L^p-EXACT CONTROLLABILITY OF PARTIAL DIFFERENTIAL EQUATIONS WITH NONLOCAL TERMS

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Abstract. The paper deals with the exact controllability of partial differential equations by linear controls. The discussion takes place in infinite dimensional state spaces since these equations are considered in their abstract formulation as semilinear equations. The linear parts are densely defined and generate strongly continuous semigroups. The nonlinear terms may also include a nonlocal part. The solutions satisfy nonlocal properties, which are possibly nonlinear. The states belong to Banach spaces with a Schauder basis and the results exploit topological methods. The novelty of this investigation is in the use of an approximation solvability method which involves a sequence of controllability problems in finite-dimensional spaces. The exact controllability of nonlocal solutions can be proved, with controls in L^p spaces, $1 < p < \infty$. The results apply to the study of the exact controllability for the transport equation in arbitrary Euclidean spaces and for the equation of the nonlinear wave equation.

1. Introduction

This paper deals with the exact controllability of nonlinear partial differential equations by means of linear controls. The discussion is led in suitable Banach spaces since we start from the abstract formulation of such equations.

The study of the exact controllability for nonlinear dynamics in infinite dimension started with the seminal paper [25] and the pioneering results in [20], [23] and [24]. This topic currently includes many contributions where the discussion also accounts of restrictions on the admissible controls, on the initial and final states, on the type of solution or on the properties of the dynamics. The main results can be found in the books [1], [4], [15] and [35]. Some very recent achievements appear in [6], [17] and [21] to which we also refer for their bibliography.

We consider the semilinear equation

$$y'(t) = Ay(t) + f(t, y(t)) + Bu(t), \qquad t \in [0, T], \;\; y(t) \in W \tag{1.1}$$

in the separable Banach space W. We assume that its linear part A is densely defined and generates the strongly continuous semigroup $\{S(t)\}_{t\geq 0}$. We take a linear control action B and treat the two cases when its domain U is a uniformly convex Banach space and when it is a Hilbert space. We consider $f\colon [0,T]\times W \to W$ satisfying suitable regularity conditions for which we refer to Section 3.

Given a control function $u(t)$, we assume that $y(t)$ satisfies equation (1.1) in its integral form that is we consider mild solutions of (1.1) (see Definition 3.6).

We require that the solution y satisfies a nonlocal condition of the following type: $y(0) = Dy + y_0$ where $y_0 \in W$ and $D\colon C([0,T], W) \to W$ may also be nonlinear.

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The map D accounts of possible restrictions on y such as, for instance, a Cauchy multi-point condition or some integral constraints (see Example 3.13). Starting from the seminal paper [8] it is increasing the interest to introduce these nonlocal restrictions since they are able to capture additional important information about the dynamics. We refer to [2] for a list of references and some recent results about this family of solutions. It is showed in [2], in particular, that equation (1.1) without control terms, admits nonlocal solutions.

We say that equation (1.1) is exactly *p-controllable* (p-controllable or simply controllable for short) (see Definition 3.7) when, for every y_0, y_1 in W, it is always possible to find $u \in L^p([0,T],U)$, with $1 < p < \infty$, such that the corresponding nonlocal solution of (1.1) further satisfies $y(T) = Dy + y_1$. In particular, when $D = 0$, we recover the classical notion of exact controllability.

In the linear case, i.e. when $f \equiv 0$, the p-controllability with $D = 0$ is equivalent to the surjectivity of the operator (see e.g. [14, p. 456])

$$G\colon L^p([0,T],U) \to W, \qquad u \longmapsto \int_0^T S(T-t)Bu(t)\,dt.$$

The nonlinear problem is generally thought to inherit the same controllability properties. This is essentially true but the restrictions on f may be very strong depending on the technique used for its study. In particular, the choice to address the controllability in Banach spaces leads to far grater difficulties than the use of Hilbert spaces. Indeed, when G is surjective, $p = 2$ and U is a Hilbert space, then G has a very natural right inverse, i.e. there exists $G^{-1}\colon W \to L^2([0,T],U)$ with $G \circ G^{-1} = Id_W$. This is the map $w \longmapsto v \in L^2([0,T],U)$ such that $\|v\| = \min\{\|u\| : G(u) = w\}$ which is well-defined, linear and bounded (see e.g. [26]). The same map, when U is an arbitrary Banach space, $v \in L^p([0,T],U)$ and $p \neq 2$, is again a right inverse of G, but it is not necessarily linear as we showed in [26]. For this reason, in our opinion, most controllability investigations involve control strategies belonging to Hilbert spaces.

To the best of our knowledge, the topological method used in this paper is new in this framework and it was previously introduced in [26] for studying the controllability of integro-differential equations. It starts from the observation that we can often assume that the state space is $W = L^r(\Omega)$ with $1 < r < \infty$ and $\Omega \subseteq \mathbb{R}^n$ sufficiently regular. It is known that $L^r(\Omega)$ admits a Schauder basis, i.e. its elements can be written as limits of suitable finite linear combinations (see Section 2). From this property, we introduce a sequence of controllability problems each one in a finite dimensional state space, hence simpler to be investigated. We then obtain the controllability of (1.1) by passing to the limit. The method can be applied under quite general regularity conditions. In particular, the map G^{-1} defined above, is always continuous when $1 < p < \infty$ and U is uniformly convex (see Proposition 2.3) and such regularity for G^{-1} is sufficient for getting the limiting procedure. This finite dimensional approach is also particularly suitable for the study of solutions which satisfy nonlocal conditions.

Our main results are Theorem 3.8, Theorem 3.9 and Theorem 3.10. In the first two contributions we state the p-controllability of (1.1), i.e. we allow u to be in $L^p([0,T],U)$ with $1 < p < \infty$ and U is a uniformly convex Banach space. Instead, in Theorem 3.10, we test our approximation technique with a control strategy in Hilbert spaces and again we get the controllability in a very general framework; in particular, we do not need any compactness concerning the operator G, a condition which could be very difficult to check in concrete situations.

The controllability is a rather strong property for partial differential models which fails in several cases, since G is never surjective when either the semigroup $\{S(t)\}_{t\geq 0}$

or the map B are compact (see [32] and also [14]). However, in some cases, the surjectivity of B or its very nature, imply that the corresponding operator G satisfies the same property. It happens, in particular, either when A generates a strongly continuous group (e.g. A is the translation group) or when A is obtained by the wave equation modeling a vibrating string (see [14, Example VI.8.9]). Our results allow to investigate the controllability associated to both these cases (see Theorem 4.1 in Section 4 and Theorem 5.1 in Section 5, respectively). The controllability of the wave equation is frequently treated with localized controls (see e.g. [20] and [37] for the case when $D = 0$). In Section 5 (Theorem 5.4) we also outline the use of our techniques in this case.

Controllability theory can also be set in the framework of multivalued dynamics occurring when the nonlinearity f or the nonlocal condition D are given by a multivalued map (see e.g. [3], [29] or [33]). We believe that our solvability method works also in this framework.

Section 2 contains some preliminary results. The abstract controllability problem is discussed in Section 3 and the main results are stated and proved there. Section 4 and 5 respectively deal with the controllability of the transport equation and the wave equation.

2. Notations and preliminary results

Let W be an infinite dimensional real Banach space with norm $\|\cdot\|$. For every $x \in W$ and $r > 0$, $B_r(x)$ is the open ball centred in x with radius r. Given a sequence $\{x_k\}_k$ in W, we write $x_k \to x$ if the sequence converges in norm to x and $x_k \rightharpoonup x$ if it converges weakly.

If $\{e_n\}_n$ is a Schauder basis for W (see the Appendix for this definition) for every $x \in W$ there exists a unique sequence of real numbers $\alpha_n = \alpha_n(x)$, $n \in \mathbb{N}$ such that

$$\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\| \to 0, \qquad \text{as } n \to \infty.$$

If $W_n = \operatorname{span}\{e_1, \dots, e_n\}$ is the n-dimensional Banach space generated by the first n vectors of the basis, we denote by $P_n : W \to W_n$ the natural projection:

$$P_n\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k e_k.$$

We refer to the Appendix for some useful properties of the sequences $\{e_n\}_n$ and $\{P_n\}_n$.

Remark 2.1. Obviously, if a Banach space admits a Schauder basis it is separable, but in [13] is proved that there exist separable Banach spaces which have no Schauder basis. However for each measurable subset $\Omega \subset \mathbb{R}^n, L^p(\Omega)$, $1 \le p < \infty$ has a monotone Schauder basis (see e.g. [18, Chap. 1.3 and 1.4]).

Definition 2.2. If X and Y are two non empty sets and $f : X \to Y$ is surjective, a *right inverse* of f is a function $\tilde{f}^{-1} : Y \to X$ such that $f \circ \tilde{f}^{-1} = id_Y$.

The following proposition shows that for every surjective, bounded, linear operator G defined in a uniformly convex Banach space, the function that select the element of minimal norm in the preimages of G is continuous. This function is a right inverse of G but in general it is not linear (Remark 2.4).

Proposition 2.3. *Let $G : E \to F$ be a surjective, bounded, linear operator, E a uniformly convex Banach space and F a Banach space. Then*

(i) *the map* $\mathcal{G} : E/\ker G \to F$ *defined by* $\mathcal{G}([u]) = G(u)$ *for every* $u \in E$ *is linear, bounded, one to one and onto;*
(ii) *there exists a continuous map* $\Pi : E/\ker G \to E$ *such that*
$$G(\Pi([u])) = G(u) \quad \text{and} \quad \|\Pi([u])\| = \min\{\|v\| : G(u) = G(v)\};$$
(iii) *the map* $\tilde{G}^{-1} = \Pi \circ \mathcal{G}^{-1}$ *is a continuous right inverse of* G *and*
$$\left\|\tilde{G}^{-1}(w)\right\| = \min\left\{\|u\| : u \in G^{-1}(w)\right\};$$
(iv) *if* E *is a Hilbert space, then* $\tilde{G}^{-1}$ *is linear.*

Proof. (i) Since G is bounded, $K = \ker G$ is a closed subspace of E and the quotient space E/K is a Banach space with the norm
$$|||[u]||| = \inf_{v\in[u]} \|v\|, \qquad [u] = \{v \in E : G(v) = G(u)\} = u + K$$
(see [31, Thm. 5.1]). Then $\mathcal{G}$ is obviously well defined, linear, one to one and onto. Moreover, by the definition of $|||\cdot|||$, $\mathcal{G}$ is bounded and $\|\mathcal{G}\| = \|G\|$.
(ii) Notice that $|||[u]||| = \min_{v\in[u]} \|v\|$ for every $u \in E$. In fact let $\{v_n\}_n \subseteq [u]$ be a minimizing sequence:
$$|||[u]||| = \lim_{n\to\infty} \|v_n\|.$$
E is uniformly convex, then reflexive, the sequence $\{v_n\}_n$ is bounded in E and $[u]$ is closed, then there exists a subsequence $\{v_{n_k}\}_k$ weakly convergent to $\bar{v} \in [u]$. By the weakly lower semicontinuity of the norm
$$|||[u]||| \le \|\bar{v}\| \le \lim_{k\to\infty} \|v_{n_k}\| = |||[u]|||.$$
Moreover, by the strict convexity of the norm in E there is a unique minimizer. Therefore we can define the function $\Pi : E/\ker G \to E$ calling $\Pi([u])$ the unique $\bar{u} \in u + K$ such that $|||[u]||| = \|\bar{u}\|$.
We have to prove that Π is continuous. Consider a sequence $([u_n])_n$ in E/K strongly convergent to $[u_0]$. Setting $\bar{u}_n = \Pi([u_n])$ and $\bar{u}_0 = \Pi([u_0])$, we have to show that $\bar{u}_n \to \bar{u}_0$. Suppose by contraddiction that there exists $\epsilon > 0$ and a subsequence $\{\bar{u}_{n_k}\}_k$ such that $\|\bar{u}_{n_k} - \bar{u}_0\| > \epsilon$ for every k.
Since
$$\|\bar{u}_{n_k}\| = |||[u_{n_k}]||| \to |||[u_0]||| = \|\bar{u}_0\|$$
the sequence $\{\bar{u}_{n_k}\}_k$ is bounded, then there exists a subsequence, denoted $\{\bar{u}_{n_k}\}_k$ for simplicity, such that $\bar{u}_{n_k} \rightharpoonup \bar{u}$. G and $\mathcal{G}$ are linear and bounded, then
$$G(\bar{u}_{n_k}) \rightharpoonup G(\bar{u}) \quad \text{and} \quad G(\bar{u}_{n_k}) = \mathcal{G}([\bar{u}_{n_k}]) \to \mathcal{G}([\bar{u}_0]) = G(\bar{u}_0)$$
as $k \to \infty$, therefore $G(\bar{u}) = G(\bar{u}_0)$, i.e. $\bar{u} \in [u_0]$. By the weakly lower semicontinuity of the norm
$$\|\bar{u}\| \le \lim_{k\to\infty} \|\bar{u}_{n_k}\| = \|\bar{u}_0\|$$
then $\bar{u} = \bar{u}_0 = \Pi([u_0])$. Now, $\|\bar{u}_{n_k}\| \to \|\bar{u}_0\|$, $\bar{u}_{n_k} \rightharpoonup \bar{u}_0$ and E is uniformly convex, then $\bar{u}_{n_k} \to \bar{u}_0$ strongly ([7] Proposition 3.32) a contradiction. Then Π is continuous.
(iii) $\tilde{G}^{-1}$ is continuous because composition of continuous functions. Moreover, for every $w \in F$ we have
$$G \circ \tilde{G}^{-1}(w) = G \circ \Pi \circ \mathcal{G}^{-1}(w) = w.$$
The last equality in (iii) is a direct consequence of the definition of Π.
(iv) the linearity of $\tilde{G}^{-1}$ is proved in [26, Proposition 2.2]. □

Remark 2.4. By the definition of $\tilde{G}^{-1}$ in the previous proposition, $\tilde{G}^{-1}$ is linear if and only if Π is. If E is not a Hilbert space, the map Π is homogeneous, but in general Π is not linear, then also the right inverse $\tilde{G}^{-1}$ is not a linear operator (see [26, Remark 2.3]).

