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UNIFORM GAUSSIAN BOUNDS FOR SUBELLIPTIC HEAT KERNELS AND AN APPLICATION TO THE TOTAL VARIATION FLOW OF GRAPHS OVER CARNOT GROUPS

LUCA CAPOGNA, GIOVANNA CITTI, AND MARIA MANFREDINI

ABSTRACT. In this paper we study heat kernels associated to a Carnot group G , endowed with a family of collapsing left-invariant Riemannian metrics σ_ϵ which converge in the Gromov-Hausdorff sense to a sub-Riemannian structure on G as $\epsilon \rightarrow 0$. The main new contribution are Gaussian-type bounds on the heat kernel for the σ_ϵ metrics which are stable as $\epsilon \rightarrow 0$ and extend the previous time-independent estimates in [16]. As an application we study well posedness of the total variation flow of graph surfaces over a bounded domain in (G, σ_ϵ) . We establish interior and boundary gradient estimates, and develop a Schauder theory which are stable as $\epsilon \rightarrow 0$. As a consequence we obtain long time existence of smooth solutions of the sub-Riemannian flow ($\epsilon = 0$), which in turn yield sub-Riemannian minimal surfaces as $t \rightarrow \infty$.

1. INTRODUCTION

Models of image processing based on total variation flow

$$\partial_t u = \operatorname{div}(\nabla u / |\nabla u|)$$

have been first introduced in by Rudin, Osher, and Fatemi in [43], in order to perform edge preserving denoising. We refer the reader to the review paper [19] where many recent applications of total variation equation in the Euclidean setting are presented. The flow $t \rightarrow u(\cdot, t)$ represents the gradient descent associated to the total variation energy $\int |\nabla u|$ and as such has the property that both the total variation of the solution $t \rightarrow u(\cdot, t)$ and the perimeter measure of fixed level sets $\{u(\cdot, t) = \text{const}\}$ are non-increasing in time. Aside from its usefulness in image processing, the flow also arises in connection with the limit of solutions of the parabolic p -Laplacian $\partial_t u_p = \operatorname{div}(|\nabla u_p|^{p-2} \nabla u_p)$ as the parameter $p \rightarrow 1^+$. In the case where the evolution of graphs $S_t = \{(x, u(x, t))\}$ is considered, i.e. $\partial_t u = \operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2})$, then in both the total variation flow and the closely related *mean curvature flow* $\partial_t u = \sqrt{1 + |\nabla u|^2} \operatorname{div}(\nabla u / \sqrt{1 + |\nabla u|^2})$, given appropriate boundary/initial conditions, global in time solutions asymptotically converge to minimal graphs. For further results, references and applications of the Euclidean total variation flow we refer the reader to [1], [2], [5] and for an anisotropic version of the flow to [36].

In this paper we study long time existence of graph solutions of the total variation flow in a special class of degenerate Riemannian ambient spaces: The so-called (sub-Riemannian) Carnot groups [26], [46]. Such spaces are nilpotent Lie groups endowed with a metric structure (G, σ_0) that arises as limit of collapsing left-invariant *tame* Riemannian structures (G, σ_ϵ) . Our motivations are twofold:

(a) On the one hand, minimal surfaces and mean curvature flow in the Carnot group setting (and in particular in the special case of the Heisenberg group) have been the subject of numerous papers in recent years, leading to partial solutions of long-standing problems such as the Pansu conjecture¹. Despite such advances little is known about explicit construction of minimal surfaces. Since the asymptotic limit of

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¹See for instance [13], [41], [37], [40], [15] [21], [14], [9], [10], [45] and references therein

graphical mean curvature flow provide minimal surfaces in the Riemannian setting it is only natural to follow the same approach in the sub-Riemannian setting. However this avenue runs into considerable (so far unsolved) technical difficulties due to the combined effect of the degeneracy of the metric and the non-divergence form aspect of the relevant PDE. In the present work we provide the basis for the construction of sub-Riemannian minimal graphs through the asymptotic behavior as $t \rightarrow \infty$ of solutions of the sub-Riemannian total variation flow. For previous work on a similar theme see the work of Pauls [40] and Cheng, Hwang and Yang [14].

(b) A new class of image processing models based on the functionality of the visual cortex have been recently introduced in Lie groups. With a generalization of the classical Bargmann transform studied by Folland (see [27]) which lifts a L^2 function to a new one, defined on the phase space, an image can be lifted to a Lie group with a sub-Riemannian metric. The choice of the Lie group depends on the geometric property of the image to be studied: in [18], and [23], [24] 2D images are lifted to surfaces in the Lie group of Euclidean motions of the plane to study geometric properties of their level lines, in [25] cardiac images are lifted in the Heisenberg group to study their deformations, in [4] moving images are lifted in the Galilei group. The image processing is performed in these groups with algorithms expressed in terms of second order subelliptic differential equations. In particular sub-Riemannian mean curvature flows or total variation flows can be applied to perform image completion or inpainting in this setting.

Our approach to the existence of global (in time) smooth solutions is based on a Riemannian approximation scheme: We study graph solutions of the total variation flow in the Riemannian spaces (G, σ_ϵ) where G is a Carnot group and σ_ϵ is a family of tame Riemannian metrics that 'collapse' as $\epsilon \rightarrow 0$ to a sub-Riemannian metric σ_0 in G . The *main technical novelties in the paper* are a series of different a-priori estimates, which are stable as $\epsilon \rightarrow 0$:

- (1) Heat kernel estimates for sub-Laplacians and their elliptic regularizations (see Proposition 2.2). These estimates are established in the setting of any Carnot group and provide a *parabolic* counterpart to the time-independent estimates proved by two of the authors in [16].
- (2) Uniform Schauder estimates for second order, non-divergence form subelliptic PDE and their *elliptic* regularizations (see Proposition 4.4). These estimates are established in the setting of any Carnot group.
- (3) Interior gradient estimates for solutions of the total variation flow (see Proposition 3.2). These estimates are established in the setting of any Carnot group.
- (4) Boundary gradient estimates for solutions of the total variation flow (see Proposition 3.4). These estimates are established only in the setting of Carnot group of step two.

We remark that (3) and (4) can also be proved, with similar arguments, for solutions of the sub-elliptic mean curvature flow². The limitation to Carnot groups of step two in (4) is due to the fact that for step three and higher there no known suitable barrier functions, or in other words, the class of known explicit minimal graphs is not sufficiently large.

In order to state our results we need to introduce some notation.

1.1. Carnot group structure. Let G be an analytic and simply connected Lie group with topological dimension n and such that its Lie algebra \mathcal{G} admits a stratification $\mathcal{G} = V^1 \oplus V^2 \oplus \dots \oplus V^r$, where $[V^1, V^j] = V^{j+1}$, if $j = 1, \dots, r-1$, and $[V^k, V^r] = 0$, $k = 1, \dots, r$. Such groups are called in [26], [28], and [46] *stratified nilpotent Lie groups*. Set $H(0) = V^1$, and for any $x \in G$ we let $H(x) = xH(0) = \text{span}(X_1, \dots, X_m)(x)$. The distribution $x \rightarrow H(x)$ is called *the horizontal sub-bundle* H .

The metric structure is given by assuming that one has a left invariant positive definite form σ_0 defined in the sub-bundle H . We fix a orthonormal horizontal frame X_1, \dots, X_m which we complete it to a basis

²For more references on the sub-Riemannian mean curvature flow see [8], [22], [12] and [35]

(X_1, \dots, X_n) of \mathcal{G} by choosing for every $k = 2, \dots, r$ a basis of V_k . If X_i belongs to V_k , then we will define its *homogeneous degree* as

$$(1.1) \quad d(i) = k.$$

We will denote by $xX = \sum_{i=1}^n x_i X_i$ a generic element of \mathcal{G} . Since the exponential map $\exp : \mathcal{G} \rightarrow G$ is a global diffeomorphism we use exponential coordinates in G , and denote $x = (x_1, \dots, x_n)$ the point $\exp(xX)$.

We also set $x_H = (x_1, \dots, x_m)$. Define non-isotropic dilations as $\delta_s(x) = (s^{d(i)} x_i)$, for $s > 0$.

We let $\nabla_0 = (X_1, \dots, X_m)$ denote the *horizontal gradient* operator. If $\phi \in C^\infty(G)$ we set $\nabla_0 \phi = \sum_{i=1}^m X_i \phi X_i$ and $|\nabla_0 \phi|^2 = \sum_{i=1}^m (X_i \phi)^2$.

We denote by (X_1, \dots, X_n) (resp. (X_1^r, \dots, X_n^r)) the left invariant (resp. right invariant) translation of the frames (X_1, \dots, X_n) .

The vectors X_1, \dots, X_m and their commutators span all the Lie algebra \mathcal{G} , and consequently verify Hörmander's finite rank condition ([32]). This allows to use the results from [39], and define a control distance $d_0(x, y)$ associated to the distribution X_1, \dots, X_m , which is called *the Carnot-Carathéodory metric* (denote by $d_{r,0}$ the corresponding right invariant distance). We call the couple (G, d_0) a *Carnot Group*.

We define a family of left invariant Riemannian metrics σ_ϵ , $\epsilon > 0$ in \mathcal{G} by requesting that

$$\{X_1^\epsilon, \dots, X_n^\epsilon\} := \{X_1, \dots, X_m, \epsilon X_{m+1}, \dots, \epsilon X_n\}$$

is an orthonormal frame. We will denote by d_ϵ the corresponding distance functions. Correspondingly we use ∇_ϵ , (resp. ∇_ϵ^r) to denote the left (resp. right) invariant gradients. In particular, if $\phi \in C^\infty(G)$ we set $\nabla_\epsilon \phi = \sum_{i=1}^m X_i^\epsilon \phi X_i^\epsilon$ and $|\nabla_\epsilon \phi|^2 = \sum_{i=1}^m (X_i^\epsilon \phi)^2$.

We conclude by recalling the expression of the left invariant vector fields in exponential coordinates (see [42])

$$(1.2) \quad X_i = \partial_i + \sum_{k=d(i)+1}^r \sum_{d(j)=k} p_{ik}^j(x) \partial_j,$$

where $p_{ik}^j(x)$ is an homogeneous polynomial of degree $k - d(i)$ and depends only on x_h , with $d(1) \leq d(h) \leq k - d(i)$.

