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Large Deviations and Additivity Principle for the Open Harmonic Process

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Abstract: We consider the boundary driven harmonic model, i.e. the Markov process associated to the open integrable XXX chain with non-compact spins. We characterize its stationary measure as a mixture of product measures. For all spin values, we identify the law of the mixture in terms of the Dirichlet process. Next, by using the explicit knowledge of the non-equilibrium steady state we establish formulas predicted by Macroscopic Fluctuation Theory for several quantities of interest: the pressure (by Varadhan's lemma), the density large deviation function (by contraction principle), the additivity principle (by using the Markov property of the mixing law). To our knowledge, the results presented in this paper constitute the first rigorous derivation of these macroscopic properties for models of energy transport with unbounded state space, starting from the microscopic structure of the non-equilibrium steady state.

1. Motivations and Informal Discussion of the Main Results

In non-equilibrium statistical physics, a major problem is to understand systems with open boundaries, in particular the structure of their stationary measure. In the literature this is often referred to as the "non-equilibrium steady state" or the "stationary non-equilibrium state". In the simplest set-up one considers one-dimensional models on a finite segment of length N which are driven out-of-equilibrium by two boundary reservoirs with densities $\rho_l > 0$, resp. $\rho_r > 0$. A paradigmatic model, for which explicit knowledge of the stationary measure is available, is the boundary-driven simple symmetric exclusion process, where one has the description of the stationary measure via the matrix-product ansatz [18]. Other models are solvable but do not exhibit the long-range correlations structure that is believed to be a distinguishing feature of non-equilibrium, such as zero-range models [1,36,39] which have a non-equilibrium steady state which is product, or the Ginzburg–Landau model [12,14], whose non-equilibrium steady state is a Gibbs measure with exponentially decaying correlations. Clearly there is urgent need

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to identify other boundary-driven models for which one has full control of the stationary state. This is especially important to extract universal large scale properties via the asymptotic analysis.

In this paper we will prove that the family of boundary-driven model introduced in [26] (called "harmonic models" because it involves harmonic numbers) admits an explicit description of the invariant measure for each system size N as a probabilistic mixture. This family of models, labelled by a parameter $s > 0$, emerged as the *integrable version* of the family of discrete Kipnis–Marchioro–Presutti models [11,33] (the two families share the same large scale behavior). The root of the exact solvability of the harmonic models can be traced back to the fact that they are related to the open integrable XXX spin chain with non-compact spins [3,23,26,35,37]. Remarkably, this spin chain is integrable for all spin values $s > 0$ and thus the whole family of harmonic processes is exactly solvable. See [25] where the moments and the stationary state were obtained in closed-form.

Our first main result is presented in Theorem 3.1, where we prove that the stationary measure of the harmonic models is a "mixture of inhomogeneous Gibbs distributions". In the equilibrium set-up (equal reservoir densities $\rho_l = \rho_r$) the reversible Gibbs distribution of the harmonic models is an homogenous product measure, the marginal at each site being given by a Negative Binomial distribution with shape parameter $2s > 0$ and mean equal to the density of the reservoirs. Theorem 3.1 tell us that in a non-equilibrium set-up (different reservoir densities $\rho_l \neq \rho_r$) the invariant measure of the harmonic models is a mixture of inhomogeneous products of Negative Binomials distributions with shape parameter $2s > 0$ and *scale parameters which are given by random variables, representing a random chemical potential at each site*. We identify the law of these random variables in terms of the symmetric Dirichlet distribution with parameter $2s > 0$ on the $(N+1)$-dimensional simplex. As it is well known, when the parameter $2s$ is an integer, the Dirichlet distribution can be expressed in terms of the order statistics of i.i.d. uniform random variables. Our result agrees with the steady state obtained in [25] (see Appendix A), and reduces to the case of [10] where the stationary measure of the harmonic model with s = 1/2 was proved to be a mixture of i.i.d. geometric random variables whose mean are the order statistics of i.i.d. uniforms.

A second motivation of this paper is the Macroscopic Fluctuation Theory (MFT) [7], which is a theory for diffusive systems proposed in recent years to describe the macroscopic properties emerging in the limit $N \to \infty$. MFT relies on the study of dynamical large deviations and states that macroscopically the behavior of a diffusive systems is dictated by two transport coefficients, the diffusivity $D(\rho)$ and the mobility $\sigma(\rho)$ depending on the system density $\rho : [0,1] \to \mathbb{R}_+$. For the simple symmetric exclusion process, for which a dynamical large deviation principle is available [34], several findings of MFT nicely match the results obtained with microscopic computations using Bethe ansatz methods. See for instance [19] for the large deviations of the density profiles in the stationary state, [15,16] for the large deviations of the current and [31] for the large deviations of the positions of tagged particles. More recently, the time-dependent solution of the MFT dynamical equations was found in [38] using integrability.

The boundary-driven harmonic models considered in this paper, labelled by a parameter $s > 0$, belongs to the class of models with constant diffusivity and convex quadratic mobility

$$D(\rho) = \frac{1}{2s} \qquad \text{and} \qquad \sigma(\rho) = \frac{\rho}{2s}\left(1 + \frac{\rho}{2s}\right). \tag{1.1}$$

Other particle models in the same class include the symmetric inclusion processes [11, 27] and the discrete Kipnis–Marchioro–Presutti models [11,33]. For all these models, the state space is non-compact and the *dynamical large deviation principle is not available*. The reason is that the stationary measures have exponential tails, and the proof of the dynamical large deviation principle, based on super-exponential replacement lemmas requires super-exponential tails of the stationary measures. This technical obstacle has so far not been overcome, so all the results based on the Macroscopic Fluctuation Theory such as in [5,6] are conditional on the solution of this (highly non-trivial) technical issue. This is also the case for the corresponding continuous models of energy transport, namely the Kipnis–Marchioro–Presutti models [11,33] (see also [13] for recent results), the Brownian energy processes [28] and the integrable heat conduction models recently introduced in [24]. We also mention [4], where a stochastic model of linear oscillators is studied and large deviations for the temperature profile in the non-equilibrium stationary state are analyzed. Therefore, for the class of models with constant diffusivity and convex quadratic mobility it is crucial to substantiate the predictions of MFT with microscopic computations, which is the second aim of this paper. In the rest of this introduction we give a summary of those MFT predictions, first formulated in [5] for the discrete Kipnis–Marchioro–Presutti model, that we prove here for the boundary-driven harmonic model.

Large deviations, pressure, additivity principle We recall that, given a sequence of random variables $(X_n)_{n\geq 1}$ taking values in the measurable space $(\mathscr{X}, \mathscr{B})$, with $\mathscr{X}$ a topological space and $\mathscr{B}$ a σ-field of subsets of $\mathscr{X}$, then we say that $(X_n)_{n\geq 1}$ satisfies a large deviation principle with rate function $I(x)$ and speed n w.r.t. a sequence of probability measure $(\mu_n)_{n\geq 1}$ if, for all $B \in \mathscr{B}$

$$\begin{aligned} -\inf_{x\in B^o} I(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mu_n(X_n \in B) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mu_n(X_n \in B) \leq -\inf_{x\in \bar{B}} I(x) \end{aligned}$$

where B^o denotes the interior of B and $\bar{B}$ its closure. Consider the empirical density profile

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \eta_i\, \delta_{\frac{i}{N}} \tag{1.2}$$

where $\delta_{\frac{i}{N}}$ is a Dirac measure in $\frac{i}{N} \in [0, 1]$ and $(\eta_i)_{i=1,\ldots,N}$ are distributed according to the invariant distribution of the boundary-driven harmonic model with parameter $s > 0$, system size $N \in \mathbb{N}$ and boundary densities $0 \leq \rho_l \leq \rho_r < \infty$ (for a precise definition of the model see Sect. 2). We introduce the space of finite positive measure $\mathscr{M}^+[0, 1]$ equipped with the weak topology and in particular the subset of absolutely continuous finite positive measures

$$\mathscr{X} = \{\mu \in \mathscr{M}^+[0, 1] : \mu(dx) = \rho(x)dx,\ \text{with}\ \rho \in L^1([0, 1], dx) : \rho(x) \geq 0\}.$$

Then, for models with transport coefficients (1.1) MFT predicts [5] that the sequence of empirical measures $(L_N)_{N\geq 1}$ satisfies a large deviation principle with speed N and rate function which is infinite when $\mu \notin \mathscr{X}$ and for $\mu(dx) = \rho(x)dx$ is given by $I(\rho)$ which is the solution of the variational problem

$$I(\rho) = \inf_{\theta} \mathscr{I}(\rho, \theta) \tag{1.3}$$

with

$$\mathscr{I}(\rho,\theta)=2s\int_0^1 dx\Big[\frac{\rho(x)}{2s}\log\frac{\frac{\rho(x)}{2s}}{\theta(x)}+(1+\frac{\rho(x)}{2s})\log\Big(\frac{1+\theta(x)}{1+\frac{\rho(x)}{2s}}\Big)-\log\Big(\frac{\theta'(x)}{\rho_r-\rho_l}\Big)\Big]. \tag{1.4}$$

The infimum in (1.3) is over increasing C^1 functions $\theta:[0,1]\to\mathbb{R}$ such that $\theta(0)=\rho_l$ and $\theta(1)=\rho_r$. As remarked in [6] this large deviation function contains a relative entropy term and a contribution related to the large deviations of the empirical profile of the order statistics of independent uniforms. We will obtain rigorously (see Theorem 5.1) this variational expression from the exact description of the stationary measure, which indeed involves the order statistics of independent uniforms. In particular the infimum in (1.3) corresponds to the contraction principle over the empirical profile of order statistics.

We will also study the pressure, which for a function $h:[0,1]\to\mathbb{R}$ is defined as

$$P(h)=\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\Big[e^{N\langle L_N,h\rangle}\Big]. \tag{1.5}$$

The pressure can be obtained from the density large deviation rate function via Legendre transformation, i.e.,

$$P(h)=\sup_\rho\Big(\int_0^1 h(x)\rho(x)dx-I(\rho)\Big).$$

One gets the variational formula

$$P(h)=\sup_\theta\mathscr{P}(h,\theta) \tag{1.6}$$

with

$$\mathscr{P}(h,\theta)=2s\int_0^1 dx\left[\log\left(\frac{1}{1+\theta(x)(1-e^{h(x)})}\right)+\log\left(\frac{\theta'(x)}{\rho_r-\rho_l}\right)\right] \tag{1.7}$$

where again the supremum in (1.6) is over increasing C^1 functions $\theta:[0,1]\to\mathbb{R}$ such that $\theta(0)=\rho_l$ and $\theta(1)=\rho_r$. This will also be rigorously proved from the exact description of the stationary measure, see Theorem 4.1. We remark that for models with constant diffusivity and convex quadratic mobility it has been shown [5] that the large deviation function of the density profile is non-convex and therefore the Legendre transform of the pressure does not reproduce the large deviation function (it rather gives its convex hull).

Finally, the variational representations predicted by MFT encode an additivity principle [5], which can be formulated either for the pressure or for the density large deviation function. For the pressure it is stated as follows. For a macroscopic system of size $(b-a)$, where $-\infty<a<b<\infty$ with boundary parameters ρ_l,ρ_r define the modified pressure

$$\widetilde{P}^{[a,b]}_{\rho_l,\rho_r}(h):=P^{[a,b]}_{\rho_l,\rho_r}(h)+2s(b-a)\log\left(\frac{\rho_r-\rho_l}{b-a}\right)$$

where

$$P^{[a,b]}_{\rho_l,\rho_r}(h)=\sup_\theta\int_a^b dx\;2s\left[\log\left(\frac{1}{1+\theta(x)(1-e^{h(x)})}\right)+\log\left(\frac{(b-a)\theta'(x)}{\rho_r-\rho_l}\right)\right].$$

Then, considering a macroscopic system of unit volume [0, 1] and two subsystems of macroscopic size $[0, x]$ and $[x, 1]$ (with $0 < x < 1$), the variational formula (1.6)–(1.7) of MFT is equivalent to the following additivity principle:

$$\widetilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) = \sup_{\rho_l \le \rho \le \rho_r} \left[\widetilde{P}^{[0,x]}_{\rho_l,\rho}(h_1) + \widetilde{P}^{[x,1]}_{\rho,\rho_r}(h_2) \right]$$

where h_1 and h_2 are the restrictions of the function h to the intervals $[0, x]$ and $[x, 1]$. Thus, the additivity principle relates the pressure of a macroscopic system of unit volume [0, 1] with boundary parameters ρ_l, ρ_r to the pressure of two subsystems, of macroscopic size $[0, x]$ and $[x, 1]$ respectively, where the first subsystem is in contact with reservoirs of parameters ρ_l, ρ and the second subsystem is in contact with reservoirs of parameters ρ, ρ_r. This will be proved in Theorem 6.1 as a consequence of the Markovian structure of the order statistics used to describe the stationary measure. The additivity principle implies that the pressure for a constant function h, which corresponds to the large deviations of the total density, completely determines the pressure for any function h. See [19] for the additivity principle of the density large deviation function of the symmetric exclusion process and [8] for a discussion of the additivity principle of the time integrated current large deviation function, and its consequences in the setting of general diffusive systems.

1.1. Relation between the main results and the existing literature. In this section we explain the significance of our results in the context of the existing literature. Starting from large deviations from the hydrodynamic limit, developed in [30], [34], the macroscopic fluctuation theory (MFT) was developed. MFT allows to compute the large deviation rate function for the density profile in non-equilibrium steady states (NESS) as a quasi-potential (see e.g. [7] for a review paper). This quasi-potential can rarely be computed explicitly. In fact, only in the cases of quadratic mobility this can be done, and then is based on an ansatz which is inspired by the exact solution of the SEP via matrix ansatz.

One of the basic assumptions underlying MFT is that there is a large deviation principle around the hydrodynamic limit. The proof of this large deviations principle requires the existence of the moment generating function $M(t)$ of the marginals of the equilibrium product measures, for all values of $t \in \mathbb{R}$. This is essential to perform a cut-off argument (see e.g. Lemma 4.2 in chapter 5 in [32]) which is a crucial step in the replacement lemma and the super-exponential estimate.

In the context of models of KMP type, a large deviation principle around the hydrodynamic limit was *formally* derived in [5]. Because the marginals of the reversible product measures for models of KMP type have exponential tails, this condition is not satisfied, and, as a consequence, the proof of the large deviation principle around the hydrodynamic limit is still an open problem, as already mentioned in Remark 3.3 of [5]. Notice that this is not a simply a "technical issue" as we can infer from the fact that since the appearance of the paper [5], no rigorous proof of large deviations around the hydrodynamic limit for KMP has been found. As a further consequence, all conclusions based on the macrosopic fluctuation theory for these models of KMP type are non-rigorous.

In [6] the formal large deviation rate function for the density profile in the NESS of the KMP model (with left reservoir parameter θ_L and right reservoir parameter θ_R) was recognized as being compatible with a mixture of exponentials whose parameters are distributed as the order statistics of N independent uniforms on $[\theta_L, \theta_R]$. The infimum appearing in the rate function could then be viewed as a contraction principle over the empirical profile of these "hidden parameters". In the same paper it was proved that the NESS of the boundary driven KMP model is *not* given by this mixture of product of exponentials.

In this paper we study the harmonic models, which is a one-parameter family of energy redistribution models similar to the KMP model. Whereas in the KMP model the energy redistribution among particles occurs uniformly at random, in the harmonic models the energy redistribution occurs with precise rates as described in Sect. 2. These rates are derived from the integrable XXX Hamiltonian with non-compact spins. From the point of view of the macroscopic fluctuation theory the KMP model and the harmonic models are indistinguishable, as they share the same macroscopic transport coefficient. However, for the harmonic models we are able to obtain an explicit description of the non-equilibrium steady state (not available for the KMP model) and prove rigorously the large deviations of the density profile and additivity principle in the non-equilibrium steady state. These agree with those formally predicted by MFT (again we notice that harmonic model has the same reversible measures of the KMP model and thus no dynamical large deviation principle around the hydrodynamic limit is available). More precisely our results show for a whole one-parameter family of models the following:

1. *The NESS is explicitly identified as a mixture of product measures.* Even if in previous papers [25], [24], factorial moments were explicitly computed, it is still a large step to compute from these the moment-generating function, and recognize from it the structure of a mixture of product measures with mixing measure the ordered Dirichlet process. It is rarely the case that a NESS is available in explicit form, and when it is the case it always led to several interesting developments, see for instance the various consequences and developments of the matrix ansatz solution for the SEP and ASEP. Previous to our work, in [10], another, independent and simpler proof is given for the case $s = 1/2$. It is not clear how to generalize this to other values of s. After our work, in [29] another development related to intertwining was considered for the general case $s > 0$.
2. From the structure of the NESS *we derive both the large deviation principle for the density profile and the pressure (i.e., the Legendre transform of the rate function).* This is more than a rigorous verification of the results predicted by the macroscopic fluctuation theory, as it also identifies a "microscopic" version of the additivity principle, due to the spatial Markov structure of the mixing measure.

