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Nonlocal Mixed Systems With Neumann Boundary Conditions

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ABSTRACT

We prove well posedness and stability in $\mathbf{L}^1$ for a class of mixed hyperbolic–parabolic nonlinear and nonlocal equations in a bounded domain with *no flow* along the boundary. While the treatment of boundary conditions for the hyperbolic equation is standard, the extension to $\mathbf{L}^1$ of classical results about parabolic equations with Neumann conditions is here achieved.

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1 | Introduction

We consider the following nonlinear and nonlocal problem on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \partial_t u + \nabla \cdot (u\, v(t, w(t))) = \alpha(t, x, w(t))u + a(t, x), \\ \partial_t w - \mu\, \Delta w = \beta(t, x, u(t), w(t))w + b(t, x), \end{cases} \quad (t, x) \in [0, T] \times \Omega \tag{1.1}$$

When $n = 2, 3$, this mixed system is motivated by a variety of predator–prey models. Indeed, for instance, $u = u(t, x)$ can be the density of a population that *chases* the other population $w = w(t, x)$. This chase is described through the nonlocal operator v, able to model the movement of u towards regions where the concentration of w is higher. A change in the sign of v allows to model the case where the population u *escapes* from w. The w population diffuses isotropically in all directions. The source terms α and β account for natality, mortality or predation, while a and b may describe controls acting on the system, which consist in the introduction of the two species at desired times and locations.

We equip problem (1.1) with the following initial data and conditions at the boundary:

$$\begin{aligned} &\begin{cases} u(0, x) = u_0(x) \\ w(0, x) = w_0(x) \end{cases} && x \in \Omega, \\ &\begin{cases} u(t, \xi) = 0 \\ \nabla w(t, \xi) \cdot \nu(\xi) = 0 \end{cases} && (t, \xi) \in]0, T[\times \partial\Omega. \end{aligned} \tag{1.2}$$

The present choice of Neumann boundary conditions for the parabolic equation is motivated by the *no flow* requirement typically suited to the physical setting considered. Recall that boundary conditions for the hyperbolic equation bear an entirely different meaning: where characteristics exit the domain, boundary conditions have to be essentially neglected to prevent that the hyperbolic problem becomes over determined. As a consequence, the first boundary condition in (1.2) does not prevent u from exiting Ω, but it ensures that no u enters the domain Ω through its boundary. Refer for instance to [1, § 4.8.1] for basic information about characteristics, or also to [2–6] for the specific treatment of boundary conditions in hyperbolic equations.

The analytical treatment of (1.1) relies on that of the separate hyperbolic and parabolic equations

$$\begin{aligned} \partial_t u + \nabla \cdot (u\, c(t,x)) &= A(t,x)\, u + a(t,x) \quad \text{and} \\ \partial_t w - \mu\, \Delta w &= B(t,x)\, w + b(t,x) \end{aligned} \tag{1.3}$$

together with the initial and boundary conditions (1.2). While general well posedness results related to the former equation are available in the literature, the treatment in $\mathbf{L}^1$ of the latter equation with Neumann boundary conditions has received far less attention and is here presented. Recall that system (1.1) equipped with homogeneous Dirichlet boundary conditions for both equations is considered in [7]. Indeed, as the classical books [8, 9], the literature offers a variety of results in $\mathbf{L}^2$, a choice which is hardly justifiable in the present physical setting. Here, on the contrary, the $\mathbf{L}^1$ norm has a clear meaning but $\mathbf{L}^1$ stability estimates were not available, not even in [8, 9], especially in the case of Neumann boundary conditions. Also, the general mixed hyperbolic–parabolic setting in [10] does not comprise the well posedness of (1.1). Therefore, we provide a definition of weak solution in $\mathbf{L}^1$ to the parabolic equation, and correspondingly, we develop a well posedness and stability theory in the $\mathbf{L}^1$ norm. In this procedure, differently from typical results in the literature, the many properties that follow from reflexivity can not be used. It is then remarkable that the weak completeness of $\mathbf{L}^1$ plays a key role in our treatment of the parabolic equation.

Note that in (1.1) the dependence of v, α and β on $u(t)$ and $w(t)$ is of a *functional* nature, allowing for *nonlocal* dependencies. Indeed, $u(t)$ and $w(t)$ in (1.1) denote the *functions* $x \mapsto u(t,x)$ and $x \mapsto w(t,x)$, both defined on all Ω.

Systems of this form arise, for instance, in predator–prey models [11] and can be used in the control of parasites, see [12, 13]. A similar mixed hyperbolic–parabolic system is considered, in one space dimension, in [14], where Euler equations substitute the balance law in (1.1). A mixed ODE—parabolic PDE predator–prey model with Neumann boundary conditions is presented in [15]: Here, the predators' movement is the superposition of a directed hunting and a random dispersion. The present model (1.1)–(1.2) is applicable also to the setting of pursuit-evasion games, similarly to [16]. Here, however, the movement of the pursuer is not purely diffusive, but it is directed towards the average gradient of the evaders' density.

A mixed hyperbolic–parabolic system motivated by population dynamics is considered in [17], where the theoretical framework is set in $\mathbf{L}^2$ and the theory of m-accretive operators is the key analytic tool. A more applied result is [18], where the description of an aneurysm leads to a mixed hyperbolic–parabolic system in 1 space dimension. Global classical solutions to a parabolic predator–prey system, under Neumann boundary conditions, are exhibited in [19], motivated by the dynamics of competing populations with repulsive chemotaxis. In the $\mathbf{L}^2$ framework, local in time well posedness of a mixed hyperbolic–parabolic system is obtained in [20], by means of a Cauchy sequence of approximate solutions. A mixed elliptic–parabolic problem also with biological motivations is studied in [21].

The next section presents the main analytical results. System (1.1) is split in the 2 Equations 1.3. The former one is dealt with by means of results mainly coming from the literature, see Section 2.1. On the contrary, in Section 2.2, we present results on the parabolic part developed *ad hoc* in $\mathbf{L}^1$ for the purposes of the present work. Finally, all proofs are deferred to Section 3.

2 | Main Results

Throughout, the following notation is used. $\mathbb{R}_+ = [0, +\infty[$. If $E \subseteq \mathbb{R}^n$, the characteristic function χ_E is defined by $\chi_E(x) = 1$ if and only if $x \in E$ and $\chi_E(x) = 0$ if and only if $x \in \mathbb{R}^n \setminus E$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B(x_0, r)$ is the open sphere centered at x_0 with radius r. For a function $u : \mathbb{R}^n \to \mathbb{R}$, we denote by $\mathrm{TV}(u)$ its total variation, see [22, definition 1.1 and definition 1.3]. For a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $D_x u$ and $D_x^2 u$ denote the first and second total derivatives of the map $x \mapsto u(t,x)$.

Fix $T > 0$. We pose the following assumptions on Ω and on the functions appearing in problem (1.1):

(Ω) Ω is a nonempty, bounded and connected open subset of $\mathbb{R}^n$, with $\mathbf{C}^{1,\gamma}$ boundary, for a $\gamma \in]0,1]$.

($\boldsymbol{v}$) $v : [0,T] \times \mathbf{L}^1(\Omega;\mathbb{R}) \to (\mathbf{C}^2 \cap \mathbf{W}^{1,\infty})(\Omega;\mathbb{R}^n)$ is such that for a constant $K_v > 0$ and for a map $k_v \in \mathbf{L}^\infty_{\mathbf{loc}}([0,T] \times \mathbb{R}_+;\mathbb{R}_+)$ non decreasing in each argument, for all $t, t_1, t_2 \in [0,T]$ and $w, w_1, w_2 \in \mathbf{L}^1(\Omega;\mathbb{R})$,

$$\begin{aligned} &\|v(t,w)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \leq K_v\, \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}; \\ &\left\|D_x v(t,w)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \leq K_v\, \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}; \\ &\left\|v(t_1,w_1) - v(t_2,w_2)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \\ &\leq K_v\left(|t_1 - t_2| + \|w_1 - w_2\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right); \\ &\left\|D_x^2 v(t,w)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^{n\times n})} \\ &\leq k_v\left(t, \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right) \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}; \\ &\left\|\nabla \cdot \left(v(t_1,w_1) - v(t_2,w_2)\right)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \\ &\leq k_v\left(t, \max_{i=1,2} \|w_i\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right) \|w_1 - w_2\|_{\mathbf{L}^1(\Omega;\mathbb{R})}. \end{aligned}$$

($\boldsymbol{\alpha}$) $\alpha : [0,T] \times \Omega \times \mathbf{W}^{1,1}(\Omega;\mathbb{R}) \to \mathbb{R}$ admits a constant $K_\alpha > 0$ such that for a.e. $t \in [0,T]$, for a.e. $x \in \Omega$ and all $w_1, w_2 \in \mathbf{W}^{1,1}(\Omega;\mathbb{R})$,

$$\sup_{x \in \Omega} |\alpha(t,x,w_1) - \alpha(t,x,w_2)| \leq K_\alpha\, \|w_1 - w_2\|_{\mathbf{L}^1(\Omega;\mathbb{R})}$$

and there exists $k_\alpha \in \mathbf{L}^1([0,T];\mathbb{R}_+)$ such that for all $w \in \mathbf{W}^{1,1}(\Omega;\mathbb{R})$ and a.e. $t \in [0,T]$,

$$\begin{aligned} \mathrm{TV}(\alpha(t,\cdot,w)) &\leq K_\alpha\left(1 + \|\nabla w\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\right) \\ \operatorname*{ess\,sup}_{x\in\Omega} |\alpha(t,x,w)| &\leq k_\alpha(t)\left(1 + \|w\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right). \end{aligned}$$

($\boldsymbol{a}$) $a \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$ and $t \mapsto \mathrm{TV}(a(t))$ is in $\mathbf{L}^1([0,T];\mathbb{R})$.

