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A well-posed non-local theory in 1D linear elastodynamics

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ABSTRACT

We show how to construct a well-posed theory of purely non-local elasticity by kernel modification. Specifically, we modify the classical Helmholtz kernel so that the constitutive boundary conditions associated with it are replaced by constraints that emerge from the natural boundary conditions of the problem at hand. The procedure is illustrated by two examples, one dealing with a statically indeterminate problem and the other concerning free vibrations of a cantilever beam. The defining feature of the method is that the modified kernel is no longer a difference kernel. This outcome is a consequence of the incorporation of the problem's boundary conditions, which affects the kernel near the boundaries and, consequently, induces a different mechanical response in dependence of the distance from those. In contrast, negligible changes are found in the interior of the material. Still, the modified kernel remains symmetric and positive definite, which property guarantees that the strain energy is quadratic and positive definite, and it complies with the impulsivity requirement, by which it reverts to the classical local theory in the limit of a vanishing non-local length-scale. Kernel modification is conceptually different from the two-phase approach under many respects, most notably because it gets away from the need to introduce extra boundary conditions besides those naturally associated with the physics of the problem.

1. Introduction

In classical theory of elasticity, the stress at a point in a material depends solely on the strain at that point through the constitutive relation. In this sense, the theory is often referred to as being purely local. Despite its great success, this assumption fails to capture the inherent size effect connected to the existence of internal length-scales associated with the material microstructure. Similarly, long-range interactions may be relevant, as a result of specific features (such as, for example, molecular interactions). Following Eringen and Kim (1974), one major failure of classical local elasticity is connected to the appearance of singularities at the tip of a crack, that is testimony of the inability of the theory to account for effects which are extremely localized and, therefore, susceptible to the granular structure of matter. Precisely to address these limitations, non-local elasticity theory was developed in two directions, sometimes referred to as *weakly* and *strongly non-local*. The weak approach introduces constitutive relations where higher order strain gradients appear, thereby conveying non-locality in an "extrapolated" sense. Examples of these are the couple stress theory (Nobili, 2021) and the large family of strain gradient theories. The strong approach, instead, relies on spatial integrals of the strain field to account for interactions acting beyond the immediate vicinity of a point. The strong form of non-local elasticity, originally introduced by Kröner (1967) and later developed by Eringen and Edelen (1972), incorporates the influence of long-range interactions within a material by featuring an internal length-scale. This capability allows the theory to capture size-dependent effects and microstructural interactions, making it particularly relevant for the analysis of nanostructures (Radi et al., 2021), composite materials, and other systems with prominent small-scale effects (Peddieson et al., 2003; Wang and Liew, 2007). This non-local formulation relies on an integral operator endowed with a kernel function that reflects attenuation over distance of the internal interactions, enabling the modelling of physical phenomena not addressed by classical elasticity or strain-gradient theories (Altenbach et al., 2011; Kahrobaiyan et al., 2011). The kernel (or attenuation) function is of prominent importance as it determines how the spatial distribution of the strain field affects stress, it captures material-specific microstructural behaviour, and it ensures mathematical consistency in both the static and in the dynamic framework. It is therefore clear that proper selection and adaptation of the kernel function are critical steps for accurate modelling and for meaningful results in non-local elasticity.

The determination of the kernel function was introduced by Eringen (1972), through the analysis of wave propagation in periodic lattice structures. Subsequently, Eringen (1983) defined multiple non-local

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kernels and established the essential characteristics that a function must satisfy to be accepted as a non-local kernel. Building on this, Picu (2002) presented an analytical procedure for deriving kernels based on interatomic potentials and demonstrated its application in computing stress in strained superlattices and wave dispersion. More recently, Aksoylu and Gazonas (2020) proposed a selection criterion for determining the kernel function that provides the best approximation to the dispersion relation. Although a wide variety of kernels have been introduced, the Helmholtz (or exponential) kernel remains the most commonly used in the literature for analysing one-dimensional non-local elasticity problems, see Alotta et al. (2022), Mikhasev and Nobili (2020) and Eroğlu and Ruta (2022).

One of the defining challenges in non-local elasticity is the appropriate formulation and treatment of the boundary conditions. Unlike classical elasticity, where the boundary conditions are direct expression of physical principles, non-local elasticity requires dealing with extra boundary conditions, named constitutive (CBCs), which come about by the selection of the kernel function (Benvenuti and Simone, 2013). In this sense, CBCs are unphysical, or, more correctly, they express the physics embedded in the kernel function. Such extra conditions usually lead to an over-determined system, which is almost inevitably ill-posed unless some compatibility condition is satisfied.

To avoid the difficulty associated with dealing with complex integro-differential equations, Eringen suggested to adopt, as kernel, the Green's function of a differential operator. Accordingly, several studies, such as Karlicic et al. (2015), Rafii-Tabar et al. (2016) and Li et al. (2015), took advantage of the Helmholtz kernel to formulate the non-local problem in purely differential form and successively applied it to various nanostructures in both statics and dynamics. Similarly, Reddy (2007) analysed the non-local behaviour using different beam theories by adopting the differential formulation of the constitutive relation. However, these researchers overlooked to impose the CBCs alongside the natural boundary condition of the problem and, as a consequence, the differential formulation turned no longer equivalent to the integral formulation.

Besides such inconsistency, a paradoxical result may appear, first identified by Peddieson et al. (2003), by which the small-scale lengthscale does not appear in the solution of an integral-based non-local beam problem. Challamel and Wang (2008) further exposed this paradox in the case of a cantilever beam and proposed to solve the issue by reversing the role of the local and of the non-local curvature in the constitutive model, which leads in fact to a strain gradient theory that sometimes goes under the name of a stress-driven model. As discussed by Zhang et al. (2022) and Romano et al. (2017), the CBCs naturally arise from the non-local integral formulation and must be carefully incorporated to ensure well-posedness. The problem was revisited by Fernandez-Saez et al. (2016) who proposed a general solution procedure that attempts to modify the load function in order that it satisfies the CBCs. However, this approach fails to circumvent the issue for it introduces two additional ill-posed integral equations (Eqs. 31 and 32). To address the ill-posedness of the purely non-local model, a mixed local/non-local theory has been adopted by several researchers, see Pisano and Fuschi (2003), Khodabakhshi and Reddy (2015), Wang et al. (2016), Eptaimeros et al. (2016) and Khaniki (2018), based upon Eringen's original work (Eringen, 1987, 2002). However, while this approach mitigates some issues, it introduces some new difficulties. Indeed, the constitutive boundary conditions (CBCs) from the non-local phase persist and require an additional phase parameter that is very difficult to assess.

An alternative approach, while keeping within the purely non-local model, was introduced by Koutsoumaris and Eptaimeros (2018) in connection with the normalization requirement for the kernel. The idea is to modify the non-local kernel so that its modulus remains unitary over finite domains. In a previous paper, Nobili and Pramanik (2025) developed over the general idea of kernel modification for statically determined problems using a cantilever beam as an illustrative example. This modification strategy relies on the substitution of the CBCs with the natural boundary conditions of the problem or combinations thereof. The solution of the integral problem was also obtained by the method of eigenfunction expansion of the integral operator and it was found to correspond to the solution of the differential model. Building upon these results, we now extend the analysis to statically indeterminate scenarios and to dynamics. In particular, we are interested in determining the properties of the modified kernel and in the features it acquires upon incorporating the natural boundary conditions of the problem. This paper demonstrates that kernel modification can be successfully applied to both statically indeterminate structures as well as to dynamics. By addressing these aspects, we establish a well-posed framework for incorporating non-locality across a broad range of scenarios, without recourse to extra boundary conditions of difficult physical interpretation.

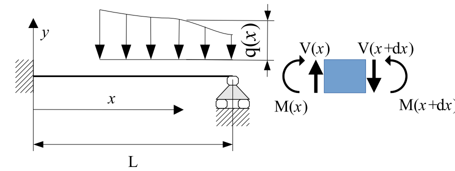

Fig. 1. A cantilever beam elastically supported at its tip and acted upon by the distributed load $q(x)$.