1.2. The total variation. The total variation flow is characterized by the fact that each point of the evolving surface graph moves in the direction of the *upward* unit normal with speed equal to the mean curvature times the volume element. In the setting of the approximating Riemannian metrics (G, σ_ϵ) and in terms of the functions $t \rightarrow u(\cdot, t) : \Omega \subset G \rightarrow \mathbb{R}$ describing the evolving graphs, the relevant equation reads:

$$(1.3) \quad \frac{\partial u_\epsilon}{\partial t} = h_\epsilon = \sum_{i=1}^n X_i^\epsilon \left(\frac{X_i^\epsilon u_\epsilon}{W_\epsilon} \right) = \sum_{i,j=1}^n a_{ij}^\epsilon (\nabla_\epsilon u_\epsilon) X_i^\epsilon X_j^\epsilon u_\epsilon$$

for $x \in \Omega$ and $t > 0$, with $u_\epsilon(x, 0) = \varphi(x)$, h_ϵ is the mean curvature of the graph of $u_\epsilon(\cdot, t)$ and

$$W_\epsilon^2 = 1 + |\nabla_\epsilon u_\epsilon|^2 = 1 + \sum_{i=1}^n (X_i^\epsilon u_\epsilon)^2 \quad \text{and} \quad a_{ij}^\epsilon(\xi) = W_\epsilon^{-1} \left(\delta_{ij} - \frac{\xi_i \xi_j}{1 + |\xi|^2} \right),$$

for all $\xi \in \mathbb{R}^n$. In the sub-Riemannian limit $\epsilon = 0$ the equation reads

$$(1.4) \quad \frac{\partial u}{\partial t} = \sum_{i=1}^m X_i \left(\frac{X_i u}{\sqrt{1 + |\nabla_0 u|^2}} \right),$$

for $x \in \Omega$ and $t > 0$, with $u(x, 0) = \varphi(x)$.

We will be concerned with uniform (in the parameter ϵ as $\epsilon \rightarrow 0$) estimates and with the asymptotic behavior of solutions to the initial value problem for the mean curvature motion of graphs over bounded domains of a Carnot group G ,

$$(1.5) \quad \begin{cases} \partial_t u_\epsilon = h_\epsilon & \text{in } Q = \Omega \times (0, T) \\ u_\epsilon = \varphi & \text{on } \partial_p Q. \end{cases}$$

Here $\partial_p Q = (\Omega \times \{t = 0\}) \cup (\partial\Omega \times (0, T))$ denotes the parabolic boundary of Q .

Given appropriate hypothesis on the data, for instance convexity of Ω and φ twice continuously differentiable, the classical parabolic theory yields existence and uniqueness for smooth solutions u_ϵ of (1.5). However classical parabolic theory will only provide estimates involving constants that degenerate as $\epsilon \rightarrow 0$ (in the transition from parabolic to degenerate parabolic regime). Our main goal consists in proving stable estimates.

Our first result consists in showing that if the initial/boundary data is sufficiently smooth then the solutions of (1.5) are Lipschitz up to the boundary uniformly in $\epsilon > 0$.

Theorem 1.1. (*Global gradient bounds*) *Let G be a Carnot group of step two, $\Omega \subset G$ a bounded, open, convex³ set and $\varphi \in C^2(\bar{\Omega})$. For $1 \geq \epsilon > 0$ denote by $u_\epsilon \in C^2(\Omega \times (0, T)) \cap C^1(\bar{\Omega} \times (0, T))$ the non-negative unique solution of the initial value problem (1.5). There exists $C = C(G, \|\varphi\|_{C^2(\bar{\Omega})}) > 0$ such that*

$$(1.6) \quad \sup_{\bar{\Omega} \times (0, T)} |\nabla_\epsilon u_\epsilon| \leq \sup_{\bar{\Omega} \times (0, T)} |\nabla_1 u_\epsilon| \leq C.$$

Remark 1.2. The hypothesis of the Theorem above can be slightly weakened by asking that G is a step two Carnot group, $\varphi \in C^2(\bar{\Omega})$ but that $\partial\Omega$ be required to be convex in the Euclidean sense only in a neighborhood of its characteristic locus $\Sigma(\partial\Omega)$ and that its mean curvature $h_\epsilon^{\partial\Omega} \leq -\delta < 0$ in $\partial\Omega \setminus \Sigma(\partial\Omega)$.

Having established Lipschitz bounds, it is easy to show that the right derivatives $X_i^r u_\epsilon$ of the solutions of (1.5) are themselves solutions of (3.1), a divergence form, degenerate parabolic PDE whose weak solutions satisfy a Harnack inequality (see [11] and Proposition 3.8 below). Consequently one obtains $C^{1,\alpha}$ interior estimates for the solution u_ϵ which are uniform in $\epsilon > 0$. At this point one rewrites the PDE in (1.5) in non-divergence form⁴ of the total variation flow equation $\partial_t u_\epsilon = a_{ij}^\epsilon(\nabla_\epsilon u_\epsilon) X_i^\epsilon X_j^\epsilon u_\epsilon$ and invokes the Schauder estimates in Proposition 4.4 to prove local higher regularity and long time existence. Since all the estimates are stable as $\epsilon \rightarrow 0$ one immediately obtains smoothness and the consequent global in time existence of the solution for the sub-Riemannian case $\epsilon = 0$.

Theorem 1.3. *In the hypothesis of Theorem 1.1 (or Remark 1.2) one has that there exists a unique solution $u_\epsilon \in C^\infty(\Omega \times (0, \infty)) \cap L^\infty((0, \infty), C^1(\bar{\Omega}))$ of the initial value problem*

$$(1.7) \quad \begin{cases} \partial_t u_\epsilon = h_\epsilon & \text{in } Q = \Omega \times (0, \infty) \\ u_\epsilon = \varphi & \text{on } \partial_p Q \end{cases}$$

and that for each $k \in \mathbb{N}$ there exists $C_k = C_k(G, \varphi, k, \Omega) > 0$ not depending on ϵ such that

$$(1.8) \quad \|u_\epsilon\|_{C^k(Q)} \leq C_k.$$

Since the estimates are uniform in ϵ and in time, and with respect to ϵ , we will deduce the following corollary:

³We say that a set $\Omega \subset G$ is convex in the Euclidean sense if $\exp^{-1}(\Omega) \subset \mathcal{G}$ is convex in the Euclidean space \mathcal{G} . In a group of step two this condition is translation invariant.

⁴this is possible since for $\epsilon > 0$ the solutions u_ϵ are sufficiently smooth

Corollary 1.4. *Under the assumptions of the Theorem 1.1, as $\epsilon \rightarrow 0$ the solutions u_ϵ converge uniformly (with all its derivatives) on compact subsets of Q to the unique, smooth solution $u_0 \in C^\infty(\Omega \times (0, \infty)) \cap L^\infty((0, \infty), C^1(\bar{\Omega}))$ of the sub-Riemannian total variation flow (1.4) in $\Omega \times (0, \infty)$ with initial data φ .*

Corollary 1.5. *Under the assumptions of Theorem 1.1, as $T \rightarrow \infty$ the solutions $u_\epsilon(\cdot, t)$ converge uniformly on compact subsets of Ω to the unique solution of the minimal surface equation*

$$h_\epsilon = 0 \quad \text{in } \Omega$$

with boundary value φ , while $u_0 = \lim_{\epsilon \rightarrow 0} u_\epsilon \in C^\infty(\Omega) \cap Lip(\bar{\Omega})$ is the unique solution of the sub-Riemannian minimal surfaces equation $h_0 = 0$ in Ω , with boundary data φ .

2. STRUCTURE STABILITY IN THE RIEMANNIAN LIMIT

The Carnot-Carathéodory metric is equivalent to a more explicitly defined pseudo-distance function, that we will call (improperly) *gauge distance*, defined as

$$(2.1) \quad |x|^{2r!} = \sum_{k=1}^r \sum_{i=1}^{m_k} |x_i|^{\frac{2r!}{d(i)}}, \quad \text{and } d_0(x, y) = |y^{-1}x|.$$

The *ball-box* theorem in [39] states that there exists $A = A(G, \sigma_0)$ such that for each $x \in G$,

$$A^{-1}|x| \leq d_0(x, 0) \leq A|x|.$$

If $x \in G$ and $r > 0$, we will denote by

$$B(x, r) = \{y \in G \mid d_0(x, y) < r\}$$

the balls in the gauge distance. For each $\epsilon > 0$ we also define the distance function d_ϵ corresponding to the Riemannian metric σ_ϵ ,

$$d_\epsilon(x, y) = \inf \left\{ \int_0^1 |\gamma'|_{\sigma_\epsilon}(s) ds \text{ with } \gamma : [0, 1] \rightarrow G \right.$$

$$\left. \text{a Lipschitz curve s. t. } \gamma(0) = x, \gamma(1) = y \right\}$$

as well as the pseudo-distance $d_{G, \epsilon}(x, y) = N_\epsilon(y^{-1}x)$ with

$$(2.2) \quad N_\epsilon^2(x) = \sum_{d(i)=1} x_i^2 + \min \left\{ \sum_{i=2}^r \left(\sum_{d(k)=i} x_k^2 \right)^{\frac{1}{i}}, \epsilon^{-2} \sum_{d(i) \geq 2} x_i^2 \right\}.$$

Note that in the definition of d_ϵ , if the curve for which the infimum is achieved happens to be horizontal then $d_\epsilon(x, y) = d_0(x, y)$. In general we have $\sup_{\epsilon > 0} d_\epsilon(x, y) = d_0(x, y)$ and it is well known⁵ that (G, d_ϵ) converges in the Gromov-Hausdorff sense as $\epsilon \rightarrow 0$ to the sub-Riemannian space (G, d_0) . The *ball-box* theorem in [39] and elementary considerations yield that there exists $A = A(G, \sigma_0) > 0$ independent of ϵ such that for all $x, y \in G$

$$(2.3) \quad A^{-1}d_{G, \epsilon}(x, y) \leq d_\epsilon(x, y) \leq Ad_{G, \epsilon}(x, y)$$

(see for example [11]).

⁵See for instance [30] and references therein

2.1. Stability of the homogenous structure as $\epsilon \rightarrow 0$. If G is a Carnot group, d_ϵ is the distance function associated to σ_ϵ , we will denote

$$B_\epsilon(x, r) = \{y \in G \mid d_\epsilon(x, y) < r\}.$$

If we denote by dx the Lebesgue measure and by $|\Omega|$ the corresponding measure of a subset Ω , then Rea and two of the authors have recently proved in [11] that

Proposition 2.1. *There is a constant C independent of ϵ such that for every $x \in G$ and $r > 0$,*

$$|B_\epsilon(x, 2r)| \leq C|B_\epsilon(x, r)|.$$

Having this property the spaces (G, d_ϵ, dx) are called *homogenous* with constant $C > 0$ independent of ϵ (see [20]).