($\boldsymbol{\beta}$) $\beta : [0,T] \times \Omega \times \mathbf{L}^1(\Omega;\mathbb{R}) \times \mathbf{L}^1(\Omega;\mathbb{R}) \to \mathbb{R}$ admits a constant $K_\beta > 0$ and a $k_\beta \in \mathbf{L}^1([0,T];\mathbb{R}_+)$ such that for a.e. $t \in [0,T]$, all $x \in \Omega$ and $u, u_1, u_2, w, w_1, w_2 \in \mathbf{L}^1(\Omega;\mathbb{R})$,

$$\begin{aligned}&\left|\beta(t,x,u_1,w_1) - \beta(t,x,u_2,w_2)\right| \\ &\le K_\beta\left(\left\|u_1-u_2\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|w_1-w_2\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right); \\ &\operatorname*{esssup}_{x\in\Omega}|\beta(t,x,u,w)| \le k_\beta(t)\,.\end{aligned}$$

($\boldsymbol{b}$) $b \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$.

Because (1.1) is the coupling of a hyperbolic and a parabolic problem, the following definition of solution to (1.1) is a sort of gluing of the definitions of solutions to the equations in (1.3):

Definition 2.1. A pair $(u,w) \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R}^2)\right)$ is a solution to problem (1.1)–(1.2) if, setting

$$\begin{aligned}c(t,x) &= v(t,w(t))(x), \\ A(t,x) &= \alpha(t,x,w(t)), \\ B(t,x) &= \beta(t,x,u(t),w(t))\,,\end{aligned}$$

the function u solves, according to Definition 2.3, the problem

$$\begin{cases}\partial_t u + \nabla\cdot(u\,c(t,x)) = A(t,x)u + a(t,x) & (t,x)\in[0,T]\times\Omega \\ u(t,\xi) = 0 & (t,\xi)\in]0,T[\times\partial\Omega \\ u(0,x) = u_0(x) & x\in\Omega\end{cases} \tag{2.1}$$

and the function w solves, according to Definition 2.5, the problem

$$\begin{cases}\partial_t w - \mu\,\Delta w = B(t,x)w + b(t,x) & (t,x)\in[0,T]\times\Omega \\ \nabla w(t,\xi)\cdot\nu(\xi) = 0 & (t,\xi)\in]0,T[\times\partial\Omega \\ w(0,x) = w_0(x) & x\in\Omega.\end{cases} \tag{2.2}$$

We now state the main result of this work: It ensures the well posedness in $\mathbf{L}^1$ of (1.1) and provides stability estimates to be used, for instance, in control problems based on (1.1).

Theorem 2.2. *Fix $T > 0$. Assume that (Ω), $(\boldsymbol{v})$, $(\boldsymbol{\alpha})$, $(\boldsymbol{a})$, $(\boldsymbol{\beta})$ and $(\boldsymbol{b})$ hold. Let $(u_0,w_0) \in (\mathbf{L}^1\cap\mathbf{BV})(\Omega;\mathbb{R})\times\mathbf{L}^1(\Omega;\mathbb{R})$. Then:*

(M1) Problem (1.1) admits a unique solution (u,w) in the sense of Definition 2.1.

(M2) The map $t\mapsto(u,w)(t)$ is in $\mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R}^2)\right)$.

(M3) If $(u_0^1,w_0^1),(u_0^2,w_0^2)\in(\mathbf{L}^1\cap\mathbf{BV})(\Omega;\mathbb{R})\times\mathbf{L}^1(\Omega;\mathbb{R})$, the corresponding solutions (u_1,w_1) and (u_2,w_2) satisfy the estimate:

$$\begin{aligned}&\left\|(u_1,w_1)(t)-(u_2,w_2)(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^2)} \\ &\le C(t)\,\left\|(u_0^1,w_0^1)-(u_0^2,w_0^2)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^2)}\end{aligned}$$

where $C\in\mathbf{L}^\infty([0,T];\mathbb{R}_+)$ depends on μ, (Ω), $(\boldsymbol{v})$, $(\boldsymbol{\alpha})$, $(\boldsymbol{\beta})$, $(\boldsymbol{a})$, $(\boldsymbol{b})$ and on $\mathrm{TV}(u_0^1)$, $\mathrm{TV}(u_0^2)$, $\left\|(u_0^1,w_0^1)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^2)}$, $\left\|(u_0^2,w_0^2)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^2)}$.

(M4) If a_1,a_2 satisfy $(\boldsymbol{a})$ and b_1,b_2 satisfy $(\boldsymbol{b})$, the corresponding solutions (u_1,w_1) and (u_2,w_2) satisfy the estimate:

$$\begin{aligned}&\left\|(u_1,w_1)(t)-(u_2,w_2)(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^2)} \\ &\le C(t)\left(\left\|a_1-a_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}+\left\|b_1-b_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right)\end{aligned}$$

where $C\in\mathbf{L}^\infty([0,T];\mathbb{R}_+)$ depends on μ, (Ω), $(\boldsymbol{v})$, $(\boldsymbol{\alpha})$, $(\boldsymbol{\beta})$, $(\boldsymbol{a})$, $(\boldsymbol{b})$ and on $\mathrm{TV}(u_0)$, $\left\|(u_0,w_0)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^2)}$.

(M5) If k_β in $(\boldsymbol{\beta})$ is bounded and $u_0\ge0$, $w_0\ge0$, $a\ge0$ and $b\ge0$, then for a.e. $t\in[0,T]$, the solution (u,w) satisfies $u(t)\ge0$ and $w(t)\ge0$.

Remark that if the various assumptions hold on the time interval $\mathbb{R}_+$, then Theorem 2.2 ensures well posedness on $\mathbb{R}_+$, with the function C appearing in (M3) and (M4) belonging to $\mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+,\mathbb{R}_+)$.

To deal with nonlocal operators on a bounded domain Ω, the modified convolution introduced in [23, § 3] is an adequate tool. For a function $\rho\in\mathbf{L}^1(\Omega;\mathbb{R})$ and a smooth kernel η, it reads

$$(\rho *_\Omega \eta)(x) = \frac{\int_\Omega \rho(y)\,\eta(x-y)\mathrm{d}y}{\int_\Omega \eta(x-y)\mathrm{d}y} \tag{2.3}$$

The quantity $(\rho\ *_\Omega\ \eta)(x)$ is an average of the values attained by ρ in Ω around x as soon as the kernel η satisfies, for instance,

($\boldsymbol{\eta}$) $\eta(x)=\tilde\eta(\|x\|)$, where $\tilde\eta\in\mathbf{C}^3(\mathbb{R}_+;\mathbb{R})$, $\mathrm{spt}\tilde\eta=[0,\ell]$, $\ell>0$, $\tilde\eta'\le0$, $\tilde\eta'(0)=\tilde\eta''(0)=0$ and $\int_{\mathbb{R}^n}\eta(\xi)\mathrm{d}\xi=1$.

It is often reasonable to assume that u hunts w moving towards areas with higher density of w, or else that u escapes from w towards regions with lower w density. Thus, v is parallel to the average gradient of w in Ω, such as

$$v(t,w) = k(t)\,\frac{\nabla(w\ *_\Omega\ \eta)}{\sqrt{1+\left\|\nabla(w *_\Omega \eta)\right\|^2}} \tag{2.4}$$

where, for instance,

$$\eta(x) = \overline{\eta}\,\left(1-(\|x\|/\ell)^4\right)^4\,\chi_{B(0,\ell)}(x) \tag{2.5}$$

Here, ℓ has the clear physical meaning of the distance, or *horizon*, at which individuals of the u population *feel* the presence of the w population. The normalization parameter $\overline{\eta}$ is chosen so that $\int_{\mathbb{R}^n}\eta(x)\mathrm{d}x=1$. A choice like (2.4) is consistent with the requirements $(\boldsymbol{v})$, as proved in [23, lemma 3.2].

2.1 | The Hyperbolic Problem

We focus on the hyperbolic problem (2.1). For completeness, we present the standard definition of solution and a detailed well posedness result based on the current literature. Precise references are provided in Section 3.1.

Definition 2.3. A function $u \in \mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$ is a solution to (2.1) if for any test function $\phi \in \mathbf{C}^1_c(]-\infty,T[\times\Omega;\mathbb{R})$,

$$\int_0^T \int_\Omega \left(u\,(\partial_t\phi + c\cdot\nabla\phi) + (A\,u + a)\phi\right)\mathrm{d}x\mathrm{d}t + \int_\Omega u_0(x)\,\phi(0,x)\mathrm{d}x = 0. \tag{2.6}$$

Theorem 2.4. *Fix $T>0$. Assume ($\boldsymbol{a}$) and*

(c) $c \in \left(\mathbf{C}^0\cap\mathbf{L}^\infty\right)([0,T]\times\Omega;\mathbb{R}^n)$, $c(t)\in\mathbf{C}^1(\Omega;\mathbb{R}^n)$ for all $t\in[0,T]$, $D_x c\in\mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R}^{n\times n})$.

(A) $A\in\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$ and $t\mapsto \mathrm{TV}(A(t))$ is in $\mathbf{L}^1([0,T];\mathbb{R})$.