2. Cantilever beam under a uniform load and elastically supported at the free end

We begin by considering a statically indeterminate problem for a cantilever beam, elastically supported at its tip, within the purely non-local theory of linear elasticity. For this structure, we develop the corresponding solution by modification of the Helmholtz (or exponential) kernel. Let us consider a cantilever beam of length L acted upon by a uniform distributed load $q(x) \equiv q_0$. Let the built-in end be located at $x = 0$ and the point support at $x = L$, exerting an elastic reaction force $R = -k_e v(L)$, where $v(x)$ is the beam transverse displacement, see Fig. 1, and k_e is the support stiffness. We assume that the beam follows the Euler–Bernoulli theory, that is, beam cross-sections remain plane and perpendicular to the beam axis and bending deformation dominates over shear. Multiplication of the second moment of inertia of the beam cross-section, denoted by I, by Young's modulus, E, produces the beam flexural rigidity $D = EI$. The corresponding bending moment is given by

$$M(x) = R(L-x) - \frac{q_0}{2}(L-x)^2, \tag{1}$$

and the support reaction force R is yet undetermined. Within the classical purely non-local elasticity theory, the constitutive law for the cantilever beam is expressed through integration of the local curvature, namely

$$\mathcal{R}[\chi](x) = D^{-1}M(x), \tag{2}$$

where we have let the integral operator

$$\mathcal{R}[\chi](x) = \int_0^L K(x,\xi)\chi(\xi)\mathrm{d}\xi, \tag{3}$$

endowed with the kernel (or attenuation) function $K(x,\xi)$, which accounts for diffusion effects within the beam. Here,

$$\chi(x) = \mathrm{d}^2 v/\mathrm{d}x^2 \tag{4}$$

represents the classical local curvature in the beam. As already anticipated, the Helmholtz kernel is adopted

$$K(x,\xi) = \frac{1}{2\epsilon}\exp\left(-\frac{|x-\xi|}{\epsilon}\right), \tag{5}$$

where ϵ is the non-local parameter related to an internal characteristic length associated with the extension of the diffusion process in the beam. As well known, this kernel $K(x, \xi)$ is the Green's function of the linear differential operator

$$\mathfrak{L} \equiv 1 - \epsilon^2 \frac{d^2}{dx^2}, \tag{6}$$

endowed with the Constitutive Boundary Conditions (CBCs)

$$\frac{dK}{dx}(0, \xi) - \epsilon^{-1} K(0, \xi) = 0, \quad \frac{dK}{dx}(L, \xi) + \epsilon^{-1} K(L, \xi) = 0. \tag{7}$$

Obviously, the bending moment $M(x)$, as defined by Eq. (1), generally fails to comply with the CBCs (7), and indeed this discrepancy stands at the root of the standard non-local integral problem (2) being ill-posed. In line with some recent literature contributions, by which kernel modification is introduced to accommodate for the presence of boundaries, Nobili and Pramanik (2025) proposed a novel approach to non-local elasticity which leads to a well-posed formulation because it relies on the modification of the kernel function, that is the Green function of a given differential operator, only in terms of boundary conditions. For statically determined problems, it was demonstrated that this approach indeed produces meaningful solutions, in the sense that they correspond to the solution of the integral operator. The modification of the kernel function is obtained by adding a suitable homogeneous solution of the linear operator $\mathfrak{L}$, in order that it becomes compliant with the natural BCs of the problem under investigation, which are immediately available for a statically determined condition. We now extend this approach to a statically indetermined scenario. Accordingly,

$$K^{\dagger}(x, \xi) = K(x, \xi) + k_0(x, \xi), \tag{8}$$

where the yet unknown correction $k_0(x, \xi)$ is such that $\mathfrak{L} k_0(x, \xi) \equiv 0$. With this modification, the constitutive law (2) may be rewritten as

$$\chi_0(x) + \int_0^L K(x, \xi) \chi(\xi) d\xi = D^{-1} M(x), \tag{9}$$

where, clearly, $\chi_0(x) = \int_0^L k_0(x, \xi) \chi(\xi) d\xi$. As already discussed in Nobili and Pramanik (2025), the constitutive relation (9) is reminiscent of the two-phase non-local model (TPNM), also introduced by Eringen and Edelen (1972), with yet important differences, the most relevant of which is that no extra boundary conditions of difficult physical interpretation are now demanded to lend a unique solution. Indeed, Eq. (9) still defines a Fredholm integral equation of the *first kind*, as opposed to the TPNM which, instead, leads to an equation of the *second kind*. Yet, kernel modification guarantees that the integral problem fulfils the boundary conditions and, as a result, it is well-posed, allowing for a physically meaningful solution.

Let us now test the kernel modification approach in this new framework. Application of the linear operator $\mathfrak{L}$ to Eq. (9), where the bending moment (1) has been introduced, immediately yields the local curvature

$$\chi(x) = D^{-1} \left[R(L - x) - \frac{1}{2} q_0 (L - x)^2 + q_0 \epsilon^2 \right], \tag{10}$$

although the support reaction R still remains unknown. Using the curvature–displacement relation (4), the displacement $v(x)$ is derived

$$v(x) = -\frac{x^2}{24D} \left[6L^2 q_0 - 4L(q_0 x + 3R) + q_0 \left(x^2 - 12\epsilon^2 \right) + 4Rx \right] + c_2 x + c_1, \tag{11}$$

where c_1 and c_2 are integration constants. We now come to the matter of boundary conditions, which read

$$v(0) = v'(0) = 0, \tag{12a}$$

$$M(L) = 0, \tag{12b}$$

$$M'(L) = k_e v(L). \tag{12c}$$

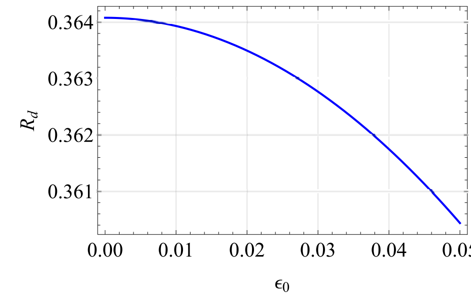

Fig. 2. Dimensionless reaction force R_d as a function of the dimensionless non-local length-scale ϵ_0 for a cantilever beam under an uniformly distributed load and elastically supported at the free end within the purely non-local elasticity theory (dimensionless support stiffness $\kappa = 100$).

Obviously, condition (12b) has already been incorporated by considering the bending moment distribution (1). Conversely, the remaining conditions are yet to be enforced and they provide the expression of the integration constants c_1 and c_2 as well as the support reaction R. Indeed, the integration constants read

$$c_1 = 0, \quad c_2 = \frac{3 q_0 L^3}{24D} \left[1 - 4\epsilon_0^2 - 8 \left(\frac{1}{3} + \frac{1}{\kappa} \right) R_d \right], \tag{13}$$

where we have let the non-dimensional reaction force $R_d = R/(q_0 L)$ that is given by

$$R_d = \frac{1}{2} \frac{\kappa}{1 + \frac{1}{3}\kappa} \left(\frac{1}{4} - \epsilon_0^2 \right), \tag{14}$$

in terms of the dimensionless support stiffness $\kappa = k_e L^3 / D$ and of the dimensionless non-local length-scale parameter $\epsilon_0 = \epsilon / L$.

Eq. (14) obviously reduces to the classical result when non-locality disappears, namely for $\epsilon_0 = 0$. However, this expression is remarkable in that the non-local parameter *decreases* the reaction force at the elastic support, whence the beam appears less rigid compared to the classical solution. Indeed, this reduction emerges because the non-local nature of the constitutive response allows the beam to distribute the support reaction more evenly, thus spreading out its contribution. Fig. 2 shows the reaction force of the elastic support with respect to dimensionless non-local length-scale ϵ_0.

Once the reaction force has been obtained through Eq. (14), it may be plugged back into the bending moment (1) to give the complete expression of the bending moment

$$M(x) = \frac{1}{2} q_0 L^2 \left(1 - \frac{x}{L} \right) \left[\frac{\kappa}{1 + \frac{1}{3}\kappa} \left(\frac{1}{4} - \epsilon_0^2 \right) - 1 + \frac{x}{L} \right]. \tag{15}$$

This expression provides a new pair of CBCs to be enforced onto the attenuation function, namely

$$M'(0) + \frac{\beta}{L} M(0) = 0, \quad M(L) = 0, \tag{16}$$

where

$$\beta = 1 + \frac{1 + \frac{1}{3}\kappa}{1 + \left(\frac{1}{12} + \epsilon_0^2 \right) \kappa}. \tag{17}$$

Indeed, Eqs. (15) are meant to replace (7) and it may be seen that no correspondence is ever achieved, regardless of the stiffness $\kappa > 0$, which suggests that this problem is always ill-posed under the theory of non-local elasticity with the standard Helmholtz kernel. The limiting case of rigid support gives $\beta = 1 + \frac{4}{1 + 12\epsilon_0^2}$, while, in the case of no support, i.e. $\kappa \to 0$, it is $\beta = 2$, irrespective of the non-local parameter ϵ_0, as it would be expected from a now statically determined problem.

