t}_{\mathbf{1}}} = \bar{\mathrm{Q}}\boldsymbol{u}_{,1} + \bar{\mathrm{R}}\dot{\boldsymbol{u}}, \quad \hat{\boldsymbol{t}_{\mathbf{2}}} = \bar{\mathrm{R}}^T\boldsymbol{u}_{,1} + \bar{\mathrm{T}}\dot{\boldsymbol{u}}. \tag{39}$$

This amounts to defining the new stress tensor $\hat{\boldsymbol{T}}$, which differs from the total stress $\boldsymbol{T}$ by the constant hydrostatic pressure $\frac{r}{m}p_f\mathbf{1}$, and corresponds to Biot's *effective stress* σ_{ij}, that is the force in excess to pressure applied to the solid per unit surface of the bulk material, see (Biot, 1956a, Eq.(3.2)) and (Biot, 1962, Eq.(3.9)). The corresponding momentum immediately follows

$$\hat{\boldsymbol{p}} = \frac{\partial L}{\partial \dot{\boldsymbol{u}}} = \hat{\boldsymbol{t}_{\mathbf{2}}}, \tag{40}$$

and it can be solved for the conjugate coordinate $\dot{\boldsymbol{u}}$ giving again (23), yet assuming that $p_f = 0$, i.e.

$$\dot{\boldsymbol{u}} = \bar{\mathrm{T}}^{-1}\left(\hat{\boldsymbol{t}_{\mathbf{2}}} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1}\right). \tag{41}$$

The possibility to invert $\bar{\mathrm{T}}$ is granted by strong ellipticity, as discussed in the Appendix. The corresponding Hamiltonian is similarly obtained from (31) letting $p_f = 0$,

$$\hat{H} = \hat{\boldsymbol{p}} \cdot \dot{\boldsymbol{u}} - \hat{L} = \tfrac{1}{2}\left(\hat{\boldsymbol{t}_{\mathbf{2}}} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1}\right) \cdot \bar{\mathrm{T}}^{-1}\left(\hat{\boldsymbol{t}_{\mathbf{2}}} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1}\right) - \tfrac{1}{2}\boldsymbol{u}_{,1} \cdot \bar{\mathrm{Q}}\boldsymbol{u}_{,1}. \tag{42}$$

The canonical equations are

$$\dot{\boldsymbol{u}} = \frac{\partial \hat{H}}{\partial \hat{\boldsymbol{t}_{\mathbf{2}}}}, \tag{43}$$

that indeed gives (41), and

$$\dot{\hat{\boldsymbol{t}_{\mathbf{2}}}} = -\frac{\partial \hat{H}}{\partial \boldsymbol{u}} = -\left[\bar{\mathrm{R}}\bar{\mathrm{T}}^{-1}\left(\hat{\boldsymbol{t}_{\mathbf{2}}} - \bar{\mathrm{R}}^T\boldsymbol{u}_{,1}\right) - \bar{\mathrm{Q}}\boldsymbol{u}_{,1}\right]_{,1}, \tag{44}$$

that corresponds to (38).

For a homogeneous material, letting the stress potential $\hat{\boldsymbol{\phi}} = \int \hat{\boldsymbol{t}}_{\mathbf{2}} \mathrm{d}x_1$ and the vector of unknowns

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{u} \\ \hat{\boldsymbol{\phi}} \end{bmatrix}, \tag{45}$$

we can write the Stroh formalism

$$\frac{\partial}{\partial x_2}\boldsymbol{\xi} = \mathrm{N}\frac{\partial}{\partial x_1}\boldsymbol{\xi}, \tag{46}$$

where N is the *fundamental elasticity block-matrix* (Ting, 1996, §6)

$$\mathrm{N} = \begin{bmatrix} \mathrm{N}_1 & \mathrm{N}_2 \\ \mathrm{N}_3 & {\mathrm{N}_1}^T \end{bmatrix}, \tag{47}$$

and we have let the 3 by 3 block-matrices

$$\mathrm{N}_1 = -\bar{\mathrm{T}}^{-1}\bar{\mathrm{R}}^T, \quad \mathrm{N}_2 = \bar{\mathrm{T}}^{-1}, \quad \mathrm{N}_3 = \bar{\mathrm{R}}\bar{\mathrm{T}}^{-1}\bar{\mathrm{R}}^T - \bar{\mathrm{Q}}. \tag{48}$$

We observe that $\boldsymbol{\xi}$ has mixed dimensions, namely length and force over length for the first and for the second vector component, respectively. Consequently, N_1 is dimensionless, while N_3 and N_2 have dimension of stress and inverse of stress (compliance), respectively.

