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Convergence of generalized Orlicz norms with lower growth rate tending to infinity

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ABSTRACT

We study convergence of generalized Orlicz energies when the lower growth-rate tends to infinity. We generalize results by Bocea–Mihăilescu (Orlicz case) and Eleuteri–Prinari (variable exponent case) and allow weaker assumptions: we are also able to handle unbounded domains with irregular boundary and non-doubling energies.

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1. Introduction

In this paper we consider Γ -convergence of a sequence of energies whose growth-rates tend to infinity. Let us first recap motivation and earlier studies. Garroni, Nesi and Ponsiglione [16] studied dielectric breakdown using Γ -convergence of power-growth functions as the exponent tends to infinity. A more complicated functional of power-type was considered by Katzourakis [24] in a model for fluorescent optical tomography. Prinari and co-authors [10,28,29] considered an abstract version with the convergence of the energy

$$I_p(u) := \int_{\Omega} \frac{1}{p} f(x, u, Du)^p dx$$

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as $p \rightarrow \infty$. A similar problem without dependence on the derivative was considered in [1,6,7] and related to applications in polycrystal plasticity. If the energy is of non-standard growth-type [25], then the Γ -limit can contain singular parts [27].

The energy I_p has been generalized to two complementary cases: Orlicz growth by Bocea and Mihăilescu [5] and variable exponent growth by Eleuteri and Prinari [15]. In this article, we consider the result in generalized Orlicz spaces, which covers both of these as special cases. Generalized Orlicz spaces (also known as Musielak–Orlicz spaces) and related PDE have been intensely studied recently, see, e.g., [12,20–23] and references therein. Both the L^∞ - and the non-standard growth energy have been related to image processing [9,11,19].

We follow the general approach of previous papers but we develop improved techniques with streamlined proofs and weaker assumptions. The tools used in [5,15] led to the unexpected assumption that ratios of the lower and upper growth-rates of the approximating functionals (denoted $\varphi_n^\pm$ and $p_n^\pm$ in these papers) are bounded. We develop more precise estimates which allow us to avoid this assumption. Indeed, we need no assumptions about the upper growth-rate and can consider non-doubling functionals as well. For instance, we can handle the energy $|Du|^{n+1/|x|}$ that was excluded from previous results. In addition, we are able to deal with open sets of finite measure, whereas previous results required regularity of the boundary. We cover both norm- and modular-type energies with much the same method (cf. Theorem 4.4). The general φ -growth highlights the exact properties that are needed for the results more clearly than the L^p -case, as a comparison of the assumptions in Theorems 4.2 and 4.4 shows, see also Example 4.5.

We start by recalling some prior results related to Young measures and generalized Orlicz spaces. In Section 3, we prove key tools for transferring results for constant exponents to the generalized Orlicz case. We conclude in Section 4 with the main convergence results.

2. Auxiliary results

Throughout the paper we always consider a domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, i.e. an open and connected set. We denote the set of measurable functions $f : \Omega \rightarrow \mathbb{R}^d$ by $L^0(\Omega, \mathbb{R}^d)$. When $d = 1$, the second set is omitted and we write $L^0(\Omega)$; the same convention is used for other function spaces. The Lebesgue and Borel measures on $\mathbb{R}^k$ are denoted $\mathcal{L}^k$ and $\mathcal{B}_k$, respectively, and $\mathcal{L}^k(A)$ is abbreviated as $|A|$.

Definition 2.1. We say that $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is *level convex* if the level-set $\{\xi \in \mathbb{R}^N : f(\xi) \leq t\}$ is convex for every $t \in \mathbb{R}$.

The following is a substitute for Jensen's inequality for level convex functions.

Lemma 2.2 (Theorem 1.2, [4]). *Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a lower semicontinuous and level convex function and let μ be a probability measure in the open set $U \subset \mathbb{R}^N$. Then*

$$f\left(\int_U u \, d\mu\right) \leq \mu\text{-ess sup}_U f \circ u$$

for every $u \in L^1_\mu(U, \mathbb{R}^N)$.

Young measures. We recall some results regarding Young measures following the presentation of Section 2.3, [15] (which is based on [30]).