As a final comment, we observe that both in [29], and [13] a class of models (which in particular includes the KMP model and the harmonic models) has been derived where the NESS is in the form of a mixture of product measures. The mixing measure is in turn the stationary distribution of a "hidden temperature" model. Notice that this does not imply per se the large deviation principle for the density profile, except in the cases of the harmonic models, where the mixture measure can be identified explicitly. In particular the identification of the mixing measure of the KMP model remains an open problem.

2. Model Definition

Denote by Ω_N the configuration space made of N-dimensional vectors $\eta = (\eta_i)_{i\in\{1,\dots,N\}}$ with non-negative integer components. We interpret the component η_i as the number of particles at site $i \in \{1, \dots, N\}$. We shall write $\delta^i \in \Omega_N$ for the vector with all components zero except in the i^{th} place, i.e.

$$(\delta^i)_j = \begin{cases} 1 \text{ if } j = i, \\ 0 \text{ otherwise.} \end{cases} \tag{2.1}$$

Definition 2.1 (*Boundary-driven harmonic process with parameter $s > 0$*, [25]). For $N \in \mathbb{N}$, we define the open symmetric harmonic process with parameter $s > 0$ and reservoir densities $0 \le \rho_l \le \rho_r < \infty$ as the continuous-time Markov chain $\{\eta(t)\,,\ t \ge 0\}$ having configuration space Ω_N and whose time-evolution is defined by the generator $\mathscr{L}$ working on functions $f : \Omega_N \to \mathbb{R}$

$$\mathscr{L} f := \mathscr{L}_1 f + \Big(\sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} f \Big) + \mathscr{L}_N f \tag{2.2}$$

where

$$\begin{aligned} (\mathscr{L}_{i,i+1} f)(\eta) := &\sum_{k=1}^{\eta_i} \varphi_s(k, \eta_i) \Big[f(\eta - k\delta^i + k\delta^{i+1}) - f(\eta) \Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi_s(k, \eta_{i+1}) \Big[f(\eta + k\delta^i - k\delta^{i+1}) - f(\eta) \Big] \end{aligned} \tag{2.3}$$

and, for $i \in \{1, N\}$,

$$\begin{aligned} (\mathscr{L}_i f)(\eta) := &\sum_{k=1}^{\eta_i} \varphi_s(k, \eta_i) \Big[f(\eta - k\delta^i) - f(\eta) \Big] \\ &+ \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\rho_i}{1+\rho_i} \right)^k \Big[f(\eta + k\delta^i) - f(\eta) \Big] \end{aligned} \tag{2.4}$$

with $\rho_1 = \rho_l$ and $\rho_N = \rho_r$. Here the function $\varphi_s : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is given by

$$\varphi_s(k, n) := \frac{1}{k} \frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n-k+1)\Gamma(n+2s)} \mathbb{1}_{\{1 \le k \le n\}} \,. \tag{2.5}$$

Remark 2.1 (*Harmonic numbers*). When the occupation of the i^{th} site is n, the function $\varphi_s(k, n)$ in (2.5) represents the rate at which k particles (with $1 \le k \le n$) jump from site i to a nearest neighbour site $i \pm 1$. One can check that

$$\sum_{k=1}^{n} \varphi_s(k, n) = \sum_{k=1}^{n} \frac{1}{k + 2s - 1} \tag{2.6}$$

which are the "shifted" harmonic numbers. In particular, for $s = 1/2$ one recovers the standard harmonic numbers, which explains the name of the process.

For a system of size N and reservoirs parameters $0 \le \rho_l \le \rho_r < \infty$ we denote by μ_{N,ρ_l,ρ_r} the invariant measure of the process $\{\eta(t),\ t \ge 0\}$ of Definition 2.1, i.e. the "non-equilibrium steady state" of the boundary-driven harmonic process with parameter $s > 0$. To alleviate the notation we do not write in the measure the dependence on the parameter s.

As a particular case, in the equilibrium set-up $\rho_l = \rho_r$, one can check that the harmonic process with parameter $s > 0$ has a reversible invariant measure given by a product of Negative Binomial distributions with shape parameter $2s$ and mean $2s\Theta$. Namely, considering the univariate probability mass function

$$\nu_\Theta(n) := \frac{1}{n!}\frac{\Gamma(2s+n)}{\Gamma(2s)}\left(\frac{\Theta}{1+\Theta}\right)^n\left(\frac{1}{1+\Theta}\right)^{2s} \qquad n \in \mathbb{N}_0, \quad \Theta \geq 0 \tag{2.7}$$

with mean

$$\sum_{n=0}^{\infty} n\nu_\Theta(n) = 2s\Theta,$$

and defining the product law

$$\mu_{N,\rho_l,\rho_l}(\eta) := \prod_{i=1}^{N} \nu_{\rho_l}(\eta_i) \qquad \eta \in \Omega_N, \quad \rho_l > 0 \tag{2.8}$$

then one has $\langle f, \mathscr{L} g\rangle = \langle \mathscr{L} f, g\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the Hilbert space $L^2(\mathbb{N}^N, \mu_{N,\rho_l,\rho_l})$.

In the non-equilibrium case ($0 \leq \rho_l < \rho_r < \infty$) the stationary measure was computed in [25] by a combination of stochastic duality and quantum inverse scattering method. Define the (scaled) factorial moment of order $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{N}_0^N$ as

$$G(\xi) = \sum_{\eta\in\mathbb{N}_0^N} \mu_{N,\rho_l,\rho_r}(\eta)\left[\prod_{i=1}^{N}\frac{\eta_i!}{(\eta_i-\xi_i)!}\cdot\frac{\Gamma(2s)}{\Gamma(2s+\xi_i)}\right]. \tag{2.9}$$

Then the following result is available:

Theorem 2.2 (Factorial moments, [25]). *Using the notation $|\eta| = \sum_{i=1}^N \eta_i$, the scaled factorial moments of the non-equilibrium steady state are given by*

$$G(\xi) = \sum_{\eta\in\mathbb{N}_0^N} \rho_r^{|\xi|-|\eta|}(\rho_\ell - \rho_r)^{|\eta|} \prod_{i=1}^{N}\binom{\xi_i}{\eta_i} f_i(\eta) \tag{2.10}$$

with

$$f_i(\eta) := \prod_{j=1}^{\eta_i} \frac{2s(N+1-i) - j + \mathscr{N}_i^+(\eta)}{2s(N+1) - j + \mathscr{N}_i^+(\eta)} \qquad \textit{and} \qquad \mathscr{N}_i^+(\eta) := \sum_{k=i}^{N} \eta_k. \tag{2.11}$$

The steady state of the boundary driven harmonic process can the be reconstructed in terms of the factorial moments (2.10) via the formula

$$\mu_{N,\rho_l,\rho_r}(\eta) = \sum_{\xi\geq\eta} G(\xi)\Big[\prod_{i=1}^{N}\frac{(-1)^{\xi_i-\eta_i}}{\xi_i!}\binom{\xi_i}{\eta_i}\frac{\Gamma(2s+\xi_i)}{\Gamma(2s)}\Big]. \tag{2.12}$$

3. The Non-equilibrium Steady State

In this section we identify the non-equilibrium steady state of the harmonic model in (2.12) as a mixture measure. In the equilibrium set-up ($\rho_l = \rho_r$) the invariant measure is reversible and is an homogeneous (Gibbs) product measure. In non-equilibrium ($\rho_l \neq \rho_r$) we shall prove that the invariant measure is a *mixture* of inhomogeneous product measures. The mixing measure is related to the the "ordered Dirichlet distribution", and, in the particular case when $2s$ is an integer, to the order statistics of i.i.d. uniform random variables.

3.1. Stationary measure as a probabilistic mixture.

Theorem 3.1 (Mixture structure of the NESS). *Let $s > 0$ and $N \in \mathbb{N}$ and assume without loss of generality that $0 \le \rho_l \le \rho_r < \infty$. Then, the non-equilibrium steady state of the open harmonic process of Definition 2.1 is equal to*

$$\mu_{N,\rho_l,\rho_r}(\eta) = \mathbb{E}\left(\prod_{i=1}^{N} \nu_{R_i}(\eta_i)\right) \tag{3.1}$$

where $\mathbb{E}$ denotes expectation w.r.t. the joint distribution of the random variables $(R_1, \ldots, R_N)$ with joint probability density

$$f_{R_1,\ldots,R_N}(\rho_1, \ldots, \rho_N) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}(\rho_r - \rho_\ell)^n} \cdot \prod_{i=1}^{N+1} (\rho_{i+1} - \rho_i)^{2s-1} \cdot 1_{\rho_\ell \le \rho_1 \le \cdots \le \rho_N \le \rho_r} \tag{3.2}$$

with $n := 2s(N+1) - 1$. More explicitely

$$\mu_{N,\rho_l,\rho_r}(\eta) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \frac{1}{(\rho_r - \rho_l)^{2s(N+1)-1}} \cdot \int_{\rho_l}^{\rho_r} d\rho_1 \int_{\rho_1}^{\rho_r} d\rho_2 \cdots \int_{\rho_{N-1}}^{\rho_r} d\rho_N \prod_{i=1}^{N+1} (\rho_i - \rho_{i-1})^{2s-1} \prod_{i=1}^{N} \frac{1}{\eta_i!} \frac{\Gamma(2s + \eta_i)}{\Gamma(2s)} \left(\frac{\rho_i}{1+\rho_i}\right)^{\eta_i} \left(\frac{1}{1+\rho_i}\right)^{2s} \tag{3.3}$$

with the convention $\rho_0 = \rho_l$ and $\rho_{N+1} = \rho_r$.

Remark 3.2. (*Connection with ordered Dirichlet distribution*) The law of the mixing measure (3.2) is related to the "ordered Dirichlet distribution". More precisely this arises from the sum of the components of the symmetric Dirichlet distribution. Let $R_0 := \rho_\ell$ and $R_{N+1} := \rho_r$, and define $V_i := R_i - R_{i-1}$ for $i = 1, \ldots, N+1$, then the joint distribution of $(V_1, \ldots, V_{N+1})$ reads

$$f_{V_1,\ldots,V_N}(v_1, \ldots, v_{N+1}) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}(\rho_r - \rho_\ell)^n} \prod_{i=1}^{N+1} v_i^{2s-1} \cdot 1_{v \in \Sigma_{N+1}} \tag{3.4}$$

which is the Dirichlet distribution on the $(N+1)$-dimensional simplex

$$\Sigma_{N+1} = \left\{ (v_1, \ldots, v_{N+1}) : \rho_\ell \le v_i \le \rho_r, \ \sum_{i=1}^{N+1} v_i = \rho_r - \rho_\ell \right\}$$

with all parameters equal to $2s > 0$. Notice that the inverse transformation is

$$R_i = \sum_{j=1}^{i} V_j \qquad \text{for } i = 1, \ldots, N$$

that allows to recover from (3.4) the distribution of the vector $(R_1, \ldots, R_N)$ given in (3.2).

In the special case $2s \in \mathbb{N}$, the mixing measure (3.2) can be characterized in terms of the law of the order statistics of i.i.d. uniform random variables.

Corollary 3.3 (NESS for $2s \in \mathbb{N}$). *For $2s \in \mathbb{N}$, the non-equilibrium steady state of the open harmonic process of Definition 2.1 is equal to*

$$\mu_{N,\rho_l,\rho_r}(\eta) = \mathbb{E}\left(\prod_{i=1}^{N} \nu_{\Theta_{2si,n}}(\eta_i) \right) \qquad \textit{with} \quad n := 2s(N+1) - 1 \tag{3.5}$$

where ν_θ is the Negative Binomial law defined in (2.7) and the expectation $\mathbb{E}$ is w.r.t. the random variables $(\Theta_{2s,n} \ldots, \Theta_{2sN,n})$ obtained as a marginal of the order statistics $\Theta_{1,n} \le \cdots \le \Theta_{n,n}$ of the independent random variables $\Theta_1, \ldots, \Theta_N$ that are i.i.d. uniform random variables on $[\rho_l, \rho_r]$.

Proof. Let $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$ be the order statistics of n i.i.d. uniform random variables $U_1, \ldots, U_n$ in $[0, 1]$. The distribution of the N-dimensional vector $(U_{2s,n}, U_{4s,n}, \ldots, U_{2sN,n})$ has probability density given by (see Lemma B.1)

$$\begin{aligned} f_{(U_{2s,n}, U_{4s,n}, \ldots, U_{2sN,n})}(u_1, \ldots, u_N) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \\ \cdot \prod_{i=1}^{N+1} (u_i - u_{i-1})^{2s-1} \cdot \mathbb{1}_{\{0 \le u_1 \le \cdots \le u_N \le 1\}} \end{aligned} \tag{3.6}$$

with the convention $u_0 = 0$ and $u_{N+1} = 1$. Let

$$\Theta_i = \rho_l + (\rho_r - \rho_l) U_i \qquad i = 1, \ldots, n \tag{3.7}$$

then $\Theta_1, \ldots, \Theta_n$ are i.i.d. uniform random variables on $[\rho_l, \rho_r]$. Denote by $\Theta_{1,n} \le \cdots \le \Theta_{n,n}$ their order statistics. Then, the random vector $(R_1, \ldots, R_N)$ defined in Theorem 3.1 coincides with $(\Theta_{2s,n} \ldots, \Theta_{2sN,n})$ obtained as a marginal of the order statistics $\Theta_{1,n} \le \cdots \le \Theta_{n,n}$. Equivalently we have that the random vector $(V_1, V_2, \ldots, V_{N+1})$ whose probability density function is given in (3.4) is distributed as

$$(V_1, V_2, \ldots, V_{N+1}) = (\Theta_{2s,n} - \Theta_{0,n}, \Theta_{4s,n} - \Theta_{2s,n}, \ldots, \Theta_{2s(N+1),n} - \Theta_{2sN,n})$$

with the convention $\Theta_{0,n} = 0$ and $\Theta_{2s(N+1),n} = 1$, which is the well-known relation between the symmetric Dirichlet distribution with parameter $2s$ on the $(N+1)$-dimensional simplex and the vector constructed from differences (with gaps $2s$) of the order statistics of $n = 2s(N+1) - 1$ i.i.d. uniform random variables on the unit interval. □

3.2. Proof of Theorem 3.1. In this section we provide a proof of Theorem 3.1. We also refer the reader to Appendix A where it is shown that the integral representation (3.3) is identical to the closed-form expression in (2.12).

3.2.1. Moment generating function The strategy to prove Theorem 3.1 is to use the moment generating function to characterize the stationary measure. Define the set

$$\mathscr{A}_{N,\rho_l,\rho_r} = \left\{ \mathsf{h} = (\mathsf{h}_1, \ldots, \mathsf{h}_N) \in \mathbb{R}^N \ : \ |\mathsf{h}_i| \le \log\left(1 + \frac{1}{\rho_r}\right) \quad \text{for } i = 1, \ldots, N \right\}. \tag{3.8}$$

For $\mathsf{h} \in \mathscr{A}_{N,\rho_l,\rho_r}$, let us denote by $\Psi_{N,\rho_l,\rho_r}(\mathsf{h})$ the moment generating function (MGF) of the non-equilibrium steady state, i.e.

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \sum_{\eta} \mu_{N,\rho_l,\rho_r}(\eta) \prod_{i=1}^{N} e^{\mathsf{h}_i \eta_i}. \tag{3.9}$$

Starting from the factorial moments (2.10) we will compute the generating function and show it coincides with the one of the law (3.5). We split the computation of the moment generating function into three steps, which are given in Proposition 3.4, Proposition 3.6 and in Proposition 3.8.