In the sequel, for vector valued functions we will consider strong measurability (or simply *measurability*) and *Bochner integrability.*

Remark 2.5. If W is separable, for functions with values in W measurability is indifferently strong and weak measurability and the integrals are indifferently Bochner or Pettis integrals (see [11, Chap. II]).

For $1 \leq p \leq \infty$, $L^p([0\,,T]\,,W)$ denotes the Banach space of equivalence classes of functions $y : [0\,,T] \to W$ such that y is measurable in $[0\,,T]$ and $\|y\|_p < +\infty$, where $\|\cdot\|_p$ is the usual norm defined by

$$\|y\|_p = \left(\int_0^T \|y(t)\|^p\, dt\right)^{\frac{1}{p}}$$

when $1 \leq p < \infty$ and

$$\|y\|_\infty = \operatorname*{ess\,sup}_{t\in[0\,,T]} \|y(t)\|$$

when $p = \infty$.

Remark 2.6. If W is a uniformly convex Banach space and $1 < p < \infty$, then even $L^p([0\,,T]\,,W)$ is uniformly convex and hence reflexive ([7]).

$\mathcal{C}([0\,,T]\,,W)$, the space of continuous functions $y : [0\,,T] \to W$, is a Banach space with the norm

$$\|y\|_\infty = \max\{\|y(t)\| : \ t \in [0\,,T]\}.$$

Let the family $\{S(t)\}_{t\geq 0}$, $S(t) : W \to W$ bounded and linear operator, be a strongly continuous semigroup (C_0-semigroup), that is:

(1) $S(0) = I$;
(2) $S(t)S(s) = S(t+s)$ for every $t, s \in [0\,,+\infty)$;
(3) the map $t \to S(t)x$ is continuous on $[0\,,+\infty)$ for every $x \in W$.

It is well known (see [28, Theorem 2.2. page 4]) that, if $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup, there exists $M \geq 1$ such that

$$\|S(t)\| \leq M \quad \text{for all } \ t \in [0\,,T]. \tag{2.1}$$

If moreover $\|S(t)\| \leq 1$ for all $t \in [0\,,+\infty)$ (or $\|S(t)\| \leq 1$ for all $t \in \mathbb{R}$), it is called a C_0-semigroup (or group) of contractions.
If $S(t)$ is defined for all $t \in \mathbb{R}$, it satisfies (1), (2) for all s,t and the map $t \to S(t)x$ is continuous on all $\mathbb{R}$ for every x, then $\{S(t)\}_{t\in\mathbb{R}}$ is a strongly continuous group.

We now recall a useful compactness result for semicompact sequences proved in [19, Corollary 5.1.1].

Definition 2.7. We say that a sequence $\{f_n\}_n \subset L^1([0\,,T]\,,W)$ is *semicompact* if it is integrably bounded and the set $\{f_n(t)\}_n$ is relatively compact for a.e. $t \in [0\,,T]$.

Theorem 2.8. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup and F be the linear operator from $L^1([0\,,T]\,,W)$ to $\mathcal{C}([0\,,T]\,,W)$ defined by*

$$F(f)(t) = \int_0^t S(t-s)f(s)\,ds, \quad f \in L^1([0\,,T]\,,W) \text{ and } t \in [0\,,T]. \tag{2.2}$$

Then, for every semicompact sequence $\{f_n\}_n \subset L^1([0\,,T]\,,W)$*, the sequence* $\{F(f_n)\}_n$ *is relatively compact in* $\mathcal{C}([0\,,T]\,,W)$ *and*

$$\text{if } f_n \rightharpoonup f_0 \text{ in } L^1([0\,,T]\,,W), \text{ then } F(f_n) \to F(f_0).$$

In order to prove that a subset of a Banach space is relatively compact it is possible to use a measure of non-compactness (m.n.c. for short). We recall that, given a non empty subset C of the Banach space W, the *Hausdorff m.n.c.* of C (see e.g. [19, Chap. 2]) is defined by

$$\chi_W(C) = \inf\left\{\epsilon > 0 : \ \exists x_1, \dots x_k \in W \text{ such that } C \subset \bigcup_{i=1}^{k} B_\epsilon(x_i)\right\}.$$

if C is bounded and $\chi_W(C) = +\infty$ if C is unbounded.
Some properties of the Hausdorff measure of non compactness that we will use in the sequel are stated in the Appendix.

3. The abstract problem

In this section we consider the control problem

$$\begin{cases} y'(t) = Ay(t) + f(t\,,y(t)) + Bu(t) \\ y(0) = Dy + y_0 \end{cases} \tag{$\mathcal{P}$}$$

where W is a reflexive Banach space with a monotone Schauder basis $\{e_n\}_n$, $A : D(A) \subset W \to W$ is a linear (not necessarily bounded) operator whose domain $D(A)$ is dense in W and generating a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on W, $f : [0\,,T]\times W \to W$, $D : \mathcal{C}([0,T],W) \to W$, $y_0 \in W$ and $B : U \to W$ is a bounded linear operator defined in a uniformly convex Banach space U.

Remark 3.1. Notice that all the results we are proving hold true also if $K = \sup\{\|P_n\|\}_n > 1$, introducing K in conditions (f_3^2) and (3.27). Nevertheless all the Banach spaces involved in our applications admit a monotone basis, therefore we chose to assume that $K = 1$ in order to simplify the notations.

We will prove the controllability of $(\mathcal{P})$ in different frameworks. In particular in Theorems 3.8 and 3.9 the control space U is a Banach space, but a strong condition on f is assumed, the weak-weak sequential continuity with respect to the second variable. Instead, Theorem 3.10 requires that U is a Hilbert space, but it applies to functions f that are Lipschitz-continuous in the second variable.
In the first two results we will consider the following assumptions on f, D and B:

- (B) the linear operator $G : L^p([0\,,T]\,,U) \to W$, $1 < p < \infty$, defined by
$$G(u) = \int_0^T S(T-t)Bu(t)\,dt;$$
is onto.
- (D_1) D is a sequentially continuous function with respect to the weak topology both in $\mathcal{C}([0,T]\,,W)$ and in W:
$$\text{if } q_n \rightharpoonup q, \text{ then } Dq_n \rightharpoonup Dq$$
and maps bounded sets into bounded sets;
- (D_2) $\displaystyle\lim_{\|q\|_\infty \to +\infty} \frac{\|Dq\|}{\|q\|_\infty} = 0;$
- (f_1) for every $y \in W$ the function $f(\cdot\,,y) : [0\,,T] \to W$ is measurable with respect to the Lebesgue measure on $[0\,,T]$ and the Borel measure on W;

(f_2) for almost every $t \in [0\,,T]$ the function $f(t\,,\cdot) : W \to W$ is sequentially continuous with respect to the weak topology in W:

$$\text{if } y_n \rightharpoonup y, \text{ then } f(t\,,y_n) \rightharpoonup f(t\,,y)$$

for a.e. $t \in [0\,,T]$;

(f_3^1) for every $N \in \mathbb{N}$ there exists $\varphi_N \in L^1(0\,,T)$ such that

$$\begin{cases} \sup\limits_{\|y\|\le N} \|f(t\,,y)\| \le \varphi_N(t), & \text{for a.e. } t \in [0\,,T], \\ \liminf\limits_{N\to\infty} \dfrac{1}{N}\displaystyle\int_0^T \varphi_N(s)\,ds = 0; \end{cases}$$

(f_3^2) there exist $\alpha, \beta \in L^1(0\,,T)$ such that

$$\begin{cases} \|f(t\,,y)\| \le \alpha(t)\|y\| + \beta(t), & \text{for a.e. } t \in [0\,,T], \text{ for every } y \in W, \\ M\|\alpha\|_1 \left(1 + M\|B\|T^{1-\frac{1}{p}}\left\|\mathcal{G}^{-1}\right\|\right) < 1, \end{cases}$$

where $\mathcal{G}^{-1}$ is defined from G as in Proposition 2.3.

Remark 3.2. (f_3^1) and (f_3^2) represent different growth conditions of the nonlinear term f. More precisely, when (f_3^2) is satisfied, let

$$\psi_N(t) := \sup_{\|y\|\le N} \|f(t,y)\| \le \alpha(t)N + \beta(t), \qquad \text{for a.e. } t \in [0\,,T], \text{ for every } y \in W$$

and compare $\{\psi_N\}_N$ with the sequence $\{\varphi_N\}_N$ appearing in (f_3^1).
First notice that $\{\|\psi_N\|_1\}_N$ is an infinite at most of the same order as N. It is exactly of the same order as N when $\|\alpha\|_1 \neq 0$ and the inequality in the definition of ψ_N is an equality; moreover in this latter case $\lim_{N\to\infty} \frac{\psi_N(t)}{N} = \alpha(t)$ a.e. on $[0, T]$.
Instead (f_3^1) implies that $\{\|\varphi_N\|_1\}_N$ has a subsequence which is bounded or an infinite of lower order as N. Moreover since $\liminf_{N\to\infty} \dfrac{\|\varphi_N\|_1}{N} = 0$, Fatou's lemma implies that $\liminf_{N\to\infty} \frac{\varphi_N}{N}$ is the zero function.
However the validity of (f_3^1) in general does not imply (f_3^2) as the following example shows. In order to obtain condition (f_3^2) from (f_3^1) we need to assume further regularity such as, for instance, that $\{\frac{\varphi_N}{N}\}_N$ is an a.e. on $[0, T]$ dominated convergent sequence.

Example 3.3. Consider the sequence $\{\varphi_N\}_N$ defined by

$$\varphi_N(t) := k!\ln(k+1), \qquad t \in [0, T],\ k! \le N < (k+1)!,\ N \ge 1.$$

Since its subsequence $\{\varphi_{N_h}\}_h$ with $N_h =: (h+1)! - 1$, $h \ge 1$ satisfies

$$\lim_{h\to\infty} \frac{1}{N_h}\int_0^T \varphi_{N_h}(t)\,dt = \lim_{h\to\infty} \frac{h!T\ln(h+1)}{(h+1)!-1} \to 0, \quad \text{as } h \to \infty,$$

then condition (f_3^1) is true for $\{\varphi_N\}_N$. Let now $\{\varphi_{N_k}\}_k$ be the subsequence with $N_k =: k!$, $k \ge 1$; since

$$\lim_{k\to\infty} \frac{1}{N_k}\int_0^T \varphi_{N_k}\,dt = T\ln(k+1) \to \infty, \qquad \text{as } k \to \infty$$

the first condition in (f_3^2) can never be satisfied (see Remark 3.2).

Proposition 3.4. *Let conditions* (f_1), (f_2), (f_3^1) *or* (f_3^2) *hold and let* $q \in \mathcal{C}([0,T]\,,W)$. *Then the function* $f(\cdot, q(\cdot)) \in L^1([0,T), W)$.

Proof. For every $q \in \mathcal{C}([0,T],W)$, there exists a sequence of step functions $\{q_n\}_n$ uniformly convergent to q in $[0,T]$. In particular

$$\text{if } q_n(t) \rightharpoonup q(t), \text{ then } f(t,q_n(t)) \rightharpoonup f(t,q(t))$$

for every $t \in [0,T]$, therefore $f(\cdot,q(\cdot))$ is measurable.
Moreover, there exists N such that $\|q\|_\infty \leq N$, then by (f_3^1) the function $f(\cdot,q(\cdot))$ is integrable in $[0,T]$. □

Proposition 3.5. *Let conditions (D_1) and (D_2) hold. Then*

$$\lim_{N\to\infty} \frac{D_N}{N} = 0, \tag{3.1}$$

where

$$D_N = \sup_{\|q\|_\infty \leq N} \|Dq\|, \qquad \text{for every } N \in \mathbb{N}.$$

Proof. By contradiction let us suppose that there exist $l > 0$ and a subsequence $\{N_k\}_k$ such that

$$\frac{D_{N_k}}{N_k} \geq l$$

for every k, which yields the existence of $\{q_{N_k}\}_k$ with $\|q_{N_k}\|_\infty \leq N_k$ and $\frac{\|Dq_{N_k}\|}{N_k} \geq \frac{l}{2}$ for every k. If $\{q_{N_k}\}_k$ is bounded, then by (D_1) we have that there exists $\overline{D}$ such that $\|Dq_{N_k}\| \leq \overline{D}$ for every k, in contradiction with $\frac{\|Dq_{N_k}\|}{N_k} \geq \frac{l}{2}$ for every k. Therefore $\{q_{N_k}\}_k$ is unbounded and we get that, for every k,

$$\frac{\|Dq_{N_k}\|}{\|q_{N_k}\|_\infty} \geq \frac{\|Dq_{N_k}\|}{N_k} \geq \frac{l}{2}$$

in contradiction with (D_2) proving (3.1). □

Definition 3.6. Let $A : D(A) \subset W \to W$ be a linear (not necessarily bounded) operator whose domain $D(A)$ is dense in W and generating a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on W. Then, given a function $g : [0,T] \times W \to W$ and a value $y_0 \in W$, we say that $y \in \mathcal{C}([0,T],W)$ is a *mild solution* of the non-local problem

$$\begin{cases} y'(t) = Ay(t) + g(t,y(t)) \\ y(0) = y_0 + Dy \end{cases}$$

if $g(\cdot,y(\cdot)) \in L^1([0,T],W)$ and

$$y(t) = S(t)(y_0 + Dy) + \int_0^t S(t-s)g(s,y(s))ds$$

for every $t \in [0,T]$.

Definition 3.7. We say that the problem $(\mathcal{P})$ is (exactly) *p-controllable* on $[0,T]$, $1 < p < \infty$, if for all $y_0, y_1 \in W$ there exist $u \in L^p([0,T],U)$ and a mild solution $y(\cdot) \in \mathcal{C}([0,T],W)$ to $(\mathcal{P})$ (see Definition 3.6 with $g(t,y(t)) := f(t,y(t)) + Bu(t)$) such that $y(T) = Dy + y_1$.