Let $\tau > 0$ and consider the space $\tilde{G} = G \times (0, \tau)$ with its product Lebesgue measure $dxdt$. In \tilde{G} define the pseudo-distance function

$$(2.4) \quad \tilde{d}_\epsilon((x, t), (y, s)) = \max(d_\epsilon(x, y), \sqrt{|t - s|}).$$

Proposition 2.1 tells us that $(\tilde{G}, \tilde{d}_\epsilon, dxdt)$ is a homogeneous space with constant independent of $\epsilon \geq 0$.

In the paper [11] it is also shown that a Poincaré inequality holds with a choice of a constant which is stable as $\epsilon \rightarrow 0$. The stability of the homogenous space structure and of the constant in the Poincaré inequality are two of the key factors in the proof of Proposition 3.8.

2.2. Stability in the estimates of the Heat Kernel. The results in this section are of independent interest and concern uniform Gaussian estimates for the heat kernel associated to elliptic regularizations of the Carnot group sub-Laplacians. They will be used in this paper in connection to the Schauder estimates and the higher regularity of the total variation flow. We will deal with non-divergence form operators similar to those in (1.3), but with constant coefficients. More precisely we will consider the operator

$$(2.5) \quad L_{\epsilon, A} = \partial_t - \sum_{i, j=1}^n a_{ij} X_i^\epsilon X_j^\epsilon$$

where $A = (a_{ij})_{i, j=1, \dots, n}$ is a symmetric, real-valued $n \times n$ matrix, such that for some choice of constants $\Lambda, C_1, C_2 > 0$ and for all $\xi \in \mathbb{R}^n$ one has

$$(2.6) \quad \Lambda^{-1} \sum_{d(i)=1} \xi_i^2 + C_1 \sum_{d(i)>1} \xi_i^2 \leq \sum_{i, j=1}^n a_{ij} \xi_i \xi_j \leq \Lambda \sum_{d(i)=1} \xi_i^2 + C_2 \sum_{d(i)>1} \xi_i^2.$$

The point here is that the constants C_1, C_2, Λ are independent of ϵ and provide Λ -uniform coercivity of (2.5) in the horizontal directions. Formally, in the sub-Riemannian limit $\epsilon \rightarrow 0$ the equation becomes

$$(2.7) \quad L_A = \partial_t - \sum_{i, j=1}^m a_{ij} X_i X_j.$$

In order to ensure that the operator $L_{\epsilon, A}$ (reps. L_A) is uniformly elliptic (reps. subelliptic), we will assume that the matrix $A = (a_{ij})$ belongs to a set of the form

$$M_\Lambda = \{A : A \text{ is a symmetric } n \times n, \text{ real valued constant matrix,}$$

$$\text{satisfying (2.6) for some choice of } C_1 \text{ and } C_2\}$$

for some fixed $\Lambda > 0$.

Heat kernel estimates in Nilpotent groups are well known (see for instance [33] and references therein). We also refer to [6] where a self contained proof is provided. In our work we will need estimates which are uniform in the variable ϵ as $\epsilon \rightarrow 0$, in the same spirit as the results in [16]. We will denote $\Gamma_{\epsilon, A}(x, t)$ the fundamental solution of (2.5), with matrix (a_{ij}) in M_Λ , and $\Gamma_A(x, t)$ the fundamental solution of (2.7).

Proposition 2.2. *There exists constants $C_\Lambda > 0$ depending on G, σ_0, Λ but independent of ϵ such that for each $\epsilon > 0$, $x \in G$ and $t > 0$ one has*

$$(2.8) \quad C_\Lambda^{-1} \frac{e^{-C_\Lambda \frac{d_\epsilon(x,0)^2}{t}}}{|B_\epsilon(0, \sqrt{t})|} \leq \Gamma_{\epsilon, A}(x, t) \leq C_\Lambda \frac{e^{-\frac{d_\epsilon(x,0)^2}{C_\Lambda t}}}{|B_\epsilon(0, \sqrt{t})|}.$$

For $s \in \mathbb{N}$ and k -tuple $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$ there exists $C_{s,k} > 0$ depending only on $k, s, G, \sigma_0, \Lambda$ such that

$$(2.9) \quad |(\partial_t^s X_{i_1}^\epsilon \cdots X_{i_k}^\epsilon \Gamma_{\epsilon, A})(x, t)| \leq C_{s,k} t^{-s-k/2} \frac{e^{-\frac{d_\epsilon(x,0)^2}{C_\Lambda t}}}{|B_\epsilon(0, \sqrt{t})|}$$

for all $x \in G$ and $t > 0$. For any $A_1, A_2 \in M_\Lambda$, $s \in \mathbb{N}$ and k -tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ there exists $C_{s,k} > 0$ depending only on $k, s, G, \sigma_0, \Lambda$ such that

$$(2.10) \quad |(\partial_t^s X_{i_1} \cdots X_{i_k} \Gamma_{\epsilon, A_1})(x, t) - (\partial_t^s X_{i_1} \cdots X_{i_k} \Gamma_{\epsilon, A_2})(x, t)| \leq \|A_1 - A_2\| C_{s,k} t^{-s-k/2} \frac{e^{-\frac{d_\epsilon(x,0)^2}{C_\Lambda t}}}{|B_\epsilon(0, \sqrt{t})|},$$

where $\|A\|^2 := \sum_{i,j=1}^n a_{ij}^2$.

Moreover, for $s \in \mathbb{N}$ and k -tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ as $\epsilon \rightarrow 0$ one has

$$(2.11) \quad X_{i_1} \cdots X_{i_k} \partial_t^s \Gamma_{\epsilon, A} \rightarrow X_{i_1} \cdots X_{i_k} \partial_t^s \Gamma_A$$

uniformly on compact sets and in a dominated way on all G .

The proof of our result is directly inspired from the proof in [16], where the time independent case was studied by two of us, and where a general procedure was introduced for handling the dependence on ϵ and obtain independent estimates in the sub-Laplacian case.

Let us consider the group $\bar{G} = G \times G$ defined in terms of n new coordinates $y \in G$, so that points of \bar{G} will be denoted $\bar{x} = (x, y)$. Denote by Y_1, \dots, Y_n a copy of the vectors X_1, \dots, X_n , defined in terms of new variables y . The vector fields (X_i) and (Y_i) are defined in the product algebra $\bar{\mathcal{G}} = \mathcal{G} \times \mathcal{G}$.

In \bar{G} we will consider two different families of sub-Riemannian structures:

- A sub-Riemannian structure determined by the choice of horizontal vector fields given by

$$(2.12) \quad (\bar{X}_1^0, \dots, \bar{X}_{m+n}^0) = (X_1, \dots, X_m, Y_1, \dots, Y_n).$$

The sub-Laplacian/heat operator associated to this structure is

$$(2.13) \quad \bar{L}_{0,A} = \partial_t - \sum_{i,j=1}^{m+n} \bar{a}_{ij} \bar{X}_i^0 \bar{X}_j^0 = \partial_t - \sum_{i,j=1}^m a_{ij} X_i X_j - \sum_{i,j=1}^n a_{ij} Y_i Y_j.$$

We complete the horizontal frame to a basis of the whole Lie algebra by adding the commutators of the vectors (X_i) : (X_{m+1}, \dots, X_n) . We denote the exponential coordinates of a point \bar{x} around a point \bar{x}_0 in terms of the full frame through the coefficients (v_i^0, w_i^0) which are defined by

$$\bar{x} = \exp\left(\sum_{i=1}^m v_i^0 X_i + \sum_{i=1}^n w_i^0 Y_i + \sum_{i=m+1}^n v_i^0 X_i\right)(\bar{x}_0).$$

- For every $\epsilon \in [0, 1)$ consider a sub-Riemannian structure determined by the choice of horizontal vector fields given by

$$(2.14) \quad (\bar{X}_1^\epsilon, \dots, \bar{X}_{m+n}^\epsilon) = (X_1, \dots, X_m, Y_1, \dots, Y_m, X_{m+1}^\epsilon + Y_{m+1}, \dots, X_n^\epsilon + Y_n).$$

The sub-Laplacian/heat operator associated to this structure is

$$\bar{L}_{\epsilon,A} = \partial_t - \sum_{i,j=1}^m a_{ij} X_i X_j - \sum_{i,j=1}^m a_{ij} Y_i Y_j - \sum_{i>m \text{ or } j>m} a_{ij} (X_i^\epsilon + Y_i)(X_j^\epsilon + Y_j).$$

Analogously, the exponential coordinates associated to $L_{\epsilon,A}$ will depend on the given horizontal frame (2.14) and the family (X_{m+1}, \dots, X_n) . The coordinates of a point \bar{x} are the coefficients v_i^ϵ and w_i^ϵ satisfying

$$(2.15) \quad \bar{x} = \exp\left(\sum_{i=1}^m v_i^\epsilon X_i + \sum_{i=1}^m w_i^\epsilon Y_i + \sum_{i=m+1}^n w_i^\epsilon (X_i^\epsilon + Y_i) + \sum_{i=m+1}^n v_i^\epsilon X_i\right)(\bar{x}_0).$$

We denote by $\bar{\Gamma}_{0,A}$ and $\bar{\Gamma}_{\epsilon,A}$ the heat kernels of the corresponding heat operators. In both structures we define associated (pseudo)distance functions $\bar{d}_0(\bar{x}, \bar{y})$ and $\bar{d}_\epsilon(\bar{x}, \bar{y})$ that are equivalent to those defined in (2.1) and that assign unit weight to the corresponding horizontal vectors while (X_{m+1}, \dots, X_n) will be weighted according to their degree $d(i)$.

Definition 2.3. For every $\epsilon > 0$ and $\bar{x}, \bar{x}_0 \in \bar{G}$ define

$$\bar{d}_\epsilon(\bar{x}, \bar{x}_0) = \sum_{i=1}^m |v_i^\epsilon| + \sum_{i=1}^m |w_i^\epsilon| + \sum_{i=m+1}^n \min(|w_i^\epsilon|, |w_i^\epsilon|^{1/d(i)}) + \sum_{i=m+1}^n |v_i^\epsilon|^{1/d(i)}$$

For $\epsilon = 0$ and $\bar{x}, \bar{x}_0 \in \bar{G}$ define

$$\bar{d}_0(\bar{x}, \bar{x}_0) = \sum_{i=1}^n (|w_i^0|^{1/d(i)} + |v_i^0|^{1/d(i)})$$

We will denote by \bar{B}_ϵ and \bar{B}_0 the corresponding metric balls.