Letting the 6 by 6 constant matrix (Ting, 1996, Eq.(5.5-7))

$$\hat{\mathrm{I}} = \begin{bmatrix} \mathrm{O} & \mathrm{I} \\ \mathrm{I} & \mathrm{O} \end{bmatrix}, \tag{49}$$

and in view of the symmetry of N_2 and N_3, one retrieves the fundamental symmetric matrix

$$\hat{\mathrm{I}}\mathrm{N} = \begin{bmatrix} \mathrm{N}_3 & {\mathrm{N}_1}^T \\ \mathrm{N}_1 & \mathrm{N}_2 \end{bmatrix} = (\hat{\mathrm{I}}\mathrm{N})^T. \tag{50}$$

Following Ting (1996, §5.5), N_2 is positive definite and $-\mathrm{N}_3$ is positive semidefinite. When looking for travelling solutions of the form $\boldsymbol{\xi} = \boldsymbol{\Xi} f(x_1 + p x_2)$, a right eigenvalue problem is retrieved

$$\mathrm{N}\boldsymbol{\Xi} = p\boldsymbol{\Xi}, \tag{51}$$

The Hamiltonian density (42) may be rewritten as the quadratic form associated with the fundamental matrix

$$\hat{H} = \tfrac{1}{2}\boldsymbol{\xi} \cdot \hat{\mathrm{I}}\mathrm{N}\boldsymbol{\xi}, \tag{52}$$

whence the first integral associated with energy conservation (given that the Lagrangian is x_2-independent) reads

$$\int_{\Sigma} \boldsymbol{\xi} \cdot \hat{\mathrm{I}}\mathrm{N}\boldsymbol{\xi} \mathrm{d}x_1 \mathrm{d}x_3 = \mathrm{const}, \tag{53}$$

in the assumption that we may decompose the domain as $B = \Sigma \times I$, where I is an interval in the x_2 coordinate. Finally, we may define the *edge impedance matrix* M as

$$\hat{\boldsymbol{\phi}} = \imath \mathrm{M}\boldsymbol{u}, \tag{54}$$

and in light of (45,46), we may write

$$\dot{\hat{\boldsymbol{\phi}}} = \imath \mathrm{M}\dot{\boldsymbol{u}} = (\imath \mathrm{M}\mathrm{N}_1 - \mathrm{M}\mathrm{N}_1\mathrm{M})\boldsymbol{u}_{,1} = \left(\mathrm{N}_3 + \imath {\mathrm{N}_1}^T\mathrm{M}\right)\boldsymbol{u}_{,1},$$

whence M satisfies the matrix equation (Fu, 2007, Eq.(4.40))

$$\mathrm{N}_3 + \imath {\mathrm{N}_1}^T\mathrm{M} - \imath \mathrm{M}\mathrm{N}_1 + \mathrm{M}\mathrm{N}_1\mathrm{M} = \mathrm{O}. \tag{55}$$

This matrix provides a very simple procedure to determine localized waves, for which $\boldsymbol{t_2} \equiv \boldsymbol{o}$ on the body surface $x_2 = 0$. Indeed, the dispersion relation is simply obtained by admitting non-trivial solutions to the system

$$\mathrm{M}\boldsymbol{u} = \boldsymbol{o},$$

hence the major obstacle lying in the way is the determination of the impedance matrix through the connection (55). This result is most simply achieved through the integral representation originally introduced by Barnett and Lothe (1974).

4. Weak reversible poroelasticity and the incompressible limit

We shall now consider the limit where the coupling effect is weaker than the elastic response. For this, we let $\tau_0 = \|\mathrm{T}\|$ be the norm of the matrix T, and we assume that $\tau_0^{-1} r^2/m = \varepsilon \ll 1$ is small. We name this condition *weak reversible poroelasticity*. In this case,

$$\bar{\mathrm{T}}^{-1} = \left(\mathbf{1} + \varepsilon\tau_0 \mathrm{T}^{-1} \boldsymbol{e_2} \otimes \boldsymbol{e_2}\right) \mathrm{T}^{-1} + \mathrm{O}(\varepsilon^2), \tag{56}$$

and

$$\zeta_1 = 1 - \varepsilon\tau_0\zeta_0^{-1} + O(\varepsilon)^2, \qquad \zeta_0 = 1/(\boldsymbol{e_2} \cdot \mathrm{T}^{-1}\boldsymbol{e_2}),$$