We will first prove the following result.

Theorem 3.8. *Let conditions (f_1), (f_2), $(f_3^1), (D_1), (D_2)$ and (B) hold. Then problem $(\mathcal{P})$ is p-controllable, $1 < p < \infty$.*

Proof. We have to prove that for all $y_0, y_1 \in W$ there exist a control $u \in L^p([0,T],U)$ and a mild solution $y \in \mathcal{C}([0,T],W)$ to $(\mathcal{P})$ such that $y(T) = Dy + y_1$.
For every $n \in \mathbb{N}$ and $N > 0$ consider the subset of $\mathcal{C}([0,T],W_n)$ defined by

$$Q_N^n = \{q \in \mathcal{C}([0,T],W_n) : \ \|q(t)\| \leq N, \ \text{ for all } t \in [0,T]\}$$

and the integral operator $\mathcal{T}_n : Q^n_N \to \mathcal{C}([0\,,T]\,,W_n)$ defined by

$$\begin{aligned}\mathcal{T}_n(q)(t) =& P_nS(t)[y_0 + P_nDq] + \int_0^t P_nS(t-s)P_nf(s\,,q(s))\,ds\\ &+ \int_0^t P_nS(t-s)B\left(\tilde{G}^{-1}(P_n(p_q))(s)\right)ds, \qquad t\in[0\,,T],\end{aligned}$$

where

$$p_q = y_1 - S(T)y_0 + [id_W - S(T)]Dq - \int_0^T S(T-s)f(s\,,q(s))\,ds \in W$$

and $\tilde{G}^{-1}$ is the continuous map satisfying $G\circ\tilde{G}^{-1} = id_W$ defined in Proposition 2.3. The proof consists of the following steps:

Step 1. For every $n\in\mathbb{N}$ we prove that $\mathcal{T}_n$ admits a fixed point $y_n : [0\,,T]\to W_n$.
Step 2. We prove that the sequence $\{y_n\}_n$ found in previous step admits a subsequence pointwise weakly converging to a continuous function $y : [0\,,T]\to W$ such that

$$y(t) = S(t)[y_0 + Dy] + \int_0^t S(t-s)f(s\,,y(s))\,ds + \int_0^t S(t-s)B\,(u(s))\,ds \tag{3.2}$$

and $y(T) = Dy + y_1$, for a control $u\in L^p([0\,,T]\,,U)$.

Step 1. The proof is based on Schauder's fixed point theorem and to apply it we shall prove that for every $n\in\mathbb{N}$

Claim 1: for every $N>0$, $\mathcal{T}_n(Q^n_N)$ is relatively compact in $\mathcal{C}([0\,,T]\,,W_n)$;
Claim 2: for every $N>0$, $\mathcal{T}_n : Q^n_N \to \mathcal{C}([0\,,T]\,,W_n)$ is continuous;
Claim 3: there exists $\bar{N}>0$, which does not depend on n, such that $\mathcal{T}_n(Q^n_{\bar{N}})\subseteq Q^n_{\bar{N}}$.

Proof of Claim 1. Fix $n\in\mathbb{N}$ and $N>0$. We have to show that every sequence $\{\mathcal{T}_n(q_m)\}_m$, $q_m\in Q^n_N$, admits a uniformly convergent subsequence. Setting $f_m(\cdot) = f(\cdot\,,q_m(\cdot))$, by (f^1_3) we have

$$\|f_m(t)\| \le \varphi_N(t), \quad \text{for a.e. } t\in[0\,,T] \text{ and for every } m\in\mathbb{N}, \tag{3.3}$$

then $\{f_m\}_m$ is weakly relatively compact in $L^1([0\,,T]\,,W)$ (see [11, p. 101] recalling that W is reflexive) and there exist $f_0\in L^1([0\,,T]\,,W)$ and a subsequence, denoted by $\{f_m\}_m$ for simplicity, such that $f_m \rightharpoonup f_0$ in $L^1([0\,,T]\,,W)$.
Since $P_n : W\to W$ is linear and bounded, the functional $\tilde{P}_n : L^1([0\,,T]\,,W)\to L^1([0\,,T]\,,W)$ defined by

$$\tilde{P}_n(g)(t) = P_ng(t), \qquad \text{for every } g\in L^1([0\,,T]\,,W) \text{ and } t\in[0\,,T]$$

is linear and bounded, in fact $\|\tilde{P}_n(g)\|_1 \le \|P_n\|\|g\|_1 \le \|g\|_1$. Therefore for every Φ in the dual of $L^1([0\,,T]\,,W)$ the map $\Psi = \Phi\circ\tilde{P}_n$ is in the dual of $L^1([0\,,T]\,,W)$ too and

$$\Phi(\tilde{P}_n(f_m)) = \Psi(f_m) \to \Psi(f_0) = \Phi(\tilde{P}_n(f_0)) \quad \text{ as } m\to\infty$$

proving that

$$\tilde{P}_n(f_m) \rightharpoonup \tilde{P}_n(f_0) \quad \text{ in } L^1([0\,,T]\,,W).$$

By (3.3) for almost every $t\in[0\,,T]$ the set $\{P_nf_m(t) : m\in\mathbb{N}\}$ is a bounded subset of the finite dimensional space W_n, then it is relatively compact in W_n and in W. Therefore, by Theorem 2.8,

$$\int_0^t S(t-s)P_nf_m(s)\,ds \to \int_0^t S(t-s)P_nf_0(s)\,ds$$

uniformly in $[0\,,T]$, then

$$\int_0^t P_n S(t-s)P_n f_m(s)\,ds \to \int_0^t P_n S(t-s)P_n f_0(s)\,ds \qquad (3.4)$$

uniformly in $[0\,,T]$.
The linear functional $\Sigma : L^1([0\,,T]\,,W) \to L^1([0\,,T]\,,W)$ defined by

$$\Sigma(g)(t) = S(T-t)g(t), \quad \text{for every } g \in L^1([0\,,T]\,,W) \text{ and } t \in [0\,,T]$$

is bounded since, by (2.1), $\|\Sigma(g)\|_1 \le M\|g\|_1$.
For every Φ in the dual of $L^1([0\,,T]\,,W)$ we have that the map $\Psi = \Phi \circ \Sigma$ is in the dual of $L^1([0\,,T]\,,W)$ too, hence, since $f_m \rightharpoonup f_0$ in $L^1([0\,,T]\,,W)$, $\Sigma(f_m) \rightharpoonup \Sigma(f_0)$ in $L^1([0\,,T]\,,W)$ and, in particular,

$$\int_0^T S(T-s)f_m(s)\,ds = \int_0^T \Sigma(f_m)(s)\,ds \rightharpoonup$$
$$\rightharpoonup \int_0^T \Sigma(f_0)(s)\,ds = \int_0^T S(T-s)f_0(s)\,ds \qquad (3.5)$$

in W. Moreover, from (D_1) we know that $\{Dq_m\}_m$ is bounded, hence weakly relatively compact. Thus there exist $l_0 \in W$ and a subsequence still denoted as the sequence such that

$$Dq_m \rightharpoonup l_0. \qquad (3.6)$$

Therefore

$$p_{q_m} \rightharpoonup p_0 = y_1 - S(T)y_0 + [id_W - S(T)]l_0 - \int_0^T S(T-s)f_0(s)\,ds \quad \text{in } W$$

and, by (A.1), $P_n p_{q_m} \to P_n p_0$ in W. By the continuity of $\tilde{G}^{-1}$ we have that

$$\alpha_m = \tilde{G}^{-1}(P_n p_{q_m}) \to \tilde{G}^{-1}(P_n p_0) = \alpha_0$$

in $L^p([0\,,T],U)$ so, using (2.1) and Hölder inequality, we obtain

$$\left\|\int_0^t P_n S(t-s)B\left(\tilde{G}^{-1}(P_n p_{q_m})(s)\right)\,ds - \int_0^t P_n S(t-s)B\left(\tilde{G}^{-1}(P_n p_0)(s)\right)\,ds\right\|$$
$$= \left\|\int_0^t P_n S(t-s)B\left(\alpha_m(s) - \alpha_0(s)\right)\,ds\right\| \le \int_0^t \|P_n S(t-s)B(\alpha_m(s)-\alpha_0(s))\|\,ds$$
$$\le \int_0^T M\|B\|\|\alpha_m(s)-\alpha_0(s)\|_U\,ds \le M\|B\|T^{1-\frac{1}{p}}\|\alpha_m - \alpha_0\|_p$$

for every $t \in [0\,,T]$. Hence

$$\int_0^t P_n S(t-s)B\tilde{G}^{-1}(P_n p_{q_m})(s)\,ds \to \int_0^t P_n S(t-s)B\tilde{G}^{-1}(P_n p_0)(s)\,ds \qquad (3.7)$$

in $\mathcal{C}([0\,,T]\,,W_n)$. Finally by (3.6) and (A.1) we get, for every $t \in [0,T]$,

$$\|P_n S(t)P_n Dq_m - P_n S(t)P_n l_0\| \le M\|P_n Dq_m - P_n l_0\| \to 0$$

when $m \to \infty$ which, together with (3.4) and (3.7), proves Claim 1.

Proof of Claim 2. Let $\{q_m\}_m$ be a sequence in Q_N^n uniformly convergent to $q \in Q_N^n$. We have to prove that $\mathcal{T}_n(q_m) \to \mathcal{T}_n(q)$ as $m \to \infty$ uniformly on $[0\,,T]$. To conclude it is sufficient to prove that $\mathcal{T}_n(q_m)(t) \to \mathcal{T}_n(q)(t)$ as $m \to \infty$ for every $t \in [0\,,T]$. In fact, by Claim 1, if $\{\mathcal{T}_n(q_m)\}_m$ converges pointwisely to $\mathcal{T}_n(q)$, every subsequence of $\{\mathcal{T}_n(q_m)\}_m$ admits a subsequence uniformly convergent to $\mathcal{T}_n(q)$. Therefore all the sequence must converge to $\mathcal{T}_n(q)$ in $\mathcal{C}([0\,,T]\,,W_n)$.

Fix $t \in [0,T]$. Since $q_m \to q$ in $\mathcal{C}([0\,,T]\,,W_n)$, in particular $q_m \rightharpoonup q$ in $\mathcal{C}([0,T],W)$ and $q_m(s) \rightharpoonup q(s)$ for every $s \in [0,t]$ (see Proposition A.8). Then by (D_1) we have that $Dq_m \rightharpoonup Dq$, hence that $S(T)Dq_m \rightharpoonup S(T)Dq$ and

$$P_nS(t)P_nDq_m \to P_nS(t)P_nDq \tag{3.8}$$

while (f_2) yields that

$$f(s,q_m(s)) \rightharpoonup f(s,q(s)) \quad \text{for a.e. } s \in [0\,,t]. \tag{3.9}$$

Since P_n is bounded and W_n is finite-dimensional,

$$P_nf(s,q_m(s)) \to P_nf(s,q(s)) \quad \text{for a.e. } s \in [0,t]$$

and the boundedness of $S(t-s)$ then yields

$$P_nS(t-s)P_nf(s,q_m(s)) \to P_nS(t-s)P_nf(s,q(s)) \quad \text{for a.e. } s \in [0\,,t].$$

Since by (2.1) and (f_3^1)

$$\|P_nS(t-s)P_nf(s,q_m(s))\| \le M\varphi_N(s),$$

by the dominated convergence theorem we can conclude that

$$\int_0^t P_nS(t-s)P_nf(s,q_m(s))\,ds \to \int_0^t P_nS(t-s)P_nf(s,q(s))ds \tag{3.10}$$

as $m \to \infty$.
Let us show that

$$\int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_{q_m}))(s)\right)ds \to \int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_q))(s)\right)ds \tag{3.11}$$

as $m \to \infty$.
By (3.9) and considerations above

$$P_nS(T-s)f(s,q_m(s)) \to P_nS(T-s)f(s,q(s)) \quad \text{for a.e. } s \in [0\,,T]$$

and

$$\int_0^T P_nS(T-s)f(s,q_m(s))\,ds \to \int_0^T P_nS(T-s)f(s,q(s))ds$$

therefore $P_n(p_{q_m}) \to P_n(p_q)$ as $m \to \infty$.
If $\mathcal{G} : L^p([0\,,T]\,,U)/\ker G \to W$ is the linear and bounded map defined in (i) of Proposition 2.3, then also $\mathcal{G}^{-1}$ is linear and bounded and

$$\mathcal{G}^{-1}(P_n(p_{q_m})) \to \mathcal{G}^{-1}(P_n(p_q)) \qquad \text{in } L^p([0\,,T]\,,U)/\ker G$$

as $m \to \infty$. By Proposition 2.3 again, $\Pi : L^p([0\,,T]\,,U)/\ker G \to L^p([0\,,T]\,,U)$ is continuous, then

$$\tilde{G}^{-1}(P_n(p_{q_m})) = \Pi\mathcal{G}^{-1}(P_n(p_{q_m})) \to \Pi\mathcal{G}^{-1}(P_n(p_q)) = \tilde{G}^{-1}(P_n(p_q)) \tag{3.12}$$

in $L^p([0\,,T]\,,U)$, as $m \to \infty$. Finally, by Hölder inequality and (3.12) we have

$$\begin{aligned}
&\left\|\int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_{q_m}))(s)\right)ds - \int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_q))(s)\right)ds\right\| \\
&\quad = \left\|\int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_{q_m}))(s) - \tilde{G}^{-1}(P_n(p_q))(s)\right)ds\right\| \\
&\quad \le \int_0^t \left\|S(t-s)B\left(\tilde{G}^{-1}(P_n(p_{q_m}))(s) - \tilde{G}^{-1}(P_n(p_q))(s)\right)\right\|ds \\
&\quad \le M\|B\|\int_0^T \left\|\tilde{G}^{-1}(P_n(p_{q_m}))(s) - \tilde{G}^{-1}(P_n(p_q))(s)\right\|ds \\
&\quad \le M\|B\|T^{1-\frac{1}{p}}\left\|\tilde{G}^{-1}(P_n(p_{q_m}) - \tilde{G}^{-1}(P_n(p_q))\right\|_p \to 0
\end{aligned}$$

as $m \to \infty$ and (3.11) is proved.