Then, for all $u_0\in\mathbf{L}^1(\Omega;\mathbb{R})$, problem (2.1) admits a unique solution in the sense of Definition 2.3. Moreover,

(H1) *For all $t\in[0,T]$,*

$$\|u(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \left(\|u_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \times \exp\left(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right). \tag{2.7}$$

(H2) *For all $t\in[0,T]$, if $u_0\in\mathbf{L}^\infty(\Omega;\mathbb{R})$,*

$$\|u(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \le \left(\|u_0\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \times \exp\left(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} + \|\nabla\cdot c\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right). \tag{2.8}$$

(H3) *If $u_0\ge 0$ and $a\ge 0$, then $u\ge 0$.*

(H4) *If $u_0^1,u_0^2\in\mathbf{L}^1(\Omega;\mathbb{R})$, then the corresponding solutions u_1,u_2 satisfy for all $t\in[0,T]$,*

$$\left\|u_1(t)-u_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le \left\|u_0^1-u_0^2\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \exp\left(\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right). \tag{2.9}$$

(H5) *If A_1,A_2 satisfy* (A) *and a_1,a_2 satisfy ($\boldsymbol{a}$), the corresponding solutions u_1 and u_2 satisfy for all $t\in[0,T]$*

$$\begin{aligned}&\left\|u_1(t)-u_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\le \exp\left(\max\left\{\|A_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))},\ \|A_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right\}\right)\\ &\quad\times\left(\|u_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|a_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\\ &\quad\times\|A_2-A_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\\ &\quad+\exp\left(\|A_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\|a_1-a_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}.\end{aligned}$$

(H6) *If $u_0\in\mathbf{BV}(\Omega;\mathbb{R})$, $c(t)\in\mathbf{C}^2(\Omega;\mathbb{R}^n)$ for all $t\in[0,T]$ and $\nabla\nabla\cdot c\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$, then the map $t\to u(t)$ is locally Lipschitz continuous in $\mathbf{L}^1(\Omega;\mathbb{R})$.*

(H7) *If $u_0\in\mathbf{BV}(\Omega;\mathbb{R})$, c_1,c_2 satisfy* (c) *and $c_1(t), c_2(t)\in\mathbf{C}^2(\Omega;\mathbb{R}^n)$ for all $t\in[0,T]$ and $\nabla\nabla\cdot c_1,\nabla\nabla\cdot c_2\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$, then the corresponding solutions u_1, u_2 satisfy*

$$\begin{aligned}&\left\|u_2(t)-u_1(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\le C\Bigg(\|D_x c_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^{n\times n}))}, \|\nabla\nabla\cdot c_1\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)},\\ &\quad\|A\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}, \int_0^t \mathrm{TV}A(\tau)\mathrm{d}\tau\Bigg)\\ &\quad\times\left(\|c_1-c_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))} + \|\nabla\cdot(c_1-c_2)\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\end{aligned}$$

where C also depends on $\|a\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}$, $\int_0^t \mathrm{TV}\,(a(\tau))\mathrm{d}\tau$ and $\|u_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})}$, $\|u_0\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}$, $\mathrm{TV}(u_0)$.

The proof is based on results from the literature detailed in Section 3.1.

2.2 | The Parabolic Problem

We focus on the $\mathbf{L}^1$ well posedness for the parabolic problem (2.2), first adapting the classical definition of weak solution, see for instance [24, § 2.3], to the case of interest here.

Definition 2.5. A function $w\in\mathbf{L}^\infty\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ with $w(t)\in\mathbf{W}^{1,1}(\Omega;\mathbb{R})$ for a.e. $t\in[0,T]$ is a *weak solution to* (2.2) if for all test functions $\phi\in\mathbf{C}^\infty_c([0,T[\times\overline{\Omega};\mathbb{R})$

$$\int_0^T\int_\Omega w\,\partial_t\phi d x\mathrm{d}t - \mu\int_0^T\int_\Omega \nabla w\cdot\nabla\phi d x\mathrm{d}t + \int_0^T\int_\Omega B\,w\,\phi d x\mathrm{d}t + \int_0^T\int_\Omega b\,\phi d x\mathrm{d}t + \int_\Omega w_0(x)\,\phi(0,x)\mathrm{d}x = 0\,. \tag{2.10}$$

Recall that a map ϕ is in $\mathbf{C}^\infty_c([0,T[\times\overline{\Omega};\mathbb{R})$ if it is the restriction to $[0,T[\times\overline{\Omega}$ of a map in $\mathbf{C}^\infty_c(\mathbb{R}\times\mathbb{R}^n;\mathbb{R})$ with support contained in $]-\infty,T[\times\mathbb{R}^n$.

A relevant consequence of the definition chosen above is the following convergence result where the weak completeness of $\mathbf{L}^1$ is essential. This Lemma is of use in a few key points in the proof of Theorem 2.7.

Lemma 2.6. *For $h\in\mathbb{N}$, let $b_h\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$, $B_h\in\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$ and $w_0^h\in\mathbf{L}^1(\Omega;\mathbb{R})$ be such that (2.2) admits the solution w_h in the sense of Definition 2.5. Moreover, assume that*

$$\begin{aligned}&\lim_{h\to+\infty} b_h = b \text{ in } \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})\,,\\ &\lim_{h\to+\infty} B_h = B \text{ in } \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))\,,\\ &\lim_{h\to+\infty} w_0^h = w_0 \text{ in } \mathbf{L}^1(\Omega;\mathbb{R})\,,\\ &\lim_{h\to+\infty} w_h = w \text{ in } \mathbf{L}^\infty\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right).\end{aligned}$$

Then:

(1) *For a.e. $t\in[0,T]$, $w(t)\in\mathbf{W}^{1,1}(\Omega;\mathbb{R})$.*

(2) $\lim_{h\to+\infty} \nabla w_h = \nabla w$ *weakly in* $\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$.

(3) w *is a solution to* (2.2) *in the sense of Definition* 2.5.

The following result differs from others found in the literature in its being set in $\mathbf{L}^1$ and in its referring to Neumann boundary conditions.

Theorem 2.7. *Let* Ω *satisfy* (Ω). *Fix* $T,\mu>0$. *Assume* $B\in\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$ *and* $b\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. *Then, for all* $w_0\in\mathbf{L}^1(\Omega;\mathbb{R})$, *problem* (2.2) *admits a unique solution in the sense of Definition* 2.5. *Moreover,*

(P1) *It also holds that* $w\in\mathbf{C}^0\big([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\big)$, $\nabla w\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$ *and*

$$\begin{aligned}\|\nabla w\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)} \le{}& \frac{1}{\mu}\|B\|_{\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\ \|w\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))}\\ &+\frac{1}{\mu}\ \|b\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})}+\frac{1}{\mu}\|w_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})}.\end{aligned} \tag{2.11}$$

(P2) *The following implicit representation formula holds:*

$$\begin{aligned}w(t,x)={}&\int_\Omega N(t,x,0,y)\ w_0(y)\mathrm{d}y\\ &+\int_0^t\int_\Omega N(t,x,s,y)(B(s,y)\ w(s,y)+b(s,y))\mathrm{d}y\mathrm{d}s\end{aligned} \tag{2.12}$$

and the Neumann function N *is defined in Proposition* 3.3.

(P3) *There exists a positive* K *depending on* μ, Ω*—hence on* n*—such that the following a priori bound holds for all* $t\in[0,T]$:

$$\begin{aligned}\|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\le{}& K\Big(\|w_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big)\\ &\times\exp\left(K\ \|B\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right).\end{aligned} \tag{2.13}$$

(P4) *If* $w_0^1,w_0^2\in\mathbf{L}^1(\Omega;\mathbb{R})$, *the corresponding solutions* w_1 *and* w_2 *satisfy for all* $t\in[0,T]$,

$$\begin{aligned}\left\|w_1(t)-w_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\le{}& K\ \left\|w_0^1-w_0^2\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\times\exp\left(K\ \|B\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right).\end{aligned} \tag{2.14}$$

(P5) *If* $B_1,B_2\in\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$, *the corresponding solutions* w_1 *and* w_2 *satisfy for all* $t\in[0,T]$,

$$\begin{aligned}&\|w_1(t)-w_2(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\le K^2\exp\left(K(\|B_1\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}+\|B_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))})\right)\\ &\quad\times\left(\|w_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right)\|B_1-B_2\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}.\end{aligned}$$

(P6) *If* $b_1,b_2\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$, *the corresponding solutions* w_1, w_2 *satisfy for all* $t\in[0,T]$,

$$\begin{aligned}\left\|w_1(t)-w_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\le{}& K\exp\left(K\ \|B\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\\ &\times\left\|b_1-b_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}.\end{aligned} \tag{2.15}$$

(P7) *Assume* $B\in\mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})$. *If* $b\ge0$ *and* $w_0\ge0$, *then* $w\ge0$.

The proof is deferred to Section 3.2.

3 | Analytical Proofs

3.1 | Hyperbolic Problem

Proof of Theorem 2.4. The existence and uniqueness of u follow from [7, proposition 3.9].

The *a priori* $\mathbf{L}^1$ and $\mathbf{L}^\infty$ bounds **(H1)** and **(H2)** are obtained in [3, lemma 4.2]. Positivity in **(H3)** is proved as in [7, lemma 3.12]. The Lipschitz continuous dependence on the initial datum **(H4)** follows from **(H1)** by linearity. The stability estimate **(H5)** is proved through the same computations as in [3, lemma 4.3], taking advantage of linearity and of **(H1)**. The continuity **(H6)** follows from [7, lemma 3.13]. The stability in **(H6)** is proved in [7, lemma 3.1].