The linear differential operator $\mathfrak{L}$, complemented with the above self-adjoint homogeneous boundary conditions (16), allows us to derive

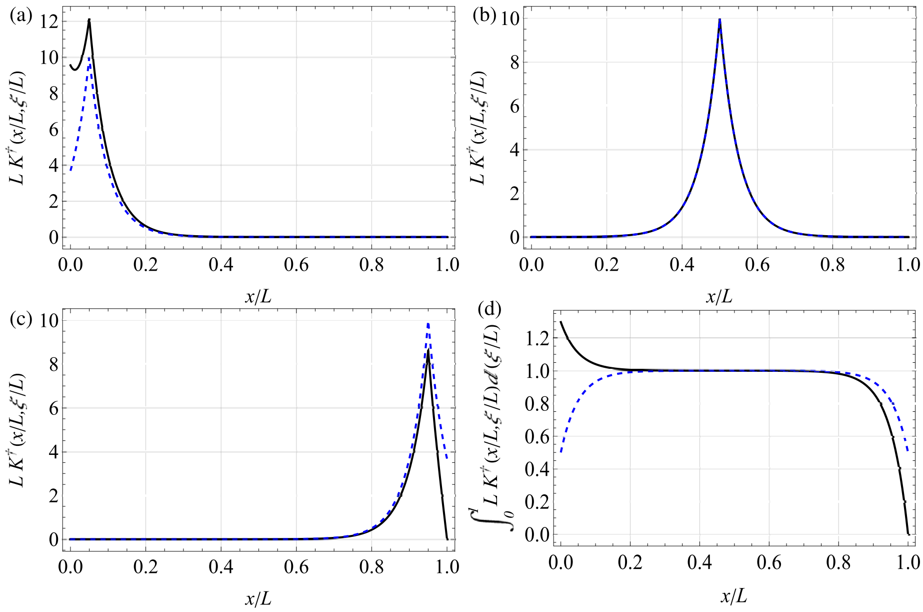

Fig. 3. Comparison of the modified kernel for a tip-supported cantilever beam (black, solid) with the Helmholtz kernel (blue, dashed). The kernels are evaluated at (a) $\xi/L = 5\%$, (b) $\xi/L = 50\%$, and (c) $\xi/L = 95\%$, with dimensionless length-scale parameter $\epsilon_0 = 0.05$ and relative support stiffness $\kappa = 100$; (d) illustrates the normalization condition applied over the finite domain $[0, L]$, while normalization over the infinite domain is always granted.

the modified kernel (8)

$$K^{\dagger}(x,\xi) = \frac{\sinh\left(\frac{1-L^{-1}|x-\xi|}{\epsilon_0}\right) + \sinh\left(\frac{1-L^{-1}(x+\xi)}{\epsilon_0}\right)}{2\epsilon\cosh\epsilon_0^{-1}}\,\frac{\epsilon_0\beta\tanh\left(\frac{\min\{x,\xi\}}{\epsilon}\right)-1}{\epsilon_0\beta\tanh\epsilon_0^{-1}-1}, \tag{18}$$

which remains symmetric with respect to its arguments. This last feature is very important, for it guarantees that the stored elastic energy is a quadratic positive-definite functional of the deformation. Furthermore, the modified kernel still satisfies the localization (or impulsivity) property, given that it is a delta sequence $\lim_{\epsilon\to 0} K^{\dagger}(x,\xi) = \delta(|x-\xi|)$, whose maximum is attained at $x = \xi$. On the contrary, it should be remarked that the new kernel is no longer a difference kernel, which means that some form of "inhomogeneity" has been introduced, in accordance with the idea that we now account for the presence of boundaries.

Fig. 3 shows the modified kernel $K^{\dagger}(x,\xi)$ as it compares with the standard Helmholtz kernel and it highlights their close resemblance inasmuch as ξ is sufficiently far from the boundaries. This outcome parallels what was already observed for statically determinate problems in Nobili and Pramanik (2025) and it supports the idea that kernel modification only really deals with the presence of the boundaries. Fig. 3 also illustrates the kernel normalization condition over the finite domain $[0, L]$ (see Appendix B for infinite domain) and again great similarity is found with the standard kernel over interior points, while significant divergence appears in the neighbourhood of the end points, see also Koutsoumaris et al. (2017). Indeed, as it was the case for statically determined situations, points near the built-in end are over-expressed, in the sense that their curvature contribution to the bending moment is emphasized, while points near the supported end are under-expressed.

Nonetheless, an important point should be raised here, which could not be observed for statically determined situations, namely the kernel function now depends, although only mildly and near the end points, on the support stiffness κ through the coefficient β. From a theoretical standpoint, this dependence may appear disconcerting, because the constitutive relation for the beam, either (2) or (9), is now affected by the support stiffness κ, which is a feature that does not belong to the beam. Even more troublesome, this dependence appears also at the built-in end point, which sits far away from the elastic support. Indeed, plugging $x = 0$ in (18), it is found

$$K^{\dagger}(0,\xi) = \frac{1}{\epsilon\cosh\epsilon_0^{-1}}\,\frac{\sinh\left(\frac{1-\xi/L}{\epsilon_0}\right)}{1-\epsilon_0\beta\tanh\epsilon_0^{-1}}, \tag{19}$$

that depends on κ through β. However, for $\epsilon_0 \ll 1$, we get the asymptotic expansion

$$K^{\dagger}(0,\xi) = \frac{1}{\epsilon}\exp\left(\frac{-\xi/L}{\epsilon_0}\right) + O(\epsilon_0\exp\epsilon_0^{-1}), \tag{20}$$

whence it is seen that the κ dependence of the modified kernel (18) at the built-in end only appears through a higher order correction term to the leading order solution (20), that is instead κ independent. This result is confirmed by the first of Fig. 3, where we see that, for $\xi > x$, $K^{\dagger}(x,\xi)$ rapidly converges to the standard Helmholtz kernel (that is independent of κ). Concerning the dependence on κ at the supported end $x = L$, one may argue that non-locality, by its very definition, should express the idea that the effect at a particular point originates from the contribution of a neighbourhood of that point, which, in the case of end points, inevitably calls into question the features adjoining the beam. In this sense, it is little surprising that the response of the end point also depends on the support stiffness, which is there located. Similarly, one may observe that the same outcome also emerges from the classical local theory and from the TPNM (see Appendix A), which takes into account the BC (12c) (alongside some extra conditions to be somehow devised) and consequently produces a bending moment distribution which equally depends on κ. Besides, this dependence on κ also appears at the built-in end point (see Fig. 4). Therefore, given the similar outcomes, it appears that this apparent incongruence connected to the κ dependence of the modified kernel is more of a semantical nature, in that the *constitutive nature* of the relation (2), or (9), should be intended in a broader sense, that includes the constitutive response of the adjoining structures, in a true non-local spirit, which goes beyond artificial boundaries between the elements of the structure. Indeed, we shall presently see that an even larger viewpoint needs to be embraced when moving from statics to dynamics.

3. Free vibrations of a cantilever beam

We now attempt to extend the procedure of kernel modification to the realm of elastodynamics, where inertia plays a significant role. In

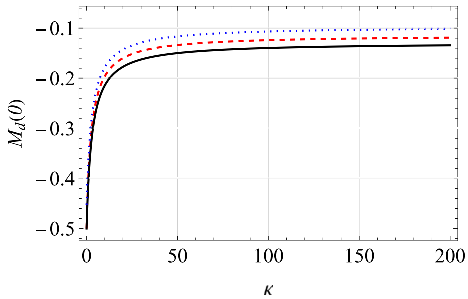

Fig. 4. Dimensionless bending moment $M_d(0) = M(0)/\left(q_0 L^2\right)$ at the built-in end, as a function of the dimensionless support stiffness κ for the purely non-local (black-solid), the two-phase non-local (red-dashed) and the purely non-local model with the asymptotic kernel (20) (blue-dotted), having set the dimensionless non-local length-scale $\epsilon_0 = 0.05$ and the phase parameter $\xi_1 = 0.1$.

the context of non-local elasticity, dynamic effects introduce additional complexity due to the interplay between non-local interactions and time-varying inertial forces. As in the case of thermoelasticity, an assumption is made concerning the speed at which stress diffusion takes place, against the speed of elastic waves in the body: For simplicity, we assume the former to be so much faster than the latter to appear, in fact, instantaneous.