Lemma 2.4. *There exists constants $C_\Lambda > 0$ depending on G, σ_0, Λ but independent of ϵ such that for each $\epsilon > 0$, $\bar{x} \in \bar{G}$ and $t > 0$ one has*

$$(2.16) \quad C_\Lambda^{-1} \frac{e^{-C_\Lambda \frac{\bar{d}_\epsilon(\bar{x}, 0)^2}{t}}}{|\bar{B}_\epsilon(0, \sqrt{t})|} \leq \bar{\Gamma}_{\epsilon,A}(\bar{x}, t) \leq C_\Lambda \frac{e^{-\frac{\bar{d}_\epsilon(\bar{x}, 0)^2}{C_\Lambda t}}}{|\bar{B}_\epsilon(0, \sqrt{t})|}.$$

For $s \in \mathbb{N}$ and k -tuple $(i_1, \dots, i_k) \in \{1, \dots, n+m\}^k$ there exists $C_{s,k} > 0$ depending only on $k, s, G, \sigma_0, \Lambda$ such that

$$(2.17) \quad |(\partial_t^s \bar{X}_{i_1}^\epsilon \cdots \bar{X}_{i_k}^\epsilon \bar{\Gamma}_{\epsilon,A})(\bar{x}, t)| \leq C_{s,k} t^{-s-k/2} \frac{e^{-\frac{\bar{d}_\epsilon(\bar{x}, 0)^2}{C_\Lambda t}}}{|\bar{B}_\epsilon(0, \sqrt{t})|}$$

for all $\bar{x} \in \bar{G}$ and $t > 0$. Moreover, for $s \in \mathbb{N}$ and k -tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ as $\epsilon \rightarrow 0$ one has

$$(2.18) \quad \bar{X}_{i_1}^\epsilon \cdots \bar{X}_{i_k}^\epsilon \partial_t^s \bar{\Gamma}_{\epsilon,A} \rightarrow \bar{X}_{i_1}^0 \cdots \bar{X}_{i_k}^0 \partial_t^s \bar{\Gamma}_{0,A}$$

uniformly on compact sets, in a dominated way on all \bar{G} .

Proof. In order to estimate the fundamental solution of the operators $\bar{L}_{\epsilon,A}$ in terms of $\bar{L}_{0,A}$, we define a volume preserving change of variables on the Lie algebra and on G : Define $\bar{F}_\epsilon : \bar{G} \rightarrow \bar{G}$ as $\bar{F}_\epsilon(\bar{x}) = \exp(T_\epsilon(\log(\bar{x})))$ with

$$T_\epsilon(\bar{X}_i^0) = \bar{X}_i^\epsilon \text{ for } i = 1, \dots, m+n.$$

By definition the relation between the distances \bar{d}_0 and \bar{d}_ϵ is expressed by the formula

$$(2.19) \quad \bar{d}_\epsilon(\bar{x}, \bar{x}_0) = \bar{d}_0(\bar{F}_\epsilon(\bar{x}), \bar{F}_\epsilon(\bar{x}_0)).$$

Analogously we also have

$$(2.20) \quad \bar{\Gamma}_{\epsilon,A}(\bar{x}, t) = \bar{\Gamma}_{0,A}(\bar{F}_\epsilon(\bar{x}), t),$$

$$X_i \bar{\Gamma}_{\epsilon, A}(\bar{x}, t) = X_i \bar{\Gamma}_{0, A}(\bar{F}_\epsilon(\bar{x}), t), \text{ for } i = 1, \dots, m$$

and

$$(X_i^\epsilon + Y_i) \bar{\Gamma}_{\epsilon, A}(\bar{x}, t) = Y_i \bar{\Gamma}_{0, A}(\bar{F}_\epsilon(\bar{x}), t), \text{ for } i = m + 1, \dots, n,$$

with similar identities holding for the iterated derivatives.

In view of the latter, assertions (2.16) and (2.17) immediately follow from the well known estimates of $\bar{\Gamma}_{0, A}$ (see for instance there references cited above or [6]). The pointwise convergence (2.18) is also an immediate consequence of (2.19) and (2.20). In order to prove the dominated convergence result we need to relate the distances \bar{d}_0 and \bar{d}_ϵ : Expressing the exponential coordinates $v_i^\epsilon, w_i^\epsilon$ in terms of v_i^0, w_i^0 one easily obtains

$$\bar{d}_\epsilon(\bar{x}, \bar{x}_0) = \sum_{i=1}^m (|v_i^0| + |w_i^0|) + \sum_{i=m+1}^n \left(|v_i^0 - \epsilon w_i^0|^{1/d(i)} + \min(|w_i^0|, |w_i^0|^{1/d(i)}) \right)$$

so that for all⁶ $\bar{x}, \bar{x}_0 \in \bar{G}$

$$(2.21) \quad \bar{d}_0(\bar{x}, \bar{x}_0) - C_0 \leq \bar{d}_\epsilon(\bar{x}, \bar{x}_0) \leq \bar{d}_0(\bar{x}, \bar{x}_0) + C_0$$

where C_0 is independent of ϵ . The latter and (2.17) imply dominated convergence on \bar{G} . \square

Lemma 2.5. *For $s \in \mathbb{N}$ and k -tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ there exists $C_{s, k} > 0$ depending only on $k, s, G, \sigma_0, \Lambda$ such that depending on G, σ_0, Λ but independent of ϵ such that for each $A_1, A_2 \in M_\Lambda$, $\epsilon > 0$, $\bar{x} \in \bar{G}$ and $t > 0$ one has*

$$(2.22) \quad \begin{aligned} & |(\partial_t^s \bar{X}_{i_1}^\epsilon \cdots \bar{X}_{i_k}^\epsilon \bar{\Gamma}_{\epsilon, A_1})(\bar{x}, t) - (\partial_t^s \bar{X}_{i_1}^\epsilon \cdots \bar{X}_{i_k}^\epsilon \bar{\Gamma}_{\epsilon, A_2})(\bar{x}, t)| \leq \\ & \leq \|A_1 - A_2\| C_{s, k} t^{-s-k/2} \frac{e^{-\frac{\bar{d}_\epsilon(\bar{x}, 0)^2}{C_\Lambda t}}}{|\bar{B}_\epsilon(0, \sqrt{t})|}. \end{aligned}$$

Proof. In view of (2.19) and (2.20) it is sufficient to establish the result for $\bar{\Gamma}_{0, A_1}$ and $\bar{\Gamma}_{0, A_2}$, thus eliminating the dependence on ϵ .

Although the vector fields \bar{X}_i^0 , $i = 1, \dots, m + n$ are not free, we can invoke the Rothschild and Stein lifting theorem [42] and lift them to a new family of free vector fields in a free Carnot group. For the sake of simplicity we will continue to use the same notation \bar{X}_i^0 , $i = 1, \dots, m + n$ to denote such family of free vector fields. To recover the desired estimate (2.22) for the original vector fields one needs to argue through projections down to the original space, exactly as done in [39] and [16]. A standard argument (see for instance [7]) yields that for every $A \in M_\Lambda$, if we set \bar{A} as in (2.13) then there exists a Lie group automorphism F_A such that

$$\left(\sum_{j=1}^{m+n} (\bar{A}^{1/2})_{ij} \bar{X}_j^0 \right) (u \circ F_A) = (\bar{X}_i u) \circ F_A,$$

and

$$\bar{\Gamma}_{0, A}(\bar{x}, t) = |\det \bar{A}^{1/2}| \bar{\Gamma}_{0, I}(F_A(\bar{x}), t)$$

where I denotes the identity matrix. The automorphism F_A is defined by

$$F_A(\bar{x}) = \exp(T_A(\log(\bar{x}))), \text{ with } T_A(\bar{X}_i^0) = \sum_{j=1}^{m+n} (A^{1/2})_{ij} \bar{X}_j^0 \text{ for } i = 1, \dots, m$$

and is extended to the whole Lie algebra as morphism (see the Appendix for more details). As in (2.15), we denote by (v_i, w_i) the canonical coordinates of $F_{A_1}(\bar{x})$ around $F_{A_2}(\bar{x})$, then the Mean Value Theorem yields

$$(\partial_t^s \bar{X}_{i_1}^0 \cdots \bar{X}_{i_k}^0 \bar{\Gamma}_I)(F_{A_1}(\bar{x}), t) - (\partial_t^s \bar{X}_{i_1}^0 \cdots \bar{X}_{i_k}^0 \bar{\Gamma}_I)(F_{A_2}(\bar{x}), t) =$$

⁶This estimate indicates the well known fact that at large scale the Riemannian approximating distances are equivalent to the sub-Riemannian distance

$$\sum_{i=1}^n (v_i X_i + w_i Y_i) (\partial_t^s \bar{X}_{i_1}^0 \cdots \bar{X}_{i_k}^0 \bar{\Gamma}_I)(\bar{y}, t).$$

By the result in the Appendix, the operators $v_i X_i + w_i Y_i$ are zero order differential operators whose coefficients can be estimated by $\|A_1 - A_2\|$. The conclusion follows by virtue of Proposition 5.1. \square

We conclude this section with the proof of the main result Proposition 2.2.

Proof of Proposition 2.2. From the definition of fundamental solution we have that

$$\Gamma_{0,A}(x, t) = \int_G \bar{\Gamma}_{0,A}((x, y), t) dy, \quad \text{and} \quad \Gamma_{\epsilon,A}(x, t) = \int_G \bar{\Gamma}_{\epsilon,A}((x, y), t) dy,$$

for any $x \in G$ and $t > 0$. In view of the (global) dominated convergence of the derivatives of $\bar{\Gamma}_{\epsilon,A}$ to the corresponding derivatives of $\bar{\Gamma}_{0,A}$ as $\epsilon \rightarrow 0$, we deduce that

$$\int_G \bar{\Gamma}_{\epsilon,A}((x, y), t) dy \rightarrow \int_G \bar{\Gamma}_{0,A}((x, y), t) dy$$

as $\epsilon \rightarrow 0$. The Gaussian estimates of $\Gamma_{\epsilon,A}$ follow from the corresponding estimates on $\bar{\Gamma}_{\epsilon,A}$ and the fact that in view of (2.21),

$$\begin{aligned} \bar{d}_\epsilon((x, y), (x_0, y_0)) &\geq \bar{d}_0((x, y), (x_0, y_0)) - C_0 \geq d_0(x, x_0) + d_0(y, y_0) - C_0 \geq \\ &\geq d_\epsilon(x, x_0) + d_\epsilon(y, y_0) - 3C_0 \end{aligned}$$

Indeed the latter shows that there exists a constant $C > 0$ depending only on G, σ_0 such that for every $x \in G$,

$$\int_G e^{-\frac{d_\epsilon^2((x,y),(x_0,y_0))}{t}} dy \leq C e^{-\frac{d_\epsilon^2(x,x_0)}{t}} \int_G e^{-\frac{d_\epsilon^2(y,y_0)}{t}} dy \leq C e^{-\frac{d_\epsilon^2(x,x_0)}{t}}.$$

The conclusion follows at once

3. GRADIENT ESTIMATES

In this section we prove Theorem 1.1. The proof is carried out in two steps: First we use the maximum principle to establish interior L^∞ bounds for the full gradient of the solution $\nabla_1 u$ of (1.5) with respect to the Lipschitz norm of u on the parabolic boundary. Next, we construct appropriate barriers and invoke the comparison principle established in [8] to prove boundary gradient estimates. The combination of the two will yield the uniform global Lipschitz bounds.