3.2.2. N-fold sums In this section we show that the moment generating function $\Psi_{N,\rho_l,\rho_r}(\mathsf{h})$ can be written, modulo multiplication by a factor, as the composition of a function $\Phi_N : \mathbb{R}^N \to \mathbb{R}$ and the map

$$\begin{aligned} c_{N,\rho_\ell,\rho_r} : \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ (\mathsf{h}_1, \ldots, \mathsf{h}_N) &\longrightarrow \left(\frac{(\rho_r - \rho_\ell)\left(1 - e^{\mathsf{h}_1}\right)}{1 + \rho_r(1 - e^{\mathsf{h}_1})}, \ldots, \frac{(\rho_r - \rho_\ell)\left(1 - e^{\mathsf{h}_N}\right)}{1 + \rho_r(1 - e^{\mathsf{h}_N})} \right) \end{aligned} \tag{3.10}$$

i.e. the i-th component of the vector $c_{N,\rho_\ell,\rho_r}(\mathsf{h})$ is given by

$$(c_{N,\rho_\ell,\rho_r}(\mathsf{h}))_i = c_{\rho_r,\rho_\ell,i}(\mathsf{h}_i) := \frac{(\rho_r - \rho_\ell)\left(1 - e^{\mathsf{h}_i}\right)}{1 + \rho_r(1 - e^{\mathsf{h}_i})}. \tag{3.11}$$

We will see that the function Φ_N for which we will obtain an explicit formula in terms of an N-fold sum, does not depend on the boundary densities ρ_l and ρ_r. The dependence on this parameters is then completely offloaded onto the map c_{N,ρ_ℓ,ρ_r}.

Proposition 3.4 (MGF, un-nested sums). *For* $\mathsf{h} \in \mathscr{A}_{N,\rho_l,\rho_r}$ *we have that*

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \prod_{i=1}^{N} \left(1 + \rho_r(1 - e^{\mathsf{h}_i})\right)^{-2s} \cdot \Phi_N(c_{N,\rho_\ell,\rho_r}(\mathsf{h})) \tag{3.12}$$

with $c_{N,\rho_\ell,\rho_r} : \mathbb{R}^N \to \mathbb{R}^N$ *defined in* (3.10)–(3.11) *and* $\Phi_N : \mathbb{R}^N \to \mathbb{R}$ *defined by*

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)} \sum_{\eta \in \mathbb{N}_0^N} \prod_{i=1}^{N} c_i^{\eta_i}$$

$$\cdot \frac{1}{\eta_i!}\frac{\Gamma(\eta_i+2s)}{\Gamma(2s)} \cdot \frac{\Gamma(2s(N+1-i)+\mathscr{N}_i^+(\eta))}{\Gamma(2s(N+2-i)+\mathscr{N}_i^+(\eta))} \tag{3.13}$$

for all $c=(c_1,\dots,c_N)\in\mathbb{R}^N$.

Proof. The moment generating function can be rewritten in terms of the scaled factorial moments as follows:

$$\begin{aligned}\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) &= \sum_\eta \left[\prod_{i=1}^N \sum_{\xi_i=0}^{\eta_i} \binom{\eta_i}{\xi_i}(e^{\mathsf{h}_i}-1)^{\xi_i}\right]\mu_{N,\rho_\ell,\rho_r}(\eta)\\ &= \sum_\xi \left[\prod_{i=1}^N \frac{1}{\xi_i!}\frac{\Gamma(2s+\xi_i)}{\Gamma(2s)}(e^{\mathsf{h}_i}-1)^{\xi_i}\right]G(\xi)\end{aligned}$$

with G as defined in (2.9) and where it has been used that $\binom{\eta_i}{\xi_i}=0$ for natural numbers $\xi_i>\eta_i$. Therefore, as a consequence of Theorem 2.2 we have

$$\begin{aligned}\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) &= \sum_\xi \left[\prod_{i=1}^N \frac{\Gamma(2s+\xi_i)}{\Gamma(2s)\cdot\xi_i!}\left(e^{\mathsf{h}_i}-1\right)^{\xi_i}\right]\sum_\eta \rho_r^{|\xi|-|\eta|}(\rho_\ell-\rho_r)^{|\eta|}\prod_{i=1}^N\binom{\xi_i}{\eta_i}f_i(\eta)\\ &= \sum_{\substack{\xi,\eta\\ \eta\le\xi}}\prod_{i=1}^N \rho_r^{\xi_i-\eta_i}(\rho_\ell-\rho_r)^{\eta_i}\binom{\xi_i}{\eta_i}\ f_i(\eta)\left(e^{\mathsf{h}_i}-1\right)^{\xi_i}\frac{\Gamma(2s+\xi_i)}{\Gamma(2s)\cdot\xi_i!}\end{aligned}$$

where we used the notation $\eta\le\xi$ to indicate that $\eta_i\le\xi_i$ for all $i\in\{1,\dots,N\}$. By exchanging the order of summations we obtain

$$\begin{aligned}\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \sum_\eta\prod_{i=1}^N&(\rho_\ell-\rho_r)^{\eta_i}\left(e^{\mathsf{h}_i}-1\right)^{\eta_i}\ f_i(\eta)\\ &\sum_{\xi_i\ge\eta_i}\binom{\xi_i}{\eta_i}\rho_r^{\xi_i-\eta_i}\left(e^{\mathsf{h}_i}-1\right)^{\xi_i-\eta_i}\frac{\Gamma(2s+\xi_i)}{\Gamma(2s)\cdot\xi_i!}.\end{aligned}$$

The sum of the ξ variables can now be performed using that for all $i\in\{1,\dots,N\}$

$$\begin{aligned}&\sum_{\xi_i\ge\eta_i}\binom{\xi_i}{\eta_i}\rho_r^{\xi_i-\eta_i}\left(e^{\mathsf{h}_i}-1\right)^{\xi_i-\eta_i}\frac{\Gamma(2s+\xi_i)}{\Gamma(2s)\cdot\xi_i!}\\ &=\frac{\Gamma(\eta_i+2s)}{\Gamma(2s)\cdot\eta_i!}\sum_{\xi_i\ge\eta_i}\frac{\Gamma(2s+\xi_i)}{\Gamma(\eta_i+2s)\cdot(\xi_i-\eta_i)!}\rho_r^{\xi_i-\eta_i}\left(e^{\mathsf{h}_i}-1\right)^{\xi_i-\eta_i}\\ &=\frac{\Gamma(\eta_i+2s)}{\Gamma(2s)\cdot\eta_i!}\sum_{k_i\ge0}\frac{\Gamma(2s+\eta_i+k_i)}{\Gamma(2s+\eta_i)\cdot k_i!}(\rho_r(e^{\mathsf{h}_i}-1))^{k_i}\\ &=\frac{\Gamma(\eta_i+2s)}{\Gamma(2s)\cdot\eta_i!}\frac{1}{(1-\rho_r(e^{\mathsf{h}_i}-1))^{\eta_i+2s}}.\end{aligned}$$

where in the last equality we have used the identity

$$\frac{1}{(1-x)^a}=\sum_{k=0}^\infty\frac{\Gamma(a+k)}{\Gamma(a)\cdot k!}x^k \qquad |x|<1. \tag{3.14}$$

Thus we arrive to

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \sum_{\eta} \prod_{i=1}^{N} (\rho_\ell - \rho_r)^{\eta_i} \left(e^{\mathsf{h}_i} - 1\right)^{\eta_i} f_i(\eta)$$
$$\frac{\Gamma(\eta_i + 2s)}{\Gamma(2s)\eta_i!} \frac{1}{(1 - \rho_r(e^{\mathsf{h}_i} - 1))^{\eta_i + 2s}}.$$

Equivalently, multiplying both sides by $\prod_{i=1}^{N} \left(1+\rho_r(1-e^{\mathsf{h}_i})\right)^{2s}$ we rewrite this identity in terms of the function Φ_N defined in (3.12) as

$$\Phi_N(c) = \sum_{\eta} \prod_{i=1}^{N} c_i^{\eta_i} \frac{\Gamma(\eta_i + 2s)}{\Gamma(2s) \cdot \eta_i!} \cdot f_i(\eta) \tag{3.15}$$

with c_i as given in (3.10). Recalling the definition of the functions f_i in (2.11) and using the convention $\mathscr{N}_{N+1}^{+}(\eta) = 0$, we write $\prod_{i=1}^{N} f_i(\eta)$ as a telescopic product

$$\prod_{i=1}^{N} f_i(\eta) = \prod_{i=1}^{N} \prod_{k=\mathscr{N}_{i+1}^{+}(\eta)}^{\mathscr{N}_i^{+}(\eta)-1} \frac{2s(N+1-i)+k}{2s(N+1)+k}.$$

As a consequence

$$\begin{aligned}\prod_{i=1}^{N} f_i(\eta) &= \frac{\Gamma(2s(N+1))}{\Gamma(2s(N+1)+\mathscr{N}_1^{+}(\eta))} \cdot \prod_{i=1}^{N} \frac{\Gamma(2s(N+1-i)+\mathscr{N}_i^{+}(\eta))}{\Gamma(2s(N+1-i)+\mathscr{N}_{i+1}^{+}(\eta))} \\ &= \frac{\Gamma(2s(N+1))}{\Gamma(2s)} \cdot \prod_{i=1}^{N} \frac{\Gamma(2s(N+1-i)+\mathscr{N}_i^{+}(\eta))}{\Gamma(2s(N+1-(i-1))+\mathscr{N}_i^{+}(\eta))}.\end{aligned}$$

Inserting this last expression in (3.15), the result of the proposition follows. □

Remark 3.5. (*MGF, nested sums*) There is a one-to-one relation between the set of configurations $\eta \in \mathbb{N}_0^N$ and the set of N-tuples $\{(m_1, \ldots, m_N) \in \mathbb{N}_0^N : m_1 \geq m_2 \geq \cdots \geq m_N \geq 0\}$. This implies that the moment generating function can also be written as nested sums. Then we have

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)} \sum_{m_1 \geq \cdots \geq m_N \geq 0} \prod_{i=1}^{N} c_i^{m_i - m_{i+1}} \frac{\Gamma(m_i - m_{i+1} + 2s)}{\Gamma(2s)(m_i - m_{i+1})!}$$
$$\cdot \frac{\Gamma(2s(N+1-i)+m_i)}{\Gamma(2s(N+2-i)+m_i)}$$

with the convention $m_{N+1} = 0$. This easily follows from Proposition 3.4 by implementing the change of variables:

$$\eta = (\eta_1, \ldots, \eta_N) \longrightarrow m = (m_1, \ldots, m_N), \qquad \text{with} \qquad m_i := \mathscr{N}_i^{+}(\eta)$$

from which one has $\eta_i(m) = m_i - m_{i+1}$.

3.2.3. N-fold integrals We proceed further by moving from a representation of the moment generating function with N sums to one involving N integrals. This will be useful to recognize the invariant distribution of the harmonic process as a mixture.

Proposition 3.6 (MGF, un-nested integrals). *We have*

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{i=1}^N t_i^{2s(N-i+1)-1}(1-t_i)^{2s-1} \left(\frac{1}{1-c_i \prod_{j=1}^i t_j} \right)^{2s}. \tag{3.16}$$

Proof. We prove that (3.16) coincides with (3.13) using again the identity (3.14). Indeed, plugging this identity in (3.16) we have

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 dt_1 \cdots \int_0^1 dt_N \times \prod_{i=1}^N t_i^{2s(N-i+1)-1}(1-t_i)^{2s-1} \sum_{\eta_i=0}^{\infty} \frac{\Gamma(2s+\eta_i)}{\Gamma(2s)\eta_i!} \left(c_i \prod_{j=1}^i t_j \right)^{\eta_i}.$$

Collecting the powers of t_i and recalling the definition $\mathcal{N}_i^+(\eta) = \sum_{k=i}^N \eta_k$ this can be rewritten as

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \sum_\eta \prod_{i=1}^N \frac{\Gamma(2s+\eta_i)}{\Gamma(2s)\eta_i!} \cdot c_i^{\eta_i} \int_0^1 t_i^{2s(N-i+1)+\mathcal{N}_i^+(\eta)-1}(1-t_i)^{2s-1} dt_i.$$

Using that for all $a, b > 0$

$$\int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

it then follows

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \sum_\eta \prod_{i=1}^N c_i^{\eta_i} \frac{\Gamma(2s+\eta_i)}{\Gamma(2s)\eta_i!} \cdot \frac{\Gamma(2s(N+1-i)+\mathcal{N}_i^+(\eta)) \cdot \Gamma(2s)}{\Gamma(2s(N+2-i)+\mathcal{N}_i^+(\eta))}$$

which reproduces (3.13) after simplifications. □

Remark 3.7. (*MGF, nested integrals*) Similarly to the discrete case (see Remark 3.5), one can also write an expression in terms of nested integrals. We have

$$\Phi_N(c) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \cdot \int_0^1 du_1 \int_{u_1}^1 du_2 \cdots \int_{u_{N-1}}^1 du_N \prod_{i=1}^{N+1} (u_i - u_{i-1})^{2s-1} \frac{1}{\left(1-c_i(1-u_i)\right)^{2s}} \tag{3.17}$$

where we recall the convention $u_0 = 0$ and $u_{N+1} = 1$. The result easily follows from Proposition 3.6 by implementing the change of variables $u_i = 1 - \prod_{j=1}^{i} t_j$. Inverting this mapping one gets

$$t_i = \frac{1-u_i}{1-u_{i-1}} \qquad \text{and} \qquad 1-t_i = \frac{u_i - u_{i-1}}{1-u_{i-1}}$$

which substituted in (3.16) yields (3.17).

3.2.4. Concluding the proof The last step in the proof of Theorem 3.1 consists in recognizing in the expression (3.17) the probability generating function of the probability measure (3.1). We recall that the moment generating function of a Negative Binomial distribution with law (2.7) is given by

$$M_\theta(\mathsf{h}) = \sum_{n=0}^{\infty} e^{\mathsf{h}n} \nu_\theta(n) = \left(\frac{1}{1+\theta(1-e^{\mathsf{h}})}\right)^{2s} \qquad \text{for } |\mathsf{h}| < \log\left(1+\tfrac{1}{\theta}\right). \quad (3.18)$$

Proposition 3.8 (MGF, mixture). *For* $\mathsf{h} \in \mathscr{A}_{N,\rho_l,\rho_r}$ *we have*

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \mathbb{E}\left[\prod_{i=1}^{N} M_{R_i}(\mathsf{h}_i)\right] \quad (3.19)$$

where the expectation is w.r.t. the law of the random vector $(R_1, \ldots, R_N)$ *whose probability density is given in* (3.2).

Proof We observe that using (3.10), namely

$$c_i = \frac{(\rho_r - \rho_\ell)\left(1-e^{\mathsf{h}_i}\right)}{1+\rho_r(1-e^{\mathsf{h}_i})}$$

we have

$$\frac{1}{1-c_i(1-u_i)} = \frac{\left(1+\rho_r(1-e^{\mathsf{h}_i})\right)}{1+\left(\rho_l+(\rho_r-\rho_l)u_i\right)(1-e^{\mathsf{h}_i})}$$

Inserting this into (3.17) and recalling the relation (3.12), the moment generating function of the non-equilibrium steady state is given by

$$\begin{aligned} \Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \int_0^1 du_1 \int_{u_1}^1 du_2 \cdots \int_{u_{N-1}}^1 du_N \\ \prod_{i=1}^{N+1}(u_i - u_{i-1})^{2s-1} \cdot \prod_{i=1}^{N} \frac{1}{\left(1+\left(\rho_l+(\rho_r-\rho_l)u_i\right)(1-e^{\mathsf{h}_i})\right)^{2s}}. \end{aligned} \quad (3.20)$$

Therefore, using (3.2) and (3.18) we obtain (3.19). □

4. Pressure

In this section we use the formula for the stationary measure proven in Sect. 3 to compute the pressure associated to the non-equilibrium steady state. We will reproduce the expression predicted by the Macroscopic Fluctuation Theory by first conditioning to a given realisation of the random local parameters and then using the large deviation properties of those local parameters. For simplicity, we will restricts to the special case $2s \in \mathbb{N}$, for which we can exploit the characterization, given in Corollary 3.3, of the non-equilibrium steady state in terms of the order statistics of i.i.d. uniform random variables. This allows us to use known result on the Large Deviation Principle for the sample paths of the order statistics, see Lemma B.5. Of course, due to the Markovian structure of the measure (proved in Theorem 3.1), the results actually hold for all $s > 0$ and one could prove this by considering the sample path large deviations of the ordered Dirichlet process.