As well known, the equation governing the dynamics of flexural beams under an external load $q(x,t)$ reads

$$\frac{\partial^2 M(x,t)}{\partial x^2} + q(x,t) = -\mu \frac{\partial^2 v(x,t)}{\partial t^2}, \tag{21}$$

where $\mu = \rho A$ is the mass density per unit length, A is the cross-sectional area and ρ the mass density per unit volume. Then, looking at free vibrations (i.e., $q(x,t) = 0$) and plugging in the extended non-local assumption (9), one gets

$$r^2 \frac{\partial^2}{\partial x^2}\left[\chi_0(x,t) + \int_0^L K(x,\xi)\chi(\xi,t)\mathrm{d}\xi\right] + c_b^{-2}\frac{\partial^2 v(x,t)}{\partial t^2} = 0, \tag{22}$$

where we have let the longitudinal wave speed $c_b = \sqrt{E/\rho}$ and the radius of gyration of the beam cross-section $r = \sqrt{I/A}$. Upon applying the linear operator $\mathfrak{L}$, Eq. (22) becomes

$$r^2 \frac{\partial^4 v(x,t)}{\partial x^4} + c_b^{-2}\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\epsilon^2}{c_b^2}\frac{\partial^4 v(x,t)}{\partial x^2 \partial t^2} = 0, \tag{23}$$

that, by the time-harmonic assumption $v(x,t) = \hat{v}(x)e^{-\iota\omega t}$, reads

$$\frac{\mathrm{d}\hat{v}^4}{\mathrm{d}x^4} + \epsilon^2 \frac{\omega^2}{r^2 c_b^2}\frac{\mathrm{d}\hat{v}^2}{\mathrm{d}x^2} - \frac{\omega^2}{r^2 c_b^2}\hat{v} = 0, \tag{24}$$

where ω is the angular frequency and we have omitted to specify that $\hat{v}$ depends on x to lighten notation. The general solution of the ODE (24) is given by

$$\hat{v}(x) = C_1 \cos\left(\alpha_1 \frac{x}{L}\right) + C_2 \sin\left(\alpha_1 \frac{x}{L}\right) + C_3 \cosh\left(\alpha_2 \frac{x}{L}\right) + C_4 \sinh\left(\alpha_2 \frac{x}{L}\right), \tag{25}$$

where C_1, C_2, C_3, C_4 are integration constants and the functions α_1 and α_2 have been defined as

$$\alpha_1 = \omega_0\epsilon_0\sqrt{1+\sqrt{1+2\omega_0^{-2}\epsilon_0^{-4}}}, \quad \alpha_2 = \omega_0\epsilon_0\sqrt{-1+\sqrt{1+2\omega_0^{-2}\epsilon_0^{-4}}}, \tag{26}$$

with $\omega_0 = \omega L^2/(\sqrt{2}rc_b)$ being the dimensionless natural frequency. We note that α_1 and α_2 satisfy the following connections

$$\alpha_1\alpha_2 = \alpha_0^2, \quad \alpha_1^2 - \alpha_2^2 = \epsilon_0^2\alpha_0^4, \tag{27}$$

having let $\alpha_0 \equiv 2^{1/4}\sqrt{\omega_0}$. In particular, in the case of purely local elasticity, $\epsilon_0 = 0$ and $\alpha_1 = \alpha_2 = \alpha_0$. It is emphasized that the case of purely local elasticity is equally obtained as $\eta = \omega_0\epsilon_0^2 \to 0$, that also includes the static limit. The dimensionless parameter η expresses the ratio between the non-local length scale over the length of elastic waves, which we expect to be indeed very small, i.e.

$$\omega_0 \ll \epsilon_0^{-2}. \tag{28}$$

Now, inverting the moment–curvature relation (9) through application of the linear differential operator $\mathfrak{L}$, we find

$$M(x,t) - \epsilon^2 \frac{\partial^2 M(x,t)}{\partial x^2} = D\chi(x,t), \tag{29}$$

and then exploiting the governing Eq. (21) to eliminate the second derivative of the bending moment, we arrive at (we recall $q(x,t) = 0$)

$$M(x,t) = -\mu\epsilon^2 \frac{\partial^2 v(x,t)}{\partial t^2} + D\frac{\partial^2 v(x,t)}{\partial x^2}, \tag{30}$$

that provides the bending moment once the displacement v is known. In particular, under the time-harmonic assumption $M(x,t) = \hat{M}(x)e^{-\iota\omega t}$, this equation becomes

$$\hat{M}(x) = \frac{2D}{L^2}\left(\omega_0^2\epsilon_0^2\,\hat{v} + \frac{1}{2}L^2\frac{\mathrm{d}^2\hat{v}}{\mathrm{d}x^2}\right). \tag{31}$$

The boundary conditions for a cantilever beam in the frequency domain read

$$\hat{v}(0) = \frac{\mathrm{d}\hat{v}}{\mathrm{d}x}(0) = 0, \quad \text{and} \quad \hat{M}(L) = \frac{\mathrm{d}\hat{M}}{\mathrm{d}x}(L) = 0, \tag{32}$$

and they may be used to determine the yet unknown constants C_i, $i \in \{1,2,3,4\}$. Indeed, a homogeneous system of linear equations is arrived at and, for a non-trivial solution to exist, the determinant of the relevant coefficient matrix must vanish, which condition yields the frequency (or secular) equation

$$\frac{\alpha_1^4+\alpha_2^4}{2\alpha_1^2\alpha_2^2} + \frac{\alpha_1^2-\alpha_2^2}{2\alpha_1\alpha_2}\sin\alpha_1\sinh\alpha_2 + \cos\alpha_1\cosh\alpha_2 = 0, \tag{33}$$

that, in light of the connections (27), also reads

$$1 + \eta^2 + \frac{\eta}{\sqrt{2}}\sin\alpha_1\sinh\alpha_2 + \cos\alpha_1\cosh\alpha_2 = 0. \tag{34}$$

As well known, this equation provides the beam natural frequencies and, in the case of local elasticity $\epsilon_0 = 0$, we retrieve the classical result

$$1 + \cos\alpha_0\cosh\alpha_0 = 0. \tag{35}$$

Indeed, this result holds in general, to leading order, in the asymptotic limit $\eta = \omega_0\epsilon_0^2 \ll 1$, for we have $\alpha_1 \sim \alpha_2 \sim \alpha_0$ and

$$\epsilon_0\alpha_0 = \sqrt[4]{2}\sqrt{\eta} \pm O(\eta^{3/2}), \tag{36}$$

whence we see that the eigenfrequencies of the non-local problem corresponds to those of the local problem to leading order. The quality of the approximation is good only for the first few modes and for $\epsilon_0 \ll 1$, as it is shown in Fig. 5.

Looking at Eq. (36), it is seen that the leading order term for α_0 does not depend on ϵ_0 and in fact it may be large for $\epsilon_0 \ll 1$ provided that ω_0 is large. From this observation, a very good approximation may be obtained, that is valid for high frequencies and that reveals the deviation from the classical theory. By observing that we may write (26) in terms of η as

$$\epsilon_0\alpha_{1,2} = \eta\sqrt{\pm 1 + \sqrt{1+2\eta^{-2}}}, \tag{37}$$

and then considering the case when $\eta \gg 1$, one gets

$$\alpha_1 = \sqrt{2}\eta\epsilon_0^{-1}\left(1 + \frac{1}{4\eta^2}\right) + O(\eta^{-3}), \quad \alpha_2 = \epsilon_0^{-1}\left(1 - \frac{1}{4\eta^2}\right) + O(\eta^{-4}), \tag{38}$$

whence, for $\epsilon_0 \ll 1$, it is $\alpha_2 \gg \alpha_1 \sim \sqrt{\omega_0}$. Accordingly, by neglecting the algebraic powers of η in Eq. (34), we get the transcendental equation

$$\frac{\eta}{\sqrt{2}}\tan\alpha_1 + 1 = 0, \tag{39}$$

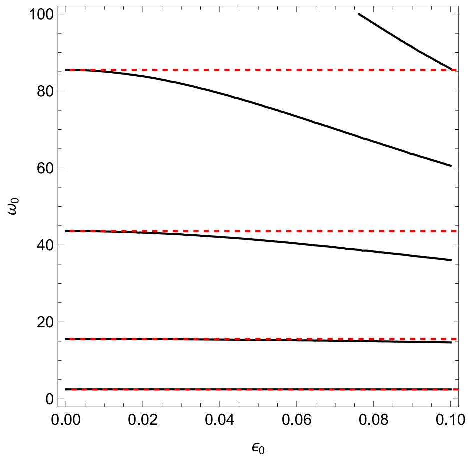

Fig. 5. Dimensionless eigenfrequency ω_0 vs. dimensionless non-local parameter ϵ_0 for a cantilever beam (solid, black) and the classical approximation (dashed, red).

