ISSN 2590-9770
The Art of Discrete and Applied Mathematics 9 (2026) #P3.10
**https://doi.org/10.26493/2590-9770.1904.de6**
(Also available at http://adam-journal.eu)

# Some bounds on the number of cyclic $k-$cycle systems of the complete graph*

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Received 15 July 2025, accepted 7 March 2026, published online 19 October 2026

**Abstract**

In a recent paper M. Buratti and M. E. Muzychuk have established some lower bounds on the number of non isomorphic cyclic Steiner Triple Systems of order $v \equiv 1 \pmod 6$. We complete their results to the case $v \equiv 3 \pmod 6$. Furthermore, we find lower bounds for the number of non isomorphic cyclic $(K_v, C_k)$-designs for each admissible pair $(v, k)$. We place somewhat emphasis on the case $k$ odd and $v \equiv 1, k \pmod{2k}$. We also find lower bounds for the number of cyclic Hamiltonian cycle systems of order $3p$, $p > 3$ a prime.

*Keywords: Cyclic cycle systems, difference systems.*

*Math. Subj. Class.: 05B05, 05B07, 05B10.*

## 1 Introduction

A $k$-cycle system of $K_v$, also called a $(K_v, C_k)$-design, is a set $\mathcal{D}$ of $k$-cycles in $K_v$ (the complete graph of order $v$) whose edges partition the edge-set of $K_v$. It is said to be *cyclic* if the vertex-set of $K_v$ is $\mathbb{Z}_v$ and for every cycle $C = (c_0, c_1, \ldots, c_{k-1})$ of $\mathcal{D}$ we also have $C + 1 := (c_0 + 1, c_1 + 1, ..., c_{k-1} + 1) \in \mathcal{D}$. In particular a 3-cycle system of $K_v$ is usually called a Steiner Triple System or $STS(v)$ for short. The admissible values of $v$, $3 \leq k \leq v$, for which there exists a $(K_v, C_k)$-design are those satisfying the following conditions: $v(v-1) \equiv 0 \pmod{2k}$ and $v \equiv 1 \pmod 2$. This is because in a $(K_v, C_k)$-design we have exactly $v(v-1)/(2k)$ cycles and the number of cycles through any given

*We are very greatful to the referee for the careful reading and valuable comments, which led to a significative improvement of the paper.

This research was carried out within the activities of INdAM–GNSAGA and was supported by the University of Modena and Reggio Emilia (FAR2025).

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vertex is $(v-1)/2$. In what follows, we will assume that these conditions are satisfied. In particular, if $k$ is a prime power these conditions are equivalent to $v \equiv 1$ or $k \pmod{2k}$.

The existence question for $(K_v, C_k)$-designs has been completely settled by Alspach and Gavlas [2] in the case of $k$ odd (see also [7]) and by Sajna [25] in the even case.

Regarding cyclic $(K_v, C_k)$-designs, existence has been proved whenever $v \equiv 1 \pmod{2k}$. This, for $k$ even, was proved in the 1960s by Kotzig [18] and by Rosa [23, 24] who also settled the cases of $k = 3, 5, 7$, [24] (for the earliest solution of $k = 3$ see [21]). The case $v \equiv 1 \pmod{2k}$ was solved by Buratti and Del Fra [9] and other existence solutions were independently found by Fu and Wu [16] and by Bryant et al. [6]. The case $v \equiv k \pmod{2k}$, that necessarily implies $k$ odd, was solved by Buratti and Del Fra in [9] and [10], with some exceptions which were solved by Vietri [26].

To resume, the main results guarantee the existence of a cyclic $(K_v, C_k)$-design whenever $v \equiv 1$ or $k \pmod{2k}$, except for the cases $(v, k) = (9, 3), (15, 15), (q, q)$, with $q$ a prime power that is not a prime. In these cases, the non-existence is ensured.

Throughout the paper, the vertex set of $K_v$ will always be assumed to be $\mathbb{Z}_v$. For each cycle $C$ of $K_v$ we denote by $Orb(C)$ and $Stab(C)$ the orbit and the stabilizer of $C$ under the natural action of $\mathbb{Z}_v$.

In a $(K_v, C_k)$-design, the *type* of a $k$-cycle $C$ of $K_v$ is the order of $Stab(C)$. Hence, the type of $C$ is a common divisor, say $d \geq 1$, of $v$ and $k$ and

$$C = (c_0, \ldots, c_{k-1}) \text{ with } c_{\frac{k}{d}+i} = c_i + g \text{ for } i \geq 0$$

where the indices are taken modulo $k$ and $g$ is a generator of the subgroup of $\mathbb{Z}_v$ of order $d$.

We call a cycle $C$ *full* or *short* depending on whether $Orb(C)$ has the corresponding property. This happens, respectively, when its type is trivial or non-trivial. In both cases, the list of partial differences from $C$ is the multiset

$$\partial C = \{\pm(c_{i+1} - c_i) : 0 \leq i \leq \frac{k}{d} - 1\}.$$

Note that if $C$ is full, then $\partial C$ has size $2k$ and it coincides with $\Delta C$, the list of differences from $C$ in the usual sense.

More generally, if $\mathcal{F} = \{C_1, C_2, \ldots, C_n\}$ is a set of $k$-cycles, the list of partial differences from $\mathcal{F}$ is the multiset $\partial\mathcal{F} = \bigcup_i \partial C_i$.

Clearly, a cyclic $(K_v, C_k)$-design is completely determined by a complete set $\mathcal{F}$ of representatives for its cycle-orbits. That is, the set of all its cycles is the union of the orbits of the cycles in $\mathcal{F}$. We refer to it as *the design arising from* $\mathcal{F}$ denoting it by $\mathcal{D}(\mathcal{F})$. It is well-known (see, e.g., [12]) that a set of $k$-cycles $\mathcal{F}$ is a complete set of representatives for the cycle-orbits of a cyclic $(K_v, C_k)$-design if and only if its list of partial differences coincides with the set $\mathbb{Z}_v^* = \mathbb{Z}_v \setminus \{0\}$. For this reason, $\mathcal{F}$ will be called a $(K_v, C_k)$-*difference system*, or simply a difference system, when $k$ and $v$ are clear from the context. Each cycle of $\mathcal{F}$ will be called a *starter* cycle.

Observe that we can always assume that each starter cycle contains 0 as a vertex. If $v \equiv 1 \pmod{2k}$ then necessarily each starter cycle is full and we have $v = 2nk + 1$, where $n$ is the number of starter cycles in each $(K_v, C_k)$-difference system. While if $v \equiv k \pmod{2k}$ then necessarily $k$ is odd and a difference system necessarily contains short starter cycles.