Theorem 4.1 (Pressure). *Let $h : [0, 1] \to \mathbb{R}$ be a smooth function. Define the pressure of the open symmetric harmonic process as*

$$P(h) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\Big[e^{\sum_{i=1}^N \eta_i h(\frac{i}{N})}\Big]. \tag{4.1}$$

Then, for $2s \in \mathbb{N}$, the pressure admits the following variational expression:

$$P(h) = \sup_{\substack{\theta:[0,1]\to\mathbb{R}_+ \\ \text{increasing} \\ \theta(0)=\rho_l \\ \theta(1)=\rho_r}} \Big[P(h,\theta) - J(\theta)\Big] \tag{4.2}$$

where

$$P(h,\theta) = 2s \int_0^1 \log\Big(\frac{1}{1+(1-e^{h(x)})\theta(x)}\Big)dx \tag{4.3}$$

and

$$J(\theta) = -2s \int_0^1 \log\Big(\frac{\theta'(x)}{\rho_r - \rho_l}\Big)dx. \tag{4.4}$$

Proof Recalling Proposition 3.8, we have

$$\mathbb{E}\Big[e^{\sum_{i=1}^N \eta_i h(\frac{i}{N})}\Big] = \mathbb{E}\left[\prod_{i=1}^N M_{\Theta_{2si,n}}\left(h\left(\tfrac{i}{N}\right)\right)\right] \tag{4.5}$$

$$= \mathbb{E}\left[\prod_{i=1}^N \left(\frac{1}{1+\Theta_{2si,n}\Big(1-e^{h(\frac{i}{N})}\Big)}\right)^{2s}\right] \tag{4.6}$$

where $n = 2s(N+1) - 1$. Introducing the sample path of the order statistics

$$\Theta_n(x) = \Theta_{\lfloor (n+1)x\rfloor+1,n} \qquad x \in [0,1]$$

with the convention $\Theta_{n+1,n} := \rho_r$, we arrive to

$$\mathbb{E}\Big[e^{\sum_{i=1}^N \eta_i h(\frac{i}{N})}\Big] = \mathbb{E}\left[\prod_{i=1}^N \left(\frac{1}{1+\Theta_n(\frac{2si}{n})\Big(1-e^{h(\frac{i}{N})}\Big)}\right)^{2s}\right]$$

$$= \mathbb{E}\left[\exp\left\{2s\sum_{i=1}^{N}\left(\log\left(\frac{1}{1+\Theta_n(\frac{i}{N})\left(1-e^{h(\frac{i}{N})}\right)}\right)+o(1)\right)\right\}\right]$$

where $o(1)$ to 0 as $N \to \infty$, uniformly.

For an increasing function $\theta : [0, 1] \to \mathbb{R}$ we define

$$P_N(h,\theta) = \frac{2s}{N}\sum_{i=1}^{N}\log\left(\frac{1}{1+\theta(\frac{i}{N})\left(1-e^{h(\frac{i}{N})}\right)}\right).$$

By using the properties of conditional expectation, this allows to rewrite the generating function of the empirical distribution as the conditional expectation of an exponential functional

$$\mathbb{E}\left[e^{\sum_{i=1}^{N}\eta_i h(\frac{i}{N})}\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\{N(P_N(h,\Theta)+o(1))\}\,|\Theta\right]\right] \tag{4.7}$$

where we denote by Θ the collection of random variables $(\Theta_n(\frac{i}{N}))_{i=1,\dots,N}$. Observe that by Riemann approximation

$$\lim_{N\to\infty} P_N(h,\theta) = P(h,\theta) = 2s\int_0^1 \log\left(\frac{1}{1+\theta(x)\left(1-e^{h(x)}\right)}\right)dx.$$

Therefore we can rewrite

$$\mathbb{E}\left[e^{\sum_{i=1}^{N}\eta_i h(\frac{i}{N})}\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\{N(P(h,\Theta)+o(1))\}\,|\Theta\right]\right] \tag{4.8}$$

Using Lemma B.5 we have that the sample path of the order statistics satisfy the LDP with good rate function

$$J(\theta) = \begin{cases} -2s\int_0^1 \log(\frac{\theta'(x)}{\rho_r-\rho_l})\,dx & \text{if } \theta \in A_{\rho_l,\rho_r} \text{ is increasing} \\ \infty & \text{otherwise} \end{cases}$$

formula (4.2) follows by applying Varadhan's lemma to the exponentially growing functional (4.8) and using that the map $\theta \mapsto P(h,\theta)$ is bounded and continuous for the profiles θ which are increasing and take value in the interval $[\rho_l, \rho_r]$. □

5. Large Deviations

In this section we prove that the sequence of empirical density measures $(L_N)_{N\geq 1}$ satisfies a LDP. One might think that knowing the pressure one could extract from it the large deviation function by using Gärtner-Ellis theorem. As we shall see and comment below this is not possible because the large deviation function is not convex. However we can obtain the large deviation function by following a direct approach that starts from the explicit knowledge of the (microscopic) stationary measure of the open harmonic model and proceed via a contraction principle. We restrict to the case $2s \in \mathbb{N}$, in order to use Lemma B.5 for the characterization of the sample path large deviations for the order statistics of i.i.d uniforms.

Theorem 5.1 (Density large deviation). *Let $2s \in \mathbb{N}$, then the empirical profiles of the open symmetric harmonic process*

$$L_N = \frac{1}{N} \sum_{i=1}^{N} \eta_i \delta_{\frac{i}{N}}$$

satisfy a large deviation principle with good rate function

$$I(\rho) = \inf_{\substack{\theta:[0,1]\to\mathbb{R}_+ \\ increasing \\ \theta(0)=\rho_l \\ \theta(1)=\rho_r}} \Big[I(\rho,\theta) + J(\theta) \Big] \tag{5.1}$$

where

$$I(\rho,\theta) = 2s \int_0^1 \left[\frac{\rho(x)}{2s} \log \frac{\rho(x)}{2s\theta(x)} + \left(1 + \frac{\rho(x)}{2s}\right) \log \left(\frac{1+\theta(x)}{1+\frac{\rho(x)}{2s}} \right) \right] dx \tag{5.2}$$

and

$$J(\theta) = -2s \int_0^1 \log \left(\frac{\theta'(x)}{\rho_r - \rho_l} \right) dx\ . \tag{5.3}$$

Before proving the theorem we add a few remarks.

Remark 5.2 The expression (5.1) coincides with the prediction of Macroscopic Fluctuation Theory with transport coefficients

$$D(\rho) = \frac{1}{2s}, \qquad \sigma(\rho) = \frac{\rho}{2s}\left(1 + \frac{\rho}{2s}\right)$$

which indeed are the transport coefficient of the harmonic model, as proved in [9]. In particular, for $s = 1/2$, we recover the transport coefficient of the discrete KMP model and the large deviation function (5.1) coincides with the one computed in [5]. There it was already remarked that the infimum over θ can be viewed as a contraction principle over a random local temperature profile given by uniform order statistics. The macroscopic fluctuation theory can strictly speaking not be applied to the KMP model, or to any of the models studied in this paper, because the proof requires superexponential tails of the marginals of the equilibrium product measures, which does not hold for any of the models in the KMP class. Therefore, even if Theorem 5.1 gives the large deviation principle for the whole class of harmonic models with parameter $2s$ integer, it does not prove yet the same for the KMP model and its generalizations. Nevertheless the macroscopic fluctuation theory predicts that these models sharing the same macroscopic transport coefficients have the same rate function.

Remark 5.3 As already remarked in [5] for the case $s = 1/2$, the rate function (5.1) is non-convex. This is at the root of the fact that the large deviation function can not be represented as the Legendre transform of a convex function. Indeed if one takes the Legendre transform of the pressure one rather obtains the convex hull of the rate function.

Remark 5.4 For the models with compact state space, such as the exclusion process, the expression for the large deviation function contains a supremum, rather then an infimum [7,19]. For the weakly asymmetric exclusion process the density large deviation has been written as a minimization problem (see formula (2.3) of [22]) and for the asymmetric exclusion process a contraction involving Brownian excursions has been considered [17].

Proof of Theorem 5.1 Preliminarily, consider an inhomogenous product measure with marginal Negative Binomials with a smooth slowly varying parameter. Thus, assume we have a measure μ_N of the form

$$\mu_N = \otimes_{i=1}^N \nu_{\theta(\frac{i}{N})} \tag{5.4}$$

where $\nu_{\theta(\frac{i}{N})}$ is the Negative Binomial measure introduced in (2.7) with mean $\theta(\frac{i}{N})$ and where $\theta : [0, 1] \to [0, \infty)$ is a smooth increasing function. We call

$$\ell_N = \frac{1}{N} \sum_{i=1}^N \eta_i \delta_{i/N} \tag{5.5}$$

the empirical density profile when η has distribution μ_N. Then, Gärtner-Ellis theorem tells us that the sequence of measures $(\ell_N)_{N\geq 1}$ satisfies a large deviation principle with a good rate function $I(\rho, \theta)$. The LDP of $(\ell_N)_{N\geq 1}$ has to be interpreted in the set of positive finite measures on $[0, 1]$ equipped with the weak topology. We have $I(\rho, \theta) = \infty$ for a measure ρ which is not absolutely continuous w.r.t. Lebesgue measure on $[0, 1]$; otherwise the rate function $I(\rho, \theta)$ is given and is obtained as the Legendre transform of the pressure

$$I(\rho, \theta) = \sup_h \left(\int \rho(x)h(x)dx - P(h, \theta) \right) \tag{5.6}$$

where

$$\begin{aligned} P(h, \theta) &= \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_{\mu_N} \left(e^{N\langle \ell_N, h\rangle} \right) \\ &= \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_{\mu_N} e^{\sum_{i=1}^N \eta_x h(i/N)} \end{aligned} \tag{5.7}$$

has been computed in (4.3). Evaluating the Legendre transform one obtains for $I(\rho, \theta)$ the expression that is given in (5.2).

The type of measures which are of interest to us, are not product measures of the form (5.4), but product measures with parameters that are themselves random variables. More precisely we have a measure of the form

$$\mu_{N,\rho_l,\rho_r} = \mathbb{E}\left(\otimes_{i=1}^N \nu_{\Theta_{2si,n}} \right) \tag{5.8}$$

where $n = 2s(N + 1) - 1$ and the additional expectation refers to the random variables $\Theta_{1,n} \leq \Theta_{2,n} \leq \cdots \leq \Theta_{n,n}$ which are the ascending order statistics of a sequence $\Theta_1 \ldots, \Theta_n$ of i.i.d. random variables with common uniform distribution on the interval $[\rho_l, \rho_r]$. Recalling the definition of the sample path of the order statistics

$$\Theta_n(x) = \Theta_{\lfloor (n+1)x\rfloor+1,n} \qquad x \in [0, 1], \qquad \text{with} \quad \Theta_{n+1,n} := \rho_r$$

the stationary measure is rewritten as

$$\mu_{N,\rho_l,\rho_r} = \mathbb{E}\left(\otimes_{i=1}^N \nu_{\Theta_n\left(\frac{i}{N+1}\right)} \right) . \tag{5.9}$$

As we know from Lemma B.5, the sample path of the order statistics of uniform random variables satisfies a large deviation principle with rate function $J(\theta)$ given in (5.3). As a consequence, the contraction principle gives that, under μ_{N,ρ_l,ρ_r}, the sequence $(L_N)_{N\geq 1}$ satisfies the large deviation principle with rate function I which is only finite on positive measures ρ of the form $\rho(x)dx$, where it is equal to

$$I(\rho) = \inf_{\substack{\theta:[0,1]\to\mathbb{R}_+ \\ \text{increasing} \\ \theta(0)=\rho_l \\ \theta(1)=\rho_r}} \Big[I(\rho,\theta) + J(\theta) \Big].$$

□

6. Additivity Principle

In this section we compare the moment generating function of system of size N to the moment generating function of two subsystems of sizes N_1, N_2 with $N_1+N_2 = N$. In the macroscopic limit (i.e. when the two subsystems are of macroscopic sizes $N_1 = Nx$ and $N_2 = N(1-x)$ with $x \in (0, 1)$) we get a rigorous proof of an additivity principle for the pressure (and similarly for the density large deviations). In the non-equilibrium set-up, an additivity principle was first established in [19] for the density profile large deviations of the non-equilibrium steady state of the symmetric exclusion process. Surprisingly, the corresponding additivity principle for the pressure of the symmetric exclusion process contained an infimum, whose physical basis remain not understood.

The pressure additivity principle proved here for the harmonic model contains instead a supremum and generalizes the one conjectured in [5] for the discrete-KMP model. Also in this section we will restrict to the case $2s$ integer. The proof relies on an integral equation (see (6.8) below) relating the partition functions of the systems of sizes N_1, N_2 and N and an application of Varadhan's lemma. The integral equation is in turn a consequence of the properties of order statistics of uniform i.i.d. random variables, in particular the Markovian structure of Lemma B.2 and the properties of conditioning of Lemmas B.3 and B.4.

As it will be discussed in Sect. 7, the additivity principle for the pressure implies that the pressure with constant test function, corresponding to the large deviations of the total density, determines completely the pressure, by approximation by piece-wise constant functions. This implies in particular that Theorem 7.1 completely determines the pressure.

6.1. The additivity principle for the pressure. In order to formulate the additivity principle, we need to generalise the definition of pressure given in (4.1) to the case of a system whose macroscopic volume is the interval $[a, b]$ and the boundary densities are $0 < \rho_a \leq \rho_b$. This is obtained by starting from a microscopic system with $\lceil (b-a)N \rceil$ sites, where $\lceil b \rceil$ represents the upper integer part of the number $b \in \mathbb{R}$, and taking the limit as $N \to \infty$

$$P^{[a,b]}_{\rho_a,\rho_b}(h) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\Big[e^{\sum_{i=1}^{N_{a,b}} \eta_i h\left(a+\frac{i}{N}\right)} \Big] \qquad \text{with} \qquad N_{a,b} = \lceil (b-a)N \rceil\,. \tag{6.1}$$

Here $h : [a, b] \to \mathbb{R}$ and $\mathbb{E}$ denotes expectation with respect to the stationary measure $\mu_{N_{a,b},\rho_a,\rho_b}$. As we did in (4.5) for the system with macroscopic unit volume, the expectation in (6.1) can be written in terms of the moment generating function:

$$\Psi^{[a,b]}_{N_{a,b},\rho_a,\rho_b}(\mathsf{h}) = \mathbb{E}\Bigg[\prod_{i=1}^{N_{a,b}} M_{\Theta_{2si,n_{a,b}}}(\mathsf{h}_i)\Bigg], \qquad \text{with} \qquad n_{a,b} = 2s(N_{a,b}+1)-1 \quad (6.2)$$

defined on vectors $\mathsf{h} \in \mathscr{A}_{N_{a,b},\rho_a,\rho_b}$. Here $\Theta_{1,n_{a,b}} \leq \Theta_{2,n_{a,b}} \leq \cdots \leq \Theta_{n_{a,b},n_{a,b}}$ is the ascending order statistics of $n_{a,b}$ independent uniform random variables on $[\rho_a, \rho_b]$ and $M_\theta(\cdot)$ is the moment generating function of a Negative Binomial distribution with parameters $(2s, \theta)$, as defined in (3.18). It then follows that

$$P^{[a,b]}_{\rho_a,\rho_b}(h) := \lim_{N\to\infty} \frac{1}{N} \log \Psi^{[a,b]}_{N_{a,b},\rho_a,\rho_b}(\mathsf{h}^{(N)}) \tag{6.3}$$

where $\mathsf{h}^{(N)}$ is the $N_{a,b}$-dimensional vector of components:

$$\mathsf{h}^{(N)}_i := h\left(a + \tfrac{i}{N}\right), \qquad \text{for} \qquad i = 1, \ldots N_{a,b}\ . \tag{6.4}$$

Furthermore, to formulate the additivity principle, we define the modified pressure

$$\widetilde{P}^{[a,b]}_{\rho_a,\rho_b}(h) := P^{[a,b]}_{\rho_a,\rho_b}(h) + 2s(b-a)\log\left(\frac{\rho_b-\rho_a}{b-a}\right). \tag{6.5}$$

In the next theorem we prove that the modified pressure satisfies the additivity principle.