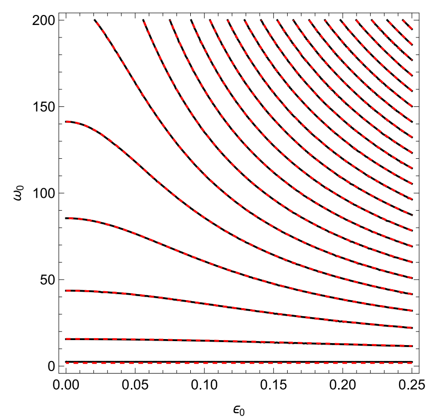

Fig. 6. Dimensionless eigenfrequency ω_0 vs. dimensionless non-local parameter ϵ_0 for the free vibrations of a non-local cantilever beam (solid, black), compared against the high frequency approximation (39) (dashed, red) in a broad range of frequencies: the difference can be barely appreciated but it is largest for the first eigenmode.

whose solutions for α_1 may be plugged into the first of Eqs. (26) to retrieve ω_0. The excellent agreement with (34) extend to a broad range of the parameter ϵ_0 and it is shown in Fig. 6. Eq. (39) may be equivalently rewritten in terms of α_1, to the same degree of approximation, as

$$\frac{\epsilon_0^2\alpha_1^2}{\sqrt{1+\epsilon_0^2\alpha_1^2}}\tan\alpha_1 + 2 = 0. \tag{40}$$

The convergence of the purely non-local to the purely local theory can be proved to occur also for the eigenfunctions, that in general read

$$\hat{v}(x) = C_1\left[\cos\left(\alpha_1\frac{x}{L}\right) - \cosh\left(\alpha_2\frac{x}{L}\right) + \frac{\alpha_2}{\alpha_3}\sin\left(\alpha_1\frac{x}{L}\right) - \frac{\alpha_1}{\alpha_3}\sinh\left(\alpha_2\frac{x}{L}\right)\right], \tag{41}$$

where,

$$\alpha_3 = \frac{\alpha_2^2\cos\alpha_1 + \alpha_1^2\cosh\alpha_2}{\alpha_2\sin\alpha_1 - \alpha_1\sinh\alpha_2} = -\frac{\alpha_2^3\sin\alpha_1 + \alpha_1^3\sinh\alpha_2}{\alpha_2^2\cos\alpha_1 + \alpha_1^2\cosh\alpha_2}, \tag{42}$$

and the constant C_1 can be determined from the initial conditions. Indeed, it is easy to prove that α_3 corresponds, to leading order in the small quantity η, to the equivalent quantity appearing in local elasticity, namely

$$\alpha_3 = \alpha_L + O(\eta), \tag{43}$$

where

$$\alpha_L = -\alpha_0\frac{\sin\alpha_0 + \sinh\alpha_0}{\cos\alpha_0 + \cosh\alpha_0}. \tag{44}$$

Hence, the eigenmodes (41) become

$$\hat{v}(x) = C_1\left\{\cos\left(\alpha_0\frac{x}{L}\right) - \cosh\left(\alpha_0\frac{x}{L}\right) + \frac{\alpha_0}{\alpha_L}\left[\sin\left(\alpha_0\frac{x}{L}\right) - \sinh\left(\alpha_0\frac{x}{L}\right)\right]\right\}, \tag{45}$$

that indeed amounts to the classical eigenfunction expression of purely local elasticity. The similarity of the first few eigenforms with those of purely local elasticity is illustrated in Fig. 7.

Finally, it only remains to determine the modified kernel which somehow incorporates the boundary conditions (32). Once the bending moment is obtained by plugging the displacement (41) into Eq. (31)

$$\hat{M}(x) = -C_1\frac{D}{L^2}\left[\alpha_2^2\cos\left(\alpha_1\frac{x}{L}\right) + \frac{\alpha_2^3}{\alpha_3}\sin\left(\alpha_1\frac{x}{L}\right) + \alpha_1^2\cosh\left(\alpha_2\frac{x}{L}\right) + \frac{\alpha_1^3}{\alpha_3}\sinh\left(\alpha_2\frac{x}{L}\right)\right], \tag{46}$$

it may be seen that the following pair of homogeneous CBCs hold

$$\hat{M}'(0) - \frac{\alpha_c}{L}\hat{M}(0) = 0, \quad \hat{M}(L) = 0, \tag{47}$$

where $\alpha_c = \frac{\alpha_1\alpha_2}{\alpha_3}$. From that, it is possible to determine χ_0 and the modified kernel follows

$$K^{\dagger}(x,\xi,\omega_0) = \frac{1}{2\epsilon}\,\frac{1 + A_1 e^{-\frac{2}{\epsilon}\min\{x,\xi\}}}{1 + A_1 e^{-\frac{2}{\epsilon_0}}}\left(1 - e^{-\frac{2}{\epsilon}(L-\max\{x,\xi\})}\right)e^{-\frac{|\xi-x|}{\epsilon}}, \tag{48}$$

where we have let

$$A_1 = \frac{1-\epsilon_0\alpha_c}{1+\epsilon_0\alpha_c}. \tag{49}$$

The above kernel is symmetric in its spatial arguments x, ξ, and it corresponds to the Green's function of the operator $\mathfrak{L}$ with the boundary conditions (47). As such, it satisfies the impulsivity property and it attains its maximum as $x \to \xi$. As in the statically indetermined case treated at the previous Section 2, the kernel is no longer a difference kernel as a result of the incorporation of the boundaries. A graphical comparison between the modified kernel $K^{\dagger}(x,\xi)$ and the standard Helmholtz kernel is presented in Fig. 8. As it was the case in statics, the modified kernel shows great similarity with the Helmholtz kernel, with the important exception of the close neighbourhood of the end points. As shown in Appendix B, the modified kernel is still normalized over an infinite domain.

Most remarkably, the modified kernel is frequency dependent through the term $A_1 = A_1(\omega_0)$, which means that the constitutive response becomes *rate dependent*. This surprising behaviour is an inevitable outcome of the first of the boundary conditions (47) being frequency dependent through the coefficient α_c. The connection between the rate dependent nature of the response and non-locality may be easily explained in consideration of the fact that the latter entails that the elastic response at a point depends on the deformation at neighbouring points, that, in turn, is described, for each eigenfrequency, by the modal shape connected to that frequency. It is therefore not surprising that the diffusion of elastic effects is affected by frequency. However, it is easy to show that this frequency dependence is generally very mild, with the usual exception of the boundary points. We first prove this by examining the asymptotic behaviour for low-frequency vibrations (or nearly-local response, i.e. $\epsilon_0 \ll 1$), namely when $\eta \ll 1$. By Eq. (43), it is, to leading order,

$$\alpha_c = \frac{\alpha_0^2}{\alpha_L}, \tag{50}$$

that is the distinctive feature of the classical local theory. Thus we can see that, for low-frequencies, the frequency dependence of the

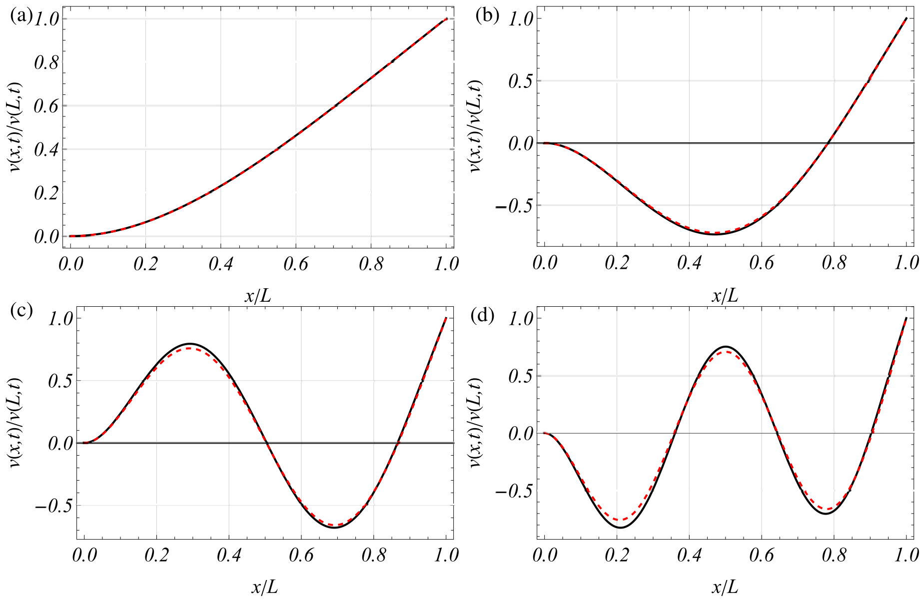

Fig. 7. Comparison of the first four eigenfunctions (a, b, c, d) for free vibrations of a cantilever beam within the purely non-local (black, solid) and the purely local theory (red, dashed), having let $\epsilon_0 = 0.05$.