3.2 | Parabolic Problem

Proof of Lemma 2.6. Using Definition 2.5, pass to the limit $h\to+\infty$ in

$$\begin{aligned}&\mu\int_0^T\int_\Omega\nabla w_h\cdot\nabla\phi\,d\,x\mathrm{d}t\\ &=\int_0^T\int_\Omega w_h\ \partial_t\phi\,d\,x\mathrm{d}t+\int_0^T\int_\Omega B_h\ w_h\ \phi\,d\,x\mathrm{d}t\\ &+\int_0^T\int_\Omega b_h\ \phi\,d\,x\mathrm{d}t+\int_\Omega w_0^h(x)\ \phi(0,x)\mathrm{d}x\,.\end{aligned}$$

By the dominated convergence theorem, for all test functions, $\phi\in\mathbf{C}_c^\infty([0,T[\times\overline{\Omega};\mathbb{R})$

$$\int_0^T\int_\Omega w_h\ \partial_t\phi\,d\,x\mathrm{d}t\underset{h\to+\infty}{=}\int_0^T\int_\Omega w\ \partial_t\phi\,d\,x\mathrm{d}t$$

[Because $w_h\to w$ in $\mathbf{L}^1$]

$$\int_0^T\int_\Omega B_h\ w_h\ \phi\,d\,x\mathrm{d}t\underset{h\to+\infty}{=}\int_0^T\int_\Omega B\ w\ \phi\,d\,x\mathrm{d}t$$

[Because $w_h\to w$ and $B_h\to B$]

$$\int_0^T\int_\Omega b_h\ \phi\,d\,x\mathrm{d}t\underset{h\to+\infty}{=}\int_0^T\int_\Omega b\ \phi\,d\,x\mathrm{d}t$$

[Because $b_h\to b$ in $\mathbf{L}^1$]

$$\int_\Omega w_0^h(x)\ \phi(0,x)\mathrm{d}x\underset{h\to+\infty}{=}\int_\Omega w_0(x)\ \phi(0,x)\mathrm{d}x$$

[Because $w_0^h\to w_0$ in $\mathbf{L}^1$]

As a consequence, we obtain that

$$\lim_{h\to+\infty}\int_0^T\int_\Omega\nabla w_h\cdot\nabla\phi\,d\,x\mathrm{d}t=c_\phi$$

for a real number c_ϕ, so that by the weak completeness of $\mathbf{L}^1$, see [25, corollary 14], there exists a map $z\in\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$

such that $\nabla w_h \underset{h\to+\infty}{\rightharpoonup} z$ in $\mathbf{L}^1$. Choose now $\psi \in \mathbf{C}_c^\infty(]0,T[\times\Omega;\mathbb{R})$:

$$
\begin{aligned}
&\int_0^T \int_\Omega w\ \nabla\psi \mathrm{d}x\mathrm{d}t \\
&= \lim_{h\to+\infty} \int_0^T \int_\Omega w_h\ \nabla\psi \mathrm{d}x\mathrm{d}t && [\text{Because } w_h \underset{h\to+\infty}{\rightharpoonup} w \text{ in } \mathbf{L}^1] \\
&= -\lim_{h\to+\infty} \int_0^T \int_\Omega \nabla w_h\ \psi \mathrm{d}x\mathrm{d}t && [\text{Because } w_h \in \mathbf{W}^{1,1}] \\
&= -\int_0^T \int_\Omega z\ \psi \mathrm{d}x\mathrm{d}t && [\text{Because } \nabla w_h \underset{h\to+\infty}{\rightharpoonup} z \text{ in } \mathbf{L}^1]
\end{aligned}
$$

proving that $w(t) \in \mathbf{W}^{1,1}(\Omega;\mathbb{R})$ for a.e. $t \in [0,T]$, proving 2.6, and $z = \nabla w \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$, proving 2.6. Hence, w satisfies the regularity requirements in Definition 2.5. As it also satisfies (2.10), w is a weak solution to (2.2) in the sense of Definition 2.5, proving 2.6 and completing the proof.

We first consider problem (2.2) with $B = 0$, namely

$$
\begin{cases}
\partial_t w - \mu\ \Delta w = b(t,x) & (t,x) \in [0,T]\times\Omega \\
\nabla w(t,\xi)\cdot \nu(\xi) = 0 & (t,\xi) \in]0,T[\ \times \partial\Omega \\
w(0,x) = w_0(x) & x \in \Omega
\end{cases}
\tag{3.1}
$$

under Condition (Ω) on Ω.

Lemma 3.1. *Fix $w_0 \in \mathbf{L}^1(\Omega;\mathbb{R})$, $b \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$ and let w solve (3.1) in the sense of Definition 2.5. Then, for all $\eta \in \mathbf{C}^\infty(\overline{\Omega};\mathbb{R})$*

$$
\begin{aligned}
&\forall\, t \in]0,T[:\ \lim_{h\to 0} \int_\Omega (w(t+h,x) - w(t,x))\ \eta(x)\mathrm{d}x = 0 \\
&t = 0 :\ \lim_{h\to 0+} \int_\Omega \left(w(h,x) - w_0(x)\right)\ \eta(x)\mathrm{d}x = 0\,.
\end{aligned}
\tag{3.2}
$$

Proof of Lemma 3.1. Fix $t \in [0,T[$ and introduce the sequence of functions

$$
\begin{aligned}
&\chi_k \in \mathbf{C}_c^\infty(\mathbb{R};[0,1])\,, \quad \mathrm{spt}\chi_k \subseteq [-1,t] \quad \text{and} \\
&\chi_k \underset{k\to+\infty}{\to} \mathbf{1}_{[0,t]} \text{ pointwise a.e. on } [0,T]\,.
\end{aligned}
$$

Then use (2.10) with $B \equiv 0$, first $\phi(s,x) = \chi_k(s+h)\ \eta(x)$ and then $\phi(s,x) = \chi_k(s)\ \eta(x)$, for a suitable $\eta \in \mathbf{C}^\infty(\overline{\Omega};\mathbb{R})$, with $t \in [0,T[$ and h sufficiently small. Taking the difference of the resulting expressions, we have

$$
\begin{aligned}
&\int_0^T \int_\Omega w(s,x)\left(\partial_t\chi_k(s+h) - \partial_t\chi_k(s)\right)\eta(x)\mathrm{d}x\mathrm{d}s \\
&= \mu \int_0^T \int_\Omega \left(\chi_k(s+h) - \chi_k(s)\right)\nabla w(s,x)\cdot\nabla\eta(x)\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^T \int_\Omega b(s,x)\left(\chi_k(s+h) - \chi_k(s)\right)\eta(x)\mathrm{d}x\mathrm{d}s \\
&\quad + \int_\Omega w_0(x)\ \left(\chi_k(0) - \chi_k(h)\right)\ \eta(x)\mathrm{d}x\,.
\end{aligned}
$$

If $t > 0$, the latter term above vanishes and in the limit $k \to +\infty$ the first equality in (3.2) follows.

When $t = 0$, the above terms reduce to

$$
\begin{aligned}
&\int_0^T \int_\Omega w(s,x)\ \partial_t\chi_k(s+h)\ \eta(x)\mathrm{d}x\mathrm{d}s \\
&= \mu \int_0^T \int_\Omega \chi_k(s+h)\ \nabla w(s,x)\cdot\nabla\eta(x)\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^T \int_\Omega b(s,x)\ \chi_k(s+h)\ \eta(x)\mathrm{d}x\mathrm{d}s - \int_\Omega w_0(x)\ \chi_k(h)\ \eta(x)\mathrm{d}x
\end{aligned}
$$

and as $k \to +\infty$ the second equality in (3.2) follows.

When the initial datum w_0 is in $\mathbf{L}^2(\Omega;\mathbb{R})$ and the source b is in $\mathbf{L}^2([0,T]\times\Omega;\mathbb{R})$, strong $\mathbf{L}^2$ continuity in time is available.

Lemma 3.2. *If $b \in \mathbf{L}^2\left([0,T];\mathbf{L}^2(\Omega;\mathbb{R})\right)$ and $w_0 \in \mathbf{L}^2(\Omega;\mathbb{R})$, then problem (3.1) admits a unique solution $w \in \mathbf{L}^2\left([0,T];\mathbf{W}^{1,2}(\Omega;\mathbb{R})\right)$ in the sense of Definition 2.5 and $\partial_t w \in \mathbf{L}^2\left([0,T];\mathbf{W}^{1,2}(\Omega;\mathbb{R})^*\right)$, so that $w \in \mathbf{C}^0\left([0,T];\mathbf{L}^2(\Omega;\mathbb{R})\right)$.*

Proof of Lemma 3.2. Use [1, problem (10.35)] and apply [1, point b) in theorem 7.104]. Note that the definition of weak solution in [1, p. 592] implies Definition 2.5 due to the density of $\mathbf{C}_c^\infty(\Omega;\mathbb{R})$ in $\mathbf{W}^{1,2}(\Omega;\mathbb{R})$ and by the dominated convergence theorem.

We introduce for later use the space $\mathcal{V}$ as the closure of $\mathbf{W}^{1,2}(\mathbb{R}\times\Omega;\mathbb{R})$ with respect to the norm

$$
\|w\|_{\mathcal{V}} = \|\nabla w\|_{\mathbf{L}^2(\mathbb{R}\times\Omega;\mathbb{R})} + \mathrm{ess\,sup}_{t\in\mathbb{R}} \|w(t)\|_{\mathbf{L}^2(\Omega;\mathbb{R})}\,.
$$

Proposition 3.3. *Let Ω satisfy (Ω) and $\mu > 0$. Then, there exists a function*

$$
N \in \mathbf{C}^0\left(\left\{(t,x,s,y) \in (\mathbb{R}\times\Omega)^2 \,:\, (t,x) \neq (s,y)\right\};\mathbb{R}\right)
$$

such that

(N1) *For all $(s,y) \in \mathbb{R}\times\Omega$,*

$$
\begin{aligned}
&(t,x) \mapsto N(t,x,s,y) \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}\times\Omega;\mathbb{R}) \quad \text{and} \\
&(t,x) \mapsto \nabla_x N(t,x,s,y) \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}\times\Omega;\mathbb{R}^n)\,.
\end{aligned}
$$

(N2) *There exist positive C, κ and K such that for all $(t,x,s,y) \in (\mathbb{R}_+\times\Omega)^2$ with $t > s$*

$$
|N(t,x,s,y)| \le C\left(1 + \frac{1}{(t-s)^{n/2}}\right)\ \exp\left(-\frac{\kappa\|x-y\|^2}{t-s}\right); \tag{3.3}
$$

$$
\int_\Omega |N(t,x,s,y)|\mathrm{d}x \le K \tag{3.4}
$$

(N3) *Fix $T > 0$. For every $w_0 \in \mathbf{L}^2(\Omega;\mathbb{R})$ and all $b \in \mathbf{C}^\infty([0,T]\times\overline{\Omega};\mathbb{R})$, the map*

$$
\begin{aligned}
w(t,x) = &\int_\Omega N(t,x,0,y)\ w_0(y)\mathrm{d}y \\
&+ \int_0^t \int_\Omega N(t,x,s,y)\ b(s,y)\mathrm{d}y\mathrm{d}s
\end{aligned}
\tag{3.5}
$$

is the unique function in $\mathcal{V}$ satisfying (2.10), with $B \equiv 0$, for all test function $\phi \in \mathbf{C}_c^\infty([0,T[\times\overline{\Omega};\mathbb{R})$.