Theorem 6.1 (Pressure additivity principle) *Let* $2s \in \mathbb{N}$, $0 < \rho_l < \rho_r$, $0 < x < 1$ *and* $h : [0,1] \to \mathbb{R}$*, then we have*

$$\widetilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) = \sup_{\rho_l \leq \theta \leq \rho_r} \left[\widetilde{P}^{[0,x]}_{\rho_l,\theta}(h_1) + \widetilde{P}^{[x,1]}_{\theta,\rho_r}(h_2)\right] \tag{6.6}$$

where $h_1 : [0,x] \to \mathbb{R}$ *and* $h_2 : [x,1] \to \mathbb{R}$ *are the restrictions of* h *respectively, to* $[0,x]$ *and to* $[x,1]$*. More generally, for* $\kappa \geq 2$ *and* $0 = x_0 \leq x_1 \leq \cdots \leq x_\kappa = 1$*, calling* $h_i : [x_{i-1}, x_i] \to \mathbb{R}$ *the restriction of* h *to* $[x_{i-1}, x_i]$*, for* $i = 1, \ldots, \kappa$*, we have*

$$\widetilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) = \sup_{\rho_0 \leq \rho_1 \leq \cdots \leq \rho_{\kappa-1} \leq \rho_\kappa} \sum_{i=1}^{\kappa} \widetilde{P}^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(h_i) \tag{6.7}$$

with the convention $\rho_0 = \rho_l$, $\rho_\kappa = \rho_r$.

Proof We prove (6.6), i.e. the case $\kappa = 2$, the case of a generic κ can be then deduced by induction. As a first step we fix two integers $N_1, N_2 \in \mathbb{N}$ such that $N_1 + N_2 = N$ and prove the following identity for the moment generating function

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}_1, \ldots, \mathsf{h}_N) = \mathbb{E}\Big(M_{\Theta_{2sN_1,n_1}}(\mathsf{h}_{N_1})\Psi_{N_1-1,\rho_l,\Theta_{2sN_1,n_1}}(\mathsf{h}_1, \ldots, \mathsf{h}_{N_1-1})\Psi_{N_2,\Theta_{2sN_1,n_2},\rho_r}(\mathsf{h}_{N_1+1}, \ldots, \mathsf{h}_N)\Big) \tag{6.8}$$

where $n_1 = 2sN_1 - 1$, $n_2 = 2s(N_2+1) - 1$. Here $\Theta_{2sN,n}$ is the $2sN^{\text{th}}$ ascending order statistics of n independent uniforms on the interval (ρ_l, ρ_r).
In order to prove (6.8) we start from Proposition 3.8 which says that, for $\mathsf{h} \in \mathscr{A}_{N,\rho_l,\rho_r}$,

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}) = \mathbb{E}\left[\prod_{i=1}^{N} M_{\Theta_{2si,n}}(\mathsf{h}_i)\right]$$

with

$$\Theta_{2si,n} = \rho_l + (\rho_r - \rho_l)U_{2si,n}, \qquad i = 1, \dots, n$$

where $U_{2si,n}$ is the $2si^{\text{th}}$ order statistics of $n = 2s(N+1) - 1$ i.i.d. random variables that are uniformly distributed on the interval $(0, 1)$. The tower property of conditional expectation implies

$$\begin{aligned} \Psi_{N,\rho_l,\rho_r}(\mathsf{h}_1, \dots, \mathsf{h}_N) &= \mathbb{E}\left(\mathbb{E}\left(\prod_{i=1}^{N} M_{\Theta_{2si,n}}(\mathsf{h}_i) \,\middle|\, \Theta_{2sN_1,n}\right)\right) \\ &= \mathbb{E}\left(M_{\Theta_{2sN_1,n}}(\mathsf{h}_{N_1})\,\mathbb{E}\left(\prod_{\substack{i=1\\ i\neq N_1}}^{N} M_{\Theta_{2si,n}}(\mathsf{h}_i) \,\middle|\, \Theta_{2sN_1,n}\right)\right). \end{aligned} \tag{6.9}$$

Now, given $\theta \in [\rho_l, \rho_r]$, the event $\{\Theta_{2sN_1,n} = \theta\}$ is equivalent to the event $\{U_{2sN_1,n} = u\}$, with $u = \frac{\theta-\rho_l}{\rho_r-\rho_l}$. Therefore, using the property of the conditional expectations of order statistics (specifically Eq. (B.6) of Lemma B.4 with $n = 2s(N+1) - 1$ and $m = 2sN_1$) we obtain

$$\begin{aligned} &\mathbb{E}\left(\prod_{\substack{i=1\\ i\neq N_1}}^{N} M_{\Theta_{2si,n}}(\mathsf{h}_i) \,\middle|\, \Theta_{2sN_1,n} = \theta\right) = \mathbb{E}\left(\prod_{\substack{i=1\\ i\neq N_1}}^{N} M_{\Theta_{2si,n}}(\mathsf{h}_i) \,\middle|\, U_{2sN_1,n} = u\right) \\ &= \mathbb{E}\left(\prod_{i=1}^{N_1-1} M_{\Theta^\star_{2si,n_1}}(\mathsf{h}_i)\right) \cdot \mathbb{E}\left(\prod_{i=1}^{N_2} M_{\tilde{\Theta}_{2si,n_2}}(\mathsf{h}_{N_1+i})\right) \end{aligned} \tag{6.10}$$

where

$$\Theta^\star_{2si,n_1} = \rho_l + (\rho_r - \rho_l)U^\star_{2si,n_1} \qquad i = 1, \dots, N_1 - 1$$

with $U^\star_{2si,n_1}$ the $2si^{\text{th}}$ order statistics of $n_1 = 2sN_1 - 1$ i.i.d. random variables uniformly distributed on the interval $(0, u)$ and similarly

$$\tilde{\Theta}_{2si,n_2} = \rho_l + (\rho_r - \rho_l)\tilde{U}_{2si,n_2} \qquad i = 1, \dots, N_2$$

with $\tilde{U}_{2si,n_2}$ the $2si^{\text{th}}$ order statistics of $n_2 = 2s(N_2 + 1) - 1$ i.i.d. random variables that are uniformly distributed on the interval $(u, 1)$. In other words, defining

$$\theta(u) = \rho_l + u(\rho_r - \rho_l), \qquad u \in [0, 1]$$

the $\{\Theta^\star_{2si,n_1}\}_{i=1,\dots,N_1-1}$ are the order statistics (sampled every $2s$ steps) of $n_1 = 2sN_1-1$ i.i.d. uniforms on $(\rho_l, \theta(u))$ and the $\{\tilde{\Theta}_{2si,n_2}\}_{i=1,\dots,N_2}$ are the order statistics (sampled every $2s$ steps) of $n_2 = 2s(N_2+1) - 1$ i.i.d. uniforms on $(\theta(u), \rho_r)$. As a consequence, combining (6.9) and (6.10), we obtain (6.8).

We further proceed by observing that, recalling (3.18), the identity (6.8) can be explicitly written as

$$\Psi_{N,\rho_l,\rho_r}(\mathsf{h}_1,\dots,\mathsf{h}_N) = \int_{\rho_l}^{\rho_r} d\theta\ \Psi_{N_1-1,\rho_l,\theta}(\mathsf{h}_1,\dots,\mathsf{h}_{N_1-1}) \cdot \Psi_{N_2,\theta,\rho_r}(\mathsf{h}_{N_1+1},\dots,\mathsf{h}_N)$$
$$\cdot\left(\frac{1}{1+(1-e^{\mathsf{h}_{N_1}})\theta}\right)^{2s} \cdot \frac{1}{\rho_r-\rho_l} \cdot f_{U_{2sN_1,n}}\left(\frac{\rho_r-\theta}{\rho_r-\rho_l}\right) \tag{6.11}$$

where $f_{U_{2sN_1,n}}$ is the probability density of the random variable $U_{2sN_1,n}$ which, from Lemma B.1, is equal to

$$f_{U_{2sN_1,n}}(u) = \frac{(2s(N+1)-1)!}{(2sN_1-1)!(2s(N_2+1)-1)!} \cdot u^{2sN_1-1}(1-u)^{2s(N_2+1)-1}\,. \tag{6.12}$$

In order to take the macroscopic limit we consider blocks of macroscopic sizes i.e. $N_1 = \lfloor Nx \rfloor$ and $N_2 = \lfloor N(1-x) \rfloor$, with $x \in (0,1)$. Now let $h : [0,1] \to \mathbb{R}$ and let $h_1 : [0,x] \to \mathbb{R}$ and $h_2 : [x,1] \to \mathbb{R}$ be the restrictions of h to $[0,x]$ and to $[x,1]$. Then by definition we have that

$$\lim_{N\to\infty} \frac{1}{N} \log \Psi^{[0,x]}_{\lfloor Nx \rfloor-1,\rho_l,\theta}\left(h\left(\tfrac{1}{N}\right),\dots,h\left(\tfrac{\lfloor Nx \rfloor-1}{N}\right)\right) = P^{[0,x]}_{\rho_l,\theta}(h_1)$$

and

$$\lim_{N\to\infty} \frac{1}{N} \log \Psi^{[x,1]}_{\lfloor N(1-x) \rfloor,\theta,\rho_r}\left(h\left(\tfrac{\lfloor N(1-x) \rfloor}{N}\right),\dots,h\left(\tfrac{N}{N}\right)\right) = P^{[x,1]}_{\theta,\rho_r}(h_2)\,.$$

Moreover, using that

$$f_{U_{2s\lfloor Nx \rfloor,n}}(u) = e^{2sN[x\log\frac{u}{x}+(1-x)\log\frac{1-u}{1-x}+o(1)]}$$

and considering (6.11) for a vector with components $\mathsf{h}_i^{(N)} := h\left(a+\frac{i}{N}\right)$ with $i = 1,\dots N_{a,b}$ we obtain

$$\Psi^{[0,1]}_{N,\rho_l,\rho_r}\left(h\left(\tfrac{1}{N}\right),\dots,h\left(\tfrac{N}{N}\right)\right)$$
$$= \int_{\rho_l}^{\rho_r} e^{N\left[P^{[0,x]}_{\rho_l,\theta}(h_1)+P^{[x,1]}_{\theta,\rho_r}(h_2)+2sx\log\frac{\theta-\rho_l}{x(\rho_r-\rho_l)}+2s(1-x)\log\frac{\rho_r-\theta}{(1-x)(\rho_r-\rho_l)}+o(1)\right]}\,d\theta.$$

Then, taking the limit as $N \to \infty$ and recalling the definition of the modified pressure, the claim (6.6) follows from the Laplace principle. □

6.2. The additivity principle for the density large deviation function. For a macroscopic system on the interval $[a,b]$ we define the modified density large deviation function with boundary parameters $0 < \rho_a < \rho_b$ as

$$\widetilde{I}^{[a,b]}_{\rho_a,\rho_b}(\rho) := I^{[a,b]}_{\rho_a,\rho_b}(\rho) - 2s(b-a)\log\left(\frac{\rho_b-\rho_a}{b-a}\right) \tag{6.13}$$

where $I^{[a,b]}_{\rho_a,\rho_b}(\cdot)$ is the large deviation function of the empirical profile

$$L_N^{[a,b]} = \frac{1}{N_{a,b}} \sum_{i=1}^{N_{a,b}} \eta_i \delta_{a+\frac{i}{N}}\,.$$

Theorem 6.2 (Large deviation additivity principle). *Let* $2s \in \mathbb{N}$. *For* $0 < x < 1$ *and* $\rho : [0,1] \to \mathbb{R}$, *we have*

$$\widetilde{I}^{[0,1]}_{\rho_l,\rho_r}(\rho) = \inf_{\rho_l \le \theta \le \rho_r} \left[\widetilde{I}^{[0,x]}_{\rho_l,\theta}(\rho_1) + \widetilde{I}^{[x,1]}_{\theta,\rho_r}(\rho_2) \right] \tag{6.14}$$

where $\rho_1 : [0,x] \to \mathbb{R}$ *and* $\rho_2 : [x,1] \to \mathbb{R}$ *are the restrictions of* ρ *respectively, to* $[0,x]$ *and to* $[x,1]$. *More generally, for* $\kappa \ge 2$ *and for* $0 = x_0 \le x_1 \le \cdots \le x_\kappa = 1$, *calling* $\rho_i : [x_{i-1}, x_i] \to \mathbb{R}$ *the restriction of* ρ *to* $[x_{i-1}, x_i]$, *for* $i = 1, \ldots, \kappa$, *we have*

$$\widetilde{I}^{[0,1]}_{\rho_l,\rho_r}(\rho) = \inf_{\theta_0 \le \theta_1 \le \cdots \le \theta_{\kappa-1} \le \rho_\kappa} \sum_{i=1}^{\kappa} \widetilde{I}^{[x_{i-1},x_i]}_{\theta_{i-1},\theta_i}(\rho_i) \tag{6.15}$$

with the convention $\theta_0 = \rho_l$, $\theta_\kappa = \rho_r$.

Proof The proof is analogous to the one of Theorem 6.1. □

7. Explicit Formulas for the Pressure and Further Results on the Additivity Principle

In this final section, we give explicit formulas for the pressure and prove equivalence between the additivity principle and the MFT variational expression. Firstly, in Sect. 7.1 we find an explicit formula for the pressure with a constant function h. In the spirit of this paper, we show how this can be achieved in two ways: either macroscopically, solving the MFT variational principle, or microscopically, using the explicit characterization of the stationary measure to produce upper and lower bounds matching in the limit $N \to \infty$. Secondly, in Sect. 7.2, using the knowledge of the pressure with a constant function h, we prove the equivalence between Theorem 4.1 (pressure MFT variational problem) and Theorem 6.1 (pressure additivity principle). Thirdly, in Sect. 7.3, we consider the finite-volume pressure P_N with a constant function h. We prove that it satisfies a recursion relation in N, which in fact can be solved for the Laplace transform. Also in this section we restrict to the case $2s \in \mathbb{N}$. In particular, we prove that the finite-volume pressure of the model with $s = 1/2$ is size-independent, i.e. it takes the same value for all system sizes N.

7.1. *The pressure with a constant function h.*

We analyse in detail the case $h(x) = \mathsf{h} \in \mathbb{R}$ for all $x \in [0,1]$.

7.1.1. *Solution of MFT variational problem*

When the function $h(\cdot)$ is constantly equal to h, the variational problem for the pressure reads

$$P^{[0,1]}_{\rho_l,\rho_r}(\mathsf{h}) = \sup_{\theta} \mathscr{P}(\mathsf{h},\theta) \tag{7.1}$$

with

$$\mathscr{P}(\mathsf{h},\theta) = 2s \int_0^1 dx \left[\log\left(\frac{1}{1 + (1 - e^{\mathsf{h}})\theta(x)} \right) + \log\left(\frac{\theta'(x)}{\rho_r - \rho_l} \right) \right] \tag{7.2}$$

and the supremum is over all functions $\theta : [0,1] \to \mathbb{R}$ monotone such that $\theta(0) = \rho_l$ and $\theta(1) = \rho_r$. In other words

$$P^{[0,1]}_{\rho_l,\rho_r}(\mathsf{h}) = \mathscr{P}(\mathsf{h},\theta_*)$$

where θ_* is defined implicitly by $\frac{\delta\mathscr{P}}{\delta\theta}\Big|_{\theta=\theta_*} = 0$. Computing the functional derivatives one gets the boundary value problem

$$\frac{1-e^{\mathsf{h}}}{1+(1-e^{\mathsf{h}})\theta_*} - \frac{\theta_*''}{(\theta_*')^2} = 0, \qquad \theta_*(0) = \rho_l,\ \theta_*(1) = \rho_r \tag{7.3}$$

whose solution is given by

$$\theta_*(x) = \frac{1}{1-e^{\mathsf{h}}}\Big[(\rho_l(1-e^{\mathsf{h}})+1)\left(\frac{\rho_r(1-e^{\mathsf{h}})+1}{\rho_l(1-e^{\mathsf{h}})+1}\right)^x - 1\Big]. \tag{7.4}$$

Plugging (7.4) in (7.2) one obtains

$$P^{[0,1]}_{\rho_l,\rho_r}(\mathsf{h}) = \mathscr{P}(\mathsf{h},\theta_*) = 2s\log\left(\frac{1}{(\rho_r-\rho_l)(1-e^{\mathsf{h}})}\log\frac{1+(1-e^{\mathsf{h}})\rho_r}{1+(1-e^{\mathsf{h}})\rho_l}\right). \tag{7.5}$$

In a similar manner, it can be proved that

$$P^{[a,b]}_{\rho_a,\rho_b}(\mathsf{h}) = 2s(b-a)\log\left(\frac{1}{(\rho_b-\rho_a)\left(1-e^{h}\right)}\cdot\log\frac{1+\rho_b(1-e^{\mathsf{h}})}{1+\rho_a(1-e^{\mathsf{h}})}\right)$$

where $P^{[a,b]}_{\rho_a,\rho_b}(\cdot)$ is the pressure for a system in the macroscopic interval $[a,b]$.