If $k = p$ is a prime, a $(K_{2np+p}, C_p)$-difference system has exactly $n$ full cycles and $(p-1)/2$ short cycles (all of type $p$) which are

$$(0, i(2n+1), 2i(2n+1), \ldots, (p-1)i(2n+1)) \quad 1 \leq i \leq (p-1)/2.$$

Two cyclic $(K_v, C_k)$-designs $\mathcal{D}$ and $\mathcal{D}'$ are *isomorphic* if there exists a permutation of $\mathbb{Z}_v$ turning $\mathcal{D}$ into $\mathcal{D}'$. In what follows we will denote by $NC(K_v, C_k)$ the number of cyclic $(K_v, C_k)$-designs up to isomorphism.

The famous Bays-Lambossy Theorem, see [3] and its generalizations proved by Pálfy, [20], and Muzychuk, [19] imply the following Proposition 1.3.

If $C = (c_0, \ldots, c_{k-1})$ is a cycle with vertices in $\mathbb{Z}_v$ and $u$ is a unit of $\mathbb{Z}_v$, we denote by $uC$ the cycle $(uc_0, \ldots, uc_{k-1})$. Similarly, given a set of cycles $\mathcal{C}$, we denote by $u\mathcal{C}$ the set $\{uC \mid C \in \mathcal{C}\}$.

We will denote by $\phi$ the Euler's totient function. The following is a specialization of the results in [3, 20].

**Theorem 1.1.** *If $\mathcal{D}$ and $\mathcal{D}'$ are two isomorphic cyclic $(K_v, C_k)$-designs with $\gcd(v, \phi(v)) = 1$, then $\mathcal{D}' = u\mathcal{D}$ for a suitable unit $u$ of $\mathbb{Z}_v$.*

The following is a specialization of the main result in [19].

**Theorem 1.2.** *The number of isomorphisms between two cyclic $(K_v, C_k)$-designs is at most equal to $\phi(v)$.*

As an immediate consequence, we have:

**Proposition 1.3.** *A set of $N$ cyclic $(K_v, C_k)$-designs guarantees $NC(K_v, C_k) \geq \lceil \frac{N}{\phi(v)} \rceil$.*

In order to establish some lower bounds for $NC(K_v, C_k)$, it could be important to produce "many" distinct cyclic $(K_v, C_k)$-designs. With this aim, Sections 2 and 4 introduce some methods useful for obtaining different cyclic $(K_v, C_k)$-designs starting from a known one. These methods are fruitful when the original design has many full cycles and few short cycles which are of type $k$ or $k/2$, see Theorems 2.2 and 4.4.

In Section 3 we focus on the Hamiltonian case and apply the method of flipping the signs (Section 2) to obtain a lower bound for $NC(K_{3p}, C_{3p})$, with $p > 3$ a prime, see Theorem 3.6.

In Section 5 we report examples of cyclic $(K_v, C_k)$-designs, $k$ odd, obtained via Skolem and split Skolem sequences, [9, 10, 26]. To these examples, we apply the methods of Sections 2 and 4 to obtain lower bounds for the number of non isomorphic cyclic Steiner triple Systems of order $v \equiv 3 \pmod 6$, thus completing the result of [11], and to obtain lower bounds for $NC(K_{2nk+r}, C_k)$, with $k \geq 5$ odd and $r = 1$ or $r = k$. See Propositions 5.1, 5.2, 5.3, 5.4, 5.5.

## 2 Flipping the signs

In this section, by a simple device – much like the one used by Buratti and Muzychuk in [11] to obtain several distinct cyclic 2-designs from a single one – we derive several cyclic $(K_v, C_k)$-designs starting from a single initial construction.

**Lemma 2.1.** *Let $C$ be a $k$-cycle in $K_v$, with $v$ odd. We have $\mathrm{Orb}(C) = \mathrm{Orb}(-C)$ if and only if $C$ is of type $k$ or $k/2$.*

*Proof.* If $\mathrm{Orb}(C) = \mathrm{Orb}(-C)$, there exists a suitable $t \in \mathbb{Z}_v$ such that $C = -C + t$. Then, for any edge $[x, y]$ of $C$, the edge $[-x + t, -y + t]$ also belongs to $C$. Since the differences from this edge are $\{x - y, y - x\}$ and the edges of $C$ with these differences are exactly those of the orbit of $[x, y]$ under $\mathrm{Stab}(C)$, we have $[-x+t, -y+t] = [x+s, y+s]$ for some $s \in \mathrm{Stab}(C)$. This necessarily implies $t = x + y + s$. Take three consecutive vertices $x, y, z$ of $C$. For what we have seen above, $t = x + y + s = y + z + s'$ for some $s, s' \in \mathrm{Stab}(C)$. Hence $z = x + \sigma$ where $\sigma = s - s' \in \mathrm{Stab}(C)$. It follows that $C = (x, y, x+\sigma, y+\sigma, x+2\sigma, y+2\sigma, \dots)$ and therefore $C$ is of type $k$ or $k/2$, according to whether $k$ is odd or even, respectively. Conversely, assume that $C$ is of type $k$ or $k/2$. In the first case we have $C = -C$. In the second case, up to translation we can write $C = (0, x, s, x + s, \dots)$ with $s$ of order $k/2$ and one checks that $C = -C + x$. □

Now, emulating the Buratti-Muzychuk construction (whose embryonic idea is in [5]), we start from a $(K_v, C_k)$-difference system $\mathcal{F}$ and generate new difference systems by switching the signs of the starter cycles in all possible ways. This yields $2^n$ difference systems, where $n = |\mathcal{F}|$. Then, by applying Lemma 2.1, we show that the number of distinct designs generated in this way is $2^m$, where $m$ is the number of cycles in $\mathcal{F}$ of type $t < k/2$.