Definition 2.3. A function $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ is called a *normal integrand* if

- f is $\mathcal{L}^N \times \mathcal{B}_d \times \mathcal{B}_k$ -measurable;
- $f(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$.

We refer to Corollary 2.10 in [15] for the next result. The measures μ_x in the lemma are the so-called Young measures.

Lemma 2.4. Suppose that Ω is bounded and $u_n \rightharpoonup u$ in $W^{1,q}(\Omega; \mathbb{R}^d)$, $q \in (1, \infty)$. Then there exists a family of probability measures $(\mu_x)_{x \in \Omega}$ on $\mathbb{R}^{Nd}$ such that

$$Du(x) = \int_{\mathbb{R}^{Nd}} \xi d\mu_x(\xi) \quad \text{for a.e. } x \in \Omega$$

and, for any normal integrand $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \rightarrow \mathbb{R}^+$,

$$\liminf_{n \rightarrow \infty} \int_{\Omega} f(x, u_n(x), Du_n(x)) dx \geq \int_{\Omega} \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi) d\mu_x(\xi) dx.$$

With the notation of the previous lemma and $v \in L^0(\Omega, \mathbb{R}^d)$, [15, Remark 2.11] states that

$$\lim_{p \rightarrow \infty} \left(\int_{\Omega} \int_{\mathbb{R}^{Nd}} f(x, v(x), \xi)^p d\mu_x(\xi) dx \right)^{\frac{1}{p}} = \operatorname{ess\,sup}_{x \in \Omega} \mu_x\text{-ess\,sup}_{\xi \in \mathbb{R}^k} f(x, v(x), \xi).$$

Combined with the inequality $\mu_x\text{-ess\,sup}_{\xi \in \mathbb{R}^k} f(x, v(x), \xi) \geq f(x, v(x), Du(x))$ from Lemma 2.2, this implies the following:

Corollary 2.5. With the assumptions and notation of the previous lemma, we have

$$\lim_{p \rightarrow \infty} \left(\int_{\Omega} \int_{\mathbb{R}^{Nd}} f(x, v(x), \xi)^p d\mu_x(\xi) dx \right)^{\frac{1}{p}} \geq \operatorname{ess\,sup}_{x \in \Omega} f(x, v(x), Du(x))$$

provided that $f(x, z, \cdot)$ is level convex for every $z \in \mathbb{R}^d$ and a.e. $x \in \Omega$.

Generalized Orlicz spaces. In this subsection, we provide background on generalized Orlicz spaces. For more details, see [12,18].

The next condition, which measures the lower growth-rate, can be stated as $\frac{f(t)}{t^p}$ being almost increasing (hence the abbreviation). We use an equivalent form which is easier to apply.

Definition 2.6. Let $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ and $p > 0$. We say that f satisfies $(\text{aInc})_p$ if there exists $L \geq 1$ such that $f(x, \lambda t) \leq L\lambda^p f(x, t)$ for a.e. $x \in \Omega$ and all $t \geq 0$, $\lambda \leq 1$.

Note that if φ satisfies $(\text{aInc})_{p_1}$, then it satisfies $(\text{aInc})_{p_2}$, for every $p_2 < p_1$. The condition $(\text{aDec})_p$, related to upper growth-rate, is defined by the same inequality when $\lambda \geq 1$, but we do not need it in this paper.

Definition 2.7. We say that $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$ is a *generalized weak Φ -function* and write $\varphi \in \Phi_w(\Omega)$ if

- $x \mapsto \varphi(x, |f(x)|)$ is measurable for every $f \in L^0(\Omega)$,
- $\varphi(x, 0) = 0$, $\lim_{t \rightarrow 0^+} \varphi(x, t) = 0$ and $\lim_{t \rightarrow \infty} \varphi(x, t) = \infty$ for a.e. $x \in \Omega$,
- $t \mapsto \varphi(x, t)$ is increasing for a.e. $x \in \Omega$,
- φ satisfies [\(aInc\)₁](#).

If φ does not depend on x , then we omit the set and write $\varphi \in \Phi_w$.

We can now define generalized Orlicz spaces. Example [2.9](#) shows that this framework covers also L^∞ -spaces without the need for special cases. This is of special importance in this article as we consider the limit when the growth-rate tends to infinity.