whence $\zeta_1 \approx 1$. Then, expanding to first order terms in ε, one gets (collecting dimensionality terms)

$$\mathrm{N}_1 = -\mathrm{T}^{-1}\mathrm{R}^T + \varepsilon\tau_0 \mathrm{T}^{-1}\boldsymbol{e_2} \otimes \boldsymbol{e_T} \tag{57a}$$

$$\mathrm{N}_2 = \mathrm{T}^{-1}\left(\mathrm{I} + \varepsilon\tau_0 \boldsymbol{e_2} \otimes \mathrm{T}^{-1}\boldsymbol{e_2}\right), \tag{57b}$$

$$\mathrm{N}_3 = \mathrm{R}\left(\mathrm{T}^{-1} + \varepsilon\tau_0 \mathrm{R}^{-1}\boldsymbol{e_T} \otimes \mathrm{R}^{-1}\boldsymbol{e_T}\right)\mathrm{R}^T - \mathrm{Q}, \tag{57c}$$

with $\boldsymbol{e_T} = \boldsymbol{e_1} - \mathrm{R}\mathrm{T}^{-1}\boldsymbol{e_2}$. It is pointed out that Eqs.(57) are indeed valid asymptotic expansions inasmuch as $\tau_0^{-1}\|\mathrm{R}\| = O(1)$ and $\tau_0^{-1}\|\mathrm{Q}\| = O(1)$ or bigger. Physically, this amounts to requiring that all elastic constants are of the same order, i.e. contrast is excluded. Formally, Eqs.(57) match the corresponding matrices in incompressible anisotropic elasticity (Fu, 2007, Eqs.(3.14-16)), provided that $\varepsilon\tau_0$ is replaced by ζ_0 and the opposite sign is taken in the incompressibility contributions, that are given by the correction term in each of Eqs.(57). Indeed, a similar expansion of Eq.(41) yields

$$\dot{\boldsymbol{u}} = \mathrm{T}^{-1}\left(\hat{\boldsymbol{t}_2} - \mathrm{R}^T\boldsymbol{u}_{,1} - p_0\boldsymbol{e_2}\right) \tag{58}$$

with

$$p_0 = \varepsilon\tau_0\left\{-\mathrm{T}^{-1}\left(\hat{\boldsymbol{t}_2} - \mathrm{R}^T\boldsymbol{u}_{,1}\right) \cdot \boldsymbol{e_2} - \boldsymbol{u}_{,1} \cdot \boldsymbol{e_1}\right\}. \tag{59}$$

Providing again that $\varepsilon\tau_0 = \zeta_0$ and p_0 is sign reversed, such equations are formally equivalent to (3.7) and (3.11) of Fu (2007), respectively giving $\dot{\boldsymbol{u}}$ and the Lagrange multiplier enforcing incompressibility for incompressible anisotropic solids. This analysis reveals that the weak poroelastic limit is similar to incompressible anisotropic elasticity, with yet two important differences. First, given that $\tau_0 \sim \zeta_0$, the condition $\varepsilon\tau_0 = \zeta_0$ can only be achieved in a correction sense and therefore incompressibility is to be intended as a perturbation from the unconstrained leading solution. Second, the sign reversal of p_0 reveals that this perturbation is taken in the opposite direction, i.e. the role of the fluid phase in the weak limit is opposite to that of the incompressibility constraint. At any rate, incompressibility cannot be achieved for the solid skeleton in the general sense.

Biot, on heuristic grounds, claims that the incompressible limit is obtained letting $m \to +\infty$ and $\alpha = r/m = 1$, see for example Biot (1962). Although, just looking at (6), it is manifest that the former condition is sufficient for fluid incompressibility, the latter needs some revision. Indeed, the condition $\alpha = 1$ merely demands that the fluid response is the same under fluid and solid volumetric changes, and therefore one may deduce that, for a given pressure p_f, it must be $\zeta - e = -\operatorname{div}(\boldsymbol{u} + \boldsymbol{w}) = p_f/m$. When the fluid phase becomes incompressible, i.e. $m \to +\infty$, one needs to specify how the pressure p_f behaves compared to m. If $p_f/m \to 0$, then zero net flow of both fluid and solid out of the control volume is approached and this limit amounts to an isochoric transformation. This line of reasoning led Biot to the concept of incompressible limit, as in Biot (1955). However, while the fluid may behave as incompressible, the foregoing analysis shows that the solid does not. In fact, the solid behaves just like an anisotropic elastic solid whose Stroh matrices (46) become unbounded as $r \sim m \to +\infty$. Besides, to support strong ellipticity (A.2), the elastic constants must also become unbounded, hence it is concluded that this limit is questionable. In fact, the actual physical regime is determined by the ratio $\tau_0\varepsilon$ of the poroelastic effect to the elastic effect. In general, when $\tau_0\varepsilon = O(1)$, the solid behaves like an ordinary anisotropic solid whose material properties are affected by the fluid phase. Instead, in the weak limit $\tau_0\varepsilon \ll 1$, the fluid acts as a perturbation to the anisotropic solid and this perturbation operates similarly to incompressibility, yet in opposing fashion, i.e. a positive pressure accompanies positive volumetric changes. Finally, when $\tau_0\varepsilon \gg 1$, the solid behaves like a perturbation of an ideal liquid with small viscosity $O(\tau_0\varepsilon)^{-1}$ given by the elastic phase.