Proof of Proposition 3.3. Here, we apply [24, theorem 3.9]; thus, we need to verify conditions (A1)–(A2) in [24]. (A1) holds by [24, (1) in examples 4.1.1] thanks to [26, theorem 9.7] which can be applied thanks to (Ω); (A2) holds by [24, (1) in examples 4.1.2], that applies in the present scalar case. We can apply [24, theorem 3.21] because also (A3) in [24] holds by [24, (3) in examples 4.1.3], because the coefficients in (3.1) are constant and (Ω) holds.

The regularity of N and the proof of **(N1)** directly follow from [24, theorem 3.9]. To prove (3.3), start from the last line in the proof of [24, theorem 3.21]:

$$\begin{aligned}|N(t,x,s,y)| &\le C \;\max\left\{1, \frac{t-s}{\mathrm{diam}(\Omega)}\right\}^{n/2} \\ &\times (t-s)^{-n/2}\;\exp\left(-\frac{\kappa\|x-y\|^2}{t-s}\right)\end{aligned}$$

which implies (3.3), up to relabeling C. Now, (3.4) is a direct consequence with $K = C\;\left(\mathrm{meas}(\Omega) + (\pi/\kappa)^{n/2}\right)$.

Consider **(N3)**. The proof that the expression (3.5) solves (3.1) in the sense of Definition 2.5 follows from [24, theorem 3.9], as well as uniqueness.

Remark 3.4. A close look at the proof of [24, theorem 3.9] shows also that $N \ge 0$. This positivity is not explicitly considered in [24] as the object of that work is a system of parabolic equations, and thus, N results to be matrix valued. This property, though basic, is not necessary in the sequel; hence, we omit its proof whose rigorous exposition might significantly lengthen this work. However, $N \ge 0$ ensures that

$$w_0 \in \mathbf{L}^2(\Omega;\mathbb{R}_+) \quad \text{and} \quad b \in \mathbf{C}^\infty([0,T]\times\overline{\Omega};\mathbb{R}_+) \;\Rightarrow\; w \ge 0\,.$$

Proposition 3.5. *Let Ω satisfy* (Ω). *Fix* $T,\mu > 0$. *For every* $w_0 \in \mathbf{L}^1(\Omega;\mathbb{R})$ *and any* $b \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$, *problem* (3.1) *admits a unique weak solution* w *in the sense of Definition 2.5. Moreover,*

(1) *w admits the representation (3.5) with N as defined in Proposition 3.3.*

(2) $w \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ *and* $\nabla w \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$.

Proof of Proposition 3.5. We split the proof in a few steps.

Step 1: Uniqueness.
Assume w_1, w_2 solve (3.1) in the sense of Definition 2.5. Then, their difference w satisfies, for all test function $\phi \in \mathbf{C}^\infty_c([0,T[\times\overline{\Omega};\mathbb{R})$,

$$\int_0^T\int_\Omega w\;\partial_t\phi\,dx\,dt - \mu\int_0^T\int_\Omega \nabla w\cdot\nabla\phi\,dx\,dt = 0. \tag{3.6}$$

First apply (3.6) with a test function depending only on time to obtain, thanks to Lemma 3.1, that for all $t\in[0,T]$,

$$\int_\Omega w(t,x)\mathrm{d}x = 0 \tag{3.7}$$

Then, fix arbitrary $a,b\in]0,T[$ with $a<b$. Introduce a sequence of test functions

$$\begin{aligned}&\chi_h \in \mathbf{C}^\infty_c(]0,T[;[0,1]), \quad \mathrm{spt}\chi_h \subseteq [a,b] \quad \text{and}\\ &\chi_h \to \mathbf{1}_{[a,b]} \text{ pointwise a.e. on } [0,T]\,.\end{aligned}$$

Let $\psi\in\mathbf{C}^\infty(\overline{\Omega};\mathbb{R})$ and apply (3.6) with $\phi = \chi_h\,\psi$, obtaining

$$\begin{aligned}0 &= \int_a^b\int_\Omega w\;\partial_t\chi_h\;\psi\mathrm{d}x\mathrm{d}t - \mu\int_a^b\int_\Omega \chi_h\;\nabla w\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\ &\underset{h\to+\infty}{=} \int_\Omega w(b,x)\mathrm{d}x - \int_\Omega w(a,x)\mathrm{d}x - \mu\int_a^b\int_\Omega\nabla w\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\ &= -\mu\int_a^b\int_\Omega \nabla w\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\end{aligned}$$

where we used (3.7). Hence, there exists a $c\in\mathbb{R}^n$ such that for a.e. $x\in\Omega$,

$$\int_a^b \nabla w(t,x)\mathrm{d}t = c \tag{3.8}$$

and, by the arbitrariness of a and b, it must be $c=0$. From (3.8), we thus obtain that for a.e. $x\in\Omega$ and for all $g\in\mathbf{L}^\infty([0,T];\mathbb{R}^n)$

$$\int_0^T \nabla w(t,x)\cdot g(t)\mathrm{d}t = 0$$

so that $\nabla w(t,x) = 0$ for all $t\in[0,T]$ and a.e. $x\in\Omega$. Together with (3.7) and the connectedness of Ω, this implies that $w = 0$, proving uniqueness.

Step 2: Approximation.
By [22, (1.8) in § 1.14], there exists a sequence $b_h \in \mathbf{C}^\infty_c(\mathbb{R}^{1+n};\mathbb{R})$ such that $b_h\to b$ in $\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$ as $h\to+\infty$. Similarly, there exists a sequence $w_0^h\in\mathbf{C}^\infty_c(\mathbb{R}^n,\mathbb{R})$ such that $w_0^h\to w_0$ in $\mathbf{L}^1(\Omega;\mathbb{R})$ as $h\to+\infty$. Note that $w_0^h\in\mathbf{L}^2(\Omega;\mathbb{R})$ for all h.
For all $h\in\mathbb{N}$, by **(N3)** in Proposition 3.3, there is a unique solution $w_h\in\mathcal{V}$ to (3.1) with source b_h and initial datum w_0^h, which is given by the representation

$$\begin{aligned}w_h(t,x) = &\int_\Omega N(t,x,0,y)\;w_0^h(y)\mathrm{d}y\\ &+\int_0^t\int_\Omega N(t,x,s,y)\;b_h(s,y)\mathrm{d}y\mathrm{d}s\,.\end{aligned} \tag{3.9}$$

Then, for $h,k\in\mathbb{N}$ with $k>h$, thanks to (3.4) we have

$$\begin{aligned}&\left\|w_k(t)-w_h(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\\ &\le \int_\Omega\int_\Omega |N(t,x,0,y)|\;\left|w_0^k(y)-w_0^h(y)\right|\mathrm{d}y\mathrm{d}x\\ &\quad+\int_\Omega\int_0^t\int_\Omega |N(t,x,s,y)|\;\left|b_k(s,y)-b_h(s,y)\right|\mathrm{d}y\mathrm{d}s\mathrm{d}x\\ &=\int_\Omega\left(\int_\Omega|N(t,x,0,y)|\mathrm{d}x\right)\;\left|w_0^k(y)-w_0^h(y)\right|\mathrm{d}y\\ &\quad+\int_\Omega\int_0^t\left(\int_\Omega|N(t,x,s,y)|\mathrm{d}x\right)\left|b_k(s,y)-b_h(s,y)\right|\mathrm{d}s\mathrm{d}y\\ &\le K\left(\left\|w_0^k-w_0^h\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}+\left\|b_k-b_h\right\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})}\right)\end{aligned}$$

proving that there exists a function $w \in \mathbf{L}^\infty\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ such that

$$\lim_{h\to+\infty} \|w_h - w\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))} = 0\,. \tag{3.10}$$

Step 3: Existence in $\mathbf{L}^\infty\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$.
To prove that w solves (3.1) in the sense of Definition 2.5, it is now sufficient to apply Lemma 2.6 with $B_h \equiv 0$.
Hence, w satisfies the regularity requirements in Definition 2.5. As it also satisfies (2.10), w is a weak solution to (3.1) in the sense of Definition 2.5, with $B \equiv 0$.

Step 4: $\mathbf{L}^1(\Omega;\mathbb{R})$ Continuity in Time.
Note that the sequence w_h defined above also satisfies $w_h \in \mathbf{L}^2\left([0,T];\mathbf{W}^{1,2}(\Omega;\mathbb{R})\right)$ and $\partial_t w_h \in \mathbf{L}^2\left([0,T];\mathbf{W}^{1,2}(\Omega;\mathbb{R})^*\right)$, so that $w_h \in \mathbf{C}^0\left([0,T];\mathbf{L}^2(\Omega;\mathbb{R})\right)$ by Lemma 3.2. Recall that $\mathbf{C}^0\left([0,T];\mathbf{L}^2(\Omega;\mathbb{R})\right) \subseteq \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$; hence, the convergence (3.10) then ensures that $w \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$.

Step 5: Representation Formula.
It is now sufficient to pass to the limit $h \to +\infty$ in (3.9) to prove that w admits the representation (3.5).

The proof is completed.

