An invariant transformation in finite hyperelasticity: Jacobian structure, boundary singularities and energy coupling

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Keywords: finite hyperelasticity, Criscione invariants, invariant transformation

A closed-form relation between the principal invariants (I_1, I_2, I_3) of the Cauchy–Green strain tensor and the invariants (K_1, K_2, K_3) of the natural strain tensor introduced by Criscione et al. [1] is derived for finite isotropic hyperelasticity, together with an analysis of its singularity structure and its implications for constitutive identification. The motivation stems from the fact that experimental data analysis is most naturally carried out in the Criscione framework, where the orthogonality of the stress basis ensures independent access to the strain energy derivatives [2]. The explicit map is obtained through the trigonometric parametrisation of the principal logarithmic stretches, and the Jacobian of the transformation is derived in closed form, with its inversion shown to reduce to a 2 $\times$ 2 algebraic problem. The transformation is examined at two levels of constitutive assumption: for general compressible materials, and under the volumetric–isochoric split, for which the Jacobian acquires an exact block-diagonal structure that decouples the volumetric and deviatoric contributions.

The singularity at $K_2 = 0$, corresponding to the undeformed or purely volumetric configuration, is shown to be removable. At the boundary $K_3 \to \pm 1$, the effective rank of the Jacobian drops from three to two: the response function $\partial W/\partial K_3$ becomes non-identifiable, as its contribution to any measurable stress component vanishes identically owing to the simultaneous vanishing of its associated basis tensor. This analytical argument emerges naturally and directly connects to the concept of energetic exhaustiveness introduced in Falope et al. [3]. The framework is illustrated through the transformation of representative strain energy functions. Classical models, when expressed in logarithmic invariants, reveal a coupling between distortion magnitude K_2 and distortion mode K_3 that is implicit in the principal invariant formulation but not directly visible; models formulated directly in (K_1, K_2, K_3) become transcendental when expressed in (I_1, I_2, I_3), reflecting an inherent asymmetry of the map. The complete derivation, singularity analysis, and transformation of representative energy functions are presented in [4].

References

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