**Theorem 2.2.** *If $m$ is the number of cycles of a $(K_v, C_k)$-difference system that are not of type $k$ or $k/2$, then $NC(K_v, C_k) \geq \lceil \frac{2^m}{\phi(v)} \rceil$.*

*Proof.* Let $\mathcal{F} = \{C_1, \dots, C_n\}$ be a $(K_v, C_k)$-difference system and without loss in generality suppose $C_1, \dots, C_m$ to be the cycles of $\mathcal{F}$ that are not of type $k$ or $k/2$. For each $m$-tuple $\sigma = (\sigma_1, \dots, \sigma_m) \in \{1, -1\}^m$ set $\mathcal{F}_\sigma = \{\sigma_1 C_1, \dots, \sigma_m C_m\} \cup \{C_{m+1}, \dots, C_n\}$. Since $C_i$ and $-C_i$ have the same list of partial differences, $\mathcal{F}_\sigma$ is a $(K_v, C_k)$-difference system itself. We show that $\mathcal{E} = \{\mathcal{D}(\mathcal{F}_\sigma) \mid \sigma \in \{1, -1\}^m\}$ has no repeated $(K_v, C_k)$-designs. In fact, assume that $\sigma$ and $\sigma'$ are distinct $m$-tuples of $\{1, -1\}^m$ so that $\sigma_i = 1$ and $\sigma'_i = -1$ for a suitable $i$, $1 \leq i \leq m$. Observe that $C_i$ is not a translate of $\sigma'_j C_j$ for $i \neq j$, $1 \leq j \leq m$, otherwise it should be $\partial C_i = \partial(\sigma'_j C_j) = \partial C_j$ contradicting the fact that $\partial\mathcal{F}$ has no repeated elements. Moreover $C_i$ cannot be a translate of $\sigma'_i C_i = -C_i$ since $Orb(C_i) \neq Orb(-C_i)$ from Lemma 2.1. We conclude that the cycle $C_i$ of $\mathcal{D}(\mathcal{F}_\sigma)$ cannot be in $\mathcal{D}(\mathcal{F}_{\sigma'})$, hence $\mathcal{D}(\mathcal{F}_\sigma)$ and $\mathcal{D}(\mathcal{F}_{\sigma'})$ are different. The assertion then follows from Proposition 1.3. □

## 3 Hamiltonian cycle systems

In this section, we would like to discuss cyclic Hamiltonian cycle systems, namely cyclic $(K_v, C_v)$-designs.

First of all, it is almost obvious that $NC(K_p, C_p) = 1$ for every odd prime $p$: in this case, the unique cyclic $(K_p, C_p)$-design is, up to isomorphism,

$$\mathcal{D} = \{(0, i, 2i, \dots, (p-1)i) \mid 1 \leq i \leq (p-1)/2\}.$$

This already highlights the enormous difference compared to 1-*rotational* $(K_v, C_v)$-designs, namely Hamiltonian cycle systems with an automorphism that fixes one vertex and cyclically permutes all the others. Indeed, it has been proved in [13] that for every odd $v \geq 13$, the number of 1-rotational $(K_v, C_v)$-designs is at least $2^{\lceil (3v-3)/8 \rceil}$. This is not too surprising considering that, despite the apparent "kinship" between cyclic cycle systems and

1-rotational cycle systems, the construction of the former seems to be much more difficult than that of the latter. In fact, while the existence question for 1-rotational cycle systems is completely settled [7, 8, 22, 27], we are currently very far from a complete solution to the analogous question for cyclic ones.

It seems unlikely that Theorem 2.2, combined with the existence results of cyclic Hamiltonian cycle systems, can yield significant bounds on the number of cyclic $(K_v, C_v)$-designs. In fact, in many cases, this combination of results is entirely ineffective. For instance, if $p$ and $q$ are distinct primes, the existence of a cyclic $(K_{pq}, C_{pq})$-design was obtained in [10] by means of a $(K_{pq}, C_{pq})$-difference system in which all cycles are of type $pq$ except for two. Using Theorem 2.2 with this difference system would yield the trivial bound $NC(K_{pq}, C_{pq}) \geq 1$. To take advantage of Theorem 2.2 effectively, one would need to construct $(K_v, C_v)$-difference systems in which the cycles of type $v$ are minimized. Using this approach, we are going to prove a non-trivial bound on the number of cyclic Hamiltonian cycle systems of order $3p$, where $p > 3$ is a prime.

To help the reader understand, it is worth recalling how to construct a Hamiltonian cycle of type $d$ in $K_v$ (which is an alternative formulation of Proposition 1.2 in [10]). Setting $v = de$, we say that a path $A = [a_0, a_1, \ldots, a_{e-1}, a_e]$ in $K_v$ is *e-admissible* if it satisfies the following conditions:

- $a_0 = 0$; $\quad a_e$ is a generator of the subgroup of $\mathbb{Z}_v$ of order $d$;
- $\{a_0, a_1, \ldots, a_{e-1}\} = \mathbb{Z}_d \pmod e$.

We have:

**Proposition 3.1.** *The Hamiltonian $v$-cycles of type $d$ are precisely those formed by the union of $d$ edge-disjoint paths $A + ia_e$, $0 \leq i \leq e-1$, where $A = [a_0, a_1, \ldots, a_{e-1}, a_e]$ is $e$-admissible.*

It is convenient to first prove the following result.

**Lemma 3.2.** *If $p > 3$ is a prime, there exists a Hamiltonian cycle $C_0$ in $K_{3p}$ of type 3 such that $\partial C$ has no repetitions and contains all non-zero elements of the subgroup of $\mathbb{Z}_{3p}$ of order $p$, that is $H = \{0, 3, 6, \ldots, 3p-3\}$.*

*Proof.* We distinguish two cases according to whether $p \equiv 1$ or 5 (mod 6).

Case 1: $p \equiv 1 \pmod 6$, say $p = 6n + 1$.

Take the path

$$A = [0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots, 3n, -3n, p]$$

which is clearly $p$-admissible, and let $C$ be the Hamiltonian cycle formed by the union of $A$, $A + p$, $A + 2p$. We have

$$\partial C = \pm\{1, 2, 3, \ldots, 6n\} \cup \pm\{9n + 1\}.$$

Given that $3p = 18n + 3$, we see that $\partial C$ does not have repetitions and that the set of non-zero elements of $H$ not appearing in it is $\{\pm(6n + 3i) \mid 1 \leq i \leq n\}$.

For $0 \leq i \leq n-1$, we replace each vertex $a_{3i+1}$ of $A$ with $a_{3i+1} + (-1)^i p$. Thus, assuming, for instance, that $n \geq 3$, we make replacements as these:

$$\begin{array}{rcrcr} 1 & \longrightarrow & 1+p & = & 6n+2 \\ -2 & \longrightarrow & -2-p & = & -6n-3 \\ 4 & \longrightarrow & 4+p & = & 6n+5 \\ -5 & \longrightarrow & -5-p & = & -6n-6 \\ & & \cdots & & \end{array}$$

In this way, we obtain a new $(p+1)$-path

$$B = [0, 6n+2, -1, 2, -6n-3, 3, -3, 6n+5, -4, 5, -6n-6, ...]$$

which is still $p$-admissible since in each substitution an element is replaced with another from the same congruence class modulo $p$.