3.1. Interior gradient estimates. Recalling that the right invariant vector fields X_j^r commute with the left invariant frame X_i , $i = 1, \dots, n$ it is easy to show through a direct computation the following result.

Lemma 3.1. *Let $u_\epsilon \in C^3(Q)$ be a solution to (1.3) and denote $v_0 = \partial_t u$, $v_i = X_i^r u$ for $i = 1, \dots, n$. Then for every $h = 0, \dots, n$ one has that v_h is a solution of*

$$(3.1) \quad \partial_t v_h = X_i^\epsilon (a_{ij} X_j v_h) = a_{ij}^\epsilon (\nabla_\epsilon u_\epsilon) X_i^\epsilon X_j^\epsilon v_h + \partial_{\xi_k} a_{ij}^\epsilon (\nabla_\epsilon u) X_i^\epsilon X_j^\epsilon u_\epsilon X_k^\epsilon v_h,$$

where

$$a_{ij}^\epsilon(\xi) = \frac{1}{\sqrt{1 + |\xi|^2}} \left(\delta_{ij} - \frac{\xi_i \xi_j}{1 + |\xi|^2} \right).$$

Note that in order to prove L^∞ bounds on the horizontal gradient of solutions of (1.5) one *cannot* invoke Lemma 3.1 with differentiation the horizontal frame, because these vector fields do not commute. The argument below bypasses this difficulty by using the algebraic relation between right and left invariant vector fields.

Proposition 3.2. *Let $u_\epsilon \in C^3(Q)$ be a solution to (1.5) with Ω bounded. There exists $C = C(G, \|\varphi\|_{C^2(\Omega)}) > 0$ such that for every compact subset $K \subset\subset \Omega$ one has*

$$\sup_{K \times [0, T]} |\nabla_1 u| \leq \sup_{\partial_p Q} (|\nabla_1 u| + |\partial_t u|),$$

where ∇_1 is the full σ_1 -Riemannian gradient.

Proof. The proof is similar to the argument in [8, Proposition 5.1] and is based on the trivial observation that left-invariant vector fields at the layer k can be written as linear combination of right-invariant vector fields at level k and left (or right) invariant vector fields at levels $k+1, k+2, \dots, r$. The first step consists in differentiating the equation in (1.5) in the direction of the highest layer V^r (which commutes with all the Lie algebra) and invoke the weak maximum principle applied to (3.1) to show

$$(3.2) \quad \sup_{K \times [0, T]} \sum_{d(i)=r} |X_i u| \leq \sup_{\partial_p Q} (|\nabla_1 u| + |\partial_t u|).$$

Next one proceeds in a similar fashion, iterating the argument by differentiating the equation in (1.5) along the right invariant vector fields in the layer V^{r-1} . The maximum principle and (3.1) yield L^∞ bounds on $X_i^r u$ with $d(X_i^r) = r-1$. Coupling such bounds with (3.2) one obtains

$$(3.3) \quad \sup_{K \times [0, T]} \sum_{d(i)=r-1} |X_i u| \leq \sup_{\partial_p Q} (|\nabla_1 u| + |\partial_t u|).$$

The proof now follows by induction on the step of the derivatives. \square

3.2. Linear barrier functions. In [8, Section 4.2] it is shown that in a step two Carnot group coordinate planes (i.e. images under the exponential of level sets of the form $x_k = 0$) solve the minimal surface equation $h_0 = 0$. In the same paper it is also shown that this may fail for step three or higher. In the construction of the barrier function we will need the following slight refinement of this result,

Lemma 3.3. *Let G be a step two Carnot group. If $f : G \rightarrow \mathbb{R}$ is linear (in exponential coordinates) then for every $\epsilon \geq 0$, the matrix with entries $X_i^\epsilon X_j^\epsilon f$ is anti-symmetric, in particular every level set of f satisfies $h_\epsilon = 0$.*

Proof. We need to show that for $f(x) = \sum_{i=1}^n a_i x_i$,

$$(3.4) \quad \sum_{i,j=1}^n \left(\delta_{ij} - \frac{X_i^\epsilon f X_j^\epsilon f}{1 + |\nabla_\epsilon f|^2} \right) X_i^\epsilon X_j^\epsilon f = 0.$$

We recall the expression (1.2) for the vector fields X_i , $d(i) = 1$ in terms of exponential coordinates $X_i = \partial_{x_i} + \sum_{d(j)=1, d(h)=2} c_{ij}^h x_j \partial_{x_h}$. The Campbell-Hausdorff formula implies the anti-symmetry relation $c_{ij}^h = -c_{ji}^h$. It is immediate to observe that, if $d(k) = 2$ one has

$$(3.5) \quad X_i(x_k) = \sum_{d(j)=1} c_{ij}^k x_j, \text{ and } X_i X_j(x_k) = c_{ji}^k, \text{ for } d(i) = d(j) = 1,$$

if either $d(i) = 2$ or $d(j) = 2$ it is easy to check $X_i^\epsilon X_j^\epsilon(x_k) = 0$ for all $k = 1, \dots, n$. Since $\left(\delta_{ij} - \frac{X_i^\epsilon f X_j^\epsilon f}{1 + |\nabla_\epsilon f|^2} \right)$ is symmetric in i, j it follows that

$$(3.6) \quad h_\epsilon = \sum_{k=1}^n a_k \sum_{i,j=1}^n \left(\delta_{ij} - \frac{X_i^\epsilon f X_j^\epsilon f}{1 + |\nabla_\epsilon f|^2} \right) X_i^\epsilon X_j^\epsilon(x_k) = 0.$$

\square

3.3. Boundary gradient estimates. We say that a set $\Omega \subset G$ is convex in the Euclidean sense if $\exp^{-1}(\Omega) \subset \mathcal{G}$ is convex in the Euclidean space \mathcal{G} . In a group of step two this condition is translation invariant.

Proposition 3.4. *Let G be a Carnot group of step two, $\Omega \subset G$ a bounded, open, convex (in the Euclidean sense) set and $\varphi \in C^2(\bar{\Omega})$. For $\epsilon > 0$ denote by $u_\epsilon \in C^2(\Omega \times (0, T)) \cap C^1(\bar{\Omega} \times (0, T))$ the non-negative unique solution of the initial value problem (1.5). There exists $C = C(G, \|\varphi\|_{C^2(\bar{\Omega})}) > 0$ such that*

$$(3.7) \quad \sup_{\partial\Omega \times (0, T)} |\nabla_\epsilon u_\epsilon| \leq \sup_{\partial\Omega \times (0, T)} |\nabla_1 u_\epsilon| \leq C.$$

We start by recalling an immediate consequence of the proof of [8, Theorem 3.3].

Lemma 3.5. *For each $\epsilon \geq 0$, if U_ϵ is a bounded subsolution and V_ϵ is a bounded supersolution of (1.5) then $U_\epsilon(x, t) \leq V_\epsilon(x, t)$ for all $(x, t) \in Q$.*

Let $u_\epsilon \in C^2(Q)$ be a solution of (1.5), and express the evolution PDE in non-divergence form

$$(3.8) \quad \partial_t u_\epsilon = h_\epsilon = \sum_{i, j=1}^n a_{ij}^\epsilon (\nabla_\epsilon u) X_i^\epsilon X_j^\epsilon u_\epsilon.$$

Set $v_\epsilon = u_\epsilon - \varphi$ so that v_ϵ solves the homogenous 'boundary' value problem

$$(3.9) \quad \begin{cases} \partial_t v_\epsilon = a_{ij}^\epsilon (\nabla_\epsilon v_\epsilon + \nabla_\epsilon \varphi) X_i^\epsilon X_j^\epsilon v_\epsilon + b^\epsilon & \text{in } Q = \Omega \times (0, T) \\ v_\epsilon = 0 & \text{on } \partial_p Q, \end{cases}$$

with $b^\epsilon(x) = a_{ij}^\epsilon (\nabla_\epsilon v_\epsilon(x) + \nabla_\epsilon \varphi(x)) X_i^\epsilon X_j^\epsilon \varphi(x)$. We define our (weakly) parabolic operator for which the function v_ϵ is a solution

$$(3.10) \quad Q(v) = a_{ij}^\epsilon (\nabla_\epsilon v_\epsilon + \nabla_\epsilon \varphi) X_i^\epsilon X_j^\epsilon v_\epsilon + b^\epsilon - \partial_t v.$$

In the following we construct for each point $p_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ a *barrier function* for Q, v_ϵ : i.e.,

Lemma 3.6. *Let G be a Carnot group of step two and $\Omega \subset G$ convex in the Euclidean sense. For each point $p_0 = (x_0, t_0) \in \partial\Omega \times (0, T)$ one can construct a positive function $w \in C^2(Q)$ such that*

$$(3.11) \quad \begin{aligned} Q(w) &\leq 0 \text{ in } V \cap Q \text{ with } V \text{ a parabolic neighborhood of } p_0, \\ w(p_0) &= 0 \text{ and } w \geq v_\epsilon \text{ in } \partial_p V \cap Q. \end{aligned}$$

Proof. In the hypothesis that Ω is convex in the Euclidean sense we have that every $x_0 \in \partial\Omega$ supports a tangent hyperplane P defined by an equation of the form $\Pi(x) = \sum_{i=1}^n a_i x_i = 0$ with $\Pi > 0$ in Ω , $\Pi(x_0) = 0$, and normalized as $\sum_{d(i)=1,2} a_i^2 = 1$. Following the standard argument (see for instance [34, Chapter 10]) we select the barrier at $(x_0, t_0) \in \partial\Omega \times (0, T)$ independent of time with

$$(3.12) \quad w = \Phi(\Pi)$$

with Φ solution of

$$(3.13) \quad \Phi'' + \nu(\Phi')^2 = 0,$$

in particular

$$(3.14) \quad \Phi(s) = \frac{1}{\nu} \log(1 + ks),$$

with k and ν chosen appropriately so that conditions (3.11) will hold. We choose a neighborhood $V = O \times (0, T)$ such that $P \cap O \cap \partial\Omega = \{x_0\}$. By an appropriate choice of k sufficiently large we can easily obtain $w_\epsilon(p_0) = 0$ and $w_\epsilon \geq v_\epsilon$ in $\partial_p V \cap Q$.