Then (3.8), (3.10) and (3.11) yield that $\mathcal{T}_n(q_m)(t) \to \mathcal{T}_n(q)(t)$ for every $t \in [0\,,T]$ proving Claim 2.

Proof of Claim 3. By (f_3^1) we have that, for every $n \in \mathbb{N}$ and for every $q \in Q_N^n$,

$$\|P_n p_q\| \le \|y_1\| + M\|y_0\| + (1+M)D_N + M\|\varphi_N\|_1, \tag{3.13}$$

where

$$D_N = \sup_{\|q\|_\infty \le N} \|Dq\|,$$

and

$$\begin{aligned}\left\|\tilde{G}^{-1}P_n(p_q)\right\|_p &= \left\|\Pi\mathcal{G}^{-1}P_n(p_q)\right\|_p = \left\|\mathcal{G}^{-1}P_n(p_q)\right\| \le \left\|\mathcal{G}^{-1}\right\| \|P_n p_q\| \\ &\le \left\|\mathcal{G}^{-1}\right\| \left(\|y_1\| + M\|y_0\| + (1+M)D_N + M\|\varphi_N\|_1\right).\end{aligned} \tag{3.14}$$

Therefore, by Hölder inequality,

$$\begin{aligned}\|\mathcal{T}_n(q)(t)\| &\le \|P_n S(t) y_0\| + \|P_n S(t) P_n D q\| \\ &\quad + \int_0^t \|P_n S(t-s) P_n f(s\,,q(s))\|\, ds \\ &\quad + \int_0^t \left\|P_n S(t-s) B\left(\tilde{G}^{-1}(P_n(p_q))(s)\right)\right\| ds \\ &\le M\|y_0\| + MD_N + M\|\varphi_N\|_1 + M\|B\|T^{1-\frac{1}{p}}\|\tilde{G}^{-1}P_n(p_q)\|_p \\ &\le C_1 + C_2\|\varphi_N\|_1 + C_3 D_N\end{aligned} \tag{3.15}$$

for every $t \in [0\,,T]$, where

$$C_1 = M\left[\|y_0\| + \|B\|T^{1-\frac{1}{p}}\left\|\mathcal{G}^{-1}\right\|\left(\|y_1\| + M\|y_0\|\right)\right],$$

$$C_2 = M\left(1 + M\|B\|T^{1-\frac{1}{p}}\left\|\mathcal{G}^{-1}\right\|\right)$$

and

$$C_3 = M[1 + \|B\|T^{1-\frac{1}{p}}\|\mathcal{G}^{-1}\|(1+M)].$$

By Proposition 3.5 and (f_3^1),

$$0 \le \liminf_{N\to\infty} \frac{1}{N} \sup_{q\in Q_N^n} \|\mathcal{T}_n(q)\|_\infty \le \liminf_{N\to\infty} \frac{C_1 + C_2\|\varphi_N\|_1 + C_3 D_N}{N} = 0,$$

therefore, there exists $\bar{N} > 0$ (independent from n) such that

$$\frac{1}{\bar{N}} \sup_{q\in Q_{\bar{N}}^n} \|\mathcal{T}_n(q)\|_\infty < 1$$

and the claim is proved.

Finally, applying Schauder's fixed point theorem we prove that for every $n \in \mathbb{N}$ there exists a fixed point y_n of $\mathcal{T}_n$.

Step 2. The sequence $\{y_n\}_n$ found in previous step satisfies

$$\begin{aligned}y_n(t) = P_n S(t)(y_0 + P_n D y_n) + &\int_0^t P_n S(t-s) P_n f(s\,,y_n(s))\, ds \\ &+ \int_0^t P_n S(t-s) B\left(\tilde{G}^{-1}(P_n(p_{y_n}))(s)\right) ds\end{aligned} \tag{3.16}$$

for every $n \in \mathbb{N}$ and $t \in [0\,,T]$. We shall prove that there exists a subsequence $\{y_{n_k}\}_k$ such that, for every $t \in [0\,,T]$, $y_{n_k}(t) \rightharpoonup y(t)$, where y satisfies (3.2) and $y(T) = y_1 + Dy$, for a control $u \in L^p([0\,,T]\,,U)$.

For every $N > 0$, let Q_N be the subset of $\mathcal{C}([0\,,T]\,,W)$ defined by

$$Q_N = \{q \in \mathcal{C}([0\,,T]\,,W) :\ \|q(t)\| \le N, \quad \text{for all } t \in [0\,,T]\}\,.$$

By Claim 3, for every $n \in \mathbb{N}$, $y_n \in Q_{\bar N}$ then, setting $g_n(t) = P_n f(t\,, y_n(t))$,

$$\|g_n(t)\| \le \varphi_{\bar N}(t) \qquad \text{for a.e. } t \in [0\,,T].$$

Hence there exist a subsequence $\{g_{n_k}\}_k$ and $g \in L^1([0\,,T]\,,W)$ such that $g_{n_k} \rightharpoonup g$ in $L^1([0\,,T]\,,W)$ and, as in (3.5),

$$\int_0^t S(t-s)g_{n_k}(s)\,ds \rightharpoonup \int_0^t S(t-s)g(s)\,ds, \quad \text{for every } t \in [0\,,T]. \tag{3.17}$$

The sequence $\{y_n\}_n$ is bounded, then by (D_1) we can suppose, possibly passing to a subsequence, that there exists $l_0 \in W$ such that $Dy_{n_k} \rightharpoonup l_0$. By Proposition A.2 (ii) and (A.3)

$$P_{n_k} S(t)(y_0 + P_{n_k} D y_{n_k}) \rightharpoonup S(t)(y_0 + l_0) \tag{3.18}$$

and

$$\int_0^t P_{n_k} S(t-s) g_{n_k}(s)\,ds \rightharpoonup \int_0^t S(t-s)g(s)\,ds, \quad \text{for every } t \in [0\,,T]. \tag{3.19}$$

Since $\tilde G^{-1}(P_{n_k}(p_{y_{n_k}})) \in L^p([0\,,T]\,,U)$, $1 < p < \infty$, and, as in (3.14),

$$\|\tilde G^{-1}(P_{n_k}(p_{y_{n_k}}))\|_p \le \left\|\mathcal{G}^{-1}\right\| \left(\|y_1\| + M\|y_0\| + (1+M)D_{\bar N} + M\|\varphi_{\bar N}\|_1\right),$$

there exist a subsequence, still denoted by n_k, and $u \in L^p([0\,,T]\,,U)$ such that $\tilde G^{-1}(P_{n_k}(p_{y_{n_k}})) \rightharpoonup u$ in $L^p([0\,,T]\,,U)$ (see Remark 2.6).
As in (3.17) we get that

$$\int_0^t S(t-s)B\left(\tilde G^{-1}(P_{n_k}(p_{y_{n_k}}))(s)\right)ds \rightharpoonup \int_0^t S(t-s)B\left(u(s)\right)ds$$

and, by (A.3),

$$\int_0^t P_{n_k} S(t-s)B\left(\tilde G^{-1}(P_{n_k}(p_{y_{n_k}}))(s)\right)ds \rightharpoonup \int_0^t S(t-s)B\left(u(s)\right)ds, \tag{3.20}$$

for every $t \in [0\,,T]$. Finally by (3.18), (3.19) and (3.20) for every $t \in [0\,,T]$ we have

$$y_{n_k}(t) \rightharpoonup y(t) = S(t)(y_0 + l_0) + \int_0^t S(t-s)g(s)\,ds + \int_0^t S(t-s)B\left(u(s)\right)ds. \tag{3.21}$$

Moreover, by (f_2) and (A.3), for almost every $t \in [0\,,T]$,

$$f(t, y_{n_k}(t)) \rightharpoonup f(t\,, y(t)) \;\text{ and then }\; g_{n_k}(t) = P_{n_k} f(t, y_{n_k}(t)) \rightharpoonup f(t\,, y(t)).$$

Therefore $f(t\,,y(t)) = g(t)$. Now, since, as in (3.15), for every $t \in [0,T]$,

$$\|y_{n_k}(t)\| \le C_1 + C_2\|\varphi_{\overline N}\|_1 + C_3 D_{\overline N},$$

where $D_{\overline N} = \sup_{\|q\|_\infty \le \overline N} \|Dq\|$, we get that $y_{n_k} \rightharpoonup y$ in $\mathcal{C}([0,T],W)$ (see Proposition A.8). Hence, by (D_1), $l_0 = Dy$ and then, by (3.21), (3.2) is satisfied.

It remains to prove that $y(T) = y_1 + Dy$. By (3.16) and the definition of $\tilde{G}^{-1}$ we have that, for every $n \in \mathbb{N}$,

$$\begin{aligned} y_n(T) =& P_n S(T)(y_0 + P_n D y_n) \\ &+ \int_0^T P_n S(T-s) P_n f(s, y_n(s))\, ds + P_n G \tilde{G}^{-1}(P_n p_{y_n}) \\ =& P_n S(T)(y_0 + P_n D y_n) + \int_0^T P_n S(T-s) P_n f(s, y_n(s))\, ds + P_n P_n p_{y_n} \\ =& P_n S(T)(y_0 + P_n D y_n) + \int_0^T P_n S(T-s) P_n f(s, y_n(s))\, ds + P_n p_{y_n} \\ =& P_n S(T)(y_0 + P_n D y_n) + \int_0^T P_n S(T-s) P_n f(s, y_n(s))\, ds + P_n y_1 \\ &- P_n S(T) y_0 + P_n(D y_n - S(T) D y_n) - P_n \int_0^T S(T-s) f(s, y_n(s))\, ds. \end{aligned}$$

Therefore, for every $k \in \mathbb{N}$,

$$\begin{aligned} y_{n_k}(T) =& P_{n_k} y_1 + P_{n_k}[D y_{n_k} + S(T)(P_{n_k} D y_{n_k} - D y_{n_k})] \\ &+ P_{n_k} \int_0^T S(T-s) \left[P_{n_k} f(s, y_{n_k}(s)) - f(s, y_{n_k}(s)) \right] ds. \end{aligned} \tag{3.22}$$

Since $D y_{n_k} \rightharpoonup Dy$, we get that $P_{n_k}[D y_{n_k} + S(T)(P_{n_k} D y_{n_k} - D y_{n_k})] \rightharpoonup Dy$. Moreover, by (f_2),

$$f(t, y_{n_k}(t)) \rightharpoonup f(t, y(t))$$

for almost every $t \in [0, T]$, then, by (A.3),

$$h_{n_k}(t) = P_{n_k} f(t, y_{n_k}(t)) - f(t, y_{n_k}(t)) \rightharpoonup 0, \qquad \text{for almost every } t \in [0, T].$$

For every $\Phi \in W^*$ and for every $s \in [0, T]$ the composition $\Phi \circ S(T-s) \in W^*$, therefore

$$\Phi\left(S(T-s) h_{n_k}(s)\right) \to 0, \qquad \text{for almost every } s \in [0, T]. \tag{3.23}$$

Moreover from (f_3^1)

$$|\Phi S(T-s) h_{n_k}(s)| \le 2\|\Phi\| M \varphi_{\bar{N}}(s), \qquad \text{for almost every } s \in [0, T]. \tag{3.24}$$

By (3.23), (3.24) and the dominated convergence theorem it follows that

$$\Phi\left(\int_0^T S(T-s) h_{n_k}(s)\, ds \right) = \int_0^T \Phi\left(S(T-s) h_{n_k}(s)\right) ds \to 0$$

for every $\Phi \in W^*$, that is

$$\int_0^T S(T-s) \left[P_{n_k} f(s, y_{n_k}(s)) - f(s, y_{n_k}(s)) \right] ds \rightharpoonup 0.$$

By passing to the weak limit in (3.22) we conclude that $y(T) = y_1 + Dy$.

□

The following result shows that controllability can be proved with the growth assumption (f_3^2) instead of (f_3^1) by adding a constraint on the linear term.

Theorem 3.9. *Let conditions (f_1), (f_2), (f_3^2), (D_1), (D_2) and (B) hold. Then problem $(\mathcal{P})$ is p-controllable, $1 < p < \infty$.*

Proof. In the proof of previous theorem the strictly sublinearity condition

$$\liminf_{N\to\infty} \frac{1}{N}\int_0^T \varphi_N(s)\,ds = 0$$

plays a role only in the proof of Claim 3. Then we can repeat all the proof (except Claim 3) with $\varphi_N(t) = \alpha(t)N + \beta(t)$.
It remains to prove that there exists $\bar{N} > 0$ such that $\mathcal{T}_n(Q^n_{\bar{N}}) \subseteq Q^n_{\bar{N}}$, for every $n \in \mathbb{N}$. Suppose, by contradiction, that for every $N \in \mathbb{N}$ there exist $\bar{n} = \bar{n}(N) \in \mathbb{N}$ and $q_N \in Q^{\bar{n}}_N$ such that $\mathcal{T}_{\bar{n}}(q_N) \notin Q^{\bar{n}}_N$. By (3.15) and $\mathcal{T}_{\bar{n}}(q_N) \notin Q^{\bar{n}}_N$ we have that

$$N < \|\mathcal{T}_{\bar{n}}(q_N)\|_\infty < C_1 + C_2(N\|\alpha\|_1 + \|\beta\|_1) + C_3 D_N,$$

with C_1, C_2 and C_3 as in the proof of Claim 3. Now dividing by N the first and the last term in the previous inequality and passing to the limit as $N \to \infty$, by (3.1) we obtain $C_2\|\alpha\|_1 \geq 1$. On the other hand, condition (f_3^2) means that $C_2\|\alpha\|_1 < 1$, a contradiction. □

Finally, in this last theorem we prove the 2-controllability of $(\mathcal{P})$ under the hypothesis that f is locally Lipschitz continuous in the second variable instead of (f_2). This proof holds when $L^p([0\,,T]\,,U)$ is a Hilbert space, that is $p = 2$ and U is a Hilbert space.