Definition 2.8. Let $\varphi \in \Phi_w(\Omega)$ and let the modular ϱ_φ be given by

$$\varrho_\varphi(f) := \int_{\Omega} \varphi(x, |f(x)|) dx$$

for $f \in L^0(\Omega)$. The set

$$L^\varphi(\Omega) := \{f \in L^0(\Omega) : \varrho_\varphi(\lambda f) < \infty \text{ for some } \lambda > 0\}$$

is called a *generalized Orlicz space*. It is equipped with the Luxemburg quasinorm

$$\|f\|_\varphi := \inf \left\{ \lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

Example 2.9. Define $\varphi_\infty \in \Phi_w(\Omega)$ by $\varphi_\infty(x, t) := \infty \chi_{(1, \infty)}(t)$. Then φ_∞ is a generalized weak Φ -function. From the definition of modular we see that

$$\varrho_{\varphi_\infty}(f) = \begin{cases} 0, & \text{if } |f| \leq 1 \text{ a.e.} \\ \infty, & \text{otherwise.} \end{cases}$$

It follows from the Luxemburg norming procedure that $\|\cdot\|_{\varphi_\infty} = \|\cdot\|_\infty$ and so $L^{\varphi_\infty}(\Omega) = L^\infty(\Omega)$.

Other examples of generalized Orlicz spaces are (ordinary) Orlicz spaces where $\varphi(x, t)$ is independent of x , variable exponent spaces $L^{p(\cdot)}$ where $\varphi(x, t) = t^{p(x)}$ [\[14\]](#), double phase spaces $\varphi(x, t) = t^p + a(x)t^q$ [\[3, 17\]](#), variable exponent double phase spaces $\varphi(x, t) = t^{p(x)} + a(x)t^{q(x)}$ [\[13\]](#) and many other variants (e.g. [\[2\]](#)). The Orlicz–Sobolev space is defined based on $L^\varphi(\Omega)$ as usual:

Definition 2.10. Let $\varphi \in \Phi_w(\Omega)$. The function $u \in L^\varphi(\Omega) \cap W_{\text{loc}}^{1,1}(\Omega)$ belongs to the *Orlicz–Sobolev space* $W^{1,\varphi}(\Omega)$, if the weak partial derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \dots, N$, belong to $L^\varphi(\Omega)$. We define a modular and quasinorm on $W^{1,\varphi}(\Omega)$ by

$$\varrho_{1,\varphi}(u) := \varrho_\varphi(u) + \sum_{i=1}^N \varrho_\varphi\left(\frac{\partial u}{\partial x_i}\right) \quad \text{and} \quad \|u\|_{1,\varphi} := \inf \left\{ \lambda > 0 : \varrho_{1,\varphi}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

3. Asymptotically sharp embeddings

The unit ball property is a fundamental relation between quasinorm and modular. Note that we do not assume lower semi-continuity of φ which complicates the relationship a bit.

Lemma 3.1 (Unit ball property, Lemma 3.2.3, [18]). *If $\varphi \in \Phi_w(\Omega)$, then*

$$\|f\|_\varphi < 1 \quad \Rightarrow \quad \varrho_\varphi(f) \leq 1 \quad \Rightarrow \quad \|f\|_\varphi \leq 1.$$

This next property anchors φ at 1. It follows from (aInc)₁ that if $\frac{1}{c} \leq \varphi(x, 1) \leq c$ for a.e. $x \in \Omega$, then φ satisfies (A0) with $\beta := \frac{1}{Lc}$.

Definition 3.2. We say that a function $\varphi \in \Phi_w(\Omega)$ satisfies (A0) if there exists $\beta \in (0, 1]$ such that $\varphi(x, \beta) \leq 1 \leq \varphi(x, \frac{1}{\beta})$ for a.e. $x \in \Omega$.

The following proposition plays a significant role in reducing our main problem to the classic Lebesgue space $L^q(\Omega)$, where we can use the results about Young measures previously presented. The embedding is well-known (see [18, Lemma 3.7.7]) but we need a version with an asymptotically sharp constant as $p \rightarrow \infty$.

Proposition 3.3. *Let Ω have finite measure, $\varphi \in \Phi_w(\Omega)$ and $c, L \geq 1$. Assume that*

- $\frac{1}{c} \leq \varphi(x, 1) \leq c$ for a.e. $x \in \Omega$;
- φ satisfies (aInc)_p with constant L for some $p \geq 1$.