Let $C_0$ be the Hamiltonian cycle formed by the union of $B$, $B+p$, $B+2p$. In $\partial C_0$ we lose some differences that were present in $\partial C$ and introduce new ones. For example, it is readily seen that the replacement of $1$ with $p+1 = 6n+2$ results in the loss of the differences $\pm 1$ and $\pm 2$ and introduces the new differences $\pm(6n+2)$ and $\pm(6n+3)$; the replacement of $-2$ with $-p-2 = -6n-3$ results in the loss of the differences $\pm 4$ and $\pm 5$ and introduces the new differences $\pm(6n+5)$ and $\pm(6n+6)$; and so on.

In the end, we see that none of the lost differences are in $H$ whereas the new differences, in addition to being pairwise distinct, are contained in the set $\{\pm(6n+i) \mid 1 \leq i \leq 3n\}$ and include all elements of $H$ that did not appear in $\partial C$. We can therefore conclude that $C_0$ is a Hamiltonian cycle as required.

Case 2: $p \equiv 5 \pmod 6$, say $p = 6n+5$.

The construction is almost identical to Case 1. Thus, we omit the details. Take the path

$$A = [0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots, 3n+2, -(3n+2), p]$$

and modify it into a new path $B$ by replacing any vertex $a_{3i+1}$ of $A$ with $a_{3i+1} + (-1)^i p$ for $0 \leq i \leq n$. Then $C_0 = B \cup (B+p) \cup (B+2p)$ is a Hamiltonian cycle as required. □

By way of illustration, we present an example with $p = 13$.

**Example 3.3.** We are working in $K_{39}$. Take the path

$$A = [0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 13]$$

and modify it into a new path $B$ as in the recipe given in the proof of the previous lemma, highlighting the replacements in boldface:

$$B = [0, \mathbf{14}, -1, 2, \mathbf{-15}, 3, -3, 4, -4, 5, -5, 6, -6, 13].$$

The list of partial differences of the Hamiltonian cycle $C_0 = B \cup (B+p) \cup (B+2p)$ is

$$\partial C_0 = \pm\{14, \underline{15}, \underline{3}, 17, \underline{18}, \underline{6}, 7, 8, \underline{9}, 10, 11, \underline{12}, 19\}.$$

It is readily seen that its elements are pairwise distinct. Also, looking at the underlined elements, we see that it contains all non-zero elements of the subgroup of $\mathbb{Z}_{39}$ of order 13.

We are now ready to give the crucial ingredient that will lead to a lower bound of $NC(K_{3p}, C_{3p})$ with $p$ prime.

**Proposition 3.4.** *For a prime $p > 3$, there exists a $(K_{3p}, C_{3p})$-difference system consisting of one cycle of type 3, $\lfloor p/6 \rfloor$ cycles of type $p$, and either 0 or 2 cycles of type $3p$, depending on whether $p \equiv 1$ or 5 modulo 6.*

*Proof.* Let $C_0$ be a Hamiltonian cycle of $K_{3p}$ of type 3 as provided by Lemma 3.2, such that $\partial C_0$ is a $2p$-set containing all non-zero elements of the subgroup $H \leq \mathbb{Z}_{3p}$ of order $p$.

Set $p = 6n + 1$ or $p = 6n + 5$ so that $n = \lfloor p/6 \rfloor$ in either case. The set $\mathbb{Z}_{3p}^* \setminus \partial C_0$ has size $(3p - 1) - 2p = p - 1$. We order its elements naturally as

$$\mathbb{Z}_{3p}^* \setminus \partial C_0 = \pm\{x_1, \ldots, x_{(p-1)/2}\} \quad \text{where } 0 < x_1 < x_2 < \cdots < x_{(p-1)/2} \leq \tfrac{3p-1}{2}.$$

Given that $\partial C_0 \cap H = \emptyset$, no $x_i$ is divisible by 3; thus, $x_i \equiv \pm 1 \pmod 3$ for each $i$. Set $Y = (y_1, y_2, \ldots, y_{(p-1)/2})$ with $y_i = x_i$ or $-x_i$ according to whether $x_i \equiv 1$ or $-1 \pmod 3$, respectively. Clearly:

$$\mathbb{Z}_{3p}^* \setminus \partial C_0 = \pm\{y_1, y_2, \ldots, y_{(p-1)/2}\}. \tag{3.1}$$

Consider the $n$ triples

$$T_0 = (y_1, y_2, y_3), \quad T_1 = (y_4, y_5, y_6), \ldots, T_{n-1} = (y_{3n-2}, y_{3n-1}, y_{3n}).$$

For $p = 6n + 1$, these triples partition $Y$. For $p = 6n + 5$, they partition $Y \setminus \{u_1, u_2\}$ where $u_1 = y_{3n+1}$ and $u_2 = y_{3n+2}$. In the latter case, it is easy to see that $u_1$ and $u_2$ are units of $\mathbb{Z}_{3p}$, so $U_i = (0, u_i, 2u_i, 3u_i, ..., (3p - 1)u_i)$ is a Hamiltonian cycle of type $3p$ with $\partial U_i = \{\pm u_i\}$ for $i = 1, 2$.

For $1 \leq i \leq n$, associate the triple $T_i = (y_{3i-2}, y_{3i-1}, y_{3i})$ with the path $A_i = [a_{i0}, a_{i1}, a_{i2}, a_{i3}]$, where

$$a_{i0} = 0; \quad a_{i1} = y_{3i-2}; \quad a_{i2} = y_{3i-2} + y_{3i-1}; \quad a_{i3} = y_{3i-2} + y_{3i-1} + y_{3i}.$$

Since all elements of $Y$ are congruent to 1 (mod 3), it follows that $a_{i3} \in H$ and $\{a_{i0}, a_{i1}, a_{i2}\} = \mathbb{Z}_3 \pmod 3$. Furthermore, it is not difficult to see that $a_{i3} \neq 0$ making it a generator of $H$. Thus, $A_i$ is 3-admissible and the union of the $p$ paths $\{A_i + ja_{i3} \mid 0 \leq j \leq p - 1\}$ forms a Hamiltonian cycle $C_i$ of $K_{3p}$ of type $p$ where, clearly, $\partial C_i = \pm T_i$. By (3.1), we have:

$$\mathbb{Z}_{3p}^* \setminus \partial C_0 = \begin{cases} \partial C_1 \cup \ldots \cup \partial C_n & \text{if } p = 6n + 1; \\ \partial C_1 \cup \ldots \cup \partial C_n \cup \partial U_1 \cup \partial U_2 & \text{if } p = 6n + 5. \end{cases}$$

The required $(K_{3p}, C_{3p})$-difference system is $\mathcal{F} = \{C_0, C_1, \ldots, C_n\}$ if $p = 6n + 1$, and $\mathcal{F} = \{C_0, C_1, \ldots, C_n, U_1, U_2\}$ if $p = 6n + 5$. □

In the following example, we explicitly construct a $(K_{39}, C_{39})$-difference system by following the instructions in the proof of the previous theorem.