Theorem 3.10. *Let U be a Hilbert space and $p = 2$ and let condition (B) holds. In addition suppose that D is continuous and satisfies (D_2) and there exists $d > 0$ such that, for every bounded sequence $\{y_n\}_n \subset \mathcal{C}([0,T],W)$*

$$\chi_W\left(\{D\{y_n\}_n\}\right) \leq d \sup_{t\in[0,T]} \chi_W\left(\{y_n(t)\}_n\right). \tag{3.25}$$

Finally assume that f satisfies (f_1), $f(\cdot\,,0) \in L^1([0\,,T]\,,W)$ and there exists $L \in L^1(0\,,T)$ such that

$$\|f(t\,,y_1) - f(t\,,y_2)\| \leq L(t)\|y_1 - y_2\| \tag{3.26}$$

for each $y_1, y_2 \in W$ and for a.e. $t \in [0\,,T]$. If

$$M(d + \|L\|_1)\left(1 + M\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\right) + dM\|B\|\sqrt{T}\|\tilde{G}^{-1}\| < 1 \tag{3.27}$$

then problem $(\mathcal{P})$ is 2-controllable.

Remark 3.11. If conditions (f_1) and (3.26) hold then f is a Carathéodory function and for every $q \in \mathcal{C}([0,T],W)$, $f(\cdot,q(\cdot))$ is measurable (q is the pointwise limit of simple functions, then $f(\cdot,q(\cdot))$ is the pointwise limit of measurable functions). Moreover, if (3.26) holds also (f_3^2) holds with $\alpha(t) = L(t)$ and $\beta(t) = \|f(t\,,0)\|$, then $f(\cdot,q(\cdot)) \in L^1([0,T],W)$.

Proof of Theorem 3.10. The structure of the proof is the same as in previous theorems. We have to prove Step 1 and Step 2 when D satisfies (3.25) and it is continuous instead of weakly continuous and (3.26) instead of (f_2) holds. Recall that, since $L^2([0\,,T]\,U)$ is a Hilbert space, by Proposition 2.3 the linear operator G admits a linear and bounded right inverse $\tilde{G}^{-1}$.
Step 1. We shall prove *Claim 1, Claim 2* and *Claim 3* as in the proof of Theorem 3.8.
Proofs of Claims 1 and 3. Notice that by (3.25) we have that D maps bounded sets into bounded sets. Hence, since in Theorem 3.8 and in Theorem 3.9 we do not use elsewhere hypothesis (D_1) and (f_2) to prove these claims, the proofs remain the same.
Proof of Claim 2. Also this proof is almost the same of the corresponding proof in Theorem 3.8. First of all notice that formula (3.8) follows also from the continuity

of D. The only main difference is in formula (3.9). If (3.26) holds we have that if $q_m \to q$ in $\mathcal{C}([0,T],W)$, then $q_m(s) \to q(s)$ for every $s \in [0,t]$ and

$$f(s,q_m(s)) \to f(s,q(s)) \quad \text{for a.e. } s \in [0\,,t].$$

The remaining part of the proof does not change.

Step 2. Let $\{y_n\}_n$ be the sequence found in previous step, satisfying (3.16). First of all we will prove that $\chi_W\left(\{y_n(t)\}_n\right) = 0$ for every $t \in [0\,,T]$, that is

$$\gamma(\{y_n\}_n) = \sup_{t\in[0\,,T]} \chi_W\left(\{y_n(t)\}_n\right) = 0. \tag{3.28}$$

For every fixed $t \in [0\,,T]$, by Proposition A.5 (ii) and (iv) we have that

$$\begin{aligned}\chi_W\left(\{y_n(t)\}_n\right) \le& \chi_W\left(\{P_nS(t)y_0\}_n\right) + \chi_W\left(\{P_nS(t)P_nDy_n\}_n\right)\\ &+ \chi_W\left(\left\{\int_0^t P_nS(t-s)P_nf(s\,,y_n(s))\,ds\right\}_n\right)\\ &+ \chi_W\left(\left\{\int_0^t P_nS(t-s)B\left(\tilde{G}^{-1}(P_n(p_{y_n}))(s)\right)ds\right\}_n\right).\end{aligned} \tag{3.29}$$

Since the sequence $\{P_nS(t)y_0\}_n$ converges to $S(t)y_0$, $\chi_W\left(\{P_nS(t)y_0\}_n\right) = 0$. As to the second and third addend, by (2.1), Proposition A.5 (v), Proposition A.6 and (3.25), we respectively get

$$\chi_W\left(\{P_nS(t)P_nDy_n\}\right) \le Md\gamma\left(\{y_n\}_n\right)$$

and

$$\begin{aligned}\chi_W\left(\{P_nS(t-s)P_nf(s\,,y_n(s))\}_n\right) \le& M\chi_W\left(\{f(s\,,y_n(s))\}_n\right)\\ \le& ML(s)\chi_W\left(\{y_n(s)\}_n\right) \le ML(s)\gamma(\{y_n\}_n)\end{aligned}$$

for every $s \in [0\,,t]$, then by Theorem A.7

$$\chi_W\left(\left\{\int_0^t P_nS(t-s)P_nf(s\,,y_n(s))\,ds\right\}_n\right) \le M\|L\|_1\gamma\left(\{y_n\}_n\right).$$

The same reasoning gives $\chi_W\{P_np_{y_n}\} \le [d + M(d + \|L\|_1)]\,\gamma(\{y_n\}_n)$. Let us consider now the linear functional $H_t : W \to W$ defined by

$$H_t(w) = \int_0^t S(t-s)B\left(\tilde{G}^{-1}(w)(s)\right)ds, \qquad w \in W, \tag{3.30}$$

then, by (2.1) and the continuity of B and $\tilde{G}^{-1}$, H_t is bounded. In fact, since $\tilde{G}^{-1}(w)$ is in $L^2([0\,,T]\,,U)$, by Cauchy-Schwarz inequality:

$$\begin{aligned}\|H_t(w)\| \le& \int_0^t \left\|S(t-s)B\left(\tilde{G}^{-1}(w)(s)\right)\right\| ds \le M\|B\| \int_0^T \|\tilde{G}^{-1}(w)(s)\|\,ds\\ \le& M\|B\|\sqrt{T}\left\|\tilde{G}^{-1}(w)\right\|_2 \le M\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\|w\|\end{aligned}$$

for every $w \in W$ and $t \in [0\,,T]$. By Proposition A.5 (v)

$$\begin{aligned}&\chi_W\left(\left\{\int_0^t P_nS(t-s)B\left(\tilde{G}^{-1}(P_n(p_{y_n}))(s)\right)ds\right\}_n\right)\\ &\le \chi_W\left(\left\{\int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_{y_n}))(s)\right)ds\right\}_n\right) = \chi_W\left(\{H_t(P_n(p_{y_n}))\}_n\right)\\ &\le M\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\chi_W\left(\{(P_n(p_{y_n}))\}_n\right)\\ &\le M\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\left[d + M(d + \|L\|_1)\right]\gamma(\{y_n\}_n).\end{aligned}$$

Therefore

$$\begin{aligned}\chi_W\left(\{y_n(t)\}_n\right) \leq & M d\gamma\left(\{y_n\}_n\right) + M\|L\|_1\gamma\left(\{y_n\}_n\right)\\ & + M\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\left[d + M(d+\|L\|_1)\right]\gamma(\{y_n\}_n)\end{aligned}$$

for every $t \in [0\,,T]$ and then

$$\gamma(\{y_n\}_n) \leq \left[M(d+\|L\|_1)\left(1+M\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\right) + dM\|B\|\sqrt{T}\|\tilde{G}^{-1}\|\right]\gamma(\{y_n\}_n)$$

which, with (3.27), imply $\gamma(\{y_n\}_n) = 0$ so $\{y_n(t)\}_n$ is relatively compact in W for every $t \in [0\,,T]$.

Let us prove that $\{y_n\}_n$ is relatively compact in $\mathcal{C}([0,T],W)$. According to Proposition A.4, it is sufficient to prove that $\{S(\cdot)P_nDy_n\}_n$, $\{F_n\}_n$ and $\{K_n\}_n$ are relatively compact in $\mathcal{C}([0,T],W)$ where

$$F_n(t) = \int_0^t S(t-s)P_nf(s\,,y_n(s))\,ds,$$

and

$$K_n(t) = \int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_np_{y_n})(s)\right)ds.$$

Since $\{y_n(t)\}_n$ is relatively compact for every $t \in [0,T]$, by (3.25) we get that $\{Dy_n\}_n$ is relatively compact in W. It means that, given any subsequence $\{Dy_{n_k}\}_k$, there exists $\overline{y} \in W$ such that, passing to a subsequence still denoted as the sequence, $Dy_{n_k} \to \overline{y}$ and, by (A.2), $P_{n_k}Dy_{n_k} \to \overline{y}$. Hence, by (2.1), we get that

$$\sup_{t\in[0,T]}\|S(t)P_{n_k}D_{n_k}y_{n_k} - S(t)\overline{y}\| \leq M\|P_{n_k}D_{n_k}y_{n_k} - \overline{y}\|,$$

i.e. $\{S(\cdot)P_{n_k}Dy_{n_k}\}_k$ is convergent, thus $\{S(\cdot)P_nDy_n\}_n$ is relatively compact in $\mathcal{C}([0,T],W)$.

Again by the relative compactness of $\{y_n(s)\}_n$ for every $s \in [0\,,T]$ and the continuity of $f(s\,,\cdot)$, $\{f(s\,,y_n(s))\}_n$ is relatively compact for almost every $s \in [0\,,T]$. Moreover by (A.2), also $\{P_nf(s\,,y_n(s))\}_n$ is relatively compact for almost every $s \in [0\,,T]$. Since $\{y_n\}_n \subset Q_{\bar{N}}$ for some $\bar{N} > 0$, by(f_3^2) (see Remark 3.11), $\{f(\cdot\,,y_n(\cdot))\}_n$ and $\{P_nf(\cdot\,,y_n(\cdot))\}_n$ are integrably bounded. Then by Theorem 2.8 we get that the sequences of functions $\{F_n\}_n$ and $\{E_n\}_n$, where

$$E_n(t) = \int_0^t S(t-s)f(s\,,y_n(s))\,ds,$$

are relatively compact in $\mathcal{C}([0,T],W)$. In particular $\{E_n(T)\}_n$ is relatively compact in W.

Consider now $\{K_n\}_n$. Since $\{Dy_n\}_n$ and $\{E_n(T)\}_n$ are relatively compact, then also $\{p_{y_n}\}_n$ is relatively compact in W. By (A.2) and the continuity of $\tilde{G}^{-1}$ we have that the sequence $\{\tilde{G}^{-1}(P_np_{y_n})\}_n$ is relatively compact in $L^2([0,T],U)$. Therefore, since B is bounded, also $\{B\big(\tilde{G}^{-1}(P_np_{y_n})(\cdot)\big)\}_n$ is relatively compact in $L^2([0,T],W)$ and, a fortiori, in $L^1([0,T],W)$. Since, for every $g_1, g_2 \in L^1([0,T],W)$,

$$\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)g_1(s)ds - \int_0^t S(t-s)g_2(s)ds\right\| \leq M\int_0^T\|g_1(s)-g_2(s)\|ds$$

we obtain that every subsequence $\{K_{n_k}\}_k$ admits a subsequence converging in $\mathcal{C}([0,T],W)$, i.e. $\{K_n\}_n$ is relatively compact in $\mathcal{C}([0,T],W)$. Then, according to above considerations, $\{y_n\}_n$ admits a subsequence, still denoted by the sequence, uniformly convergent to a function $y : [0\,,T] \to W$, hence by (3.26) and (A.2)

$$f(t\,,y_n(t)) \to f(t\,,y(t)) \quad\text{and}\quad P_nf(t\,,y_n(t)) \to f(t\,,y(t)) \qquad \text{for a.e. } t \in [0\,,T].$$

Therefore by the dominated convergence theorem, (A.2) and the continuity of D,

$$P_n \int_0^t S(t-s)P_n f(s\,,y_n(s))\,ds \to \int_0^t S(t-s)f(s\,,y(s))\,ds, \qquad \text{for all } t \in [0\,,T],$$

$$Dy_n \to Dy \qquad \text{and} \qquad P_n p_{y_n} \to p_y.$$

Then, by (A.2) and the continuity of H_t, $P_n H_t(P_n(p_{y_n})) \to H_t(p_y)$, that is

$$P_n \int_0^t S(t-s)B\left(\tilde{G}^{-1}(P_n(p_{y_n}))(s)\right)ds \to \int_0^t S(t-s)B\left(\tilde{G}^{-1}(p_y)(s)\right)ds$$

for every $t \in [0\,,T]$. Then the limit function y satisfies

$$y(t) = S(t)[y_0 + Dy] + \int_0^t S(t-s)f(s\,,y(s))\,ds + \int_0^t S(t-s)B\left(\tilde{G}^{-1}(p_y)(s)\right)ds$$

and

$$\begin{aligned} y(T) &= S(T)[y_0 + Dy] \\ &\quad + \int_0^T S(T-s)f(s\,,y(s))\,ds + \int_0^T S(T-s)B\left(\tilde{G}^{-1}(p_y)(s)\right)ds \\ &= S(T)[y_0 + Dy] + \int_0^T S(T-s)f(s\,,y(s))\,ds + G\left(\tilde{G}^{-1}(p_y)\right) \\ &= S(T)[y_0 + Dy] + \int_0^T S(T-s)f(s\,,y(s))\,ds + p_y = y_1 + Dy. \end{aligned}$$

□

Remark 3.12. Notice that we assume condition (D_2) only in order to simplify the notations. Indeed all theorems can be proved under the weaker condition

$$\limsup_{\|q\|_\infty \to \infty} \frac{\|Dq\|}{\|q\|_\infty} = l < \infty$$

with respectively $C_3 l < 1$ in Theorem 3.8 and

$$C_2\|\alpha\|_1 + C_3 l < 1 \tag{3.31}$$

in Theorem 3.9 and Theorem 3.10 (where we recall that $p = 2$ and $\alpha = L$), where

$$C_2 = M\left(1 + M\|B\|T^{1-\frac{1}{p}}\left\|\mathcal{G}^{-1}\right\|\right) \quad \text{and} \quad C_3 = M[1 + \|B\|T^{1-\frac{1}{p}}\|\mathcal{G}^{-1}\|(1+M)].$$

Notice that, in Theorem 3.10, if $d = l$, then (3.27) and (3.31) are the same condition.