Then $L^\varphi(\Omega) \hookrightarrow L^p(\Omega)$ and

$$\|\cdot\|_{L^p} \leq (2L(|\Omega| + c))^{\frac{1}{p}} \|\cdot\|_\varphi.$$

Proof. Since $\frac{1}{c} \leq \varphi(x, 1) \leq c$ and (aInc)_p imply (A0), the embedding holds [18, Lemma 3.7.7]. By the unit ball property (Lemma 3.1), the norm inequality $\|u\|_{L^p} \leq C\|u\|_\varphi$ follows once we prove that

$$\varrho_\varphi\left(\frac{C|u|}{\|u\|_p}\right) = \int_\Omega \varphi\left(x, \frac{C|u|}{\|u\|_p}\right) dx > 1.$$

Let us derive a lower bound for φ . By (aInc)_p,

$$\varphi(x, t) \geq \frac{1}{L} t^p \varphi(x, 1) \geq \frac{1}{Lc} t^p$$

when $t \geq L^{1/p} \geq 1$. If we subtract $\frac{1}{c}$ from the right-hand side, we obtain a function which is negative when $t < L^{1/p}$ and therefore a suitable lower bound also when t is in this range. It follows that

$$\varphi(x, t) \geq \frac{1}{Lc} t^p - \frac{1}{c}$$

for every $t \geq 0$. Applying this in the modular, we see that

$$\int_\Omega \varphi\left(x, \frac{C|u|}{\|u\|_p}\right) dx \geq \int_\Omega \frac{1}{Lc} \left(\frac{C|u|}{\|u\|_p}\right)^p - \frac{1}{c} dx = \frac{C^p}{Lc} - \frac{|\Omega|}{c}.$$

Thus the desired norm-inequality holds if the right-hand side is greater than 1, which is equivalent to $C^p > L(|\Omega| + c)$. $\square$

The next proposition essentially takes care of the limsup-part of the estimate in the main results. The almost increasing assumption is satisfied (with $L = 1$) for example when φ_n^{1/p_n} is convex.

Proposition 3.4. *Let Ω have finite measure, $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $c, L \geq 1$. Assume for $n \in \mathbb{N}$ that*

- $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ for a.e. $x \in \Omega$;
- φ_n satisfies $(aInc)_{p_n}$ with constant L for some $p_n \geq 1$.

If $p_n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \|u\|_{\varphi_n} = \|u\|_\infty$ for all $u \in L^\infty(\Omega)$.

Proof. If $\|u\|_\infty = 0$, then $u = 0$ and there is nothing to prove. It suffices to consider the case $\|u\|_\infty = 1$, since the general case can be reduced to it by the normalization $\tilde{u} := \frac{u}{\|u\|_\infty}$.

By the definition of limit, we have to find for every $\varepsilon \in (0, 1)$ a number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{1+\varepsilon} \leq \|u\|_{\varphi_n} \leq \frac{1}{1-\varepsilon}$$

for all $n \geq n_0$. By the unit ball property (Lemma 3.1) this inequality follows from

$$\varrho_{\varphi_n}((1-\varepsilon)u) \leq 1 < \varrho_{\varphi_n}((1+\varepsilon)u).$$

Fix $\varepsilon \in (0, 1)$. By $(aInc)_{p_n}$, $\varphi_n(x, 1-\varepsilon) \leq L(1-\varepsilon)^{p_n} \varphi_n(x, 1) \leq Lc(1-\varepsilon)^{p_n}$. Since $|u| \leq \|u\|_\infty = 1$ a.e. and φ_n is increasing, we conclude that

$$\varrho_{\varphi_n}((1-\varepsilon)u) = \int_{\Omega} \varphi_n(x, (1-\varepsilon)|u|) dx \leq \int_{\Omega} \varphi_n(x, 1-\varepsilon) dx \leq Lc(1-\varepsilon)^{p_n} |\Omega| \xrightarrow{n \rightarrow \infty} 0.$$

Thus we can find n_0 such that $\varrho_{\varphi_n}((1-\varepsilon)u) < 1$ for all $n \geq n_0$.