**Example 3.5.** Take the cycle $C_0$ constructed in Example 3.3. We have

$$\mathbb{Z}_{39}^* \setminus \partial C_0 = \pm\{1, 2, 4, 5, 13, 16\}.$$

Hence we have

$$Y = (1, -2, 4, -5, 13, 16).$$

Consider the two triples $T_1 = (1, -2, -4)$, $T_2 = (-5, 13, 16)$ partitioning $Y$ and take the paths $A_i$ "arising from $T_i$" for $i = 0, 1$:

$$A_1 = [0, 1, 1-2, 1-2+4] = [0, 1, -1, 3],$$

$$A_2 = [0, -5, -5+13, -5+13+16] = [0, -5, 8, 24].$$

We get a $(K_{39}, C_{39})$-difference system $\mathcal{F} = \{C_0, C_1, C_2\}$ where
$C_1 = \bigcup_{j=0}^{12}(A_1 + 3j)$ and $C_2 = \bigcup_{j=0}^{12}(A_2 + 24j)$.

**Theorem 3.6.** *If $p > 3$ is a prime and $n = \lfloor p/6 \rfloor$, we have NC$(K_{3p}, C_{3p}) \geq \lceil \frac{2^n}{p-1} \rceil$.*

*Proof.* It is enough to combine Theorem 2.2 with Proposition 3.4. □

The first prime $p$ for which the preceding corollary leads to a non-trivial bound is 31: $NC(K_{93}, C_{93}) \geq 2$. Of course, our bound becomes increasingly significant as $p$ grows. For example, we obtain $NC(K_{3\cdot 61}, C_{3\cdot 61}) \geq 18$ and $NC(K_{3\cdot 101}, C_{3\cdot 101}) \geq 656$.

## 4 Use of simple orderings

Given a full cycle $C$ in $K_v$ presented as $C = (0, c_1, \ldots, c_{k-1})$, it is useful to consider the $k$-sequence of its *clockwise differences*:

$$\vec{\Delta}C = (c_1 - 0, c_2 - c_1, \ldots, c_{k-1} - c_{k-2}, 0 - c_{k-1}).$$

We have the following straightforward observation:

**Remark 4.1.** If two full cycles $C$ and $C'$ are in the same orbit, then $\vec{\Delta}C$ and $\vec{\Delta}C'$ are dihedrally equivalent (that is, one can be obtained from the other by a cyclic shift or a reversal).

Moreover, we denote by $\vec{\Delta}^*C$ the $(k-1)$-sequence obtained from the $k$-sequence of the clockwise differences of $C$ by removing its last term $0 - c_{k-1}$. In other words, $\vec{\Delta}^*C$ is the sequence of differences of the directed path $[0, c_1, \ldots, c_{k-1}]$. For any reordering $\omega = (\omega_1, ..., \omega_{k-1})$ of $\vec{\Delta}^*C$ we define $C(\omega) = (0, s_1, s_2, \ldots, s_{k-1})$ as the circuit in $K_v$ where $s_i = \omega_1 + \cdots + \omega_i$ is the $i$-th partial sum of $\omega$ for $1 \leq i \leq k-1$.

Following the standard terminology in the study of *group sequenceability* [17], we will say that the ordering $\omega$ is *simple* if $C(\omega)$ is a cycle; that is, if the partial sums of $\omega$ are all distinct.

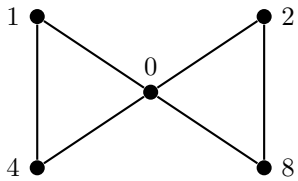
It is important to note that $\Delta C(\omega) = \Delta C$ for every simple ordering $\omega$ of $\vec{\Delta}^*C$. We also note that the $(k-1)!$ orderings of $\vec{\Delta}^*C$ are all simple if and only if there is no zero-sum subset of $\vec{\Delta}^*C$.

In the following example, we show how a zero-sum subset of $\vec{\Delta}^*C$ can give rise to non simple orderings.

**Example 4.2.** Consider the hexagon $H = (0, 1, 3, 6, 2, 8)$ in $K_{13}$. We have

$$\vec{\Delta}H = (1, 2, 3, 9, 6, 5) \quad \text{and thus} \quad \vec{\Delta}^* H = (1, 2, 3, 9, 6).$$

Among the $5! = 120$ possible orderings of $\vec{\Delta}^* H$, exactly 84 are simple. Among the 36 non-simple orderings, we find, for instance, $\omega = (1, 3, 9, 2, 6)$. Indeed, the resulting circuit $H(\omega) = (0, 1, 4, 0, 2, 8)$ is a *bowtie* rather than a cycle:




We need the following observation.

**Lemma 4.3.** *Let $C = (0, c_1, \ldots, c_{k-1})$ be a full $k$-cycle in $K_v$ and let $\omega$, $\omega'$ be two distinct simple orderings of $\vec{\Delta}^* C$. Then $C(\omega)$ and $C(\omega')$ belong to distinct orbits.*

*Proof.* Suppose that $\mathrm{Orb}(C(\omega)) = \mathrm{Orb}(C(\omega'))$. By Remark 4.1, the sequences of clockwise differences $\vec{\Delta}C(\omega)$ and $\vec{\Delta}C(\omega')$ must be dihedrally equivalent. These sequences are given by

$$\vec{\Delta}C(\omega) = (\omega_1, \omega_2, \ldots, \omega_{k-1}, \ -c_{k-1})$$

$$\vec{\Delta}C(\omega') = (\omega'_1, \omega'_2, \ldots, \omega'_{k-1}, \ -c_{k-1}).$$

Since both sequences share the same final term $-c_{k-1}$, their dihedral equivalence necessarily implies that the $(k-1)$-subsequences $\omega$ and $\omega'$ are either identical or reverse of each other. The former case is excluded by the assumption $\omega \neq \omega'$. Thus, we must have $\omega' = rev(\omega) = (\omega_{k-1}, \omega_{k-2}, \ldots, \omega_2, \omega_1)$. The resulting cycle $C(\omega')$ is

$$C(\omega') = (0, \ \omega_{k-1}, \ \omega_{k-1} + \omega_{k-2}, \ \omega_{k-1} + \omega_{k-2} + \omega_{k-3}, \ \ldots, c_{k-1})$$

Given that $c_{k-1} = \omega_1 + \omega_2 + \cdots + \omega_{k-1}$, it follows that

$$C(\omega') - c_{k-1} = (-c_{k-1}, -s_{k-2}, \ldots, -s_2, -s_1, 0)$$

where $s_i$ is the $i$-th partial sum of $\omega$. Thus, we see that $C(\omega') - c_{k-1} = -C(\omega)$. This implies that $C(\omega')$ is in the same orbit as $-C(\omega)$. However, by Lemma 2.1, $C(\omega)$ and $-C(\omega)$ are in distinct orbits, which contradicts our initial assumption. The assertion follows. □