Example 3.13. Let D be a linear and bounded operator from $\mathcal{C}([0\,,T]\,,W)$ into W. Then D maps bounded sets into bounded sets and it is continuous both with respect to the strong topology in the domain and in the codomain, and with respect to the weak topology in the domain and in the codomain. Moreover

$$\limsup_{\|q\|_\infty \to \infty} \frac{\|Dq\|}{\|q\|_\infty} = l = \|D\| < \infty.$$

In this case we can choose $d = \|D\|$ and then, by previous remark, Theorem 3.10 holds without additional assumptions. Consider now

$$D_1(q) = \sum_{i=1}^m \beta_i q(t_i), \qquad \text{with } t_1 < t_2 < ... < t_m \text{ and } \beta_1, ..., \beta_m \in \mathbb{R},$$

and

$$D_2(q) = \int_0^T \beta(s)q(s)ds, \qquad \text{with } \beta \in L^1([0\,,T]\,,\mathbb{R}).$$

Let $\{y_n\}_n \subset \mathcal{C}([0\,,T]\,,W)$ be bounded. By Proposition A.5 (iv) and (v) and Theorem A.7, we respectively have

$$\chi_W(\{D_1 y_n\}_n) \leq \sum_{i=1}^{n} |\beta_i| \chi_W(\{y_n(t_i)\}_n) \leq \sum_{i=1}^{n} |\beta_i| \gamma(\{y_n\}_n)$$

and

$$\chi_W(\{D_2 y_n\}_n) \leq \int_0^T |\beta(s)| \chi_W(\{y_n(s)\}_n) ds \leq \|\beta\|_1 \gamma(\{y_n\}_n),$$

where γ is defined in (3.28), implying that they both satisfy (3.25) with d equal to $\sum_{i=1}^m |\beta_i|$ and $\|\beta\|_1$ respectively.

4. p-controllability of a nonlocal transport equation

As an application of the previous results we will prove the p-controllability, $1 < p < \infty$, for a problem of the form

$$\begin{cases} y_t(t\,,x) + a \cdot \nabla y(t,x) \\ \quad = y(t,x) g\left(t, x, \int_{\mathbb{R}^n} k(x,\xi) y(t,\xi) d\xi\right) + b(x) u(t\,,x), & x \in \mathbb{R}^n,\ t \in [0\,,T], \\ y(0,x) = y_0(x) + \sum_{i=1}^m \beta_i y(t_i, x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $a \in \mathbb{R}^n$, $t_1, ..., t_m \in [0\,,T]$, $\beta_1, ..., \beta_m \in \mathbb{R}$ and $y_0 \in L^r(\mathbb{R}^n)$, $1 < r < \infty$.

Equation in (4.1) is a nonlinear version of the known transport equation (see e.g. [12] and [34]) which is still intensely studied because of its several applications such as the transport of particles, the study of traffic flows etc. Notice that such equations are usually considered in $L^r(\mathbb{R}^n)$ with $r \neq 2$, and hence their corresponding abstract formulations are studied in suitable Banach spaces. In some cases (see e.g. [9] and [10]) the nonlinear term contains a nonlocal part as in (4.1).

Using the notation introduced in Section 3, the Banach space $U = W = L^r(\mathbb{R}^n)$. In order to rewrite problem (4.1) in abstract form, we identify y and u respectively with functions $t \mapsto y(t,\cdot)$ and $t \mapsto u(t,\cdot)$. We look for a solution $y \in \mathcal{C}([0\,,T]\,,L^r(\mathbb{R}^n))$ associated to the control $u \in L^p([0\,,T]\,,L^r(\mathbb{R}^n))$ (see Definition 3.7). Notice that p and r may be distinct.

Consider the following assumptions on functions $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g : [0\,,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^n \to \mathbb{R}$:

- (k_1) $k \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n)$;
- (k_2) there exists $\psi \in L^{r'}(\mathbb{R}^n)$ such that $|k(x,\xi)| \leq \psi(\xi)$ for every $x \in \mathbb{R}^n$ and a.e. $\xi \in \mathbb{R}^n$, where $\frac{1}{r} + \frac{1}{r'} = 1$;
- (g_1) $g(\cdot\,,\cdot\,,q) : [0\,,T] \times \mathbb{R}^n \to \mathbb{R}$ is measurable for every $q \in \mathbb{R}$;
- (g_2) $g(t\,,x\,,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $t \in [0\,,T]$ and $x \in \mathbb{R}^n$;
- (g_3) there exists $\alpha \in L^1(0,T)$ such that $|g(t\,,x\,,q)| \leq \alpha(t)$ for a.e. $t \in [0\,,T]$, $x \in \mathbb{R}^n$ and every $q \in \mathbb{R}$;
- (b) b is measurable and there exists $b_1, b_2 > 0$ such that $b_1 \leq |b(x)| \leq b_2$ for a.e. $x \in \mathbb{R}^n$;
- (c) the following estimate is satisfied

$$\left(1 + \frac{b_2}{b_1}\right)\left(\|\alpha\|_1 + \sum_{i=1}^m |\beta_i|\right) < 1 - \frac{b_2}{b_1} \sum_{i=1}^m |\beta_i|.$$

We obtain the following result

Theorem 4.1. *Consider problem* (4.1) *with* $a \in \mathbb{R}^n$, $t_1, ..., t_m \in [0\,,T]$, $\beta_1, ..., \beta_m \in \mathbb{R}$ *and* $y_0 \in L^r(\mathbb{R}^n)$, $1 < r < \infty$. *Assume conditions* (k_1)-(k_2), (g_1)-(g_3), (b) *and* (c). *Then* (4.1) *is p-controllable, with* $1 < p < \infty$.

Proof. In order to apply Theorem 3.9 we have to define A, f, B and D and show that hypothesis (f_1), (f_2), (f_3^2) , (B), (D_1) and (D_2) hold.
The operator $A : D(A) \to L^r(\mathbb{R}^n)$, with

$$D(A) = W^{1,r}(\mathbb{R}^n),$$

is the linear operator $Az = -a \cdot \nabla z$. A is the generator of the $C_0 -$ group of contractions ($M = 1$ in (2.1))

$$S(t)z(x) = z(x - ta) \tag{4.2}$$

and it is clearly not compact ([34, Example 4.4.1 and Theorem 4.4.1]).
The function $f : [0\,, T] \times L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ is

$$f(t, z)(x) = z(x)g\left(t\,, x\,, \int_{\mathbb{R}^n} k(x, \xi)z(\xi)\, d\xi\right).$$

We claim that f is well defined, i.e. that for every $t \in [0\,, T]$ and $z \in L^r(\mathbb{R}^n)$, $f(t\,, z)$ is in $L^r(\mathbb{R}^n)$.
First of all notice that, by (k_2), for every $x \in \mathbb{R}^n$ and for every $z \in L^r(\mathbb{R}^n)$ the function $k(x\,, \cdot)z(\cdot)$ is integrable in $\mathbb{R}^n$. Moreover, given $x_m \to x_0$, from (k_1) and (k_2) it respectively follows that

$$k(x_m, \xi)z(\xi) \to k(x_0, \xi)z(\xi)$$

and

$$|k(x_m, \xi)z(\xi)| \le \psi(\xi)|z(\xi)|$$

for a.e. $\xi \in \mathbb{R}^n$ and every $m \in \mathbb{N}$, thus the dominated convergence theorem implies that the function

$$x \to \int_{\mathbb{R}^n} k(x, \xi)z(\xi)d\xi \tag{4.3}$$

is continuous. Since from (g_1) and (g_2) we obtain that $g(t\,, \cdot\,, \cdot)$ is a Carathéodory function, we can conclude that the function

$$x \to g\left(t, x, \int_{\mathbb{R}^n} k(x, \xi)z(\xi)d\xi\right)$$

is measurable (see Remark 3.11). From (g_3), we get that, for every $z \in L^r(\mathbb{R}^n)$,

$$|f(t\,, z)(x)| \le \alpha(t)|z(x)|, \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for a.e. } t \in \mathbb{R}^n \tag{4.4}$$

and so $f(t\,, z) \in L^r(\mathbb{R}^n)$ and the claim is proved.
Notice that (4.4) also implies the first condition in (f_3^2).
We now prove that, for every $z \in L^r(\mathbb{R}^n)$, the map $f(\cdot\,, z) : [0\,, T] \to L^r(\mathbb{R}^n)$ is measurable. Since $L^r(\mathbb{R}^n)$ is separable it is enough to prove that it is weakly measurable (see Remark 2.5). Since g is a Carathéodory function, from (4.3) we get that, for every $\varphi \in L^{r'}(\mathbb{R}^n)$, the map

$$(t, x) \to \varphi(x)z(x)g\left(t, x, \int_{\mathbb{R}^n} k(x, \xi)z(\xi)d\xi\right) \tag{4.5}$$

is globally measurable (see Remark 3.11). Moreover, from (g_3), for a.e. $t \in [0, T]$ and $x \in \mathbb{R}^n$,

$$\left|\varphi(x)z(x)g\left(t, x, \int_{\mathbb{R}^n} k(x, \xi)z(\xi)d\xi\right)\right| \le \alpha(t)|\varphi(x)||z(x)|,$$

thus the map defined in (4.5) is integrable over $[0, T] \times \mathbb{R}^n$. Therefore by Fubini's theorem the map

$$t \to \langle \varphi\,, f(t\,, z)\rangle = \int_{\mathbb{R}^n} \varphi(x)z(x)g\left(t, x, \int_{\mathbb{R}^n} k(x, \xi)z(\xi)d\xi\right) dx$$

is measurable.

We now claim that for a.e. $t \in [0\,,T]$, the function $f(t\,,\cdot)$ is weakly sequentially continuous in $L^r(\mathbb{R}^n)$. Indeed, let $z_n \rightharpoonup z_0$ in $L^r(\mathbb{R}^n)$, then from (k_2) we have that

$$\int_{\mathbb{R}^n} k(x,\xi)z_n(\xi)\,d\xi \to \int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)\,d\xi$$

for every $x \in \mathbb{R}^n$. Therefore, (g_2) implies that for every given $\varphi \in L^{r'}(\mathbb{R}^n)$

$$\varphi(x)g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_n(\xi)d\xi\right) \to \varphi(x)g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)d\xi\right) \tag{4.6}$$

for a.e. $x \in \mathbb{R}^n$. From (g_3) we also get that for every $z \in L^r(\mathbb{R}^n)$, it holds that

$$\left|\varphi(x)g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z(\xi)d\xi\right)\right| \le \alpha(t)|\varphi(x)|$$

i.e. the function $x \to \varphi(x)g\Big(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)d\xi\Big)$ belongs to $L^{r'}(\mathbb{R}^n)$, moreover, by the dominated convergence theorem, the sequence

$$\psi_n(x) = \varphi(x)\left[g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_n(\xi)d\xi\right) - g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)d\xi\right)\right]$$

converges to 0 in $L^{r'}(\mathbb{R}^n)$.
Hence, from the boundedness of $\{\|z_n\|_r\}_n$, we get that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\varphi(x)\left[z_n(x)g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_n(\xi)d\xi\right) - z_0(x)g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)d\xi\right)\right]dx\right| \\
&\quad\le \left|\int_{\mathbb{R}^n} z_n(x)\psi_n(x)dx\right| + \left|\int_{\mathbb{R}^n}\varphi(x)[z_n(x)-z_0(x)]g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)d\xi\right)dx\right| \\
&\quad\le \|z_n\|_r\|\psi_n\|_{r'} + \left|\int_{\mathbb{R}^n}\varphi(x)[z_n(x)-z_0(x)]g\left(t,x,\int_{\mathbb{R}^n} k(x,\xi)z_0(\xi)d\xi\right)dx\right| \to 0
\end{aligned}$$

obtaining that

$$z_n(\cdot)g\Big(t,\cdot,\int_{\mathbb{R}^n} k(\cdot,\xi)z_n(\xi)d\xi\Big) - z_0(\cdot)g\Big(t,\cdot,\int_{\mathbb{R}^n} k(\cdot,\xi)z_0(\xi)d\xi\Big) \rightharpoonup 0$$

in $L^r(\mathbb{R}^n)$ i.e. the claimed result holds.
The operators $B : L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ and $D : \mathcal{C}([0,T],L^r(\mathbb{R}^n)) \to L^r(\mathbb{R}^n)$ are respectively defined as

$$(Bz)(x) = b(x)z(x)$$

and

$$Dy = \sum_{i=1}^m \beta_i y(t_i).$$

The operator B maps $L^r(\mathbb{R}^n)$ in itself since $b \in L^\infty(\mathbb{R}^n)$. Moreover B is bounded and $\|B\| \le b_2$. Finally, according to (b), B is invertible, $(B^{-1})w(x) = \frac{1}{b(x)}w(x)$ and $\|B^{-1}\| \le \frac{1}{b_1}$.
Now, given $w \in L^r(\mathbb{R}^n)$, consider $v : [0,T] \to L^r(\mathbb{R}^n)$ defined as $v(t) = \frac{1}{T}B^{-1}S(t-T)w$. Clearly $v \in L^p([0,T],L^r(\mathbb{R}^n))$ and

$$\|v\|_p \le \frac{1}{T^{1-\frac{1}{p}}b_1}\|w\|_r.$$

Moreover

$$\begin{aligned}
Gv =& \int_0^T S(T-t)Bv(t)\,dt = \frac{1}{T}\int_0^T S(T-t)BB^{-1}S(t-T)w\,dt \\
=& \frac{1}{T}\int_0^T S(T-t)S(t-T)w\,dt = \frac{1}{T}\int_0^T w\,dt = w,
\end{aligned}$$

by the definition of C_0- group. Therefore G is surjective and

$$\|\mathcal{G}^{-1}\| \le \frac{\|v\|_p}{\|w\|_r} \le \frac{1}{T^{1-\frac{1}{p}}b_1}.$$