5. Conclusions

When deriving the Stroh-like formulation of a mechanical system, one is confronted with the crucial step of designating the right variable pairs, which unlock the full potential of the formalism. Recently, Fu (2007) pointed out that energy conjugation is really the guiding tool which drives such designation, thus getting away from guess-working and problem intuition, which may not suffice in complex situations. Indeed, the Stroh formalism is really a canonical formalism in the Hamiltonian sense, where a coordinate is treated in time-like fashion. In this paper, we adopt this viewpoint to deal with Biot's reversible poroelasticity, that dispenses with dissipation and occurs in the absence of a fluid pressure gradient. This is the same framework as thermoelasticity of perfect conductors, the pressure playing the role of temperature. Although this framework is insufficient to deal with any poroelastic problem, it may well provide the starting point for the general formulation. Also, it investigates the most useful setting for specimen testing. Spotlight is here set on emphasizing the canonical approach and the features it brings out. Two formulations are derived: the first accounts for both the solid and the fluid and it possesses, besides energy conservation, translational invariance with respect to the fluid velocity. This feature, that is a result of the absence of a pressure gradient, reveals constraints on the conjugate variables. To avoid dealing with such constraints, a second approach is developed that is restricted to the solid skeleton only. The corresponding Stroh formulation matches anisotropic elasticity where, however, the Stroh matrices incorporate fluid coupling. Besides, strong ellipticity warrants their positive definite character. Energy conservation and the impedance matrix follow naturally. The special case of weak poroelasticity, whereby fluid-solid coupling is weaker than the elastic response, is also investigates and shows remarkable similarities with incompressible anisotropic elasticity with yet two important differences, namely incompressibility acts as a small perturbation with opposite sign. This analysis leads to reconsider the incompressible limit originally introduced by Biot, that seems to show some inconsistencies.

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Conflict of Interest statement

The author has no conflict of interest to declare.

Data Availability

This paper makes use of no data.

Appendix: Strong ellipticity in reversible poroelasticity

We now discuss the role of strong ellipticity in poroelasticity in the absence of dissipation. For a reversible process, we have a uniform pressure distribution p_f and the motion equation for the solid skeleton is given by Eq.(10a) where we write inertia explicitly (i.e. it is not hidden inside the Stroh matrices)

$$c_{ijkl}u_{k,lj} - r\zeta_{,j}\delta_{ij} = \rho\ddot{u}_i. \tag{A.1}$$

As well known (Edmondson and Fu, 2009), strong ellipticity may be equally retrieved demanding that the speed v of *any amplitude* propagating body wave in *any direction* is real (and positive, without loss of generality). To this aim, let's assume $\boldsymbol{u} = \boldsymbol{\alpha} e^{\imath(\boldsymbol{\beta}\cdot\boldsymbol{x} - vt)}$, whence

$$e = \alpha_k\beta_k e^{\imath(\boldsymbol{\beta}\cdot\boldsymbol{x} - vt)}, \quad \zeta = m^{-1}\left(p_f + r\alpha_k\beta_k e^{\imath(\boldsymbol{\beta}\cdot\boldsymbol{x} - vt)}\right).$$

Then, Eq.(A.1) becomes

$$c_{ijkl}\alpha_k\beta_l\beta_j - \frac{r^2}{m}\delta_{ij}\alpha_k\beta_k\beta_j = \rho v^2\alpha_i,$$

which, multiplied through by α_i and summed over i, gives

$$\left(c_{ijkl} - \frac{r^2}{m}\delta_{ij}\delta_{kl}\right)\alpha_i\alpha_k\beta_l\beta_j = \rho v^2\alpha_i\alpha_i > 0,$$

for any $\boldsymbol{\alpha}, \boldsymbol{\beta}$ different from zero. This is a variant of the incompressibility constraint. In particular, letting $\boldsymbol{\beta} = \boldsymbol{e_2}$, one gets that

$$\bar{\mathrm{T}} = \mathrm{T} - \frac{r^2}{m}\boldsymbol{e_2} \otimes \boldsymbol{e_2} \text{ is positive definite,} \tag{A.2}$$

and therefore Eq.(40) may be solved.