Corollary 3.6. *Let* (Ω) *hold. Fix* $T, \mu > 0$. *Let* $w_0^1, w_0^2 \in \mathbf{L}^1(\Omega;\mathbb{R})$ *and* $b_1, b_2 \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. *Call* w_1, w_2 *the corresponding solutions to (3.1). Then, for all* $t \in [0,T]$,

$$\begin{aligned} &\left\|w_1(t) - w_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\le K\left(\left\|w_0^1 - w_0^2\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b_1 - b_2\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \end{aligned} \tag{3.11}$$

where K *is as in* **(N2)** *of Proposition 3.3.*

Simply apply **(1)** of Proposition 3.5 to the difference $w_2 - w_1$ and exploit the linearity of (3.1).

Proof of Theorem 2.7. We split the proof in a few steps.

Step 1: Problem (2.2) admits a unique solution on $[0,T]$ satisfying (P1) and (P2).
Consider the operators

$$\begin{aligned} \Lambda : \ \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right) &\to \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}) \\ w &\mapsto \Lambda\, w \\ \text{where}(\Lambda\, w)(t,x) &= B(t,x)\; w(t,x) + b(t,x) \end{aligned} \tag{3.12}$$

and, with reference to Definition 2.5,

$$\begin{aligned} \Phi : \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}) &\to \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right) \\ \beta &\mapsto w \\ \text{where } &\begin{cases} \partial_t w - \mu\ \Delta w = \beta(t,x) & (t,x) \in [0,T]\times\Omega \\ \nabla w(t,\xi)\cdot \nu(\xi) = 0 & (t,\xi) \in]0,T[\times\partial\Omega \\ w(0,x) = w_0(x) & x \in \Omega\,. \end{cases} \end{aligned}$$

Let us precise that Λ is well defined, meaning that if $w \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ then, by the assumptions on B and b, $\Lambda(w) \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. Similarly, Φ is well defined by Proposition 3.5.
By the assumption on B, there exist times t_i for $i = 0, \ldots, m$ such that $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_m = T$ and

$$\int_{t_i}^{t_{i+1}} \|B(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\mathrm{d}s < \frac{1}{2\,K} \tag{3.13}$$

with K as in (3.4) in Proposition 3.3. Clearly, thanks to the continuity proved in Proposition 3.5, w in $\mathbf{L}^\infty\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ with $w(t) \in \mathbf{W}^{1,1}(\Omega;\mathbb{R})$ for a.e. $t \in [0,T]$ is a weak solution to (2.2) if and only if it is a fixed point of $\Phi{\circ}\Lambda$ in $\mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$. To construct such fixed point, we apply Banach fixed point theorem iteratively in each of the spaces $\mathbf{C}^0\left([t_i,t_{i+1}];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ for $i = 0, \ldots, m-1$.
Notice that for all $w_1, w_2 \in \mathbf{C}^0\left([t_i,t_{i+1}];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ with $w_1(t_i) = w_2(t_i)$, for all $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} &\|\Phi{\circ}\Lambda(w_1)(t) - \Phi{\circ}\Lambda(w_2)(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\le K\int_{t_i}^{t}\|\Lambda(w_1)(s) - \Lambda(w_2)(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\,\mathrm{d}s && \text{[By Corollary 3.6]} \\ &= K\int_{t_i}^{t}\int_\Omega |B(s,y)|\,\left|w_1(s,y) - w_2(s,y)\right|\,\mathrm{d}y\mathrm{d}s && \text{[By (3.12)]} \\ &\le K\int_{t_i}^{t}\|B(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\int_\Omega \left|w_1(s,y) - w_2(s,y)\right|\,\mathrm{d}y\mathrm{d}s \\ &\le K\int_{t_i}^{t}\|B(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\,\mathrm{d}s\ \|w_1 - w_2\|_{\mathbf{C}^0([t_i,t_{i+1}];\mathbf{L}^1(\Omega;\mathbb{R}))} \\ &\le \frac{1}{2}\ \|w_1 - w_2\|_{\mathbf{C}^0([t_i,t_{i+1}];\mathbf{L}^1(\Omega;\mathbb{R}))}\,. && \text{[By (3.13)]} \end{aligned}$$

An iterated application of Banach fixed point theorem ensures the existence of $w_* \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\right)$ such that $w_* = \Phi \circ \Lambda(w_*)$.
Define $\tilde{b} = B\ w_* + b$, so that $\tilde{b} \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R})$. By construction, w_* solves

$$\begin{cases} \partial_t w - \mu\ \Delta w = \tilde{b}(t,x) & (t,x) \in [0,T]\times\Omega \\ \nabla w(t,\xi)\cdot\nu(\xi) = 0 & (t,\xi) \in]0,T[\times\partial\Omega \\ w(0,x) = w_0(x) & x \in \Omega \end{cases}$$

in the sense of Definition 2.5. Hence, $w_*(t) \in \mathbf{W}^{1,1}(\Omega;\mathbb{R})$ for a.e. $t \in [0,T]$ and $\nabla w \in \mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)$ by **(2)** in Proposition 3.5. This shows that w_* is a weak solution to (2.2) in the sense of Definition 2.5 on $[0,T]$ and **(P1)** holds. To prove the bound (2.11), by (2.10), we have

$$\begin{aligned} &\left|\int_0^T\int_\Omega [w \quad -\mu\,\nabla w]\begin{bmatrix}\partial_t\phi \\ \nabla\phi\end{bmatrix}\mathrm{d}x\mathrm{d}t\right| \\ &\le \left|\int_0^T\int_\Omega (B\,w + b)\phi d\,x\mathrm{d}t\right| + \left|\int_\Omega w_0\,\phi(0,\cdot)\mathrm{d}x\right| \\ &\le \Big(\ \|B\|_{\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\,\|w\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))} \\ &\quad + \|b\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})} + \left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\ \Big)\ \|\phi\|_{\mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R})} \end{aligned}$$

which together with

$$\|\nabla w\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)} \le \frac{1}{\mu}\,\|[w \quad -\mu\,\nabla w]\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^{n+1})}$$

completes the proof of (2.11).

By construction, **(P2)** holds. Step 1 is proved.

Step 2: (P3) holds.
Using (2.12), for all $t \in [0, T]$, by (3.4)

$$\begin{aligned}\|w(t)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le K\Bigg(\|w_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \int_0^t \int_\Omega (|B(s,y)|\ |w(s,y)| \\ + |b(s,y)|)\,\mathrm{d}y\mathrm{d}s\Bigg) \\ \le K\Bigg(\|w_0\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} \\ + \int_0^t \|B(s)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\ \|w(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\mathrm{d}s\Bigg).\end{aligned}$$

An application of Gronwall lemma leads to (2.13).

Step 3: (P4) holds.
This is an immediate consequence of **(P3)** by linearity.

Step 4: (P4) and (P5) hold.
Using (2.12), compute:

$$\begin{aligned}&\left\|w_1(t) - w_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\le K\int_0^t \int_\Omega \left|B_1(s,y)\,w_1(s,y) - B_2(s,y)\,w_2(s,y)\right|\mathrm{d}y\mathrm{d}s \\ &\quad + K\int_0^t \int_\Omega \left|b_1(s,y) - b_2(s,y)\right|\mathrm{d}y\mathrm{d}s \\ &\le K\int_0^t \int_\Omega \left|B_1(s,y)\,w_1(s,y) - B_1(s,y)\,w_2(s,y)\right|\mathrm{d}y\mathrm{d}s \\ &\quad + K\int_0^t \int_\Omega \left|B_1(s,y)\,w_2(s,y) - B_2(s,y)\,w_2(s,y)\right|\mathrm{d}y\mathrm{d}s \\ &\quad + K\int_0^t \int_\Omega \left|b_1(s,y) - b_2(s,y)\right|\mathrm{d}y\mathrm{d}s \\ &\le K\int_0^t \left\|B_1(s)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \left\|w_1(s) - w_2(s)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\mathrm{d}s \\ &\quad + K\left\|B_1 - B_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} \left\|w_2\right\|_{\mathbf{C}^0([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} \\ &\quad + K\left\|b_1 - b_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}.\end{aligned}$$

By Gronwall lemma and using **(P3)**,

$$\begin{aligned}&\left\|w_1(t) - w_2(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\ &\le K\Big(\left\|B_1 - B_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} \left\|w_2\right\|_{\mathbf{C}^0([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} \\ &\quad + \left\|b_1 - b_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \\ &\quad \times \exp\left(K\left\|B_1\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \\ &\le K\left\|b_1 - b_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} \exp\left(K\left\|B_1\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \\ &\quad + K^2\left\|B_1 - B_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\Big(\left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|b_2\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \\ &\quad \times \exp\left(K\left(\left\|B_1\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} + \left\|B_2\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)\right),\end{aligned}$$

which implies (2.7) and (2.15).

Step 5: (P7) holds with $B \in \mathbf{L}^\infty([0, T] \times \Omega; \mathbb{R})$.

By [22, (1.8) in § 1.14], there exists a sequence $b_h \in \mathbf{C}_c^\infty(\mathbb{R}^{1+n}; \mathbb{R}_+)$ such that $b_h \to b$ in $\mathbf{L}^1([0, T] \times \Omega; \mathbb{R})$ as $h \to +\infty$. Similarly, there exists a sequence $w_0^h \in \mathbf{C}_c^\infty(\mathbb{R}^n, \mathbb{R}_+)$ such that $w_0^h \to w_0$ in $\mathbf{L}^1(\Omega; \mathbb{R})$ as $h \to +\infty$. Note that $w_0^h \in \mathbf{L}^2(\Omega; \mathbb{R})$ for all h. Call w_h the solution to

$$\begin{cases}\partial_t w_h - \mu\ \Delta w_h = B(t,x) w_h + b_h(t,x) & (t,x) \in [0,T] \times \Omega \\ \nabla w_h(t,\xi) \cdot \nu(\xi) = 0 & (t,\xi) \in]0,T[\times \partial\Omega \\ w(0,x) = w_0^h(x) & x \in \Omega.\end{cases}$$

in the sense of the definition given in [1 , p. 592]. By [1, Remark 10.19], $w_h \ge 0$.