**Theorem 4.4.** *Let $\{C_1, \ldots, C_m\}$ be the set of full cycles of a $(K_v, C_k)$-difference system and let $o_i$ be the number of simple orderings of $\vec{\Delta}^* C_i$, $1 \leq i \leq m$. Then we have $NC(K_v, C_k) \geq \lceil \frac{o_1 \ldots o_m}{\phi(v)} \rceil$.*

*Proof.* Let $\mathcal{F}$ be a $(K_v, C_k)$-difference system and let $\{C_1, \ldots, C_m\} \subseteq \mathcal{F}$. Let $O = O_1 \times \cdots \times O_m$ where $O_i$ is the set of all simple orderings of $\vec{\Delta}^* C_i$ for $1 \leq i \leq m$. For each $m$-tuple $\sigma = (\sigma_1, \ldots, \sigma_m) \in O$, let $\mathcal{F}_\sigma$ be the set obtained from $\mathcal{F}$ by replacing $C_i$ with $C_i(\sigma_i)$ for $1 \leq i \leq m$. Since $\Delta C_i(\sigma_i) = \Delta C_i$ for any $\sigma_i \in O_i$, it is evident that $\mathcal{F}_\sigma$ is a $(K_v, C_k)$-difference system.

Given that $|O| = o_1 o_2 \ldots o_m$, the assertion follows from Proposition 1.3 provided we show that for any distinct $\sigma, \sigma' \in O$, the designs arising from $\mathcal{F}_\sigma$ and $\mathcal{F}_{\sigma'}$ are distinct.

If $\sigma \neq \sigma'$, then $\sigma_i \neq \sigma'_i$ for some $i$. The cycle $C(\sigma_i)$, which belongs to the design arising from $\mathcal{F}_\sigma$, cannot belong to the design arising from $\mathcal{F}_{\sigma'}$. Otherwise, we would have $C(\sigma_i) \in \mathrm{Orb}(C(\sigma'_j))$ for some $j$. This would imply $\Delta C(\sigma_i) = \Delta C(\sigma'_j)$, and consequently $\Delta C_i = \Delta C_j$. Since different cycles of $\mathcal{F}$ have disjoint lists of differences, it must be $i = j$. Thus, $C(\sigma_i) \in \mathrm{Orb}(C(\sigma'_i))$, which means that $\sigma_i$ and $\sigma'_i$ are two distinct simple orderings of $\vec{\Delta}^* C_i$ whose resulting cycles belong to the same orbit. This contradicts Lemma 4.3 and the assertion follows. □

It is easy to recognize that in the case $k = 3$ the Theorems 2.2 and 4.4 are exactly the same.

Now, let us apply Theorem 4.4 in cases $k = 4, 5, 6$.

**Corollary 4.5.** *We have* $NC(K_{8n+1}, C_4) \geq \lceil \frac{6^n}{\phi(8n+1)} \rceil$.

*Proof.* Look at the $(K_{8n+1}, C_4)$-difference system given in [9], that is $\mathcal{F} = \{C_1, \ldots, C_n\}$ with $C_i = (0, i, -n, 4n + i)$, $1 \leq i \leq n$. Each cycle $C_i$ is full and $\vec{\Delta}^* C_i$ has 3! simple orderings, so the assertion follows from Theorem 4.4. □

**Corollary 4.6.** *We have* $NC(K_v, C_5) \geq \lceil \frac{24^n}{\phi(v)} \rceil$ *for every* $v = 10n + r$, *with* $r = 1$ *or* $5$.

*Proof.* We have already observed in the Introduction that in a $(K_v, C_5)$-difference system, with $v = 10n + r$, with $r = 1$ or 5, there are exactly $n$ full 5-cycles. For each full cycle $C$ all the 24 orderings of $\vec{\Delta}^* C$ are simple, so the assertion follows from Theorem 4.4. □

**Corollary 4.7.** *We have* $NC(K_{12n+1}, C_6) \geq \lceil \frac{120^{n-1}84}{\phi(12n+1)} \rceil$.

*Proof.* Look at the $(K_{12n+1}, C_6)$-difference system given in [9], that is $\mathcal{F} = \{C_1, \ldots, C_n\}$ with $C_i = (0, i, -n, i + n, -2n, i + 5n)$ for $1 \leq i \leq n$. We have $C_n = (0, n, -n, 2n, -2n, 6n)$, hence $\vec{\Delta}^* C_n = (n, -2n, 3n, -4n, 8n)$ and we see that $\{n, 3n, -4n\}$ is the unique zero-sum subset of $\vec{\Delta}^* C_n$ . An easy calculation shows that the number of simple orderings of $\vec{\Delta}^* C_n$ is 84. For each other cycle $C_i$ of $\mathcal{F}$ the sequence $\vec{\Delta}^* C_i$ does not have zero-sum subsets, and then it has 5! simple orderings. Applying Theorem 4.4 we obtain the result. □

It is evident that the bound provided by Theorem 4.4 is, in general, significantly smaller than $NC(K_v, C_k)$. Let us examine what happens in the "small" cases where $(v, k) \in \{(7, 3), (9, 4), (11, 5), (15, 5), (13, 6), (15, 7)\}$, in which $NC(K_v, C_k)$ can be easily and precisely calculated by a computer.

For each of these pairs, a difference system is a singleton consisting of a full $k$-cycle $C$ except for the case $(v,k)=(15,5)$, where a difference system consists of a full pentagon $C$ and the two short cycles $(0,3i,6i,9i,12i)$ with $i=1,2$. Thus, with some abuse of language, in each of these cases, we will say that a cyclic $(K_v,C_k)$-design arises from the $k$-cycle $C$.

For $(v,k)\in\{(7,3),(9,4)\}$ the bound provided by Theorem 4.4 coincides with the actual value of $NC(K_v,C_k)$. Indeed, we have $NC(K_7,C_3)=NC(K_9,C_4)=1$: up to isomorphism, the unique cyclic $(K_7,C_3)$-design is the one arising from the triangle $(0,1,3)$, while the unique cyclic $(K_9,C_4)$-design is the one arising from the quadrangle $(0,1,5,2)$.