7.1.2. Matching upper and lower bound In this section we consider the moment generating function evaluated in a point with components all equal to each others, i.e. $(\mathsf{h},\ldots,\mathsf{h})$, with $\mathsf{h}\in\mathbb{R}$. For this observable we introduce the notation $\Psi^{(1)}_{N,\rho_l,\rho_r}:\mathbb{R}\to\mathbb{R}$ for the one-variable function

$$\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) := \Psi_{N,\rho_l,\rho_r}(\mathsf{h},\ldots,\mathsf{h}). \tag{7.6}$$

From Proposition 3.8 and Corollary 3.3 we know that, thanks to the mixture structure of the non-equilibrium steady state, this can be written as

$$\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = \mathbb{E}\left[\prod_{i=1}^N M^{2s}_{\Theta_{2si,n}}(\mathsf{h})\right]$$

where we recall that, for $2s\in\mathbb{N}$, $M^{2s}_\theta(\cdot)$ is the generating function of a Negative Binomial of parameters $2s$ and θ, i.e.

$$M^1_\theta(\mathsf{h}) = \frac{1}{1+\theta(1-e^{\mathsf{h}})} \qquad\text{and}\qquad M^{2s}_\theta(\mathsf{h}) = (M^1_\theta(\mathsf{h}))^{2s}. \tag{7.7}$$

Notice that we added the superscript $2s$ in the notation for this generating function because in what follows it will be crucial to distinguish the case of general $2s\neq 1$ and $2s=1$. In the following theorem we will prove that the logarithm of $\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h})$ divided by N converges, in the limit as $N\to\infty$, to the solution of the variational problem for the pressure given in (7.5). We will restrict to the case $2s\in\mathbb{N}$.

Theorem 7.1 (Pressure, constant function h). *For all $s > 0$ with $2s \in \mathbb{N}$, $\mathsf{h} \in \mathbb{R}$ we have that*

$$\lim_{N\to\infty} \frac{1}{N} \log \Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = 2s \log \left(\frac{1}{(\rho_r - \rho_l)(1 - e^{\mathsf{h}})} \log \frac{1 + (1 - e^{\mathsf{h}})\rho_r}{1 + (1 - e^{\mathsf{h}})\rho_l} \right) = P^{[0,1]}_{\rho_l,\rho_r}(\mathsf{h}). \tag{7.8}$$

Proof Consider first $2s = 1$. In this case, because $n := 2s(N+1) - 1 = N$, the joint distribution of $(U_{2s,n}, \ldots, U_{2sN,n})$ is simply the joint distribution of the order statistics $(U_{1,N}, \ldots, U_{N,N})$. As a consequence, the corresponding variables $\Theta_{1,N} \ldots, \Theta_{N,N}$ defined in (3.7) are the order statistics of N uniforms on the interval $[\rho_l, \rho_r]$. Let us consider N independent uniform random variables on the interval $[\rho_l, \rho_r]$, denoted $\Theta_1, \ldots, \Theta_N$ as in Eq. (3.7). Then for every smooth function g we have that in distribution,

$$\prod_{i=1}^{N} g(\Theta_{i,N}) = \prod_{i=1}^{N} g(\Theta_i)$$

because in the product of all the N terms the ordering does not matter. As a consequence,

$$\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = \mathbb{E}\left(\prod_{i=1}^{N} M^1_{\Theta_{i,N}}(\mathsf{h}) \right) = \mathbb{E}\left(\prod_{i=1}^{N} M^1_{\Theta_i}(\mathsf{h}) \right) = \left[\mathbb{E}\left(M^1_{\Theta_1}(\mathsf{h}) \right) \right]^N$$

where in the last step we used independence of the Θ_i. Since

$$\mathbb{E}(M^1_{\Theta_1}(\mathsf{h})) = \frac{1}{\rho_r - \rho_l} \int_{\rho_l}^{\rho_r} \frac{d\rho}{1 + \rho(1 - e^{\mathsf{h}})} = \frac{1}{(\rho_r - \rho_l)(1 - e^{\mathsf{h}})} \log \frac{1 + (1 - e^{\mathsf{h}})\rho_r}{1 + (1 - e^{\mathsf{h}})\rho_l}, \tag{7.9}$$

we immediately get the result for the infinite pressure

$$\lim_{N\to\infty} \frac{1}{N} \log \Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = \log \left(\frac{1}{(\rho_r - \rho_l)(1 - e^{\mathsf{h}})} \log \frac{1 + (1 - e^{\mathsf{h}})\rho_r}{1 + (1 - e^{\mathsf{h}})\rho_l} \right).$$

To deal with the general case, first notice that the joint distribution of $(\Theta_{2s,n}, \ldots \Theta_{2sN,n})$ can be obtained as follows. We consider $n := 2s(N+1) - 1$ independent uniforms $(\Theta_1, \ldots, \Theta_n)$ on the interval $[\rho_l, \rho_r]$ and denote by $(\Theta_{1,n}, \ldots, \Theta_{n,n})$ the ordered vector. Moreover, $M^{2s}_\theta(\mathsf{h}) = \left(M^1_\theta(\mathsf{h})\right)^{2s}$ and therefore

$$\prod_{i=1}^{N} M^{2s}_{\Theta_{2si,n}}(\mathsf{h}) = \prod_{i=1}^{N} \left(M^1_{\Theta_{2si,n}}(\mathsf{h}) \right)^{2s}.$$

We notice that for h fixed, the function $\theta \to M^1_\theta(\mathsf{h})$ is non-decreasing and bounded from above and below by positive constants, i.e.,

$$0 < c_1 \leq M^1_\theta(\mathsf{h}) \leq c_2 < \infty.$$

As a consequence,

$$\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = \mathbb{E}\left(\prod_{i=1}^{N} M^{2s}_{\Theta_{2si,N}}(\mathsf{h})\right) = \mathbb{E}\left(\prod_{i=1}^{N}\left(M^{1}_{\Theta_{2si,n}}(\mathsf{h})\right)^{2s}\right)$$
$$\geq \mathbb{E}\left(\prod_{i=1}^{2sN} M^{1}_{\Theta_{i,n}}(\mathsf{h})\right)$$

where the last inequality follows from the fact that $M^1_{\cdot}(\mathsf{h})$ is non-decreasing and for $i = 1, \dots, N$

$$\Theta_{2si,n} \geq \Theta_{j,n} \qquad \text{when} \qquad 2s(i-1) < j \leq 2si.$$

Considering the log, dividing by N and taking the $N \to \infty$ limit on both sides, we have

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h})\right) \geq \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left(\prod_{i=1}^{2sN} M^{1}_{\Theta_{i,n}}(\mathsf{h})\right)$$
$$= \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left(\prod_{i=1}^{2s(N+1)-1} M^{1}_{\Theta_{i,n}}(\mathsf{h})\right)$$

where the last identity follows from the boundedness of $M^1_{\cdot}$, which is used to add $2s-1$ terms in the product. As for $s = 1/2$ we can now remove the order and use the independence of the Θ_i, $i = 1, \dots, n$

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h})\right) \geq \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left(\prod_{i=1}^{2s(N+1)-1} M^{1}_{\Theta_{i,n}}(\mathsf{h})\right)$$
$$= \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left(\prod_{i=1}^{2s(N+1)-1}\left(M^{1}_{\Theta_i}(\mathsf{h})\right)\right)$$
$$= \lim_{N\to\infty}\frac{1}{N}\log\left(\mathbb{E}\left(M^{1}_{\Theta_1}(\mathsf{h})\right)\right)^{2s(N+1)-1}$$
$$= 2s\log\left(\frac{1}{(\rho_r-\rho_l)(1-e^{\mathsf{h}})}\log\frac{1+(1-e^{\mathsf{h}})\rho_r}{1+(1-e^{\mathsf{h}})\rho_l}\right)$$

where the last identity follows from (7.9).

The idea to obtain a matching upper bound is similar. Now, for $i = 1, \dots, N$, we consider $2si \leq j \leq 2s(i+1)$ so that $\Theta_{2si,n} \leq \Theta_{j,n}$ implies

$$\mathbb{E}\left(\prod_{i=1}^{N}\left(M^{1}_{\Theta_{2si,n}}(\mathsf{h})\right)^{2s}\right) \leq \mathbb{E}\left(\prod_{i=2s}^{2s(N+1)-1} M^{1}_{\Theta_{i,n}}(\mathsf{h})\right).$$

since $M^1_{\cdot}(\mathsf{h})$ is non-decreasing. As before, in the limit we can consider the full product from $i = 1, \dots, 2s(N+1)-1$ by adding the first $2s-1$ terms so that we can replace the ordered variables $\Theta_{i,n}$ with the corresponding non ordered ones Θ_i and use their independence to conclude the proof. $\square$

Remark 7.2 (*Case $s = 1/2$*) In the course of the previous proof, we have proven, in particular that, for the case $s = 1/2$ the generating function $\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h})$ can be written in the power form $(\Psi^{(1)}_{1,\rho_l,\rho_r}(\mathsf{h}))^N$, and more precisely, (7.9) tells us that

$$\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = \left(\frac{1}{(\rho_r-\rho_l)(1-e^{\mathsf{h}})}\log\frac{1+(1-e^{\mathsf{h}})\rho_r}{1+(1-e^{\mathsf{h}})\rho_l}\right)^N$$

As a consequence

$$\frac{1}{N}\log\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h}) = \log\left(\frac{1}{(\rho_r-\rho_l)(1-e^{\mathsf{h}})}\log\frac{1+(1-e^{\mathsf{h}})\rho_r}{1+(1-e^{\mathsf{h}})\rho_l}\right). \tag{7.10}$$

In other words, for $s = 1/2$ the finite volume pressure does not depend on N and it coincides with the pressure at infinite volume.

7.2. Equivalence between additivity principle and variational problem. In this section we will prove the fact that the modified pressure

$$\tilde{P}^{[a,b]}_{\rho_a,\rho_b}(h) := P^{[a,b]}_{\rho_a,\rho_b}(h) + 2s(b-a)\log\left(\frac{\rho_b-\rho_a}{b-a}\right) \tag{7.11}$$

satisfies the additivity principle (6.7), combined with the continuity of $\tilde{P}^{[a,b]}_{\rho_a,\rho_b}$ with respect to convergence in L^1 and with formula (7.6) that gives an explicit expression of the action of $P^{[a,b]}_{\rho_a,\rho_b}(h)$ on constant functions $h(x) = \mathsf{h}$ for all $x \in [a,b]$, allows to identify the pressure functional $P^{[0,1]}_{\rho_l,\rho_r}$ on a generic function $h : [0,1] \to 1$, $h \in C^1$ as the solution of the variational problem:

$$\begin{aligned} P^{[0,1]}_{\rho_l,\rho_r}(h) &= \tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) - 2s\log(\rho_r-\rho_l) \\ &= 2s \cdot \sup_\theta \int_0^1 dx \left[\log\left(\frac{\theta'(x)}{\rho_r-\rho_l}\right) + \log\left(\frac{1}{1+\theta(x)(1-e^{h(x)})}\right)\right]. \end{aligned} \tag{7.12}$$

7.2.1. Variational problem implies additivity principle Consider $0 = x_0 < x_1 < \cdots < x_\kappa = 1$. Assume $h(x) = \sum_{i=1}^{\kappa} h_i(x)\mathbb{1}_{\{[x_{i-1},x_i]\}}(x)$ for $x \in [0,1]$ where h_i is the restriction of h to the interval $[x_{i-1}, x_i]$. Then the MFT variational problem can be written as follows:

$$P^{[0,1]}_{\rho_l,\rho_r}(h) = \sup_\theta \sum_{i=1}^{\kappa}\int_{x_{i-1}}^{x_i} 2s\left[\log\left(\frac{1}{1+(1-e^{h_i(x)})\theta(x)}\right) + \log\left(\frac{\theta'(x)}{(\rho_r-\rho_l)}\right)\right]dx$$

where the supremum is over monotonic C^1 functions $\theta : [0,1] \to \mathbb{R}$ such that $\theta(0) = \rho_l$ and $\theta(1) = \rho_r$. Equivalently we can write

$$\begin{aligned} P^{[0,1]}_{\rho_l,\rho_r}(h) = &\sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} \\ &\sum_{i=1}^{\kappa}\sup_{\theta_i}\int_{x_{i-1}}^{x_i} 2s\left[\log\left(\frac{1}{1+(1-e^{h_i(x)})\theta_i(x)}\right) + \log\left(\frac{\theta_i'(x)}{\rho_r-\rho_l}\right)\right]dx \end{aligned}$$

where the i^{th} supremum is now over monotone C^1 functions $\theta_i : [x_{i-1}, x_i] \to \mathbb{R}$ such that $\theta_i(x_{i-1}) = \rho_{i-1}$ and $\theta_i(x_i) = \rho_i$.