Using **(P4)** and **(P6)**, we have

$$\begin{aligned}\left\|w_h(t) - w_k(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \le K e^{K\|B\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}} \\ \times \left(\left\|w_0^h - w_0^k\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|b_h - b_k\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right)\end{aligned}$$

hence, there exists a $w \in \mathbf{L}^\infty\left([0, T]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$ such that $w_h \to w$ in $\mathbf{L}^\infty\left([0, T]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$.

Apply now Lemma 2.6 to prove that w is a solution to (2.2) in the sense of Definition 2.5. By construction, $w \ge 0$.

The proof is completed.

3.3 | Mixed Problem

Proof of Theorem 2.2. Define u_1 and w_1 as solutions to

$$\begin{cases}\partial_t u_1 = a(t,x) & (t,x) \in [0,T] \times \Omega \\ u_1(t,\xi) = 0 & (t,\xi) \in]0,T[\times\partial\Omega \\ u_1(0,x) = u_0(x) & x \in \Omega\end{cases}$$

$$\begin{cases}\partial_t w_1 - \mu\ \Delta w_1 = b(t,x) & (t,x) \in [0,T] \times \Omega \\ \nabla w_1(t,\xi) \cdot \nu(\xi) = 0 & (t,\xi) \in]0,T[\times\partial\Omega \\ w_1(0,x) = w_0(x) & x \in \Omega\end{cases}$$

in the sense of Definition 2.3 and of Definition 2.5. Clearly, u_1 and w_1 are well defined by Theorem 2.3 and Theorem 2.7. Define recursively, for $i \in \mathbb{N} \setminus \{0\}$, u_{i+1} as solution to

$$\begin{cases}\partial_t u_{i+1} + \nabla \cdot \left(u_{i+1}\ c_i(t,x)\right) \\ \qquad = A_i(t,x)\ u_{i+1} + a(t,x) & (t,x) \in [0,T] \times \Omega \\ u_{i+1}(t,\xi) = 0 & (t,\xi) \in]0,T[\times \partial\Omega \\ u_{i+1}(0,x) = u_0(x) & x \in \Omega\end{cases} \tag{3.14}$$

in the sense of Definition 2.3 and w_{i+1} as solution to

$$\begin{cases}\partial_t w_{i+1} - \mu\ \Delta w_{i+1} = B_i(t,x)\ w_{i+1} + b(t,x) & (t,x) \in [0,T] \times \Omega \\ \nabla w_{i+1}(t,\xi) \cdot \nu(\xi) = 0 & (t,\xi) \in]0,T[\times\partial\Omega \\ w_{i+1}(0,x) = w_0(x) & x \in \Omega\end{cases} \tag{3.15}$$

in the sense of Definition 2.5, where for $i \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned}c_i(t,x) &= v\left(t, w_i(t)\right)(x) \\ A_i(t,x) &= \alpha\left(t, x, w_i(t)\right)\end{aligned} \qquad B_i(t,x) = \beta\left(t, x, u_i(t), w_i(t)\right) \tag{3.16}$$

We aim to prove that (u_i, w_i) are well defined for all $i \in \mathbb{N} \setminus \{0,1\}$ and constitute a Cauchy sequence with respect to the $\mathbf{C}^0\big([0,T];\mathbf{L}^1(\Omega;\mathbb{R}^2)\big)$ distance as soon as T is sufficiently small.

Step 1: c_i satisfies (c) in Theorem 2.4. We first check the continuity. By (3.16) and ($\boldsymbol{v}$), we have

$$\begin{aligned}
&\left\|c_i(t,x) - c_i(t_0,x_0)\right\| \\
\leq &\left\|c_i(t,x) - c_i(t_0,x)\right\| + \left\|c_i(t_0,x) - c_i(t_0,x_0)\right\| \\
= &\left\|v\big(t,w_i(t)\big)(x) - v\big(t_0,w_i(t_0)\big)(x)\right\| \\
&+ \left\|v\big(t_0,w_i(t_0)\big)(x) - v\big(t_0,w_i(t_0)\big)(x_0)\right\| \\
\leq &K_v\Big(\left|t-t_0\right| + \left\|w_i(t) - w_i(t_0)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\Big) \\
&+ \left\|v\big(t_0,w_i(t_0)\big)(x) - v\big(t_0,w_i(t_0)\big)(x_0)\right\|
\end{aligned}$$

where the terms $\left\|w_i(t) - w_i(t_0)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}$ and $\left\|v\big(t_0,w_i(t_0)\big)(x_0) - v\big(t_0,w_i(t_0)\big)(x_0)\right\|$ vanish as $t \to t_0$ and $x \to x_0$ by the continuity of w_i and that of $v\big(t_0,w_i(t_0)\big)$, ensured by ($\boldsymbol{v}$).

To prove the $\mathbf{L}^\infty$ boundedness, observe that by ($\boldsymbol{v}$)

$$\begin{aligned}
\left\|c_i(t,x)\right\| &\leq \left\|v\big(t,w_i(t)\big)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \\
&\leq K_v \left\|w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \leq K_v \left\|w_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))}.
\end{aligned} \tag{3.17}$$

The regularity $c_i(t) \in \mathbf{C}^1(\Omega;\mathbb{R}^n)$ directly follows from ($\boldsymbol{v}$). Concerning the boundedness of $D_x c_i$, observe that by ($\boldsymbol{v}$), $\left\|D_x c_i\right\|_{\mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R}^{n\times n})} \leq K_v \left\|w_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))}$.

Step 2: A_i satisfies (A) in Theorem 2.3. Compute:

$$\begin{aligned}
&\left\|A_i\right\|_{\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))} \\
= &\int_0^T \operatorname{ess\,sup}_{x\in\Omega}\left|\alpha\big(t,x,w_i(t)\big)\right|\mathrm{d}t && \text{[By (3.16)]} \\
\leq &\int_0^T k_\alpha(t)\Big(1 + \left\|w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\Big)\mathrm{d}t && \text{[By } (\alpha)\text{]} \\
\leq &\left\|k_\alpha\right\|_{\mathbf{L}^1(0,T];\mathbb{R})}\Big(1 + \left\|w_i\right\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))}\Big)
\end{aligned}$$

proving that $A_i \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$. Moreover,

$$\begin{aligned}
&\int_0^T \mathrm{TV}\big(A_i(t)\big)\mathrm{d}t \\
\leq &\int_0^T K_\alpha\Big(1 + \left\|\nabla w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\Big)\mathrm{d}t && \text{[By } (\alpha)\text{]} \\
= &K_\alpha\Big(T + \left\|\nabla w_i\right\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)}\Big), && \text{[By (2) in Proposition (3.5)]}
\end{aligned}$$

proving that (A) holds.

Step 3: $B_i \in \mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))$.Using ($\boldsymbol{\beta}$) compute:

$$\begin{aligned}
\left\|B_i\right\|_{\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))} &= \int_0^T \operatorname*{ess\,sup}_{x\in\Omega}\left|\beta\big(\tau,x,u_i(\tau),w_i(\tau)\big)\right|\mathrm{d}\tau \\
&\leq \left\|k_\beta\right\|_{\mathbf{L}^1([0,T];\mathbb{R})}
\end{aligned} \tag{3.18}$$

proving Step 3.

Hence, Theorem 2.4 and Theorem 2.7 apply, ensuring that the sequence (u_i, w_i) can be recursively defined. The next steps aim at proving that (u_i, w_i) is a Cauchy sequence.

Step 4: w_i is bounded in $\mathbf{L}^\infty\big([0,T];\mathbf{L}^1(\Omega;\mathbb{R})\big)$ uniformly in i. By (2.13) and (3.18), we have

$$\begin{aligned}
\left\|w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &\leq K\Big(\left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|b\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\Big) \\
&\quad\times \exp\Big(K\left\|B_{i-1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\Big) \\
&\leq K_w
\end{aligned}$$

where we set

$$\begin{aligned}
K_w = K&\Big(\left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \left\|b\right\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})}\Big) \\
&\times \exp\Big(K\left\|k_\beta\right\|_{\mathbf{L}^1([0,T];\mathbb{R})}\Big).
\end{aligned} \tag{3.19}$$

Step 5: Bounds uniform in i. Note that $c_i(t) \in \mathbf{C}^2(\Omega;\mathbb{R}^n)$ by ($\boldsymbol{v}$) and (3.16). Moreover,

$$\begin{aligned}
&\left\|c_i(t)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^n)} \\
\leq &K_v\left\|w_i\right\|_{\mathbf{C}^0([0,t];\mathbf{L}^1(\Omega;\mathbb{R}))} && \text{[By (3.17)]} \\
\leq &K_v\ K_w; && \text{[By (3.18)]} \\
&\left\|D_x c_i\right\|_{\mathbf{L}^\infty([0,T]\times\Omega;\mathbb{R}^{n\times n})} \\
\leq &K_v\left\|w_i\right\|_{\mathbf{C}^0([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))} && \text{[By } (v)\text{]} \\
\leq &K_v\ K_w; && \text{[By (3.19)]} \\
&\left\|\nabla\nabla\cdot c_i\right\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R}^n)} \\
= &\int_0^T \left\|\nabla\nabla\cdot v\big(t,w_i(t)\big)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\mathrm{d}t && \text{[By (3.16)]} \\
\leq &\int_0^T \left\|D_x^2 v\big(t,w_i(t)\big)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\mathrm{d}t \\
\leq &\int_0^T k_v\Big(t,\left\|w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\Big)\left\|w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\mathrm{d}t && \text{[By } (v)\text{]}
\end{aligned}$$

which is bounded because $k_v \in \mathbf{L}^\infty_{\mathbf{loc}}([0,T]\times\mathbb{R}_+;\mathbb{R}_+)$.