According to Example 3.13, since D is a linear and bounded operator, it is weakly continuous and

$$\limsup_{\|y\|_\infty\to\infty} \frac{\|D(y)\|_r}{\|y\|_\infty} = \|D\| = \sum_{i=1}^m |\beta_i|.$$

Hence, by condition (c) also assumption (3.31) of Remark 3.12 is satisfied and the existence of a solution is proved. □

5. 2-controllability of a nonlinear wave equation

In this part we investigate the controllability of the nonlinear wave equation with nonlocal conditions of integral type. Consider

$$\begin{cases} z_{tt}(t\,,x) = & z_{xx}(t\,,x)+g(t,z(t,x))+Bu(t)(x), \quad t\in[0,T],\ x\in[0,1],\\ z(t,0)= & z(t,1)=0, \quad t\in[0,T],\\ z(0,x)= & \int_0^T \eta(t)z(t,x)\,dt+z_0(x), \quad x\in[0,1],\\ z_t(0,x)= & \int_0^T \theta(t)z_t(t,x)\,dt+z_1(x), \quad x\in[0,1], \end{cases} \tag{5.1}$$

with $\eta,\theta\in L^1(0\,,T)$, $z_0\in H_0^1(0\,,T)$, $z_1\in L^2(0\,,1)$. The equation in (5.1) models the vibrating string clipped at the end points $x=0,1$. The nonlinear term $g\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ denotes a feedback control. The equation contains an additional control term given by the bounded linear operator $B\colon U\to L^2(0,1)$ defined in the control Hilbert space U and $u(\cdot)\in L^2([0,T],U)$. The case when $\eta=\theta=0$ and the equation in (5.1) is multivalued, corresponding to a more general feedback control strategy, was discussed in [27]. We claim that the techniques introduced in Section 3, on which we base the present discussion, could be easily extended in order to support a multivalued analysis as in [27]. In problem (5.1) we assume nonlocal integral initial conditions (see also [30]). Alternatively, it is possible to consider multipoint initial conditions as in [16] (see Remark 5.5).
We assume that

- (a) $g(\cdot,q):[0,T]\to\mathbb{R}$ is measurable, for all $q\in\mathbb{R}$ and $g(\cdot,0)\in L^1(0,T)$;
- (b) $|g(t,q_2)-g(t,q_1)|\le L(t)|q_2-q_1|$ for $q_1,q_2\in\mathbb{R}$ and a.a. $t\in[0,T]$ with $L\in L^1(0,T)$;
- (c) the following estimate is satisfied

$$\|L\|_1(1+\Lambda)+d(1+2\Lambda)<1$$

with $\Lambda=\|B\|\sqrt{T}\|\mathcal{G}^{-1}\|$ and $d=\sqrt{\|\eta\|_1{}^2+\|\theta\|_1{}^2}$, G is defined in (5.7), its right inverse $\mathcal{G}^{-1}$ exists and it is linear by Proposition 2.6 *(iv)*;
- (d) B is surjective.

In the following, by a change of variables, we transform (5.1) into the first order problem (5.5) in abstract spaces and apply Theorem 3.10. Condition (d) then aims to guarantee the 2-controllability of the associated linear problem (5.6) (see Proposition (5.2)). We investigate this case in Theorem 5.1.
In controllability of nonlinear wave equations controls are frequently located in an open subset of the state space, i.e.

$$B\colon L^2(\ell_1,\ell_2)\to L^2(0,1), \qquad \text{with } Bh=h\mathbf{1}_{(\ell_1,\ell_2)}, \tag{5.2}$$

where $\mathbf{1}_\omega$ is the characteristic function of the set ω (see [37] and references therein). In this case a constraint on the control time T is required, in order to get the controllability of the associated linear problem (see [36] and also Proposition 5.3). We discuss this case in Theorem 5.4.

Theorem 5.1. *Consider problem* (5.1) *with* $\eta, \theta \in L^1(0\,,T), z_0 \in H_0^1(0\,,T), z_1 \in L^2(0\,,1)$ *and* $B\colon U \to L^2(0,1)$ *bounded and linear with* U *a Hilbert space. Assume conditions* (*a*)-(*d*). *Then* (5.1) *is* 2*-controllable.*

Proof. Problem (5.1) can be written in its abstract form as

$$\begin{cases} \omega''(t) = A\omega(t) + f(t,\omega(t)) + Bu(t), & t \in [0,T] \\ \omega(0) = \displaystyle\int_0^T \eta(t)\omega(t)\,dt + z_0, & \omega'(0) = \displaystyle\int_0^T \theta(t)\omega'(t)\,dt + z_1 \end{cases} \tag{5.3}$$

with $\omega(t) \in L^2(0\,,1)$ for every $t \in [0\,,T]$. The linear term A is the abstract formulation of the Laplace operator with $D(A) = \{\omega \in H^2(0\,,1)\,;\, \omega(0) = \omega(1) = 0\}$. The function $f\colon [0,T] \times L^2(0\,,1) \to L^2(0,1)$ is defined by $f(t,\omega)(x) = g(t,\omega(x))$ for a.e. $x \in (0\,,1)$; f is well-posed by (a) and (b) since $|f(t,\omega)(x)| \le L(t)|\omega(x)| + |g(t,0)|$ for all $\omega \in L^2(0,1)$ and a.e. $t \in [0\,,T]$ and $x \in [0\,,1]$.

Following [34, Section 4.6] we transform (5.3) into a first order system in the Hilbert space $\mathcal{E} := H_0^1(0,1) \times L^2(0,1)$ with the inner product

$$\langle p, q\rangle = \int_0^1 p_0'(x)q_0'(x)\,dx + \int_0^1 p_1(x)q_1(x)\,dx, \tag{5.4}$$

$p = (p_0\,, p_1), q = (q_0\,, q_1) \in \mathcal{E}$.
We set

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, \qquad D(\mathcal{A}) = D(A) \times H_0^1(0\,,1)$$

and consider the functions $\mathcal{F} : [0,T] \times \mathcal{E} \to \mathcal{E}$, $\mathcal{B} : U \to \mathcal{E}$ and $\mathcal{D}\colon \mathcal{C}([0,T],\mathcal{E}) \to \mathcal{E}$ respectively defined by

$$\begin{aligned} \mathcal{F}(t,y) =&\ \mathcal{F}(t,(y_0,y_1)) = \begin{pmatrix} 0 \\ f(t,y_0) \end{pmatrix} \\ \mathcal{B}u =&\ \begin{pmatrix} 0 \\ Bu \end{pmatrix} \\ \mathcal{D}(y) =&\ \mathcal{D}(y_0,y_1) = (D_\eta(y_0)\,, D_\theta(y_1)) \quad \text{where} \\ &\ D_\eta(y_0) = \textstyle\int_0^T \eta(t)y_0(t)\,dt \ \text{ and } \ D_\theta(y_1) = \int_0^T \theta(t)y_1(t)\,dt. \end{aligned}$$

Notice that $\mathcal{B}$ is linear and bounded with $\|\mathcal{B}\| = \|B\|$. It is easy to see that problem (5.3) can be written as

$$\begin{cases} y'(t) = \mathcal{A}y(t) + \mathcal{F}(t,y(t)) + \mathcal{B}u(t), \ t \in [0,T] \\ y(0) = \mathcal{D}(y) + z \end{cases} \tag{5.5}$$

$z = (z_0\,, z_1)$. In [34, Theorem 4.6.2] it is proved that $\mathcal{A}$ is the generator of a C_0-group of contractions on $\mathcal{E}$, $\{\mathcal{S}(t)\}_{t\in\mathbb{R}}$.

Consider the linear problem associated to (5.5)

$$\begin{cases} y'(t) = \mathcal{A}y(t) + \mathcal{B}u(t), & t \in [0,T] \\ y(0) = \bar{y} \end{cases} \tag{5.6}$$

$\bar{y} \in \mathcal{E}$. The following result is proved in [14, Example VI - 8.10].

Proposition 5.2. *If operators* A, B, $\mathcal{A}$ *and* $\mathcal{B}$ *are defined as above, and* B *is surjective, then the linear control system* (5.6) *is* 2*-controllable.*

We prove that problem (5.5) satisfies the assumptions of Theorem 3.10. This implies the 2-controllability of the initial problem (5.1).

First we show condition (B). According to (d), by Proposition 5.2 for every $\hat{y} \in \mathcal{E}$ there exists $\hat{u} \in L^2([0,T],U)$ such that the corresponding solution y to (5.6) satisfies $y(T) = \hat{y}$. This is equivalent to assume that

$$G\colon L^2([0,T],U) \to \mathcal{E}, \qquad u \longmapsto G(u) = \int_0^T \mathcal{S}(T-t)\mathcal{B}u(t)\,dt \tag{5.7}$$

is surjective, hence condition (B) is satisfied.

We consider now the linear operator $\mathcal{D}$ and prove that it is bounded. Let $y = (y_0, y_1)$ in $C([0,T],\,\mathcal{E})$ be such that

$$\|y\|_\infty = \max_{t\in[0,T]} \sqrt{\|y_0(t)'\|_2{}^2 + \|y_1(t)\|_2{}^2} \leq 1.$$

Notice that

$$\left\| \int_0^T \theta(t) y_1(t)\,dt \right\|_2 \leq \int_0^T |\theta(t)| \|y_1(t)\|_2\,dt \leq \max_{t\in[0,T]} \|y_1(t)\|_2 \|\theta\|_1 \leq \|\theta\|_1.$$

In order to obtain a similar estimate for $\int_0^T \eta(t)y_0(t)\,dt$ we need the following claim

$$\left(\int_0^T \eta(t) y_0(t)\,dt \right)' = \int_0^T \eta(t) y_0(t)'\,dt, \quad y = (y_0, y_1) \in C([0,T],\,\mathcal{E}). \tag{5.8}$$

Indeed, by (5.8) we have that

$$\begin{aligned} \left\| \int_0^T \eta(t) y_0(t)\,dt \right\|_{H_0^1} &= \left\| \left(\int_0^T \eta(t) y_0(t)\,dt \right)' \right\|_2 \\ &= \| \int_0^T \eta(t) y_0'(t)\,dt \|_2 \\ &\leq \max_{t\in[0,T]} \|y_0'(t)\|_2 \|\eta\|_1 \leq \|\eta\|_1. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathcal{D}y\|_{\mathcal{E}} &= \sqrt{\left\| \left(\int_0^T \eta(t) y_0(t)\,dt \right)' \right\|_2^2 + \left\| \int_0^T \eta(t) y_1(t)\,dt \right\|_2^2} \\ &\leq \sqrt{\|\eta\|_1{}^2 + \|\theta\|_1{}^2}, \end{aligned} \tag{5.9}$$

implying that $\mathcal{D}$ is bounded, provided that (5.8) is satisfied.

In particular (see Example 3.13) by (5.9) we obtain that

$$\limsup_{\|y\|_\infty \to \infty} \frac{\|\mathcal{D}y\|}{\|y\|_\infty} = \|\mathcal{D}\| \leq \sqrt{\|\eta\|_1{}^2 + \|\theta\|_1{}^2}.$$

Now we prove that, for every $u \in \mathcal{C}([0,T], H_0^1(0\,,1))$,

$$\left(\int_0^T u(t)dt \right)' = \int_0^T u(t)'dt. \tag{5.10}$$

By the definition of Bochner integral, there exists a sequence of simple functions $\varphi_n : [0,T] \to H_0^1(0\,,1)$ such that

$$\int_0^T \|\varphi_n(t)' - u(t)'\|_2\,dt = \int_0^T \|\varphi_n(t) - u(t)\|_{H_0^1}\,dt \to 0.$$

In particular this implies that

$$\int_0^T \varphi_n(t)'\,dt \to \int_0^T u(t)'\,dt \quad \text{in } L^2(0\,,1) \tag{5.11}$$

and

$$\int_0^T \varphi_n(t)\,dt \to \int_0^T u(t)\,dt \quad \text{in } H_0^1(0\,,1)$$

and the latter yields that

$$\left(\int_0^T \varphi_n(t)\,dt\right)' \to \left(\int_0^T u(t)\,dt\right)' \quad \text{in } L^2(0\,,1). \tag{5.12}$$

Since φ_n is a simple function, then for every $n \in \mathbb{N}$

$$\varphi_n(t) = \sum_{i=1}^{m_n} \chi(E_i^n) v_i^n, \quad v_1^n, ..., v_{m_n}^n \in H_0^1(0\,,1) \text{ and } E_1^n, ..., E_{m_n}^n \subset [0\,,T].$$

Therefore

$$(\varphi_n(t))' = \sum_{i=1}^{m_n} \chi(E_i^n)(v_i^n)'.$$

Then (5.10) follows from (5.11), (5.12) and

$$\left(\int_0^T \varphi_n(t)\,dt\right)' = \left(\sum_{i=1}^{m_n} \lambda(E_i^n) v_i^n\right)' = \sum_{i=1}^{m_n} \lambda(E_i^n)(v_i^n)' = \int_0^T \varphi_n(t)'\,dt.$$

Finally (5.8) holds by (5.10) with $u(t) = \eta(t) y_0(t)$.

Since $\mathcal{D}$ is linear and bounded, as in Example 3.13 condition (3.25) is satisfied with

$$d = \sqrt{\|\eta\|_1^2 + \|\theta\|_1^2}.$$

We investigate now the properties of $\mathcal{F}$. Fix $y \in \mathcal{E}$ and consider first $\mathcal{F}(\cdot, y)\colon [0,T] \times \mathcal{E} \to \mathcal{E}$; its measurability can be proved as in previous Section 4, by means of the weak measurability and Fubini's Theorem. Notice, in particular that, since $\mathcal{E}$ is a Hilbert space, $\varphi \in (\mathcal{E})'$ corresponds to $p \in \mathcal{E}$ and $\varphi(y) = \langle p\,, y\rangle$ as defined in (5.4).