To deal with the second inequality, we consider $A := \{(1+\varepsilon)|u| > 1 + \frac{\varepsilon}{2}\}$. Since $\|u\|_\infty = 1$, it follows that $|A| > 0$. By $(aInc)_{p_n}$,

$$\varphi_n(x, 1 + \frac{\varepsilon}{2}) \geq \frac{1}{L}(1 + \frac{\varepsilon}{2})^{p_n} \varphi_n(x, 1) \geq \frac{1}{Lc}(1 + \frac{\varepsilon}{2})^{p_n}.$$

The inequality $1 < \varrho_{\varphi_n}((1+\varepsilon)u)$ follows for large n from the estimate

$$\int_{\Omega} \varphi_n(x, (1+\varepsilon)|u|) dx \geq \int_A \varphi_n(x, 1 + \frac{\varepsilon}{2}) dx \geq \frac{1}{Lc}(1 + \frac{\varepsilon}{2})^{p_n} |A| \xrightarrow{n \rightarrow \infty} \infty. \quad \square$$

The previous two results used the assumption $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ which is stronger than (A0). This is because (A0) is not sufficient, as the next example demonstrates.

Example 3.5. Define $\varphi_n \in \Phi_w$ by $\varphi_n(t) := (a_n t)^n$, where $\beta \leq a_n \leq \frac{1}{\beta}$. Then φ_n satisfies (A0) with constant β . If $a_n \rightarrow a \neq 1$, then $\|\cdot\|_{\varphi_n} \rightarrow \|\cdot\|_{\varphi_{a,\infty}} = a\|\cdot\|_\infty$, where $\varphi_{a,\infty}(t) := \infty \chi_{(1/a, \infty)}(t)$. Therefore, the conclusion of the previous proposition does not hold in this case. If $a > 1$, then the inequality $\|u\|_{\varphi_n} \leq c_n \|u\|_\infty$ of Proposition 3.3 similarly does not hold with any constant $c_n \rightarrow 1$.

The second example shows that the assumption that each φ_n satisfy $(aInc)_{p_n}$ with the same constant L is also needed for the conclusion.

Example 3.6. Define $\varphi \in \Phi_w$ by $\varphi(t) := \max\{0, 2t - 1\} + \varphi_\infty(t)$. We consider the constant sequence $\varphi_n = \varphi$, whose Luxemburg quasinorms converge to $\|\cdot\|_\varphi$, not $\|\cdot\|_\infty$. Since $\varphi_n(1) = 1$, the first condition of Proposition 3.4 is satisfied. Let (p_n) be a sequence converging to ∞ . We show that φ_n satisfies $(aInc)_{p_n}$ with a constant $L_n \geq 1$. If $t \leq \frac{1}{2}$, then $\varphi(t) = 0$, and if $t > 1$, then $\varphi(t) = \infty$. Thus it is enough to show the second inequality in the following when $\frac{1}{2} \leq \lambda t \leq t \leq 1$:

$$\varphi_n(\lambda t) = 2\lambda t - 1 \leq (2t - 1)\lambda \leq L_n \lambda^{p_n} (2t - 1) = L_n \lambda^{p_n} \varphi_n(t).$$

This inequality holds when $L_n \geq \lambda^{1-p_n}$. Since $\lambda \geq \frac{1}{2}$, the choice $L_n := 2^{p_n-1}$ works for $(aInc)_{p_n}$.

Note that both examples are of Orlicz type as the dependence of φ on x is not used here.

4. Main theorems

Let $\varphi \in \Phi_w(\Omega)$ and $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \rightarrow \mathbb{R}$. We define $E_\varphi, F_\varphi : L^1(\Omega, \mathbb{R}^d) \rightarrow [0, \infty]$ by

$$E_\varphi(u) := \begin{cases} \varrho_\varphi(f(\cdot, u, Du)) & \text{if } u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$F_\varphi(u) := \begin{cases} \|f(\cdot, u, Du)\|_\varphi & \text{if } u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

The abbreviations E_∞ and F_∞ refer to the case $\varphi = \varphi_\infty$ from Example 2.9, related L^∞ .