To verify $Q(w_\epsilon) \leq 0$ we begin by observing that w satisfies

$$(3.15) \quad Q(w) = \Phi' a_{ij}^\epsilon (\nabla_e w + \nabla_\epsilon \varphi) X_i^\epsilon X_j^\epsilon \Pi + \frac{\Phi''}{(\Phi')^2} \mathcal{F} + b_\epsilon,$$

with $\mathcal{F} = a_{ij}^\epsilon (\nabla_e w + \nabla_\epsilon \varphi) X_i^\epsilon w X_j^\epsilon w$.

We will show:

$$(Claim\ 1) \quad a_{ij}^\epsilon (\nabla_e w + \nabla_\epsilon \varphi) X_i^\epsilon X_j^\epsilon \Pi \leq 0$$

$$(Claim\ 2) \quad \frac{\Phi''}{(\Phi')^2} \mathcal{F} + b_\epsilon \leq 0$$

in a parabolic neighborhood of p_0 .

The first claim holds with an equality as a_{ij}^ϵ is symmetric and $X_i^\epsilon X_j^\epsilon \Pi$ is anti-symmetric in view of Lemma 3.3. To establish (Claim 2) we first note that Lemma 3.3 implies

$$\begin{aligned} \frac{\epsilon^2}{2} &\leq \max\left(\sum_{d(i)=1} a_i^2, \epsilon^2 \sum_{d(k)=1} a_k^2\right) \leq |\nabla_\epsilon \Pi| = \\ &= \sum_{d(i)=1} \left(a_i + \sum_{d(k)=2, d(j)=1} c_{ij}^k a_k x_j\right)^2 + \epsilon^2 \sum_{d(k)=2} a_k^2 \leq C(G)(1 + \epsilon^2), \end{aligned}$$

for some constant $C(G) > 0$. Consequently, for $\Phi' \gg 1$ sufficiently large one finds

$$(3.16) \quad \mathcal{F} \geq \frac{|\nabla_\epsilon w|^2}{(1 + |\nabla_\epsilon w + \nabla_\epsilon \varphi|^2)^{3/2}} \geq C(G) \frac{|\nabla_\epsilon w|^2}{(1 + |\Phi'|^2 + |\nabla_\epsilon \varphi|^2)^{3/2}} \geq C(G)\epsilon^2 > 0,$$

with $C(G) > 0$ a constant depending only on G (not always the same along the chain of inequalities). In view of the definition of b_ϵ and (3.13) with an appropriate choice of $\nu = \nu(G, \epsilon, \phi) > 0$ and $k = k(G, \phi) \gg 1$ in (3.14), we conclude

$$(3.17) \quad \frac{\Phi''}{(\Phi')^2} \mathcal{F} + b_\epsilon \leq \left(\frac{\Phi''}{(\Phi')^2} + \nu\right) \mathcal{F} = 0.$$

□

In view of Lemma 3.5, a comparison with the barrier constructed above yields that

$$(3.18) \quad 0 \leq \frac{v_\epsilon(x, t)}{\text{dist}_{\sigma_1}(x, x_0)} \leq \frac{w(x, t)}{\text{dist}_{\sigma_1}(x, x_0)} \leq C(k, \nu),$$

in $V \cap Q$, with $\text{dist}_{\sigma_1}(x, x_0)$ being the distance between x and x_0 in the Riemannian metric σ_1 , concluding the proof of the boundary gradient estimates.

The proof of Theorem 1.1 now follows immediately from Proposition 3.2 and Proposition 3.4.

Having established uniform global Lipschitz bounds one now notes that equation (3.1) satisfies *horizontal* coercivity conditions uniformly in $\epsilon > 0$. Such conditions are among the main hypothesis of the Harnack inequality in [11]. In this paper, G. Rea and two of the authors have proved that given a homogenous structure and a Poincaré inequality, then a sub-elliptic analogue of Aronson and Serrin's Harnack inequality [3] for quasilinear parabolic equations holds. As a consequence one obtains for some $\alpha \in (0, 1)$ that the solutions u_ϵ to (1.5) satisfy $C^{1, \alpha}$ Hölder estimates, uniform in $\epsilon \in (0, 1)$,

Definition 3.7. Let $0 < \alpha < 1$, $Q \subset \mathbb{R}^{n+1}$ and u be defined on Q . We say that $u \in C_{\epsilon, X}^\alpha(Q)$ if there exists a positive constant M such that for every $(x, t), (x_0, t_0) \in Q$

$$(3.19) \quad |u(x, t) - u(x_0, t_0)| \leq M \tilde{d}_\epsilon^\alpha((x, t), (x_0, t_0)).$$

We put

$$\|u\|_{C_{\epsilon, X}^\alpha(Q)} = \sup_{(x, t) \neq (x_0, t_0)} \frac{|u(x, t) - u(x_0, t_0)|}{\tilde{d}_\epsilon^\alpha((x, t), (x_0, t_0))} + \sup_Q |u|.$$

Iterating this definition, if $k \geq 1$ we say that $u \in C_{\epsilon, X}^{k, \alpha}(Q)$ if for all $i = 1, \dots, m$ $X_i \in C_{\epsilon, X}^{k-1, \alpha}(Q)$. Where we have set $C_{\epsilon, X}^{0, \alpha}(Q) = C_{\epsilon, X}^\alpha(Q)$.

Corollary 3.8. (*Interior $C_X^{1, \alpha}$ estimates*) In the hypothesis of the previous results, letting K be a compact set $K \subset\subset Q$, there exist constants $\alpha \in (0, 1)$ and $C = C(K, \alpha) > 0$ such that for all $i = 1, \dots, n$ one has that $v = X_i^\epsilon u$ satisfies

$$\|v\|_{C_{\epsilon, X}^\alpha(K)} + \|\nabla_\epsilon v\|_{L^2(K)} \leq C,$$

uniformly in $\epsilon \in (0, 1)$.

4. REGULARITY PROPERTIES IN THE $C^{k, \alpha}$ SPACES

In this section we will prove uniform estimates for solution of (1.3) in the $C_{\epsilon, X}^{k, \alpha}$ Hölder spaces. This is accomplished by using the uniform Gaussian bounds established in Section 2.2 to develop new uniform Schauder estimates for solutions of second order sub-elliptic differential equations in non divergence form

$$L_\epsilon u \equiv \partial_t u - \sum_{i, j=1}^n a_{ij}^\epsilon(x, t) X_i^\epsilon X_j^\epsilon u = 0,$$

in a cylinder $Q = \Omega \times (0, T)$ that are stable as $\epsilon \rightarrow 0$. As usual we will make use of the associated linear, constant coefficient *frozen* operator:

$$L_{\epsilon, (x_0, t_0)} \equiv \partial_t - \sum_{i, j=1}^n a_{ij}^\epsilon(x_0, t_0) X_i^\epsilon X_j^\epsilon,$$

where $(x_0, t_0) \in Q$.

As a direct consequence of the definition of fundamental solution one has the following representation formula

Lemma 4.1. *Let w be a smooth solution to $L_\epsilon w = f$ in $Q \subset \mathbb{R}^{n+1}$. For every $\phi \in C_0^\infty(Q)$,*

$$(4.1) \quad (w\phi)(x, t) = \int_Q \Gamma_{(x_0, t_0)}^\epsilon((x, t), (y, \tau)) (L_{\epsilon, (x_0, t_0)} - L_\epsilon)(w\phi)(y, \tau) dy d\tau + \\ + \int_Q \Gamma_{(x_0, t_0)}^\epsilon((x, t), (y, \tau)) (f\phi + wL_\epsilon\phi + 2 \sum_{i, j=1}^n a_{ij}^\epsilon(y, \tau) X_i^\epsilon w X_j^\epsilon \phi)(y, \tau) dy d\tau,$$

where we have denoted by $\Gamma_{(x_0, t_0)}^\epsilon$ the heat kernel for of $L_{\epsilon, (x_0, t_0)}$.

We explicitly note that for $\epsilon > 0$ fixed the operator $L_{\epsilon, (x_0, t_0)}$ is uniformly parabolic. Its heat kernel can be studied through standard singular integrals theory in the corresponding Riemannian balls. Hence, as noted in [29], one is allowed to differentiate twice the kernels defined in (4.1) with respect to any right or left invariant vector field.

Proposition 4.2. *Let $0 < \alpha < 1$ and w be a smooth solution of $L_\epsilon w = f \in C_{\epsilon, X}^\alpha(Q)$ in the cylinder Q . Let K be a compact sets such that $K \subset\subset Q$, set $2\delta = d_0(K, \partial_p Q)$ and denote by K_δ the δ -tubular neighborhood of K . Assume that there exists a constant $C > 0$ such that for every $\epsilon \in (0, 1)$*

$$\|a_{ij}^\epsilon\|_{C_{\epsilon, X}^\alpha(K_\delta)} \leq C.$$

There exists a constant $C_1 > 0$ depending on δ, α, C and the constants in Proposition 2.2 such that

$$\|w\|_{C_{\epsilon, X}^{2, \alpha}(K)} \leq C_1 \left(\|f\|_{C_{\epsilon, X}^\alpha(K_\delta)} + \|w\|_{C_{\epsilon, X}^{1, \alpha}(K_\delta)} \right).$$

Proof. The proof follows the outline of the standard case, as in [29, Theorem 4, Chapter 3], and rests crucially on the Gaussian estimates proved in Proposition 2.2, and the fact that the functions $(L_{\epsilon, (x_0, t_0)} - L_\epsilon)(w\phi)$, and $(f\phi + wL_\epsilon\phi + 2a_{ij}^\epsilon X_i^\epsilon w X_j^\epsilon \phi)$ are Hölder continuous.