We now write the right hand side above in terms of the pressures of each interval, i.e.

$$\begin{aligned} P^{[0,1]}_{\rho_l,\rho_r}(h) = \sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} \sum_{i=1}^{\kappa} \Bigg[\sup_{\theta_i} \int_{x_{i-1}}^{x_i} 2s \left[\log\left(\frac{1}{1+(1-e^{h_i(x)})\theta_i(x)}\right) \right. \\ \left. + \log\left(\frac{(x_i - x_{i-1})\theta_i'(x)}{\rho_i - \rho_{i-1}}\right)\right] dx \\ + 2s(x_i - x_{i-1}) \log\left(\frac{\rho_i - \rho_{i-1}}{(x_i - x_{i-1})(\rho_r - \rho_l)}\right)\Bigg]. \end{aligned} \tag{7.13}$$

Recall the definition (cf. Sect. 1) of the pressure of the volume $[a, b]$ with boundary parameters ρ_a, ρ_b as

$$P^{[a,b]}_{\rho_a,\rho_b}(h) = \sup_{\theta} \int_a^b 2s \left[\log\left(\frac{1}{1+(1-e^{h(x)})\theta(x)}\right) + \log\left(\frac{(b-a)\theta'(x)}{\rho_b - \rho_a}\right)\right] dx$$

where the supremum is over monotone C^1 functions $\theta : [a, b] \to \mathbb{R}$ such that $\theta(a) = \rho_a$ and $\theta(b) = \rho_b$. Then (7.13) can be written as

$$\begin{aligned} &P^{[0,1]}_{\rho_l,\rho_r}(h) \\ &= \sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} \sum_{i=1}^{\kappa} \left[P^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(h_i) + 2(x_i - x_{i-1}) \log\left(\frac{\rho_i - \rho_{i-1}}{(x_i - x_{i-1})(\rho_r - \rho_l)}\right)\right]. \end{aligned}$$

As a consequence, we obtain that the modified pressure (7.11) fulfills the additivity principle

$$\tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) = \sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} \sum_{i=1}^{n} \tilde{P}^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(h_i)\,. \tag{7.14}$$

7.2.2. Additivity principle implies variational problem For any C^1 function $h : [0, 1] \to \mathbb{R}$ we can produce a discretization by fixing a sequence of piecewise constant functions $h^{(\kappa)} : [0, 1] \to \mathbb{R}$ defined as follows:

$$h^{(\kappa)}(x) = \sum_{i=1}^{\kappa} \mathsf{h}_i \cdot \mathbb{1}_{\{[x_{i-1},x_i)\}}(x), \qquad \kappa \in \mathbb{N}, \quad \mathsf{h}_1, \ldots, \mathsf{h}_\kappa \in \mathbb{R} \tag{7.15}$$

where

$$x_i = \tfrac{i}{\kappa} \qquad \text{and} \quad \mathsf{h}_i := h(x_i) = h\left(\tfrac{i}{\kappa}\right) \tag{7.16}$$

so that

$$h^{(\kappa)}(x) = h\left(\tfrac{\lceil \kappa x \rceil}{\kappa}\right). \tag{7.17}$$

Then we have that $h^{(\kappa)}$ converges to h in L^1. We can define an analogous approximation for any C^1 function $\theta : [0, 1] \to \mathbb{R}$ that is non-decreasing and such that $\theta(0) = \rho_l$ and $\theta(1) = \rho_r$. We do it by defining the piecewise constant functions $\theta^{(\kappa)} : [0, 1] \to \mathbb{R}$

$$\theta^{(\kappa)}(x) = \sum_{i=1}^{\kappa} \rho_i \cdot \mathbb{1}_{\{[x_{i-1},x_i)\}}(x), \qquad \text{for} \quad \rho_i := \theta(x_i) = \theta\left(\tfrac{i}{\kappa}\right) \tag{7.18}$$

so that $\rho_l = \rho_0 < \rho_1 < \cdots < \rho_\kappa = \rho_r$ and

$$\theta^{(\kappa)}(x) = \theta\left(\frac{\lceil \kappa x \rceil}{\kappa}\right). \tag{7.19}$$

We assume that the modified pressure (7.11) satisfies the additivity principle (6.7) and apply this property to the case in which the function h is the piecewise constant function $h^{(\kappa)}$:

$$\tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h^{(\kappa)}) = \sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} \sum_{i=1}^{\kappa} \tilde{P}^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(\mathsf{h}_i) \tag{7.20}$$

where

$$\tilde{P}^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(\mathsf{h}_i) := P^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(\mathsf{h}_i) + 2s(x_i - x_{i-1}) \log\left(\frac{\rho_i - \rho_{i-1}}{x_i - x_{i-1}}\right). \tag{7.21}$$

We can use now formula (7.6) which gives the pressure functional on constant functions

$$P^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(\mathsf{h}_i) = 2s(x_i - x_{i-1}) \log\left(\frac{1}{(\rho_i - \rho_{i-1})\left(1 - e^{\mathsf{h}_i}\right)} \cdot \log \frac{1 + \rho_i(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right)$$

from which we compute

$$\tilde{P}^{[x_{i-1},x_i]}_{\rho_{i-1},\rho_i}(\mathsf{h}_i) = 2s(x_i - x_{i-1}) \log\left(\frac{1}{(x_i - x_{i-1})\left(1 - e^{\mathsf{h}_i}\right)} \cdot \log \frac{1 + \rho_i(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right). \tag{7.22}$$

Using (7.20) and (7.16) we have

$$\begin{aligned}\tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h^{(\kappa)}) &= \sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} 2s \sum_{i=1}^{\kappa} (x_i - x_{i-1}) \\ &\qquad \log\left(\frac{1}{(x_i - x_{i-1})\left(1 - e^{\mathsf{h}_i}\right)} \cdot \log \frac{1 + \rho_i(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right) \\ &= \sup_{\rho_l=\rho_0<\rho_1<\cdots<\rho_\kappa=\rho_r} \frac{2s}{\kappa} \sum_{i=1}^{\kappa} \log\left(\frac{\kappa}{\left(1 - e^{\mathsf{h}_i}\right)} \cdot \log \frac{1 + \rho_i(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right).\end{aligned} \tag{7.23}$$

Writing

$$\log\left(\frac{1 + \rho_i(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right) = \log\left(1 + \frac{(\rho_i - \rho_{i-1})(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right) \tag{7.24}$$

and approxamiting

$$\rho_i - \rho_{i-1} = \theta\left(\frac{\lceil \kappa x \rceil}{\kappa}\right) - \theta\left(\frac{\lceil \kappa x \rceil - 1}{\kappa}\right) = \frac{1}{\kappa}\theta'(x) + o\left(\frac{1}{\kappa}\right) \qquad \text{for} \qquad x_{i-1} \le x < x_i \tag{7.25}$$

and

$$\mathsf{h}_i = h\left(\frac{\lceil \kappa x \rceil}{\kappa}\right) = h(x) + o(1) \qquad \text{for} \qquad x_{i-1} \le x < x_i \tag{7.26}$$

we get

$$\frac{(\rho_i - \rho_{i-1})(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})} = \frac{1}{\kappa} \cdot \frac{\theta'(x)(1 - e^{h(x)})}{1 + \theta(x)(1 - e^{h(x)})} + o\left(\frac{1}{\kappa}\right) \qquad \text{for} \qquad x_{i-1} \le x < x_i \tag{7.27}$$

and, as a consequence, taking the Taylor expansion of $\log(1 + x)$ we obtain

$$\log\left(1 + \frac{(\rho_i - \rho_{i-1})(1 - e^{\mathsf{h}_i})}{1 + \rho_{i-1}(1 - e^{\mathsf{h}_i})}\right) = \frac{1}{\kappa} \cdot \frac{\theta'(x)(1 - e^{h(x)})}{1 + \theta(x)(1 - e^{h(x)})} + o\left(\frac{1}{\kappa}\right) \qquad \text{for} \qquad x_{i-1} \le x < x_i \ . \tag{7.28}$$

Substituting this in (7.23) and taking the limit as $\kappa \to \infty$, via convergence of the Riemann sum to the corresponding integral we obtain that

$$\tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) = \lim_{\kappa \to \infty} \tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h^{(\kappa)}) = 2s \cdot \sup_{\theta} \int_0^1 \log\left(\frac{\theta'(x)}{1 + \theta(x)(1 - e^{h(x)})}\right) dx \tag{7.29}$$

where the first identity follows from the continuity of the modified pressure functional with respect to convergence of function in L^1. Now, using again (7.11) we conclude that

$$\begin{aligned} P^{[0,1]}_{\rho_l,\rho_r}(h) &= \tilde{P}^{[0,1]}_{\rho_l,\rho_r}(h) - 2s \log(\rho_r - \rho_l) \\ &= 2s \cdot \sup_{\theta} \int_0^1 dx \left[\log\left(\frac{\theta'(x)}{\rho_r - \rho_l}\right) + \log\left(\frac{1}{1 + \theta(x)(1 - e^{h(x)})}\right)\right]. \end{aligned} \tag{7.30}$$

7.3. Finite volume. In what follows we show that the moment generating function Ψ_{N,ρ_l,ρ_r} has another expression which differs from the ones in terms of N-fold sums and N-folds integrals of Sects. 3.2.2 and 3.2.3. To some extent this expression is more clear because it only relies on finite sums.

7.3.1. Recurrence relation We start from the integral equation (6.11) relating partition functions of different sizes and specialise it to the case $N_1 = 1$ and $N_2 = N - 1$. This becomes

$$\begin{aligned} \Psi_{N,\rho_l,\rho_r}(\mathsf{h}_1, \ldots, \mathsf{h}_N) = \int_0^1 du \Big(\frac{1}{1 + (1 - e^{\mathsf{h}_1})\theta(u)}\Big)^{2s} \Psi_{N-1,\theta(u),\rho_r}(\mathsf{h}_2, \ldots, \mathsf{h}_N) \\ \frac{\Gamma(2s(N+1))}{\Gamma(2s)\Gamma(2sN)} u^{2s-1}(1 - u)^{2sN-1}. \end{aligned} \tag{7.31}$$

Thanks to the relation (3.12) between Ψ_{N,ρ_l,ρ_r} and Φ_N we can turn (7.31) in a recurrence relation for the function Φ_N, namely

$$\begin{aligned} \Phi_N(c_1, \ldots, c_N) = \int_0^1 du \Big(\frac{1}{1 - (1 - u)c_1}\Big)^{2s} \Phi_{N-1}\Big((1 - u)c_2, \ldots, (1 - u)c_N\Big) \\ \frac{\Gamma(2s(N+1))}{\Gamma(2sN)\Gamma(2s)} u^{2s-1}(1 - u)^{2sN-1}. \end{aligned} \tag{7.32}$$

Changing the integration variable to $t = 1 - u$ one obtains

$$\Phi_N(c_1, \ldots, c_N) = \frac{1}{B(2sN, 2s)} \int_0^1 dt \left(\frac{1}{1 - tc_1}\right)^{2s} t^{2sN-1}(1 - t)^{2s-1} \Phi_{N-1}(tc_2, \ldots, tc_N) \tag{7.33}$$

where $B(2sN, 2s) = \frac{\Gamma(2sN)\Gamma(2s)}{\Gamma(2s(N+1))}$ is the Beta function.
Choosing a constant function $(\mathsf{h}, \ldots, \mathsf{h})$ corresponds to choosing a vector $c_{N,\rho_l,\rho_r}(\mathsf{h})$ (see (3.10)) with constant components $\left(c_{N,\rho_l,\rho_r}(\mathsf{h})\right)_i = c \in \mathbb{R}$ for $i = 1, \ldots, N$. For convenience we use the notation $\Phi_N^{(1)}$ for the function:

$$\Phi_N^{(1)}(c) := \Phi_N(c, \ldots, c) \tag{7.34}$$

then, specialising (7.33) to the case $c_1 = \cdots = c_N = c \in \mathbb{R}$ we deduce the following recurrence relation on $\Phi_N^{(1)}$

$$\begin{aligned}\Phi_N^{(1)}(c) &= \frac{1}{B(2sN, 2s)} \int_0^1 dt \left(\frac{1}{1-ct}\right)^{2s} t^{2sN-1}(1-t)^{2s-1}\Phi_{N-1}^{(1)}(tc) \\ &= \mathbb{E}\left[\left(\frac{1}{1-c\mathfrak{B}}\right)^{2s} \Phi_{N-1}^{(1)}(c\mathfrak{B})\right]\end{aligned} \tag{7.35}$$

where the random variable $\mathfrak{B}$ is distributed as a Beta$(2sN,\ 2s)$. Now we will see that it is possible to turn the integral in the right hand side of (7.35) into a convolution. To this aim, we perform the following change of variables $c = 1 - e^{-2v}$ and define the random variable $\mathfrak{Z}$ via the relation $c\mathfrak{B} = 1 - e^{-2\mathfrak{Z}}$. Then the density function of $\mathfrak{Z}$ is

$$f_{\mathfrak{Z}}(z) = \frac{1}{B(2sN, 2s)} \left(\frac{1-e^{-2z}}{1-e^{-2v}}\right)^{2sN-1} \left(\frac{e^{-2z}-e^{-2v}}{1-e^{-2v}}\right)^{2s-1} \frac{2e^{-2z}}{1-e^{-2v}}\ , \tag{7.36}$$

which allows to rewrite the recurrence relation in (7.35) as

$$\Phi_N^{(1)}(1-e^{-2v}) = \mathbb{E}\left[\left(e^{2\mathfrak{Z}}\right)^{2s} \Phi_{N-1}^{(1)}(1-e^{-2\mathfrak{Z}})\right]. \tag{7.37}$$

Using the density function of $\mathfrak{Z}$, the expression above can be conveniently rewritten as

$$\begin{aligned}&\frac{e^{v(2s-1)}(1-e^{-2v})^{2s(N+1)-1}}{2^{2sN}} \cdot \Phi_N^{(1)}(1-e^{-2v}) \\ &= \frac{1}{B(2sN, 2s)} \int_0^v dz\ \frac{e^{z(2s-1)}(1-e^{-2z})^{2sN-1}}{2^{2s(N-1)}} \cdot \Phi_{N-1}^{(1)}(1-e^{-2z})\,(\sinh(v-z))^{2s-1}\ .\end{aligned} \tag{7.38}$$

Defining the l.h.s. above as

$$\mathrm{G}_N(v) := \frac{e^{v(2s-1)}(1-e^{-2v})^{2s(N+1)-1}}{2^{2sN}} \cdot \Phi_N^{(1)}(1-e^{-2v}) \tag{7.39}$$

allows to read the recurrence relation as a convolution, i.e.

$$\mathrm{G}_N(v) = \frac{1}{B(2sN, 2s)} \int_0^v dz\ \mathrm{G}_{N-1}(z)\,(\sinh(v-z))^{2s-1}\ , \tag{7.40}$$

with $G_0(v) = 2^{2s-1}\,(\sinh(v))^{2s-1}$. Iterating $N+1$ times, we can write G_N as

$$\mathrm{G}_N(v) = 2^{2s-1}\frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}}\,(g * \cdots * g)(v) \tag{7.41}$$

where the convolution is taken $N+1$ times and $g(v) = (\sinh(v))^{2s-1}$.

7.3.2. Pressure via inverse Laplace transform In the previous section we have seen how the recurrence relation for the function $\Phi_N^{(1)}$ in (7.37) simplifies in a convolution relation for the function G_N in (7.41). Therefore, we denote by $\widehat{g}(\alpha) = \mathscr{L}\{g(v)\}(\alpha)$ the Laplace transform of the function $g(v)$ so that, when considering the Laplace transform on both sides of (7.41), we get

$$\widehat{\mathrm{G}}_N(\alpha) = 2^{2s-1}\frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}}\left(\widehat{g}(\alpha)\right)^{N+1} \ . \tag{7.42}$$

Computing the Laplace transform of $g(v)$, allows to explicitly write

$$\widehat{\mathrm{G}}_N(\alpha) = \ 2^{2s-1}\frac{\Gamma(2s(N+1))}{2^{2s(N+1)}}\left(\frac{\Gamma\left(\frac{\alpha+1-2s}{2}\right)}{\Gamma\left(\frac{\alpha+1+2s}{2}\right)}\right)^{N+1} \ . \tag{7.43}$$

At this point it is clear that anti transforming $\widehat{\mathrm{G}}_N(\alpha)$ and using Eq. (7.39), one can explicitly get an expression for the finite volume pressure for all N. This is the content of the next proposition.