While Theorem 4.4 gives $NC(K_{11},C_5)\geq 3$, we have, in fact, $NC(K_{11},C_5)=4$: up to isomorphism, the cyclic $(K_{11},C_5)$-designs are those arising from the following four pentagons:

$$(0,1,10,4,8);\quad (0,1,10,4,7);\quad (0,1,10,3,8);\quad (0,1,5,10,8).$$

As in the previous case, Theorem 4.4 gives $NC(K_{15},C_5)\geq 3$, but $NC(K_{15},C_5)=4$: up to isomorphism, the four cyclic $(K_{15},C_5)$-designs are those arising from:

$$(0,1,14,6,11);\quad (0,1,14,4,8);\quad (0,1,14,6,10);\quad (0,1,14,4,11).$$

While Theorem 4.4 gives $NC(K_{13},C_6)\geq 7$, the actual value is $NC(K_{13},C_6)=16$: up to isomorphism, the cyclic $(K_{13},C_6)$-designs are those arising from the following sixteen hexagons:

$$(0,1,3,6,2,8),\quad (0,1,3,6,11,4),\quad (0,1,3,6,12,4),\quad (0,1,3,6,12,8),$$
$$(0,1,3,8,11,4),\quad (0,1,3,9,5,8),\quad (0,1,3,9,12,4),\quad (0,1,3,12,2,8),$$
$$(0,1,3,12,5,8),\quad (0,1,4,2,9,5),\quad (0,1,4,11,6,2),\quad (0,1,4,11,7,2),$$
$$(0,1,4,11,9,5),\quad (0,1,4,12,8,2),\quad (0,1,7,3,5,8),\quad (0,1,8,11,6,2).$$

A significantly larger gap exists between the exact value of $NC(K_{15},C_7)$ and the bound of Theorem 4.4. While the best bound by applying Theorem 4.4 is $NC(K_{15},C_7)\geq 55$, a simple computer search allowed us to determine that the actual value is 336.

## 5 Use of the Skolem sequences

In this section, making use of the *Skolem-type sequences*, we will compute several lower bounds for $NC(K_v,C_k)$ with $k$ odd.

A *Skolem sequence* of order $n$ is a sequence of $n$ integers $(s_1,\dots,s_n)$ such that $\bigcup_{i=1}^{n}\{s_i,s_i+i\}=\{1,2,\dots,2n+1\}\setminus\{k\}$ where either $k=2n+1$ or $k=2n$ according to whether $n\equiv 0,1 \pmod 4$, or $n\equiv 2,3 \pmod 4$. In the last case, one usually speaks of a *hooked Skolem sequence*. For the rest of the paper, we will simply talk about a Skolem sequence according to the appropriate residue class of $n$. It is well known that a Skolem sequence of order $n$ exists for every positive $n$, [14]. Moreover, for a long time, it was known that $Sk(n)$, the number of distinct Skolem sequences of order $n$, is bounded from below by $2^{\lfloor n/3\rfloor}$, [1]. A tighter lower bound was obtained in [15] in Corollaries 3.3. and 3.4, stating that $Sk(n)\geq 6.492^{(2n-1)/7}$ for sufficiently large $n\equiv 4 \pmod 7$.

A *split Skolem sequence* of order $n$ is a sequence of $n$ integers $(s_1, s_2, \ldots, s_n)$ such that $\bigcup_{i=1}^{n}\{s_i, s_i + i\} = \{1, 2, \ldots, 2n + 2\} \setminus \{n + 1, k\}$ with either $k = 2n + 2$ or $k = 2n + 1$ according to whether $n \equiv 0, 3 \pmod 4$ or $n \equiv 1, 2 \pmod 4$. In the last case one usually speaks of a *split-hooked Skolem sequence*. Split Skolem sequences and split-hooked Skolem sequences are also known as *Rosa sequences* and *hooked Rosa sequences*, respectively. In what follows, we will simply talk about a split Skolem sequence (or a Rosa sequence) according to the appropriate residue class of $n$. It is well known that a split Skolem sequence of order $n$ exists for every positive $n > 1$, [14].

We use the notation $\mathrm{Ro}(n)$ for the number of distinct split Skolem sequences of order $n$. For a long time, [1], [4], it was known that $\mathrm{Ro}(n)$ is essentially bounded below by $2^{\lfloor n/3 \rfloor}$. A tighter lower bound was obtained in [15] in Corollaries 3.2 and 3.5, stating that for sufficiently large $t \equiv 0, 3 \pmod 4$, there are more than $6.492^{2t+1}$ split Skolem sequences of order $7t + 3$ and more than $6.492^{2t+1}$ split-hooked Skolem sequences of order $7t + 5$.

In this section, given a $(K_v, C_k)$-difference system $\mathcal{F}$, we will consider the list $\Lambda(\mathcal{F})$ of all the lists of partial differences of its cycles:

$$\Lambda(\mathcal{F}) = \{\partial C_1, \ldots, \partial C_n\}.$$

### 5.1 Steiner Triple Systems

Note that a $(K_v, C_3)$-design is equivalent to a $2 - (v, 3, 1)$ design. Thus, given that the number of cyclic $2 - (v, k, 1)$ designs has been denoted by $NC(v, k)$ in [11], we will write here $NC(v, 3)$ rather than $NC(K_v, C_3)$.

Some lower bounds for $NC(6n + 1, 3)$ were established in Proposition 3.1 of [11]. In this section, we complete the result and give some lower bounds for $NC(6n + 3, 3)$.

Let $S = (s_1, \ldots, s_n)$ be a split Skolem sequence of order $n$. The set $\mathcal{F} = \{C_1, C_2, \ldots, C_n\} \cup \{B\}$, where $C_i = (0, i, s_i + i + n)$, $i = 1, \ldots, n$ and $B = (0, 2n + 1, 4n + 2)$ is a difference system which determines a cyclic $STS(6n + 3)$. In fact, we have:

$$\partial\mathcal{F} = \{\pm i, \pm(s_i + n), \pm(s_i + i + n) \mid i = 1, \ldots, n\} \cup \{\pm(2n + 1)\} = \mathbb{Z}^*_{6n+3}.$$

Moreover, each starter cycle $C_i$ is full, while the starter cycle $B$ is of type 3.

We can prove the following

**Proposition 5.1.** *We have:*

- $NC(6n + 3, 3) \geq 3.35^n$ *for any sufficiently large* $n \equiv 3, 5, 24, 26 \ (mod\ 28)$.
- $NC(6n + 3, 3) \geq 2.49^n$ *for any sufficiently large* $n \not\equiv 3, 5, 24, 26 \ (mod\ 28)$.