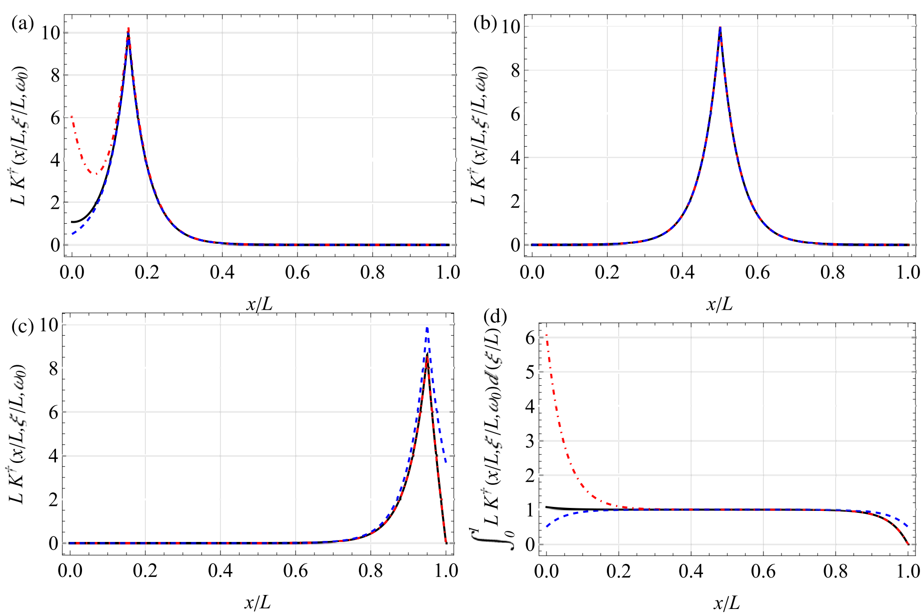

Fig. 8. Comparison of the modified kernel for a vibrating cantilever beam (black, solid) at the first eigenfrequency $\omega_0 = 2.48887$ and at the 10th eigenfrequency $\omega_0 = 359.333$ (red, dot-dashed), against the Helmholtz kernel (blue, dashed), that is frequency independent, over the domain $[0, L]$, with $L = 1$ for convenience. The kernels are evaluated at (a) $\xi/L = 5\%$, (b) $\xi/L = 50\%$, and (c) $\xi/L = 95\%$, with the dimensionless length-scale parameter $\epsilon_0 = 5\%$; (d) illustrates the normalization condition applied over the finite domain $[0, L]$, while normalization over the infinite domain is always granted.

kernel is the same as that given by the purely local theory, which is a correspondence that is in fact demanded as a good requisite of the theory. Besides, it is interesting to point out that the BCs for a cantilever beam in elastostatics are still given by (47) with $\alpha_c = -1$, and this is indeed retrieved here as the static limit. In contrast, for large frequencies and $\epsilon_0 \ll 1$, from Eqs. (38) we get, to leading order,

$$\alpha_3 \sim -\alpha_1 \quad \text{and} \quad \alpha_c = -\alpha_2 \sim -\epsilon_0^{-1} \tag{51}$$

whence the BCs (47) become *frequency independent*. Consequently, we obtain the asymptotic expansion for A_1

$$A_1 \sim \frac{1-\epsilon_0\alpha_2}{1+\epsilon_0\alpha_2} = 6 + O(\eta^{-2}), \tag{52}$$

which shows that, within this asymptotic limit (valid for high frequencies, as we already saw), the kernel becomes effectively frequency independent at every point in space, to leading order.

Yet, frequency dependence can be shown to be mild in general, even for low frequencies, at the cost of forsaking the neighbourhood of the end points. Indeed, looking at (48), the frequency-dependent factor at the denominator $1 + A_1(\omega_0)\exp(-2/\epsilon_0)$ is very close to 1, given that the exponential factor is usually very small for physically meaningful values of the length-scale parameter ϵ_0. Likewise, at the numerator, the term $1 + A_1(\omega_0)\exp(-\frac{2}{\epsilon}\min\{x,\xi\})$ cancels with the denominator for $x = \xi = L$, that is at the right end point (and this clearly follows from the fact that the second of the BCs (47) is frequency independent). In

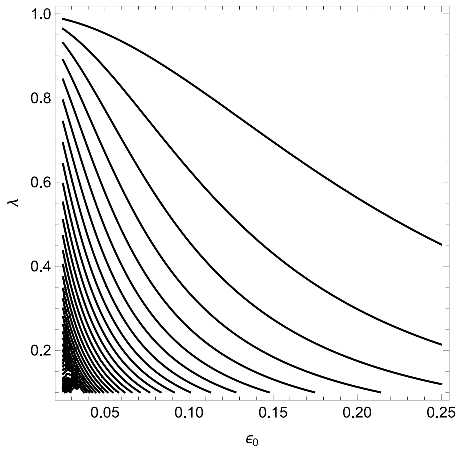

Fig. 9. Spectrum of the integral operator (3) with kernel (48), for $\alpha_c = -1.3768$ that corresponds to the first eigenfrequency for the cantilever beam (with $\epsilon_0 = 0.05$).

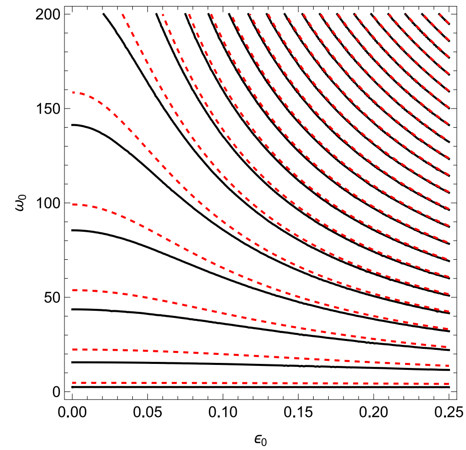

Fig. 10. Spectrum of the integral operator (3) (solid, black) against the spectrum of the cantilever beam (dashed, red) through the connection (59), with $\epsilon_0 = 0.05$.

general, this term depends on the spatial variable x/L (or ξ/L) and its contribution is maximum at the left boundary and rapidly decays, as a result of the exponential term. Looking at the first subfigure in Fig. 8, the frequency dependence is evident for $\xi < x \ll 1$, that is at points near the built-in end. Conversely, when $\min\{x,\xi\}$ is sufficiently far from the left boundary (as depicted in the second and third sub-figures), frequency dependence becomes negligible throughout the domain. The fourth sub-figure in Fig. 8 shows the normalization condition over the finite interval $[0, L]$ and it is again seen that frequency dependence is confined to the vicinity of the left boundary. As it was the case for statics, the right boundary is under-express while interior points are equally expressed. In contrast to statics, the left point may be under- or over-expressed depending on the eigenfrequency under scrutiny.

3.1. Connection between the integral operator spectrum and the eigenfrequencies

The eigenvalues $\{\lambda_n\}$ of the operator (3), where the kernel function is given by (48), are defined through the solution of the Fredholm integral equation of the *first kind*

$$\int_0^L K^\dagger(x,\xi)\psi(\xi)\mathrm{d}\xi = \lambda\psi(x), \tag{53}$$

and excluding the trivial solution $\psi(x) \equiv 0$. As it will be shortly shown, eigenvalues are countable and they may be ordered from large to low in the infinite sequence $\{\lambda_n\}$. To every eigenvalue λ_n, the corresponding eigenfunction $\psi_n(x)$ is associated. By applying the operator $\mathcal{L}$, the problem is transformed into an eigenvalue problem for the second order ODE

$$\psi'' + \Lambda^2\psi = 0, \quad \Lambda = \frac{1}{\epsilon_0}\sqrt{\frac{1-\lambda}{\lambda}}, \tag{54}$$

subjected to the BCs (47), i.e.

$$\psi'(0) - \frac{\alpha_c}{L}\psi(0) = 0, \quad \psi(L) = 0. \tag{55}$$

This problem admits nontrivial solutions provided that λ is a solution of the transcendental equation

$$\frac{\Lambda}{\alpha_c} + \tan\Lambda = 0, \tag{56}$$

which gives the spectrum of the integral operator. This is plotted in Fig. 9, having set for α_c the value obtained for the first eigenvibration of the cantilever. It appears that the eigenvalues $\lambda_n \in (0,1)$ are all positive, which fact proves that the modified kernel is associated to a positive definite operator. This property extends to all α_c values, which are in fact negative for every eigenmode.