Using (3.19) and ($\boldsymbol{v}$), prepare for later use:

$$\begin{aligned}
&\left\|c_{i+1} - c_i\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))} \\
\leq &\int_0^t \left\|v\big(\tau,w_{i+1}(\tau)\big) - v\big(\tau,w_i(\tau)\big)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R}^n)}\mathrm{d}\tau \\
\leq &K_v\left\|w_{i+1} - w_i\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}. \\
&\left\|\nabla\cdot(c_{i+1} - c_i)\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))} \\
\leq &\int_0^t \left\|\nabla\cdot\Big(v\big(\tau,w_{i+1}(\tau)\big) - v\big(\tau,w_i(\tau)\big)\Big)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\mathrm{d}\tau \\
\leq &\int_0^t k_v\big(\tau,K_w\big)\left\|w_{i+1}(\tau) - w_i(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\mathrm{d}\tau \\
= &\left\|k_v\right\|_{\mathbf{L}^\infty([0,t]\times[0,K_w];\mathbb{R})}\left\|w_{i+1} - w_i\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}.
\end{aligned}$$

Passing to the A_i:

$$\begin{aligned}
&\left\|A_i\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty\Omega;\mathbb{R}))} \\
&= \int_0^t \operatorname*{ess\,sup}_{x\in\Omega} \left|\alpha\left(\tau, x, w_i(\tau)\right)\right| \mathrm{d}\tau && \text{[By (3.16)]} \\
&\le \int_0^t k_\alpha(\tau)\left(1 + \left\|w_i(\tau)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}\right)\mathrm{d}\tau && \text{[By } (\alpha)\text{]} \\
&\le \left(1 + K_w\right)\left\|k_a\right\|_{\mathbf{L}^1([0,t];\mathbb{R})} ; && \text{[By (3.19)]} \\
&\int_0^t \mathrm{TV}\left(A_i(\tau)\right)\mathrm{d}\tau \\
&\le K_a\left(1 + \left\|\nabla w_i\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R}^n)}\right) && \text{[By Step 2]} \\
&\le K_a + \frac{K_\alpha}{\mu}\left\|B_{i-1}\right\|_{\mathbf{L}^1([0,T];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\left\|w_i\right\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1(\Omega;\mathbb{R}))} \\
&\qquad + \frac{K_\alpha}{\mu}\ \|b\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})} + \frac{K_\alpha}{\mu}\ \left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} && \text{[By (2.11)]} \\
&\le K_a + \frac{K_\alpha}{\mu}\ K_w\ \left\|k_\beta\right\|_{\mathbf{L}^1([0,T];\mathbb{R})} && \text{[By (3.18), (3.19)]} \\
&\qquad + \frac{K_\alpha}{\mu}\ \|b\|_{\mathbf{L}^1([0,T]\times\Omega;\mathbb{R})} + \frac{K_\alpha}{\mu}\ \left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} ; \\
&\left\|A_{i+1} - A_i\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} \\
&= \int_0^t \left\|A_{i+1}(\tau) - A_i(\tau)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\mathrm{d}\tau \\
&= \int_0^t \left\|\alpha\left(\tau, \cdot, w_{i+1}(\tau)\right) - \alpha\left(\tau, \cdot, w_i(\tau)\right)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\mathrm{d}\tau && \text{[By (3.16)]} \\
&\le K_a\ \left\|w_{i+1} - w_i\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} . && \text{[By } (\alpha)\text{]}
\end{aligned}$$

Step 6: (u_i, w_i) is a Cauchy sequence in $\mathbf{C}^0\left([0, \Delta T]; \mathbf{L}^1(\Omega; \mathbb{R}^2)\right)$ for ΔT small. Thanks to the above bounds uniform in i, all the constants appearing in **(H5)** and **(H7)** are bounded uniformly by quantities depending on Ω, u_0, a, b and by the constants in the assumptions (Ω), $(\boldsymbol{v})$, $(\boldsymbol{\alpha})$, $(\boldsymbol{a})$, $(\boldsymbol{\beta})$, $(\boldsymbol{b})$. Hence, **(H5)** and **(H7)** yield

$$\begin{aligned}
&\left\|u_{i+1}(t) - u_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\le \mathcal{O}(1)\left(\left\|c_i - c_{i-1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))}\right. \\
&\quad \left. + \left\|\nabla\cdot\left(c_i - c_{i-1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}^n))}\right) \\
&\quad + \mathcal{O}(1)\left(\left\|A_i - A_{i-1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right) \\
&\le \mathcal{O}(1)\ \left\|w_i - w_{i-1}\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})} .
\end{aligned}$$

To compute the distance $\left\|w_{i+1}(t) - w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})}$, apply **(P5)** thanks to Step 3 and get

$$\begin{aligned}
&\left\|w_{i+1}(t) - w_i(t)\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} \\
&\le K^2\left(\left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \\
&\quad \times e^{K\left(\left\|B_i\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} + \left\|B_{i-1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))}\right)} \\
&\quad \times \left\|B_i - B_{i-1}\right\|_{\mathbf{L}^1([0,t];\mathbf{L}^\infty(\Omega;\mathbb{R}))} && \text{[By (P5)]} \\
&\le K^2\ K_\beta\left(\left\|w_0\right\|_{\mathbf{L}^1(\Omega;\mathbb{R})} + \|b\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) \\
&\quad \times e^{2K\left\|k_\beta\right\|_{\mathbf{L}^1([0,t];\mathbb{R})}} && \text{[By (3.18)]} \\
&\quad \times \left(\left\|u_i - u_{i-1}\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right. \\
&\quad \left. + \left\|w_i - w_{i-1}\right\|_{\mathbf{L}^1([0,t]\times\Omega;\mathbb{R})}\right) . \quad \text{[By } (\beta)\text{]}
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\left\|u_{i+1} - u_i\right\|_{\mathbf{C}^0([0,\Delta T];\mathbf{L}^1(\Omega;\mathbb{R}))} + \left\|w_{i+1} - w_i\right\|_{\mathbf{C}^0([0,\Delta T];\mathbf{L}^1(\Omega;\mathbb{R}))} \\
&\le \mathcal{O}(1)\left(\left\|u_i - u_{i-1}\right\|_{\mathbf{L}^1([0,\Delta T]\times\Omega;\mathbb{R})} + \left\|w_i - w_{i-1}\right\|_{\mathbf{L}^1([0,\Delta T]\times\Omega;\mathbb{R})}\right) \\
&\le \mathcal{O}(1)\ \Delta T\left(\left\|u_i - u_{i-1}\right\|_{\mathbf{C}^0([0,\Delta T];\mathbf{L}^1(\Omega;\mathbb{R}))}\right. \\
&\quad \left. + \left\|w_i - w_{i-1}\right\|_{\mathbf{C}^0([0,\Delta T];\mathbf{L}^1(\Omega;\mathbb{R}))}\right)
\end{aligned}$$

proving that for ΔT small, (u_i, w_i) is a Cauchy sequence in $\mathbf{C}^0\left([0, \Delta T]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$. Call (u, w) the corresponding limit.

Step 7: Problem (1.1) admits a global solution in the sense of Definition 2.1. Note that by $(\boldsymbol{v})$, $(\boldsymbol{\alpha})$ and $(\boldsymbol{\beta})$, we can pass to the $\mathbf{C}^0\left([0, \Delta T]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$ limit also in (3.16). Hence, the dominated convergence theorem allows to pass to the limit in (2.6), so that Definition 2.3 applies to u. A further use of Lemma 2.6 allows to pass to the limit also in (2.10), proving the existence of a solution on the time interval $[0, \Delta T]$.

Further iterations of the above procedure yield a solution, say (u, w), to (1.1) in the sense of Definition 2.1. Call $[0, T_*[$, for a $T_* > \Delta T$, the biggest time interval on which (u, w) can be extended.

Define $B(t, x) = \beta(t, x, u(t), w(t))$. By $(\boldsymbol{\beta})$, $B \in \mathbf{L}^1\left([0, T_*]; \mathbf{L}^\infty(\Omega; \mathbb{R})\right)$. Hence, Theorem 2.7 ensures that problem (2.2) admits a solution w on $[0, T_*]$ in the sense of Definition 2.5. By **(P1)**, $w \in \mathbf{C}^0\left([0, T_*]; \mathbf{L}^1(\Omega; \mathbb{R})\right)$.

Define $c(t, x) = v(t, w(t))(x)$, $A(t, x) = \alpha(t, x, w(t))$ and repeat the same computations as in Step 1 and Step 2 to obtain that problem (2.1) admits as solution u on $[0, T_*]$ in the sense of Definition 2.3. By **(H6)**, u is continuous on $[0, T_*]$.

Thus, (u, w) is extended up to time T_*. If $T_* < T$, the above procedure can be repeated with reference to problem (1.1) with initial datum $(u, w)(T_*)$ assigned at time T_*, obtaining an extension of (u, w) beyond time T_*, hence contradicting the maximality of T_*, unless $T_* = T$. This proves the global existence stated in (M1).

The continuity at (M2) follows from **(H6)** and **(P1)**.

The continuous dependence and stability estimates (M3), and (M4) follow through long and tedious computations based on the estimates obtained so far. More precisely, to prove (M3), use **(H4)**, **(H5)**, **(H7)**, **(P4)**, **(P5)** and repeat the computations in Step 3, 4, 5 and 6. To prove (M4), the procedure is entirely similar, also using **(P6)**.

The proof of (M5) is a direct consequence of **(H3)** and **(P7)**.

Author Contributions

Rinaldo M. Colombo: conceptualization, methodology, writing – review and editing, writing – original draft, formal analysis. **Elena Rossi:** conceptualization, methodology, writing – review and editing, writing – original draft, formal analysis. **Abraham Sylla:** conceptualization, methodology, writing – review and editing, writing – original draft, formal analysis.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing not applicable to this article as no data set were generated nor analysed during the current study.

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