Since $\|\mathcal{F}(t, 0)\|_{\mathcal{E}} = |g(t, 0)|$, by condition (a) we obtain that $\mathcal{F}(\cdot\,, 0) \in L^1([0, T], \mathcal{E})$.

In order to show the lipschitzianity condition (3.26) of $\mathcal{F}(t, \cdot)$ fix $t \in [0, T]$ and let $y_1, y_2 \in \mathcal{E}$ with $y_1 = (y_0^1, y_1^1)$ and $y_2 = (y_0^2, y_1^2)$. We obtain that

$$\|\mathcal{F}(t, y_2) - \mathcal{F}(t, y_1)\|_{\mathcal{E}} = \|(0, f(t, y_0^2)) - (0, f(t, y_0^1))\|_{\mathcal{E}} = \|g(t, y_0^2(\cdot)) - g(t, y_0^1(\cdot))\|_2.$$

According to (b) we have that

$$\|\mathcal{F}(t, y_2) - \mathcal{F}(t, y_1)\|_{\mathcal{E}} \le L(t)\|y_0^2 - y_0^1\|_2 \le L(t)\|y_2 - y_1\|_{\mathcal{E}}.$$

Finally, property (3.27) is verified, because it is exactly condition (c).

Notice that condition (D_2) is not satisfied in this case, but, according to Remark 3.12 and Example 3.13, since $d = l = \sqrt{\|\eta\|_1^2 + \|\theta\|_1^2}$ also condition (3.31) is the same condition (c).

All the assumptions of Theorem 3.10 are then satisfied and then problem (5.1) is 2-controllable. □

By comparing the present discussion on problem (5.1) with the related one in [27] we notice that we are able to deal with nonlocal solutions but also that we can avoid any condition on $\chi_U\{G^{-1}(\Omega(t))\}$ (see [27, (C)]) with $\Omega \subset \mathcal{E}$ bounded (see [27, (C)]) which could be very difficult to check.

At last when the map $B\colon L^2(0,1) \to L^2(0,1)$ is defined by $Bz(x) = b(x)z(x)$ with b as in condition (b) of Section 4, then $\|B\| = b_2$.

The following result is proved in [36].

Proposition 5.3. *The linear problem*

$$\begin{cases} z_{tt}(t\,,x) = & z_{xx}(t\,,x) + u(t,x)\mathbf{1}_{(\ell_1,\ell_2)}, \quad t \in [0,T], \; x \in [0,1], \\ z(t,0) = & z(t,1) = 0, \quad t \in [0,T], \\ z(0,x) = & z_0(x) \text{ and } z_1(x), \quad x \in [0,1], \end{cases} \tag{5.13}$$

with $z_0 \in H_0^1(0\,,T)$*,* $z_1 \in L^2(0\,,1)$ *and* $u \in L^2((\ell_1,\ell_2) \times [0,T])$ *is* 2*-controllable in time* $T > 2\max\{\ell_1, 1-\ell_2\}$*.*

We remark that the controllability of (5.13) and of the corresponding first order system as in (5.6) are equivalent. Therefore, the following result can be proved, by means of a similar reasoning as before.

Theorem 5.4. *Consider problem* (5.1) *with* $\eta,\theta \in L^1(0\,,T)$*,* $z_0 \in H_0^1(0\,,T)$*,* $z_1 \in L^2(0\,,1)$ *and let* B *as in* (5.2)*. Assume (a)-(c). Then* (5.1) *is* 2*-controllable for* $T > 2\max\{\ell_1, 1-\ell_2\}$*.*

The result in Theorem 5.4 can be compared with [37, Theorem 1] where $\eta = \theta = 0$ and $g = g(z)$ is a C^1 function but with less restrictive growth conditions at infinity than in (b).

Remark 5.5. In problem (5.1) we can also replace the integral initial condition with a multipoint condition as in the previous section:

$$\begin{cases} z(0,x) = & \sum_{i=1}^m \alpha_i z(t_i,x), \quad x \in [0,1] \\ z_t(0,x) = & \sum_{i=1}^m \beta_i z_t(t_i,x), \quad x \in [0,1] \end{cases}$$

$t_1, ..., t_m \in [0\,,T]$, $\alpha_1, \ldots \alpha_m, \beta_1, \ldots, \beta_m \in \mathbb{R}$. In this case

$$\mathcal{D}(y_0\,,y_1) = \left(\sum_{i=1}^m \alpha_i y_0(t_i)\,, \sum_{i=1}^m \beta_i y_1(t_i) \right).$$

Appendix A

In this appendix we recall some standard definitions and properties about Schauder basis and measures of non-compactness. The last result, instead, deals with the weak convergence in the space of continuous functions.

Definition A.1. A sequence $\{e_n\}_n$ of vectors in a Banach space W is a *Schauder basis* for W if for every $x \in W$ there exists a unique sequence of real numbers $\alpha_n = \alpha_n(x)$, $n \in \mathbb{N}$ such that

$$\left\| x - \sum_{i=1}^n \alpha_i e_i \right\| \to 0, \quad \text{as } n \to \infty.$$

Given a Schauder basis $\{e_n\}_n$ for W, every $\alpha_n : W \to \mathbb{R}$ is linear.

The following three propositions contains some useful properties of the sequence $\{P_n\}_n$ of the natural projections (see Section 2). We refer to [22, Proposition 1.a.2, Proposition 1.b.1 and Theorem 1.b.5] for the proof of Proposition A.2.

Proposition A.2. *Let* W *be a Banach space with a basis* $\{e_n\}_n$*,* $\{\alpha_n\}_n$ *be the associated sequence of functionals, and* $\{P_n\}_n$ *be the sequence of the natural projections. Then*

(i) *there exists* $K \geq 1$ *such that*

$$\|P_n(x)\| \leq K\|x\|$$

for all $n \in \mathbb{N}$ *and* $x \in W$*;*

(ii) *for every* $x \in W$, $\|P_n(x) - x\| \to 0$ *as* $n \to \infty$*;*
(iii) *for every* $n \in \mathbb{N}$, $\alpha_n \in W^*$*;*
(iv) *if* W *is reflexive, then* $\{\alpha_n\}_{n\in\mathbb{N}}$ *is a Schauder basis for* W^* *and*

$$\phi = \sum_{n=1}^{\infty} \phi(e_n)\alpha_n$$

for all $\phi \in W^*$*;*
(v) *if* W *is reflexive, then the operator* $P_n^* : W^* \to W^*$ *defined by* $P_n^*(\phi)(x) = \phi(P_n(x))$*, for every* $\phi \in W^*$ *and* $x \in W$*, is the natural projection of* W^* *in* $W_n^* = \operatorname{span}\{\alpha_1, \dots, \alpha_n\}$*.*

When in (i) of the previous proposition $K = 1$ the Schauder basis $\{e_n\}_n$ is said *monotone*. This means that for a monotone Schauder basis $\|P_n\| = 1$ for every $n \in \mathbb{N}$.
Trivially, if W is a separable Hilbert space every orthonormal system of W is a monotone Schauder basis.

Proposition A.3. *Let* W *and* P_n *be as in Proposition A.2, and let* $\{x_k\}_k$ *be a sequence in* W*. The following properties then hold.*

$$\text{If } x_k \rightharpoonup x \text{ then } P_n(x_k) \to P_n(x), \text{ as } k \to \infty, \text{ for every } n \in \mathbb{N}, \tag{A.1}$$

and

$$\text{if } x_k \to x \text{ then } P_k(x_k) \to x, \text{ as } k \to \infty. \tag{A.2}$$

Moreover,

$$\text{if } W \text{ is reflexive and } x_k \rightharpoonup x \text{ then } P_k(x_k) \rightharpoonup x, \text{ as } k \to \infty. \tag{A.3}$$

Proof. (A.1) is an immediate consequence of (iii) in the previous proposition.
Since

$$\|P_k(x_k) - x\| \le \|P_k(x_k) - P_k(x)\| + \|P_k(x) - x\| \le K\|x_k - x\| + \|P_k(x) - x\|,$$

(A.2) follows from (ii) in the previous proposition.
As to (A.3), let $\{x_k\}_k$ be a sequence weakly convergent in W. For every fixed $\phi \in W^*$, we have to prove that $\lim_{k\to\infty} \phi(P_k(x_k)) = \phi(x)$. By (ii) in Proposition A.2 applied to P_k^* (defined in (v)),

$$P_k^*(\phi) \to \phi \quad \text{strongly in } W^*,$$

then

$$\begin{aligned} |\phi(P_k(x_k)) - \phi(x)| &= |P_k^*(\phi)(x_k) - \phi(x)| \\ &\le |P_k^*(\phi)(x_k) - \phi(x_k)| + |\phi(x_k) - \phi(x)| \\ &\le \|P_k^*(\phi) - \phi\|\|x_k\| + |\phi(x_k) - \phi(x)| \to 0 \end{aligned}$$

as $k \to \infty$. □

Proposition A.4. *Let* W *and* P_n *be as in Proposition A.2. Let* $\{q_n\}_n$ *be a sequence converging to* q *in* $\mathcal{C}([0,T], W)$*. Then the sequence* $\{P_n q_n\}_n$ *converges to* q *in* $\mathcal{C}([0,T], W)$*.*

Proof. Since $\{q_n\}_n$ is uniformly convergent it is equicontinuous and $q_n(t) \to q(t)$ for every $t \in [0,T]$. Therefore, by (A.2), $P_n q_n(t) \to q(t)$ for every $t \in [0,T]$. Moreover, by Proposition A.2 (i), $\{P_n q_n\}_n$ is equicontinuous, in fact

$$\|P_n q_n(t_1) - P_n q_n(t_2)\| \le K\|q_n(t_1) - q_n(t_2)\| \qquad \text{for every } t_1, t_2 \in [0,T].$$

Therefore, by Ascoli-Arzelá Theorem, every sequence $\{P_{n_k} q_{n_k}\}_k$ has a subsequence which converges to q in $\mathcal{C}([0,T], W)$ and this proves the statement. □

The following two propositions and Theorem A.7 deal with some useful properties of the Hausdorff m.n.c. χ_W on the Banach space W introduced in Section 2. The proof of Proposition A.5 easily follows by the definition of m.n.c.

Proposition A.5. *If $\chi_W : \mathcal{P}(W) \to [0\,,+\infty]$ is the Hausdorff m.n.c. defined above then*

(i) $\chi_W(C) = 0$ *if and only if C is relatively compact;*
(ii) *if $C_1 \subset C_2 \subset W$ then $\chi_W(C_1) \le \chi_W(C_2)$;*
(iii) *for every $C_1, C_2 \subset W$, $\chi_W(C_1 \cup C_2) \le \max\{\chi_W(C_1)\,,\chi_W(C_2)\}$;*
(iv) *for every $C_1, C_2 \subset W$, $\chi_W(C_1 + C_2) \le \chi_W(C_1) + \chi_W(C_2)$;*
(v) *if V is a Banach space and $\Phi : W \to V$ is a Lipschitz function with constant L, then for every $C \subseteq W$, $\chi_V(\Phi(C)) \le L\chi_W(C)$.*

Proposition A.6. *Let W be a Banach space with a monotone Schauder basis and $\{P_n\}_n$ be the sequence of the natural projections. Then for every sequence $\{w_n\}_n$ in W*

$$\chi_W(\{P_n w_n\}_n) \le \chi_W(\{w_n\}_n).$$

Proof. If $\{w_n\}_n$ is unbounded the inequality is trivial.
Otherwise, given $\eta > \chi_W(\{w_n\}_n)$ we shall prove that $\chi_W(\{P_n w_n\}_n) < \eta$. By the definition of χ_W, there exists $0 < \epsilon < \eta$ and $x_1, \ldots, x_k \subset W$ such that

$$\{w_n\}_n \subset \bigcup_{i=1}^{k} B_\epsilon(x_i).$$

By (ii) of Proposition A.2 there exists $\bar{n} \in \mathbb{N}$ such that $\|x_i - P_n x_i\| < \eta - \epsilon$ for every $n \ge \bar{n}$ and $i = 1, \ldots, k$. Therefore for every $n \ge \bar{n}$ given $i \in \{1, \ldots, k\}$ such that $\|w_n - x_i\| < \epsilon$ we have

$$\|P_n w_n - x_i\| \le \|P_n w_n - P_n x_i\| + \|P_n x_i - x_i\| < \|w_n - x_i\| + \eta - \epsilon < \eta$$

and then

$$\{P_n w_n\}_n \subset \bigcup_{i=1}^{k} B_\eta(x_i) \cup \bigcup_{m=1}^{\bar{n}} B_\eta(P_m w_m).$$

□

Theorem A.7. [19, Theorem 4.1.2] *Let $\{S(t)\}_{t\ge 0}$ be a C_0-semigroup and F be the linear operator* (2.2)*. Suppose that the sequence $\{f_n\}_n \subset L^1([0\,,T]\,,W)$ is integrably bounded and there exists $q \in L^1(0\,,T)$ such that*

$$\chi_W(\{f_n(t)\}_n) \le q(t), \qquad \text{for a.e. } t \in [0\,,T].$$

Then

$$\chi_W(\{F(f_n)(t)\}_n) \le M \int_0^t q(s)\,ds, \qquad \text{for every } t \in [0\,,T],$$

where M is the constant in (2.1)*.*

At last, in the following proposition we deal with the weak convergence in the space of continuous functions.

Proposition A.8. *(see* [5, Theorem 4.3]*) Let W be a Banach space. A sequence $\{y_n\}_n \subset \mathcal{C}([0, T]\,,W)$ weakly converges to an element $y \in \mathcal{C}([0, T]\,,W)$ if and only if*

(1) *there exists $N > 0$ such that, for every $n \in \mathbb{N}$ and $t \in [0, T]$, $\|y_n(t)\| \le N$;*
(2) *for every $t \in [0, T]$, $y_n(t) \rightharpoonup y(t)$.*

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