Proposition 7.3 (Closed formula). *For $2s \in \mathbb{N}$, a closed formula for $\Phi_N^{(1)}$ given in terms of a finite sum is*

$$\Phi_N^{(1)}(c) = \left(\frac{2}{c}\right)^{2s(N+1)-1}\Gamma(2sN+2s) \sum_{j=0}^{2s-1}\sum_{k=0}^{N}\frac{(-\log(1-c))^{N-k}}{2^{N-k}(N-k)!k!}\,(1-c)^j\,\phi_{j,k}(\alpha_j) \tag{7.44}$$

where

$$\phi_{j,k}(\alpha_j) = \sum_{j_0+j_1+\cdots+j_{2s-1}=k}\binom{k}{j_0,\,j_1,\,\ldots,\,j_{2s-1}}\prod_{\substack{i=0\\ i\neq j}}^{2s-1}(-1)^{j_i}\,\frac{(N+j_i)!}{N!}(2i-2j)^{-N-j_i-1}. \tag{7.45}$$

Proof First we compute the inverse Laplace transform of $\widehat{\mathrm{G}}_N(\alpha)$ then we use (7.39) to get the expression above. In order to invert the Laplace transform we notice that

$$\frac{\Gamma\left(\frac{\alpha-2s+1}{2}\right)}{\Gamma\left(\frac{\alpha+2s+1}{2}\right)} = \prod_{i=0}^{2s-1}\frac{2^{2s}}{(\alpha-(2s-1)+2i)}$$

in other words, $\widehat{\mathrm{G}}_N(\alpha)$ has $2s$ poles, all with multiplicity $N+1$ namely $\alpha_i = 2s-1-2i$ for $i = 0,\ldots,2s-1$. The inverse Laplace transform of a rational function can be computed (see for example formula (21) of [21]); in our case

$$\widehat{\mathrm{G}}_N(\alpha) = 2^{2s-1}\Gamma(2s(N+1))\prod_{i=0}^{2s-1}\left(\frac{1}{\alpha-(2s-1)+2i}\right)^{N+1}$$

has inverse Laplace transform

$$\mathrm{G}_N(v) = 2^{2s-1}\Gamma(2sN+2s)\sum_{j=0}^{2s-1}\sum_{k=0}^{N}\frac{v^{N-k}}{(N-k)!k!}\phi_{j,k}(\alpha_j)e^{\alpha_j v}$$

$$\text{where} \qquad \phi_{j,k}(\alpha) = \frac{\partial^k}{\partial\alpha^k}\prod_{\substack{i=0\\i\neq j}}^{2s-1}\left(\frac{1}{\alpha-\alpha_i}\right)^{N+1}.$$

Now we show an explicit formula for the factors $\phi_{j,k}(\alpha_j)$, which can be computed using the general Leibniz rule for the product of functions, i.e.

$$\phi_{j,k}(\alpha) = \sum_{j_0+j_1+\cdots+j_{2s-1}=k}\binom{k}{j_0,\, j_1,\, \ldots,\, j_{2s-1}}\prod_{\substack{i=0\\i\neq j}}^{2s-1}\frac{\partial^{j_i}}{\partial\alpha^{j_i}}\left(\frac{1}{\alpha-\alpha_i}\right)^{N+1}$$

where $\binom{k}{j_0, j_1, \ldots, j_{2s-1}}$ is the multinomial coefficient and the $j_i{}^{th}$ derivative with respect to α is

$$\frac{\partial^{j_i}}{\partial\alpha^{j_i}}\left(\frac{1}{\alpha-\alpha_i}\right)^{N+1} = (-1)^{j_i}\frac{(N+j_i)!}{N!}(\alpha-\alpha_i)^{-N-1-j_i}$$

so that

$$\phi_{j,k}(\alpha) = \sum_{j_0+j_1+\cdots+j_{2s-1}=k}\binom{k}{j_0,\, j_1,\, \ldots,\, j_{2s-1}}\prod_{\substack{i=0\\i\neq j}}^{2s-1}(-1)^{j_i}\frac{(N+j_i)!}{N!}(\alpha-\alpha_i)^{-N-j_i-1}. \tag{7.46}$$

All in all, recalling that the residues are $\alpha_j = 2s-1-2j$ we get an explicit expression for $\mathrm{G}_N(v)$,

$$\mathrm{G}_N(v) = 2^{2s-1}\Gamma(2sN+2s)\sum_{j=0}^{2s-1}\sum_{k=0}^{N}\frac{v^{N-k}\, e^{(2s-2j-1)v}}{(N-k)!k!}\phi_{j,k}(\alpha_j).$$

The expression in Eq. (7.44) is then obtained from (7.39) setting $c = 1-e^{-2v}$ and rewriting for $\Phi_N^{(1)}(c)$,

$$\Phi_N^{(1)}(c) = \frac{2^{2sN}}{c^{2s(N+1)-1}}(1-c)^{s-1/2}\,\mathrm{G}_N\left(-\frac{1}{2}\log(1-c)\right). \tag{7.47}$$

□

We now show how the above computation for the moment generating function specialises for the first two cases $s = 1/2$ and $s = 1$.
Case $s = 1/2$. In this first case one can check that Eq. (7.43) simplifies to

$$\widehat{\mathrm{G}}_N(\alpha) = \frac{N!}{\alpha^{N+1}}$$

and its inverse Laplace transform is

$$\mathrm{G}_N(v) = v^N.$$

Recalling Eq. (7.39) and considering again the change of variable $c = 1 - e^{-2v}$, we obtain

$$\Phi_N^{(1)}(c) = \left(-\frac{1}{c}\log(1-c)\right)^N . \tag{7.48}$$

Notice that, using the map in (3.11) with $s = 1/2$ and constant h we recover the expression (7.10) for the MGF $\Psi^{(1)}_{N,\rho_l,\rho_r}(\mathsf{h})$.

Case $s = 1$. This is the first non-trivial value of s, notice that for all $s \neq 1/2$ the Laplace transform we compute depends on exponential functions and computations are more involved. We proceed as before. From (7.43) we can write

$$\widehat{\mathrm{G}}_N(\alpha) = 2\,\Gamma(2N+2)\left(\frac{1}{\alpha^2-1}\right)^{N+1} .$$

The poles of $\widehat{\mathrm{G}}_N$ are $\alpha_0 = -1$ and $\alpha_1 = 1$, while its Laplace inverse is

$$\mathrm{G}_N(v) = 2\,\Gamma(2N+2)\sum_{k=0}^{N}\frac{v^{N-k}}{k!(N-k)!}\left[\phi_{0,k}(-1)e^{-v} + \phi_{1,k}(1)e^{v}\right]$$

where

$$\phi_{0,k}(-1) = (-1)^k\frac{(N+k)!}{N!}(-2)^{-N-1-k} \qquad \text{and} \qquad \phi_{1,k}(1) = (-1)^k\frac{(N+k)!}{N!}(2)^{-N-1-k} .$$

This leads to

$$\mathrm{G}_N(v) = 2\,\Gamma(2N+2)\sum_{k=0}^{N} v^{N-k}\frac{(N+k)!}{N!k!(N-k)!}(-1)^k 2^{-N-1-k}\left[e^{v} + e^{-v}(-1)^{N+1+k}\right]$$

and using Eq. (7.47) we obtain

$$\Phi_N^{(1)}(c) = \frac{\Gamma(2N+2)}{c^{2N+1}}\sum_{k=0}^{N}(-1)^N\frac{(N+k)!}{N!k!(N-k)!}(\log(1-c))^{N-k}\left[1+(1-c)(-1)^{N+k+1}\right] .$$

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article and no financial or proprietary interests in any material discussed in this article.

A. Comparison and Integral Representation of Moments

In this appendix we show that (2.12) coincides with the integral representation of the steady state in (3.3). To do so, we consider the integral representation of the factorial moments that follows immediately from inserting (3.3) into (2.9) and using (3.14). It reads

$$\begin{aligned} G(\xi) = \mathscr{N}(N,s) \cdot \int_{\rho_l}^{\rho_r} d\theta_1 \int_{\theta_1}^{\rho_r} d\theta_2 \\ \cdots \int_{\theta_{N-1}}^{\rho_r} d\theta_N \Big[\prod_{i=1}^{N+1} (\theta_i - \theta_{i-1})^{2s-1} \Big] \Big[\prod_{i=1}^{N} \theta_i^{\xi_i} \Big], \end{aligned} \tag{A.1}$$

with the normalisation

$$\mathscr{N}(N,s) = \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} (\rho_r - \rho_l)^{1-2s(N+1)} . \tag{A.2}$$

To show that (A.1) coincides with the factorial moments of Theorem 2.2, we consider the auxiliary function

$$\mu''(\eta) = \sum_{\xi \geq \eta} \Big[\prod_{i=1}^{N} \frac{(-\rho_r)^{\eta_i - \xi_i}}{\eta_i !} \binom{\eta_i}{\xi_i} \frac{\Gamma(2s+\eta_i)}{\Gamma(2s)} \Big] G(\xi) , \tag{A.3}$$

as introduced in [25]. It is written in terms of the integrals as

$$\begin{aligned} \mu''(\eta) = \mathscr{N}(N,s) \Big[\prod_{i=1}^{N} \frac{1}{\eta_i !} \frac{\Gamma(2s+\eta_i)}{\Gamma(2s)} \Big] \int_{\rho_l}^{\rho_r} d\theta_1 \int_{\theta_1}^{\rho_r} d\theta_2 \\ \cdots \int_{\theta_{N-1}}^{\rho_r} d\theta_N \Big[\prod_{i=1}^{N+1} (\theta_i - \theta_{i-1})^{2s-1} \Big] \Big[\prod_{i=1}^{N} (\theta_i - \rho_r)^{\eta_i} \Big] . \end{aligned} \tag{A.4}$$

These integrals can be evaluated explicitly for any spin s and length N. Introducing the variables $u_i = \theta_i - \rho_r$ and $u_0 = \rho_l - \rho_r$ we get

$$\begin{aligned}\mu''(\eta) = \Big[\prod_{i=1}^{N} \frac{1}{\eta_i!}\frac{\Gamma(2s+\eta_i)}{\Gamma(2s)}\Big] \frac{\Gamma(2s(N+1))}{\Gamma(2s)^{N+1}} \frac{(-1)^{(2s+1)N}}{u_0^{2s(N+1)-1}} \\ \cdot \int_0^{u_0} du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{N-1}} du_N \\ u_N^{2s-1}\Big[\prod_{i=1}^{N} u_i^{\eta_i}(u_i - u_{i-1})^{2s-1}\Big].\end{aligned} \tag{A.5}$$

Now using repeatedly the integral formula of the Beta function

$$\int_0^x dy\, y^a (y-x)^b = (-1)^b x^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\,. \tag{A.6}$$

that holds for $a, b > -1$, we find

$$\begin{aligned}\mu''(\eta) = (\rho_l - \rho_r)^{|\eta|}\Big[\prod_{i=1}^{N} \frac{1}{\eta_i!}\frac{\Gamma(2s+\eta_i)}{\Gamma(2s)}\Big] \frac{\Gamma(2s(N+1))}{\Gamma(2s(N+1)+|\eta|)} \\ \cdot \prod_{k=1}^{N} \frac{\Gamma(2s(N-k+1)+\sum_{i=k}^{N}\eta_i)}{\Gamma(2s(N-k+1)+\sum_{i=k+1}^{N}\eta_i)}\,,\end{aligned} \tag{A.7}$$

which coincides with [25, (6.3)].

B. Order Statistics of Uniform Random Variables

In the following lemmata, we recall a few facts about the order statistics of i.i.d. uniforms on the unit interval. See [2], [40] for more details.

Lemma B.1 (Marginals) *Let $U_1, \dots, U_n$ denote n independent uniforms on* $[0, 1]$ *and denote their ascending order statistics by $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$. Let $1 \le n_1 \le n$ then the marginal probability density of the $U_{n_1,n}$ is*

$$f_{U_{n_1,n}}(u_1) = \frac{n!}{(n_1-1)!(n-n_1)!} \cdot u_1^{n_1-1}(1-u_1)^{n-n_1} \cdot 1\!\!1_{\{0\le u_1\le 1\}}. \tag{B.1}$$

For a given $1 \le k \le n$ this generalizes as follows: if $1 \le n_1 < \cdots < n_k \le n$ then the joint probability density of $(U_{n_1,n}, U_{n_2,n}, \dots, U_{n_k,n})$ is

$$\begin{aligned}&f_{(U_{n_1,n},U_{n_2,n},\dots,U_{n_k,n})}(u_1, u_2, \dots, u_k) \\ &= n!\left[\prod_{i=1}^{k+1} \frac{(u_i - u_{i-1})^{n_i - n_{i-1}-1}}{(n_i - n_{i-1} - 1)!}\right] 1\!\!1_{\{0\le u_1\le\cdots\le u_k\le 1\}}\end{aligned} \tag{B.2}$$

where we used the convention $n_0 = 0$, $n_{k+1} = n + 1$, $u_0 = 0$ and $u_{k+1} = 1$.

It is easy to see that the sequence of order statistics of continuous random variables is Markov.

Lemma B.2 (Markov property). *Let $U_1, \dots, U_n$ denote n independent uniforms on* $[0, 1]$ *and denote their ascending order statistics by $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$. Then the order statistics forms a Markov chain, i.e. for all $1 \le m \le n$, the sets of order statistics $(U_{1,n}, \dots, U_{m-1,n})$ and $(U_{m+1,n}, \dots, U_{n,n})$ become conditionally independent if $U_{m,n}$ is fixed. Therefore for the joint densities we may write*

$$
\begin{aligned}
&f_{U_{1,n},\dots,U_{m-1,n},U_{m+1,n},\dots,U_{n,n}\,|\,U_{m,n}}(u_1,\dots,u_{m-1},u_{m+1},\dots,u_n\,|\,u_m)\\
&\quad = f_{U_{1,n},\dots,U_{m-1,n}\,|\,U_{m,n}}(u_1,\dots,u_{m-1}\,|\,u_m)\\
&\qquad \cdot f_{U_{m+1,n},\dots,U_{n,n}\,|\,U_{m,n}}(u_{m+1},\dots,u_n\,|\,u_m).
\end{aligned} \tag{B.3}
$$

We also have the following important result: the conditional distribution of the order statistics (conditioned on another order statistic) is related to the distribution of order statistics from a (smaller) population whose distribution function is a truncated form of the original distribution function.

Lemma B.3 (Left/right truncation). *Let $U_1, \dots, U_n$ denote n independent uniforms on* $[0, 1]$ *and denote their ascending order statistics by $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$. Then, for $1 \le m \le n$ and $u_m \in (0, 1)$, the conditional distribution of $(U_{1,n}, \dots, U_{m-1,n})$, given that $U_{m,n} = u_m$ is the same as the distribution of the order statistics $(U^\star_{1,m-1}, \dots, U^\star_{m-1,m-1})$ obtained from a sample of size $m-1$ from a population whose distribution is uniform on* $[0, u_m]$*, i.e.*

$$
\begin{aligned}
&f_{U_{1,n},\dots,U_{m-1,n}\,|\,U_{m,n}}(u_1,\dots,u_{m-1}\,|\,u_m)\\
&\quad = f_{U^\star_{1,m-1},\dots,U^\star_{m-1,m-1}}(u_1,\dots,u_{m-1}).
\end{aligned} \tag{B.4}
$$

Similarly, the conditional distribution of $(U_{m+1,n}, \dots, U_{n,n})$, given that $U_{m,n} = u_m$ is the same as the distribution of the order statistic $(\tilde U_{1,n-m}, \dots, \tilde U_{n-m,n-m})$ obtained from a sample of size $n-m$ from a population whose distribution is uniform on $[u_m, 1]$*, i.e.*

$$
\begin{aligned}
&f_{U_{m+1,n}\dots,U_{n,n}\,|\,U_{m,n}}(u_{m+1},\dots,u_n\,|\,u_m)\\
&\quad = f_{\tilde U_{1,n-m},\dots,\tilde U_{n-m,n-m}}(u_{m+1},\dots,u_n).
\end{aligned} \tag{B.5}
$$

Combining together Lemmas B.2 and B.3 we obtain the following property for the conditional distribution of the order statistics of i.i.d. uniform random variables on the interval $[0, 1]$.

Lemma B.4 (Conditional distribution). *With the same hypotheses and notations of Lemmas B.2 and B.3 we have*

$$
\begin{aligned}
&f_{U_{1,n},\dots,U_{m-1,n},U_{m+1,n}\dots,U_{n,n}\,|\,U_{m,n}}(u_1,\dots,u_{m-1},u_{m+1},\dots,u_n\,|\,u_m)\\
&\quad = f_{U^\star_{1,m-1},\dots,U^\star_{m-1,m-1}}(u_1,\dots,u_{m-1})\\
&\qquad \cdot f_{\tilde U_{1,n-m},\dots,\tilde U_{n-m,n-m}}(u_{m+1},\dots,u_n).
\end{aligned} \tag{B.6}
$$

Finally, we recall the following large deviation result for the sample paths of the order statistics. Let $U_1, \dots, U_n$ be a random i.i.d. sample from a uniform distribution on $[0, 1]$, and let $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$ denote the order statistics obtained from this sample. Using the convention $U_{n+1,n} := 1$, we define the sample path of the order statistics by

$$
U_n(t) = U_{\lfloor (n+1)t \rfloor + 1,n} \qquad \text{for all } t \in [0, 1]
$$

where $\lfloor y \rfloor$ denotes the largest integer that is smaller or equal to y. Then we have the following functional Large Deviation Principle (LDP) for the sample paths of the order statistics.

Lemma B.5 (Sample path large deviation for order statistics, [20]). *Let* $D[0,1]$ *denote the space of càdlàg functions on the unit interval, equipped with Skorohod topology. Let* $A_{0,1} \subset D[0,1]$ *denote the closed set of non-decreasing functions* $f : [0,1] \to \mathbb{R}$ *such that* $f(x) \geq 0$ *and* $f(1) = 1$. *Then the sample paths* $U_n(\cdot)$ *satisfy the large deviation principle with rate function*

$$J(u) = \begin{cases} -\int_0^1 \log(u'(x))dx & \text{if } u \in A_{0,1} \text{ is increasing} \\ \infty & \text{otherwise} \end{cases}$$

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