References

Barnett, D., Lothe, J., 1974. Consideration of the existence of surface wave (Rayleigh wave) solutions in anisotropic elastic crystals. Journal of physics F: Metal physics 4, 671.

Barnett, D.M., 2000. Bulk, surface, and interfacial waves in anisotropic linear elastic solids. International Journal of Solids and Structures 37, 45–54.

Berryman, J.G., Wang, H.F., 2000. Elastic wave propagation and attenuation in a double-porosity dual-permeability medium. International Journal of Rock Mechanics and Mining Sciences 37, 63–78.

Biot, M.A., 1955. Theory of elasticity and consolidation for a porous anisotropic solid. Journal of applied physics 26, 182–185.

Biot, M.A., 1956a. Theory of deformation of a porous viscoelastic anisotropic solid. Journal of Applied physics 27, 459–467.

Biot, M.A., 1956b. Thermoelasticity and irreversible thermodynamics. Journal of applied physics 27, 240–253.

Biot, M.A., 1962. Mechanics of deformation and acoustic propagation in porous media. Journal of applied physics 33, 1482–1498.

Biot, M.A., Willis, D.G., 1957. The elastic coefficients of the theory of consolidation. Journal of Applied Mechanics 24, 594–601.

Chadwick, P., Smith, G., 1977. Foundations of the theory of surface waves in anisotropic elastic materials. Advances in applied mechanics 17, 303–376.

Corapcioglu, M.Y., Tuncay, K., 1996. Propagation of waves in porous media, in: Advances in porous media. Elsevier. volume 3, pp. 361–440.

Dullien, F.A., 2012. Porous media: fluid transport and pore structure. Academic press.

Edmondson, R., Fu, Y., 2009. Stroh formulation for a generally constrained and pre-stressed elastic material. International Journal of Non-Linear Mechanics 44, 530–537.

Fu, Y., 2003. Existence and uniqueness of edge waves in a generally anisotropic elastic plate. Quarterly Journal of Mechanics and Applied Mathematics 56, 605–616.

Fu, Y., 2005. An explicit expression for the surface-impedance matrix of a generally anisotropic incompressible elastic material in a state of plane strain. International Journal of Non-Linear Mechanics 40, 229–239.

Fu, Y., 2007. Hamiltonian interpretation of the Stroh formalism in anisotropic elasticity. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 463, 3073–3087.

Fu, Y., Brookes, D., 2006. Edge waves in asymmetrically laminated plates. Journal of the Mechanics and Physics of Solids 54, 1–21.

Fu, Y., Kaplunov, J., 2012. Analysis of localized edge vibrations of cylindrical shells using the stroh formalism. Mathematics and mechanics of solids 17, 59–66.

Hwu, C., 2003. Stroh-like formalism for the coupled stretching–bending analysis of composite laminates. International Journal of Solids and Structures 40, 3681–3705.

Lopatnikov, S.L., Cheng, A.D., 2004. Macroscopic lagrangian formulation of poroelasticity with porosity dynamics. Journal of the Mechanics and Physics of Solids 52, 2801–2839.

Nobili, A., Radi, E., 2022. Hamiltonian/stroh formalism for anisotropic media with microstructure. Philosophical Transactions of the Royal Society A 380, 20210374.

Norris, A.N., Grinfeld, M.A., 1995. Nonlinear poroelasticity for a layered medium. The Journal of the Acoustical Society of America 98, 1138–1146.

Pramanik, D., Manna, S., Nobili, A., 2024. Theory of elastic wave propagation in a fluid saturated multiporous medium with multi-permeability. Proceedings of the Royal Society of London Under review.

Sharma, M., 2010. Piezoelectric effect on the velocities of waves in an anisotropic piezo-poroelastic medium. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 466, 1977–1992.

Stroh, A., 1958. Dislocations and cracks in anisotropic elasticity. Philosophical magazine 3, 625–646.

Stroh, A., 1962. Steady state problems in anisotropic elasticity. Journal of Mathematics and Physics 41, 77–103.

Ting, T., 1996. Anisotropic elasticity: theory and applications. 45, Oxford University Press on Demand.