Motivated by the high-frequency asymptotics (40), we let

$$\Lambda = \alpha_1 + \tfrac{1}{2}\pi, \tag{57}$$

so that Eq. (56) becomes

$$\frac{\alpha_3}{\alpha_1\alpha_2}(\alpha_1 + \tfrac{1}{2}\pi)\tan\alpha_1 - 1 = 0, \tag{58}$$

and we have the connection between λ and ω_0

$$\lambda = \frac{1}{1+\epsilon_0^2(\frac{1}{2}\pi+\alpha_1)^2}. \tag{59}$$

Eq. (58) is plotted in Fig. 10 against the cantilever eigenfrequency spectrum and it is seen that they are very similar for high frequencies. This means that, for high frequencies, the integral operator spectrum corresponds to the cantilever spectrum through the connection (59). This similarity extends to the integral operator eigenfunctions

$$\psi_n(x) = \frac{\Lambda}{\alpha_c}\cos\left(\Lambda\frac{x}{L}\right) + \sin\left(\Lambda\frac{x}{L}\right), \tag{60}$$

which, under the connection (57), correspond to the eigenforms (41). The fact that the integral operator and the cantilever spectra are not the same in general depends on the presence of a differential operator in the governing Eq. (22), which embodies an integro-differential problem.

4. Conclusions

The non-local theory of elasticity has long received considerable attention in the literature, in view of its potential to describe structures at the nano-scale and to overcome some well-known limits of the classical local theory, in particular the appearance of singularities at the crack tips. However, the purely non-local theory, in the form described by Eringen, suffers from a major shortcoming, in that it often leads to ill-posed problems. In this paper, we show how to circumvent this deficiency by modification of the Helmholtz kernel. Through this approach, the natural boundary conditions of the problem are incorporated into the kernel function, in the place of the original constitutive boundary conditions (CBCs). In so doing, we obtain a double advantage, in that the problem becomes well-posed and, at the same time, we dispense with the original CBCs, that are unphysical. As a result of this procedure, the modified kernel is no longer of the difference type, although we show that the consequent inhomogeneity of the elastic response arises only near the boundaries. We believe this to be well within the spirit of non-locality, that supports the idea that any boundary should have an influence on its neighbouring points. Yet, the modified kernel remains symmetric and positive definite, which properties guarantee that a quadratic positive-definite strain energy functional is defined. The procedure is illustrated by two examples: one dealing with a cantilever beam elastically supported at its tip, and the other regarding free vibrations of a cantilever beam. Within the standard non-local theory, both examples would lead to an ill-posed problem. In the first example, we observe that the modified kernel now bears a connection with the stiffness of the support. This feature is shared with the classical local theory and with the two-phase theory and, in fact, it is milder for the non-local theory. Similarly, the second example reveals that the modified kernel is weakly rate-dependent (in fact frequency dependent). Consideration of the integral operator spectrum shows that, unlike classical local elasticity, this no longer corresponds to the structure's eigenfrequencies, as a result of the problem being integro-differential in nature. Yet, a very good asymptotic connection between those spectra could be found.

CRediT authorship contribution statement

Dipendu Pramanik: Writing – original draft, Investigation. **Andrea Nobili:** Writing – review & editing, Writing – original draft, Supervision, Investigation, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Two-phase non-local elasticity theory

In this Section we outline the solution to the supported cantilever problem within the two-phase non-local elasticity model (TPNM). According to this theory, the constitutive law is expressed as a linear combination of the local and of the non-local curvature, the latter being obtained as in the left-hand side (LHS) of Eq. (2). Thus, the constitutive law (2) is now replaced by

$$\xi_1 \chi(x) + \xi_2 \int_0^L K(x,\xi)\chi(\xi)\mathrm{d}\xi = D^{-1} M(x), \tag{A.1}$$

where $0 \leq \xi_{1,2} \leq 1$ and $\xi_1 + \xi_2 = 1$ are the phase parameters (i.e. volume fractions) corresponding to the local and to the non-local phase, respectively. Upon applying the linear operator $\mathfrak{L}$, we get

$$\chi(x) - \xi_1 \epsilon^2 \chi''(x) = D^{-1}\mathfrak{L} M(x), \tag{A.2}$$

that is a second order ODE, as opposed to the algebraic relation (10). As a consequence, an extra pair of BCs, named constitutive, is required, beyond the natural ones given by Eqs. (12). For the constitutive boundary conditions (CBCs), we choose

$$\xi_1 \left\{\chi'(0) - \epsilon^{-1}\chi(0)\right\} = D^{-1}\left\{M'(0) - \epsilon^{-1} M(0)\right\}, \tag{A.3}$$

$$\xi_1 \left\{\chi'(L) + \epsilon^{-1}\chi(L)\right\} = D^{-1}\left\{M'(L) + \epsilon^{-1} M(L)\right\}, \tag{A.4}$$

that dispense with the integral term. Using the expression for the bending moment (1), and the linear operator (6), we obtain the curvature

$$\chi(x) = c_1 \exp\left(\frac{x/L}{\epsilon_0\sqrt{\xi_1}}\right) + c_2 \exp\left(-\frac{x/L}{\epsilon_0\sqrt{\xi_1}}\right)$$
$$+ \frac{q_0 L^2}{D}\left[R_d\left(1 - \frac{x}{L}\right) - \frac{1}{2}\left(1 - \frac{x}{L}\right)^2 + \epsilon_0^2\left(1 - \xi_1\right)\right], \tag{A.5}$$

where the constants c_1, c_2 are can be derived by using CBCs (A.3), (A.4). Then, using the boundary conditions (12), the dimensionless reaction force is derived as

$$R_d = \frac{3\kappa}{8}\,\frac{4\left(1-\xi_1\right)\epsilon_0 R_1 - 4\left(1-\xi_1\right)\epsilon_0^2 R_2 - R_3 + \left(1+\sqrt{\xi_1}\right)^2 e^{\frac{2}{\sqrt{\xi_1}\epsilon_0}}}{\left(\sqrt{\xi_1}+1\right)^2 e^{\frac{2\sqrt{\xi_1}}{\epsilon_0}} R_5 - \left(1-\sqrt{\xi_1}\right)^2 R_6 + \left(1-\xi_1\right)\sqrt{\xi_1} R_7}, \tag{A.6}$$

where,

$$R_1 = \left(1+\sqrt{\xi_1}\right) e^{\frac{2}{\sqrt{\xi_1}\epsilon_0}} - \left(1-\sqrt{\xi_1}\right), \tag{A.7}$$

$$R_2 = \left(e^{\frac{1}{\sqrt{\xi_1}\epsilon_0}} - 1\right)\left(-\sqrt{\xi_1} + 2\xi_1 + \left(\sqrt{\xi_1} + 2\xi_1 - 1\right) e^{\frac{1}{\sqrt{\xi_1}\epsilon_0}} - 1\right), \tag{A.8}$$

$$R_3 = \left(1-\sqrt{\xi_1}\right)^2 + 8\left(1-\xi_1\right)^2\sqrt{\xi_1}\epsilon_0^3\left(e^{\frac{1}{\sqrt{\xi_1}\epsilon_0}} - 1\right)^2, \tag{A.9}$$

$$R_5 = \kappa + 3 - 3\kappa\epsilon_0\left(1-\sqrt{\xi_1}\right)\left(\sqrt{\xi_1}\left(2\epsilon_0+1\right)\epsilon_0 - \epsilon_0 - 1\right), \tag{A.10}$$

$$R_6 = \kappa + 3 + 3\kappa\epsilon_0\left(1+\sqrt{\xi_1}\right)\left(\sqrt{\xi_1}\left(2\epsilon_0+1\right)\epsilon_0 + \epsilon_0 + 1\right), \tag{A.11}$$

$$R_7 = 12\kappa\epsilon_0^2\left(1+\epsilon_0\right) e^{\frac{1}{\sqrt{\xi_1}\epsilon_0}}. \tag{A.12}$$

Comparing Eqs. (A.5) with (10), we already see that the bending moment depends on the support stiffness, as it was the case for the purely non-local theory.