Remark 4.1. It follows from Theorems 4.2 and 4.4 that (F_{φ_n}) and (E_{φ_n}) Γ -converge to F_∞ and E_∞ respectively, with respect to the L^1 -weak topology. Specifically, we prove the $\liminf$ -property of Γ -convergence and show that we can use a constant recovery sequence; we refer the reader to [8] for the definitions related to Γ -convergence which are not otherwise needed here.

Theorem 4.2. Assume that Ω has finite measure. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \rightarrow \mathbb{R}$ be a normal integrand such that

- $f(x, u, \cdot)$ is level convex for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^d$;
- $f(x, u, \xi) \geq \alpha |\xi|^\gamma$ for some $\alpha, \gamma > 0$, a.e. $x \in \Omega$ and every $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}$.

Let $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $c, L \geq 1$ and assume that

- $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ for a.e. $x \in \Omega$;
- φ_n satisfies $(aInc)_{p_n}$ with constant L for some $p_n \geq 1$.

If $p_n \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\limsup_{n \rightarrow \infty} F_{\varphi_n}(u) \leq F_\infty(u) \leq \liminf_{n \rightarrow \infty} F_{\varphi_n}(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$.

Proof. For simplicity we abbreviate $F_n := F_{\varphi_n}$, $n \in \mathbb{N}$. Let us first show that

$$\limsup_{n \rightarrow \infty} F_n(u) \leq F_\infty(u)$$

for $u \in L^1(\Omega, \mathbb{R}^d)$. If $F_\infty(u) = \infty$, this is clear. Otherwise, $F_\infty(u) < \infty$ so $f(\cdot, u, Du) \in L^\infty(\Omega)$ and Proposition 3.4 gives that

$$\lim_{n \rightarrow \infty} F_n(u) = \lim_{n \rightarrow \infty} \|f(\cdot, u, Du)\|_{\varphi_n} = \|f(\cdot, u, Du)\|_\infty = F_\infty(u).$$

We next deal with the liminf-inequality. Let $(u_n)_n \subset L^1(\Omega, \mathbb{R}^d)$ converge weakly to $u \in L^1(\Omega, \mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n \rightarrow \infty} F_n(u_n) = \lim_{n \rightarrow \infty} F_n(u_n) = M < \infty.$$

Then every subsequence of $(u_n)_n$ also has limit M . Recall that $p_n \rightarrow \infty$. Fix $q > \gamma$ and let $n_0 \in \mathbb{N}$ be such that

$$p_n \geq \frac{q}{\gamma} \quad \text{and} \quad F_n(u_n) \leq M + 1 \quad \text{for all } n \geq n_0.$$

From $p_n \geq \frac{q}{\gamma}$ it follows that φ_n satisfies (aInc) $_{q/\gamma}$ with the same constant L as in the assumption. By Proposition 3.3,

$$\|f(\cdot, u_n, Du_n)\|_{q/\gamma} \leq \underbrace{(2L(|\Omega| + c))^\frac{\gamma}{q}}_{=: C_q} \|f(\cdot, u_n, Du_n)\|_{\varphi_n} \leq C_1(M + 1). \quad (4.3)$$

for every $n \geq n_0$. We note that $C_q \searrow 1$ as $q \rightarrow \infty$.

We use this inequality and the assumed lower bound on f to estimate the norm of the gradient:

$$\|Du_n\|_q^\gamma = \| |Du_n|^\gamma \|_{q/\gamma} \leq \frac{1}{\alpha} \|f(\cdot, u_n, Du_n)\|_{q/\gamma} \leq \frac{C_1}{\alpha} (M + 1).$$

It follows that $(\|Du_n\|_q)_n$ is bounded. Then, up to a subsequence (depending on q), $(Du_n)_n$ weakly converges to a function w in $L^q(\Omega, \mathbb{R}^{Nd})$. Since $(u_n)_n$ weakly converges to u in $L^1(\Omega, \mathbb{R}^d)$, w is the distributional gradient of u . Fix an open ball $B \subset \Omega$. We use a version of the Poincaré inequality where L^q -integrability is required only for the gradient. By the theorem in Section 1.5.2 [26, p. 35], we obtain that

$$\begin{aligned} \|u_n\|_{L^q(B, \mathbb{R}^d)} &\leq \|u_n - (u_n)_B\|_{L^q(B, \mathbb{R}^d)} + \|(u_n)_B\|_{L^q(B, \mathbb{R}^d)} \\ &\leq c(N, B) \|Du_n\|_{L^q(B, \mathbb{R}^d)} + |B|^{\frac{1}{q}-1} \|u_n\|_{L^1(B, \mathbb{R}^d)}. \end{aligned}$$

Since $(\|Du_n\|_q)_n$ is bounded and $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$, the right hand side is uniformly bounded. Thus reflexivity in $L^q(B, \mathbb{R}^d)$ yields that $(u_n)_n$ has a weakly convergent subsequence.