Choose a parabolic sphere⁷ $B_{\epsilon, \delta} \subset\subset K$ where $\delta > 0$ will be fixed later and a cut-off function $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ identically 1 on $B_{\epsilon, \delta/2}$ and compactly supported in $B_{\epsilon, \delta}$. This clearly implies that for some constant $C > 0$ depending only on G and σ_0 , $|\nabla_\epsilon \phi| \leq C\delta^{-1}$, $|L_\epsilon \phi| \leq C\delta^{-2}$, in Q . Next, invoking (4.1) one has that for every multi-index $I = (i_1, i_2) \in \{1, \dots, m\}^2$ and for every $(x_0, t_0) \in B_{\epsilon, \delta}$

$$(4.2) \quad \begin{aligned} X_{i_1}^\epsilon X_{i_2}^\epsilon (w\phi)(x_0, t_0) &= \int_Q X_{i_1}^\epsilon X_{i_2}^\epsilon \Gamma_{(x_0, t_0)}^\epsilon(\cdot, (y, \tau))|_{(x_0, t_0)} (L_{\epsilon, (x_0, t_0)} - L_\epsilon)(w\phi)(y, \tau) dy d\tau + \\ &+ \int_Q X_{i_1}^\epsilon X_{i_2}^\epsilon \Gamma_{(x_0, t_0)}^\epsilon(\cdot, (y, \tau)) \left(f\phi + wL_\epsilon\phi + 2 \sum_{i, j=1}^n a_{ij}^\epsilon X_i^\epsilon w X_j^\epsilon \phi \right) (y, \tau) dy d\tau. \end{aligned}$$

The uniform Hölder continuity of a_{ij}^ϵ , with Proposition 2.2 and Lemma 2.5 yield

$$\begin{aligned} &|X_{i_1}^\epsilon X_{i_2}^\epsilon \Gamma_{(x, t)}^\epsilon((x, t), (y, \tau)) - X_{i_1}^\epsilon X_{i_2}^\epsilon \Gamma_{(x_0, t_0)}^\epsilon((x_0, t_0), (y, \tau))| \\ &\leq C \tilde{d}_\epsilon^\alpha((x, t), (x_0, t_0)) \frac{(\tau - t_0)^{-1} e^{-\frac{d_\epsilon(x_0, y)^2}{C_\Lambda(\tau - t_0)}}}{|B_\epsilon(0, \sqrt{\tau - t_0})|}, \end{aligned}$$

with $C > 0$ independent of ϵ . In view of the latter, using basic singular integral properties (see [26]) and proceeding as in [29, Theorem 2, Chapter 4], we obtain

$$(4.3) \quad \begin{aligned} &\left\| \int X_{i_1}^\epsilon X_{i_2}^\epsilon \Gamma_{(x_0, t_0)}^\epsilon(\cdot, (y, \tau)) (L_\epsilon - L_{\epsilon, (x_0, t_0)})(w\phi)(y, \tau) dy d\tau \right\|_{C_{\epsilon, X}^\alpha(B_{\epsilon, \delta})} \\ &\leq C_1 \left\| (L_\epsilon - L_{\epsilon, (x_0, t_0)})(w\phi) \right\|_{C_{\epsilon, X}^\alpha(B_{\epsilon, \delta})} \\ &= C_1 \sum_{i, j} \left\| (a_{ij}^\epsilon(x_0, t_0) - a_{ij}^\epsilon(\cdot)) X_j^\epsilon X_j^\epsilon (w\phi) \right\|_{C_{\epsilon, X}^\alpha(B_{\epsilon, \delta})} \\ &\leq \tilde{C}_1 \delta^\alpha \|a_{ij}^\epsilon\|_{C_{\epsilon, X}^\alpha(B_{\epsilon, \delta})} \|w\phi\|_{C_{\epsilon, X}^{2, \alpha}(B_{\epsilon, \delta})}, \end{aligned}$$

where C_1 , and \tilde{C}_1 are stable as $\epsilon \rightarrow 0$. Similarly, if ϕ is fixed, the Hölder norm of the second term in the representation formula (4.2) is bounded by

$$(4.4) \quad \begin{aligned} &\left\| \int X_{i_1}^\epsilon X_{i_2}^\epsilon \Gamma_{(x_0, t_0)}^\epsilon((x_0, t_0), (y, \tau)) (f\phi(y, \tau) + wL_\epsilon\phi(y, \tau) + 2a_{ij}^\epsilon X_i^\epsilon w X_j^\epsilon \phi) dy d\tau \right\|_{C_{\epsilon, X}^\alpha(B_{\epsilon, \delta})} \\ &\leq C_2 \left(\|f\|_{C_{\epsilon, X}^\alpha(K_\delta)} + \frac{C}{\delta^2} \|w\|_{C_{\epsilon, X}^{1, \alpha}(K_\delta)} \right). \end{aligned}$$

⁷That is a sphere in the group $\tilde{G} = G \times \mathbb{R}$ in the pseudo-metric \tilde{d}_ϵ defined in (2.4).

From (4.2), (4.3) and (4.4) we deduce that

$$\|w\phi\|_{C_{\epsilon, X}^{2, \alpha}(B_\delta)} \leq \tilde{C}_2 \delta^\alpha \|w\phi\|_{C_{\epsilon, X}^{2, \alpha}(B_\delta)} + C_2 \left(\|f\|_{C_{\epsilon, X}^\alpha(K_\delta)} + \frac{C}{\delta^2} \|w\|_{C_{\epsilon, X}^{1, \alpha}(K_\delta)} \right).$$

Choosing δ sufficiently small we prove the assertion on the fixed sphere $B_{\epsilon, \delta}$. The conclusion follows from a standard covering argument. \square

A direct calculation shows the following

Lemma 4.3. *Let $k \in N$ and consider a k -tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$. There exists a differential operator B of order $k + 1$, depending on horizontal derivatives of $a_{i_j}^\epsilon$ of order at most k , such that*

$$X_{i_1}^\epsilon \cdots X_{i_k}^\epsilon (L_{\epsilon, (x_0, t_0)} - L_\epsilon) = \sum_{i, j=1}^n \left(a_{i_j}^\epsilon - a_{i_j}^\epsilon(x_0, t_0) \right) X_{i_1}^\epsilon \cdots X_{i_k}^\epsilon X_i^\epsilon X_j^\epsilon + B.$$

Proposition 4.4. *Let w be a smooth solution of $L^\epsilon w = f$ on Q . Let K be a compact sets such that $K \subset\subset Q$, set $2\delta = d_0(K, \partial_p Q)$ and denote by K_δ the δ -tubular neighborhood of K . Assume that there exists a constant $C > 0$ such that*

$$\|a_{i_j}^\epsilon\|_{C_{\epsilon, X}^{k, \alpha}(K_\delta)} \leq C,$$

for any $\epsilon \in (0, 1)$. There exists a constant $C_1 > 0$ depending on α, C, δ , and the constants in Proposition 2.2, but independent of ϵ , such that

$$\|w\|_{C_{\epsilon, X}^{k+2, \alpha}(K)} \leq C_1 \left(\|f\|_{C_{\epsilon, X}^{k, \alpha}(K_\delta)} + \|w\|_{C_{\epsilon, X}^{k+1, \alpha}(K_\delta)} \right).$$

Proof. The proof is similar to the previous one for $k = 1$. We note that differentiating the representation formula (4.1) along any k -tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$, and using Lemma 4.3, yields

$$\begin{aligned} X_{i_1}^\epsilon \cdots X_{i_k}^\epsilon (w\phi)(x, t) &= \int_Q \Gamma_{(x_0, t_0)}^\epsilon((x, t), (y, \tau)) \left(a_{i_j}^\epsilon - a_{i_j}^\epsilon(x_0, t_0) \right) X_{i_1}^\epsilon \cdots X_{i_k}^\epsilon X_i^\epsilon X_j^\epsilon (w\phi)(y, \tau) dy d\tau \\ &+ \int_Q \Gamma_{(x_0, t_0)}^\epsilon((x, t), (y, \tau)) B(w\phi)(y, \tau) dy d\tau \\ &+ \int_Q \Gamma_{(x_0, t_0)}^\epsilon((x, t), (y, \tau)) X_{i_1}^\epsilon \cdots X_{i_k}^\epsilon \left(f\phi(y, \tau) + wL_\epsilon\phi(y, \tau) + 2 \sum_{i, j=1}^n a_{i_j}^\epsilon X_i^\epsilon w X_j^\epsilon \phi \right) dy d\tau, \end{aligned}$$

where ϕ is as in the proof of Proposition 4.2 and B is a differential operator of order $k + 1$. The conclusion follows by further differentiating the previous representation formula along two horizontal directions $X_{j_1}^\epsilon X_{j_2}^\epsilon$ and arguing as in the proof of Proposition 4.2. \square

Proof of Theorem 1.3. We will prove by induction that for every $k \in N$ and for every compact set $K \subset\subset Q$ there exists a positive constant C such that

$$(4.5) \quad \|u_\epsilon\|_{C_{\epsilon, X}^{k, \alpha}(K)} \leq C,$$

for every $\epsilon > 0$. The assertion is true if $k = 2$, by Corollary 3.8 and Proposition 4.2. If (4.5) is true for $k + 1$, then the coefficients in (1.3) satisfy $a_{i_j}^\epsilon \in C_{\epsilon, X}^{k, \alpha}$ uniformly as $\epsilon \in (0, 1)$ and (4.5) thus holds at the level $k + 2$ by virtue of Proposition 4.4. \square

5. APPENDIX

Let us assume that G be a free nilpotent Lie group, and let $\mathcal{G} = V^1 \oplus \dots \oplus V^r$ its associated Lie algebra. Denote by X_1, \dots, X_m a basis of the first layer of the Lie algebra. and by X_{m+1}, \dots, X_n a list of vectors which complete a basis of the tangent space. If $u(x)$ is an homogeneous polynomial of order p and X is a differential operator of degree q we will call $u(x)X$ differential operator of degree $q - p$. If $a_{ij} = (A)_{ij}$ is a real valued, constant coefficient, $m \times m$ positive definite matrix, one can define a Lie algebra automorphism

$$T_A : V^1 \rightarrow V^1, \text{ with } T_A(X_i) = \sum_{j=1}^m a_{ij} X_j$$

for every $i = 1, \dots, m$ and extend it to the whole Lie algebra as a morphism

$$T_A[X_i, X_j] = [T_A(X_i), T_A(X_j)] = \sum_{h,k=1}^m a_{ih} a_{jk} [X_h, X_k].$$

Note that T_A can be represented as a block matrix $\bar{A} = A_1 \oplus A_2 \cdots \oplus A_r$, with $A_1 = A$ and where the block A_j act on vectors of degree j , and its coefficients are polynomial of order j of the coefficients of A_1 . In particular $T_A(X)$ and X have the same degree. Via the exponential map T_A induces a group automorphism on the whole group

$$F_A : G \rightarrow G, \text{ defined by } F_A(x) = \exp(T_A(\log(x))).$$

In terms of exponential coordinates (around the origin)

$$(5.1) \quad x = \exp\left(\sum_{i=1}^n v_i X_i\right)(0)$$

one has

$$F_A(x) = \exp\left(\sum_{i=1}^n v_i T_A(X_i)\right)(0) = \exp\left(\sum_{s=1}^r \sum_{d(i)=d(j)=s} v_i (A_s)_{ij} X_j\right)(0),$$

where $d(i)$ denotes the homogenous degree defined as in (1.1). Since $v_i X_i$ is a zero order homogeneous operator then also $v_i T_A(X_i)$ is a differential operator of order zero.