Appendix B. Normalization of the modified non-local kernel

If we consider a cantilever beam of length $L-a$, with its ends located at $x=a$ and $x=L$, then the constitutive boundary conditions (CBCs) given in Eq. (16) can be rewritten as

$$M'(a) + \frac{\beta_g}{L-a} M(a) = 0, \quad M(L) = 0, \tag{B.1}$$

where

$$\beta_g = 1 + \frac{1 + \left(\frac{1}{3} + a_0^2\frac{a_0-3}{6}\right)\kappa}{1 + \left(1+a_0\right)\left(\frac{1}{12} + a_0\frac{a_0-4}{12} + \epsilon_0^2\right)\kappa}, \quad a_0 = \frac{a}{L}. \tag{B.2}$$

Thus, the generalized kernel (18) can be rewritten in the domain $[a, L]$ as

$$K^\dagger(x,\xi) = \frac{\sinh\left(\frac{(1-a_0) - L^{-1}|x-\xi|}{\epsilon_0}\right) + \sinh\left(\frac{(1-a_0) - L^{-1}(x+\xi)}{\epsilon_0}\right)}{2\epsilon\cosh\left(\frac{1-a_0}{\epsilon_0}\right)}$$
$$\frac{\epsilon_0\beta_g\tanh\left(\frac{\min\{x,\xi\}-a}{\epsilon}\right) - 1 + a_0}{\epsilon_0\beta_g\tanh\left(\frac{1-a_0}{\epsilon_0}\right) - 1 + a_0}. \tag{B.3}$$

The area of the above kernel $K^\dagger(x,\xi)$ over the domain $\xi \in [a, L]$ is given by

$$\int_a^L K^\dagger(x,\xi)\mathrm{d}\xi = 2\sinh\left(\frac{1 - L^{-1}x}{2\epsilon_0}\right)$$
$$\frac{(1-a_0)\sinh\left(\frac{1+L^{-1}x-2a_0}{2\epsilon_0}\right) + \beta_g\epsilon_0\left\{\cosh\left(\frac{1-L^{-1}x}{2\epsilon_0}\right) - \cosh\left(\frac{1+L^{-1}x-2a_0}{2\epsilon_0}\right)\right\}}{\left(1-a_0\right)\cosh\left(\frac{1-a_0}{\epsilon_0}\right) - \beta_g\epsilon_0\sinh\left(\frac{1-a_0}{\epsilon_0}\right)}. \tag{B.4}$$

However, the area over the whole domain $(-\infty, \infty)$ is given by

$$\lim_{\substack{a\to-\infty \\ L\to\infty}} \int_a^L K^\dagger(x,\xi)\mathrm{d}\xi = \lim_{L\to\infty}\left(1 - e^{-\frac{L-x}{\epsilon}}\right) = 1. \tag{B.5}$$

Thus, the modified kernel satisfies the unity property (normalization condition) in the infinite domain, which is a fundamental requirement for any non-local kernel; see Koutsoumaris and Eptaimeros (2018, §2.2).

Data availability

No data was used for the research described in the article.

References

Aksoylu, B., Gazonas, G.A., 2020. On the choice of kernel function in nonlocal wave propagation. J. Peridyn. Nonlocal Model. 2 (4), 379–400.

Alotta, G., Di Paola, M., Pinnola, F.P., 2022. An unified formulation of strong non-local elasticity with fractional order calculus. Meccanica 57 (4), 793–805.

Altenbach, H., Maugin, G.A., Erofeev, V., 2011. Mechanics of Generalized Continua. vol. 7, Springer.

Benvenuti, E., Simone, A., 2013. One-dimensional nonlocal and gradient elasticity: Closed-form solution and size effect. Mech. Res. Commun. 48, 46–51.

Challamel, N., Wang, C., 2008. The small length scale effect for a non-local cantilever beam: a paradox solved. Nanotechnology 19 (34), 345703.

Eptaimeros, K., Koutsoumaris, C.C., Tsamasphyros, G., 2016. Nonlocal integral approach to the dynamical response of nanobeams. Int. J. Mech. Sci. 115, 68–80.

Eringen, A.C., 1972. Linear theory of nonlocal elasticity and dispersion of plane waves. Internat. J. Engrg. Sci. 10 (5), 425–435.

Eringen, A.C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. 54 (9), 4703–4710.

Eringen, A.C., 1987. Theory of nonlocal elasticity and some applications. Res Mech. 21 (4), 313–342.

Eringen, A., 2002. Nonlocal Continuum Field Theories. Springer New York, NY.

Eringen, A.C., Edelen, D., 1972. On nonlocal elasticity. Internat. J. Engrg. Sci. 10 (3), 233–248.

Eringen, A.C., Kim, B.S., 1974. Stress concentration at the tip of crack. Mech. Res. Commun. 1 (4), 233–237.

Eroğlu, U., Ruta, G., 2022. Approximate closed-form solutions for vibration of nano-beams of local/non-local mixture. Meccanica 57 (12), 3033–3049.

Fernandez-Saez, J., Zaera, R., Loya, J., Reddy, J., 2016. Bending of Euler–Bernoulli beams using eringen's integral formulation: A paradox resolved. Internat. J. Engrg. Sci. 99, 107–116.

Kahrobaiyan, M., Asghari, M., Rahaeifard, M., Ahmadian, M., 2011. A nonlinear strain gradient beam formulation. Internat. J. Engrg. Sci. 49 (11), 1256–1267.

Karlicic, D., Murmu, T., Adhikari, S., McCarthy, M., 2015. Non-Local Structural Mechanics. John Wiley & Sons.

Khaniki, H.B., 2018. Vibration analysis of rotating nanobeam systems using Eringen's two-phase local/nonlocal model. Phys. E: Low- Dimens. Syst. Nanostruct. 99, 310–319.

Khodabakhshi, P., Reddy, J., 2015. A unified integro-differential nonlocal model. Internat. J. Engrg. Sci. 95, 60–75.

Koutsoumaris, C.C., Eptaimeros, K., 2018. A research into bi-Helmholtz type of nonlocal elasticity and a direct approach to Eringen's nonlocal integral model in a finite body. Acta Mech. 229, 3629–3649.

Koutsoumaris, C.C., Eptaimeros, K., Tsamasphyros, G., 2017. A different approach to Eringen's nonlocal integral stress model with applications for beams. Int. J. Solids Struct. 112, 222–238.

Kröner, E., 1967. Elasticity theory of materials with long range cohesive forces. Int. J. Solids Struct. 3 (5), 731–742.

Li, C., Yao, L., Chen, W., Li, S., 2015. Comments on nonlocal effects in nano-cantilever beams. Internat. J. Engrg. Sci. 87, 47–57.

Mikhasev, G., Nobili, A., 2020. On the solution of the purely nonlocal theory of beam elasticity as a limiting case of the two-phase theory. Int. J. Solids Struct. 190, 47–57.

Nobili, A., 2021. Asymptotically consistent size-dependent plate models based on the couple-stress theory with micro-inertia. Eur. J. Mech. A Solids 89, 104316.

Nobili, A., Pramanik, D., 2025. A well-posed theory of linear non-local elasticity. Int. J. Eng. Sci. 215, 104314.

Peddieson, J., Buchanan, G.R., McNitt, R.P., 2003. Application of nonlocal continuum models to nanotechnology. Internat. J. Engrg. Sci. 41 (3–5), 305–312.

Picu, R., 2002. On the functional form of non-local elasticity kernels. J. Mech. Phys. Solids 50 (9), 1923–1939.

Pisano, A.A., Fuschi, P., 2003. Closed form solution for a nonlocal elastic bar in tension. Int. J. Solids Struct. 40 (1), 13–23.

Radi, E., Bianchi, G., Nobili, A., 2021. Bounds to the pull-in voltage of a MEMS/NEMS beam with surface elasticity. Appl. Math. Model. 91, 1211–1226.

Rafii-Tabar, H., Ghavanloo, E., Fazelzadeh, S.A., 2016. Nonlocal continuum-based modeling of mechanical characteristics of nanoscopic structures. Phys. Rep. 638, 1–97.

Reddy, J., 2007. Nonlocal theories for bending, buckling and vibration of beams. Internat. J. Engrg. Sci. 45 (2–8), 288–307.

Romano, G., Barretta, R., Diaco, M., de Sciarra, F.M., 2017. Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. Int. J. Mech. Sci. 121, 151–156.

Wang, Q., Liew, K., 2007. Application of nonlocal continuum mechanics to static analysis of micro-and nano-structures. Phys. Lett. A 363 (3), 236–242.

Wang, Y., Zhu, X., Dai, H., 2016. Exact solutions for the static bending of Euler-Bernoulli beams using Eringen's two-phase local/nonlocal model. Aip Adv. 6 (8).

Zhang, P., Schiavone, P., Qing, H., 2022. Local/nonlocal mixture integral models with bi-Helmholtz kernel for free vibration of Euler-Bernoulli beams under thermal effect. J. Sound Vib. 525, 116798.