We have found a subsequence, denoted still by $(u_n)_n$, with $u_n \rightharpoonup u$ in $W^{1,q}(B, \mathbb{R}^d)$. Let $(\mu_x)_{x \in B}$ be the family of probability measures (the “Young measure”) from Lemma 2.4 corresponding to this subsequence. We use the first inequality from (4.3) in B and Lemma 2.4 for the normal integrand $f^{q/\gamma}$ to conclude that

$$\begin{aligned} \liminf_{n \rightarrow \infty} F_n(u_n) &\geq \liminf_{n \rightarrow \infty} \frac{1}{C_q} \left(\int_B f(x, u_n(x), Du_n(x))^\frac{q}{\gamma} dx \right)^\frac{\gamma}{q} \\ &\geq \frac{1}{C_q} \left(\int_B \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi)^\frac{q}{\gamma} d\mu_x(\xi) dx \right)^\frac{\gamma}{q}. \end{aligned}$$

Take the limit inferior on the right-hand side as $q \rightarrow \infty$ and use $C_q \rightarrow 1$ as well as Corollary 2.5:

$$\begin{aligned} \liminf_{n \rightarrow \infty} F_n(u_n) &\geq \liminf_{q \rightarrow \infty} \left(\int_B \int_{\mathbb{R}^{Nd}} f(x, u(x), \xi)^{\frac{q}{\gamma}} d\mu_x(\xi) dx \right)^{\frac{\gamma}{q}} \\ &\geq \operatorname{ess\,sup}_{x \in B} f(x, u(x), Du(x)) = \sup_{x \in B \setminus A} f(x, u(x), Du(x)), \end{aligned}$$

where $A \subset B$ is the exceptional set of measure zero related to the essential supremum. We can write $\Omega = \cup_{i \in \mathbb{N}} B_i$ as a countable union of balls B_i and then we obtain an estimate for the supremum in $\Omega \setminus \cup_{i \in \mathbb{N}} A_i \subset \cup_{i \in \mathbb{N}} (B_i \setminus A_i)$, where A_i is the exceptional set corresponding to B_i . Since also $\cup_{i \in \mathbb{N}} A_i$ has measure zero, this gives the essential supremum in Ω . Thus we have proved the lim inf-inequality and completed the proof. $\square$

We conclude with a similar result for the modular-based energy functional. We use the notation $\varphi^-(1) := \operatorname{ess\,inf}_{x \in \Omega} \varphi(x, 1)$ and $\varphi^+(1) := \operatorname{ess\,sup}_{x \in \Omega} \varphi(x, 1)$.

Theorem 4.4. *Assume that Ω has finite measure. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{Nd} \rightarrow \mathbb{R}$ be a normal integrand such that*

- $f(x, u, \cdot)$ is level convex for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^d$;
- $f(x, u, \xi) \geq \alpha |\xi|^\gamma$ for some $\alpha, \gamma > 0$, a.e. $x \in \Omega$ and every $(u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{Nd}$.

Let $\varphi_n \in \Phi_w(\Omega)$ for $n \in \mathbb{N}$ and $L \geq 1$ and assume that

- $\limsup_{n \rightarrow \infty} \varphi_n^+(1) = 0$ and $\liminf_{n \rightarrow \infty} \varphi_n^-(1)^{1/p_n} \geq 1$;
- φ_n satisfies (aInc) $_{p_n}$ with constant L for some $p_n \geq 1$.