Proposition 5.1. *Let $(w_i)_{i=1, \dots, n}$ denote the canonical coordinates of $F_A(x)$ around $F_B(x)$, and $(v_j)_{j=1, \dots, n}$ the coordinates of x around 0, as in (5.1). There exists $M \in \mathbb{N}$ depending only on the group structure; constants c_1, \dots, c_M depending only on the group structure; zero-order differential operators Y_1, \dots, Y_M depending only on the group structure, on the coefficients $(v_j)_{j=1, \dots, n}$ and their derivatives along vector fields up to order $r - 1$ (but independent on A and B), and a constant $C = C(\|A\|, \|B\|, G) > 0$ such that*

$$(5.2) \quad \sum_{i=1}^n w_i X_i = \sum_{l=1}^M c_l Y_l \quad \text{and } |c_l| \leq C \|A - B\| \text{ for } l = 1, \dots, M.$$

Proof. By definition

$$\sum_{j=1}^n w_j X_j = \sum_{i=1}^n v_i T_A(X_i) * \left(-\sum_{i=1}^n v_i T_B(X_i)\right)$$

where $*$ is the Baker-Campbell-Hausdorff operation (see for instance [42] and references therein). This formula allows to represent the left-hand side as a finite⁸ sum of terms involving commutators (up to order

⁸since the group is nilpotent

r) of the two terms on the right-hand side. Let us consider separately the terms in this representation starting with those involving no commutators, i.e.

$$\sum_{i=1}^n v_i T_A(X_i) - \sum_{i=1}^n v_i T_B(X_i) = \sum_{s=1}^r \sum_{d(i)=d(j)=s} v_i \left((A_s)_{ij} - (B_s)_{ij} \right) X_j.$$

Clearly this expression is in the same form as (5.2). Next we consider the second term in the Baker-Campbell-Hausdorff sum, i.e. that involving commutators of the form

$$\begin{aligned} & \left[\sum_{i=1}^n v_i T_A(X_i), \sum_{h=1}^n v_h T_B(X_h) \right] = \\ & \sum_{p,s=1}^r \sum_{d(i)=d(j)=d(h)=d(k)=s} (A_s)_{ij} (B_p)_{hk} \left[v_i X_j, v_h X_k \right] = \\ & \sum_{p,s=1}^r \sum_{d(i)=d(j)=d(h)=d(k)=s} \frac{1}{2} \left((A_s)_{ij} (B_p)_{hk} - (A_p)_{hk} (B_s)_{ij} \right) \left[v_i X_j, v_h X_k \right]. \end{aligned}$$

It is evident that the coefficients of the commutators above are Lipschitz functions of A and B , vanishing for $A = B$, and consequently the whole sum satisfies (5.2). To conclude the proof we note that all further terms of the Baker-Campbell-Hausdorff sum involve operators as in (5.2) and zero order operators of the form $\sum_{i=1}^n v_i T_A(X_i)$ or $-\sum_{i=1}^n v_i T_B(X_i)$. Such commutators give rise to differential operators of zero order whose coefficients are polynomials in A and B and vanish when $A = B$. \square

REFERENCES

- [1] F. Andreu, C. Ballester, V. Caselles and J. M. Mazón, The Dirichlet problem for the total variation flow. *J. Funct. Anal.* 180 (2001), no. 2, 347-403.
- [2] F. Andreu, V. Caselles and J. M. Mazón Ruiz, Parabolic quasilinear equations minimizing linear growth functionals, *Progr. Math.*, 223, Birkhauser, Basel, 2004
- [3] D. G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, *Arch. Rational Mech. Anal.*, 25 (1967), 81-122.
- [4] D. Barbieri, G. Citti, G. Cocci and A. Sarti, A cortical-inspired geometry for contour perception and motion integration, submitted paper.
- [5] G. Bellettini, V. Caselles and M. Novaga, The total variation flow in \mathbb{R}^N , *J. Differential Equations* 184 (2002), no. 2, 475-525.
- [6] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Uniform Gaussian estimates of the fundamental solutions for heat operators on Carnot groups, *Adv. Differential Equations*, 7 (2002), 1153-1192.
- [7] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Fundamental solutions for non-divergence form operators on stratified groups, *Trans. Amer. Math. Soc.* 356 (2004), no. 7, 2709-2737.
- [8] L. Capogna and G. Citti, Generalized mean curvature flow in Carnot groups, *Comm. Partial Differential Equations* 34 (2009), no. 7-9, 937-956.
- [9] L. Capogna, G. Citti and M. Manfredini, Regularity of non-characteristic minimal graphs in the Heisenberg group H^1 , *Indiana Univ. Math. J.* 58 (5) (2009) 2115-2160.
- [10] L. Capogna, G. Citti and M. Manfredini, Smoothness of Lipschitz intrinsic minimal graphs in the Heisenberg group H^n , $n > 1$ *J. Reine Angew. Math. (Crelle's Journal)* (2010), 648, 75-110.
- [11] L. Capogna, G. Citti and G. Rea, A sub-elliptic analogue of Aronson-Serrin Harnack inequality, preprint (2011).
- [12] L. Capogna, G. Citti and C. Senni Guidotti Magnani, Sub-Riemannian heat kernels and mean curvature flow of graphs, preprint (2012).
- [13] L. Capogna, D. Danielli, J. Tyson and S. Pauls, An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem, Birkhauser, 2007.
- [14] J.-H. Cheng, J.-F. Hwang and P. Yang, Regularity of C^1 smooth surfaces with prescribed p -mean curvature in the Heisenberg group, *Math. Ann.* 344 (2009), no. 1, 1-35.
- [15] J.H. Cheng and J.F. Hwang, A. Malchiodi and P. Yang, Minimal surfaces in pseudohermitian geometry, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 4 (2005), 129-177.

- [16] G. Citti and M. Manfredini, Uniform estimates of the fundamental solution for a family of hypoelliptic operators, *Potential Anal.* 25 (2006), no. 2, 147-164.
- [17] G. Citti, A. Pascucci and S. Polidoro, On the regularity of solutions to a nonlinear ultraparabolic equation arising in mathematical finance. *Differential Integral Equations* 14 (2001), no. 6, 701-738.
- [18] G. Citti and A. Sarti, A cortical based model of perceptual completion in the Roto-Translation space, *Journal of Mathematical Analysis and Vision*, (2006), 24, 307 - 326.
- [19] T. Chan, S. Esedoglu, F. Park and A. Yip, Total Variation Image Restoration: Overview and Recent Developments *Handbook of Mathematical Models in Computer Vision*, (2006), 17-31.
- [20] R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certains espaces homogenes. (french) Etude de certaines intégrales singulières, *Lecture Notes in Mathematics*, Vol. 242, Springer-Verlag, 1971.
- [21] D. Danielli, N. Garofalo, D.M. Nhieu and S.D. Pauls, Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group H^1 , *J. Differ. Geom.* (2009), 81(2), 251-295.
- [22] N. Dirr, S. Dragoni and M. von Renesse, Evolution by mean curvature flow in sub-Riemannian geometries, *Communications on Pure and Applied Mathematics*, 9 (2), (2010) pp. 307-326.
- [23] R. Duits and E. M. Franken, Left invariant parabolic evolution equations on $SE(2)$ and contour enhancement via invertible orientation scores, part I: Linear left-invariant diffusion equations on $SE(2)$, *Quarterly of Applied Mathematics*, AMS, (2010), 68, 255-292.
- [24] R. Duits and E. Franken, Left invariant parabolic evolution equations on $SE(2)$ and contour enhancement via invertible orientation scores, part II: Nonlinear left-invariant diffusion equations on invertible orientation scores, *Quarterly of Applied Mathematics*, AMS, (2010), 68, 293-331.
- [25] R. Duits, H. Fuhr, B.J. Janssen, M. Bruurmijn, L.M.J. Florack and H.A.C.van Assen, Evolution Equations on Gabor Transforms and their Applications, *Applied and Computational Harmonic Analysis*, accepted for publication, to appear
- [26] G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.* 2 (1975), no. 13, 161-207.
- [27] G. B. Folland, *Harmonic Analysis in Phase Space*, *Annals of Mathematics Studies*, 122, Princeton Univ. Press, Princeton, 1989.
- [28] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, New Jersey, 1982.
- [29] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964 xiv+347 pp.
- [30] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhauser Boston Inc., Boston, MA, 1999, Edited by J. LaFontaine and P. Pansu. Based on the 1981. French original, Translated from the French by Sean Michael Bates.
- [31] A. A. Grigor'yan, The heat equation on noncompact Riemannian manifolds. (Russian) *Mat. Sb.* 182 (1991), no. 1, 55-87; translation in *Math. USSR-Sb.* 72 (1992), no. 1, 47-77.
- [32] L. Hormander, Hypoelliptic second order differential equations, *Acta Math.* (1967), no. 119, 147-171.
- [33] D. S. Jerison and A. Sanchez-Calle, Estimates for the heat kernel for a sum of squares of vector fields, *Indiana Univ. Math. J.* 35 (1986), no. 4, 835-854.
- [34] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
- [35] F. Ferrari, J. Manfredi and Q. Liu, On the horizontal Mean Curvature Flow for Axisymmetric surfaces in the Heisenberg Group, preprint (2012).
- [36] J. S. Moll, The anisotropic total variation flow. (English summary) *Math. Ann.* 332 (2005), no. 1, 177-218.
- [37] R. Monti and M. Rickly, Convex isoperimetric sets in the Heisenberg group, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 8 (2009), no. 2, 391-415.
- [38] J. Moser, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.* 17 (1964), 101-134.
- [39] A. Nagel, E. M. Stein, and S. Wainger, Balls and metrics defined by vector fields. I. Basic properties, *Acta Math.* 155 (1985), no. 1-2, 103-147.
- [40] S.D. Pauls: Minimal surfaces in the Heisenberg group, *Geom. Dedic.*, (2004), 104, 201-231.
- [41] M. Ritoré and C. Rosales, Area-stationary surfaces in the Heisenberg group H^1 , *Adv. Math.* 219 (2008), 633-671.
- [42] L. P. Rothschild and E. M. Stein, Hypoelliptic differential operators and nilpotent groups, *Acta Math.* 137 (1976), no. 3-4, 247-320.
- [43] L. I. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D* 60 (1992), 259-268.
- [44] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, *Internat. Math. Res. Notices* 1992 (1992), no. 2, 27-38.
- [45] F. Serra Cassano and D. Vittone, Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in Heisenberg groups, preprint (2010).

- [46] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

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