If $p_n \rightarrow \infty$ as $n \rightarrow \infty$, then

$$\limsup_{n \rightarrow \infty} E_n(u) \leq E_\infty(u) \leq \liminf_{n \rightarrow \infty} E_n(u_n)$$

for all $u, u_n \in L^1(\Omega, \mathbb{R}^d)$ with $u_n \rightharpoonup u$ in $L^1(\Omega, \mathbb{R}^d)$.

Proof. We again abbreviate $E_n := E_{\varphi_n}$, $n \in \mathbb{N}$. Let us first show that

$$\limsup_{n \rightarrow \infty} E_n(u) \leq E_\infty(u)$$

for any $u \in L^1(\Omega, \mathbb{R}^d)$. If $E_\infty(u) = \infty$, this is clear. Otherwise, $E_\infty(u) = 0$ so $\|f(\cdot, u, Du)\|_\infty \leq 1$. Thus we obtain that

$$\limsup_{n \rightarrow \infty} E_n(u) = \limsup_{n \rightarrow \infty} \int_\Omega \varphi_n(x, f(\cdot, u, Du)) dx \leq \limsup_{n \rightarrow \infty} |\Omega| \varphi_n^+(1) = 0$$

and the inequality follows.

To deal with the lim inf-inequality, let $(u_n)_n \subset L^1(\Omega, \mathbb{R}^d)$ converge weakly to $u \in L^1(\Omega, \mathbb{R}^d)$. Without loss of generality, we assume that

$$\liminf_{n \rightarrow \infty} E_n(u_n) = \lim_{n \rightarrow \infty} E_n(u_n) = M < \infty.$$

Define $\hat{\varphi}_n \in \Phi_w(\Omega)$ by $\hat{\varphi}_n(x, t) := \frac{\varphi_n(x, t)}{\varphi_n(x, 1)}$. Let $n_0 \in \mathbb{N}$ be such that $E_n(u_n) \leq M + 1$ for all $n \geq n_0$, so that

$$\int_{\Omega} \hat{\varphi}_n(x, f(x, u_n, Du_n)) dx \leq \frac{M + 1}{\varphi_n^-(1)}.$$

Since $\hat{\varphi}_n$ satisfies $(\text{aInc})_{p_n}$ with the same constant L , this yields by Corollary 3.2.10 of [18] that

$$\|f(\cdot, u_n, Du_n)\|_{\hat{\varphi}_n} \leq \max \left\{ (L \varrho_{\hat{\varphi}_n}(f(\cdot, u_n, Du_n)))^{\frac{1}{p_n}}, 1 \right\} \leq \max \left\{ \left(\frac{L(M + 1)}{\varphi_n^-(1)} \right)^{\frac{1}{p_n}}, 1 \right\},$$

and consequently

$$\limsup_{n \rightarrow \infty} \|f(\cdot, u_n, Du_n)\|_{\hat{\varphi}_n} \leq \max \left\{ \limsup_{n \rightarrow \infty} \varphi_n^-(1)^{-\frac{1}{p_n}}, 1 \right\} = 1.$$

Since $\hat{\varphi}_n$ satisfies the requirements of Theorem 4.2 with the same L and $c = 1$, it follows that $F_{\infty}(u) \leq 1$, i.e. $u \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^d)$ and $\|f(\cdot, u, Du)\|_{\infty} \leq 1$. Thus $E_{\infty}(u) \leq \varrho_{\infty}(1) = 0$, and hence the lim inf-inequality holds. $\square$

The assumptions of Theorem 4.4 differ from Theorem 4.2. The next example shows that the condition $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ is not sufficient in the latter theorem.

Example 4.5. Let $p_n := n$ for all n . Then $\varphi_n(t) := t^n$ satisfies $(\text{aInc})_n$ and $\frac{1}{c} \leq \varphi_n(x, 1) \leq c$ but not the condition $\limsup_{n \rightarrow \infty} \varphi_n^+(1) = 0$ of the previous theorem. Moreover, the claim of the theorem also does not hold: if f and u are such that $f(x, u, Du) = 1$ a.e., then $E_n(u) = |\Omega|$ for every $n \in \mathbb{N}$ but $E_{\infty}(u) = 0$ so that

$$\limsup_{n \rightarrow \infty} E_n(u) > E_{\infty}(u).$$

On the other hand, the energy $\varphi_n(t) := \frac{1}{n}t^n$ does satisfy the conditions in previous theorem.

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