*Proof.* Let $\mathcal{F}$ be the difference system described above. Take the $n$-tuples $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{1, -1\}^n$ and let $\mathcal{F}_\sigma = \{\sigma_1 C_1, \ldots, \sigma_n C_n\} \cup \{B\}$. We know from Theorem 2.2 that $\mathcal{E} = \{\mathcal{D}(\mathcal{F}_\sigma) \mid \sigma \in \{1, -1\}^n\}$ contains $2^n$ different $\mathrm{STS}(6n + 3)$s. Now observe that if $S$ and $S'$ are two different split Skolem sequences of the same order $n$ and if $\mathcal{F}$ and $\mathcal{F}'$ are the related difference systems, then $\mathcal{D}(\mathcal{F}) \neq \mathcal{D}(\mathcal{F}')$ since $\Lambda(\mathcal{F}) \neq \Lambda(\mathcal{F}')$. We can conclude that the number of distinct cyclic $\mathrm{STS}(6n + 3)$s is at least $2^n \times \mathrm{Ro}(n)$. If $n \equiv 0, 3 \pmod 4$ and $n = 7t + 3$, then for sufficiently large $t \equiv 0, 3 \pmod 4$ we have: $n \equiv 3, 24 \pmod{28}$, $2t+1 = (2n+1)/7$ and $\mathrm{Ro}(n) > 6.492^{(2n+1)/7}$. If $n \equiv 1, 2 \pmod 4$, $n \neq 1$ and $n = 7t + 5$, then for sufficiently large $t \equiv 0, 3 \pmod 4$ we have $n \equiv 5, 26 \pmod{28}$, $2t + 1 = (2n - 3)/7$ and $\mathrm{Ro}(n) > 6.492^{(2n-3)/7}$. We can state:

If $n \equiv 3, 24 \pmod{28}$, then:

$$NC(6n+3,3) > \frac{2^n \times 6.492^{(2n+1)/7}}{\phi(6n+3)} \geq \frac{2^n \times 6.492^{(2n+1)/7}}{6n+2}$$

If $n \equiv 5, 26 \pmod{28}$, then:

$$NC(6n+3,3) > \frac{2^n \times 6.492^{(2n-3)/7}}{\phi(6n+3)} \geq \frac{2^n \times 6.492^{(2n-3)/7}}{6n+2}$$

Numeric calculations show that the above fractions are both greater than $3.35^n$ starting from $n \geq 444$.

In all the other cases, we use the bound $\frac{2^n \times 2^{\lfloor n/3 \rfloor}}{\phi(6n+3)} \geq \frac{2^n \times 2^{\lfloor n/3 \rfloor}}{6n+2}$. In this case, numerical calculations show that this fraction is greater than $2.49^n$ starting from $n \geq 702$. □

### 5.2 Pentagon systems

Lower bounds for $NC(K_{10n+1}, C_5)$ and $NC(K_{10n+5}, C_5)$ are separately computed when $n$ is sufficiently large.

#### 5.2.1 The case $NC(K_{10n+1}, C_5)$

The construction of a difference system $\mathcal{F}$ of a cyclic $(K_{10n+1}, C_5)$-design is given in [9]. More precisely, if $n > 1$, let $S = (s_1, \ldots, s_n)$ be a Skolem sequence of order $n$, then we have $\mathcal{F} = \{C_1, \ldots, C_n\}$, where $C_i = (0, s_i + i, i, -2n, i + 3n)$ with $C_1$ replaced by $C_1' = (0, s_1 + 1, 1, 5n + 1, 2n)$ if the Skolem sequence is hooked, i.e. in the case $n \equiv 2, 3 \pmod 4$.

The known bounds on the number $Sk(n)$ of Skolem sequences yield the following:

**Proposition 5.2.** *We have:*

- $NC(K_{10n+1}, C_5) \geq 29.7^n$ *for any sufficiently large* $n \not\equiv 4$ *(mod* 7*);*
- $NC(K_{10n+1}, C_5) \geq 38.4^n$ *for any sufficiently large* $n \equiv 4$ *(mod* 7*).*

*Proof.* Let $\mathcal{F}$ be the $(K_{10n+1}, C_5)$-difference system obtained in [9] as described above. Each cycle $C_i$ is full and there are 4! simple orderings of $\vec{\Delta}^* C_i$. Following the notation of Theorem 4.4, the set $\{\mathcal{F}_\sigma \mid \sigma \in O\}$ contains $24^n$ difference systems. Following the proof of Theorem 4.4, we know that these difference systems give rise to $24^n$ distinct cyclic $(K_{10n+1}, C_5)$-designs. Now observe that if $S$ and $S'$ are two different Skolem sequences of the same order $n$, and if $\mathcal{F}$ and $\mathcal{F}$' are the related difference systems obtained as described above, then $\mathcal{D}(\mathcal{F}) \neq \mathcal{D}(\mathcal{F}')$ since $\Lambda(\mathcal{F}) \neq \Lambda(\mathcal{F}')$. This also implies that the set of $24^n$ designs arising from $\mathcal{F}$ is different from the set of $24^n$ distinct designs arising from $\mathcal{F}'$. From Proposition 1.3 we conclude that the number of non isomorphic cyclic $(K_{10n+1}, C_5)$-designs is at least $\frac{24^n \cdot Sk(n)}{\phi(10n+1)}$. We known that $Sk(n) \geq 2^{\lfloor n/3 \rfloor}$ whenever $n \not\equiv 4 \pmod 7$, while $Sk(n) \geq 6.492^{(2n-1)/7}$ whenever $n \equiv 4 \pmod 7$.

In the first case, we obtain: $NC(K_{10n+1}, C_5) \geq \frac{24^n \cdot 2^{\lfloor n/3 \rfloor - 1}}{5n}$ and an easy calculation shows that this number is bigger than $24^n$ for $n \geq 27$ and it is bigger than $29.7^n$ for $n \geq 570$.

In the second case, we obtain $NC(K_{10n+1}, C_5 5) \geq \dfrac{24^n \cdot 6.492^{(2n-1)/7}}{10n}$. An easy calculation shows that this number is bigger than $31.35^n$ for $n \geq 25$ and it is bigger than $38.4^n$ for $n \geq 116$. □

### 5.2.2 The case $NC(K_{10n+5}, C_5)$

The construction of a $(K_{10n+5}, C_5)$-difference system is given in [9] (see Theorem 3.2 and Theorem 4.1). More precisely, set $m = 2n + 1$ and let $H = \{0, m, 2m, 3m, 4m\}$.

If $n > 1$, let $\mathcal{F}' = \{C_1, \ldots, C_n\}$, with $C_i = (c_{i1}, c_{i2}, c_{i3}, c_{i4}, , c_{i5})$ defined by the following rules:

$$c_{ij} : \begin{cases} c\frac{m(j-1)}{2} & \text{for } j \text{ odd, } j \neq 5 \\ m\left(2 - \dfrac{j}{2}\right) - i & \text{for } j \text{ even,} \end{cases}$$

$$c_{i5} = r_i + (2m - n - 1),$$

where $(r_1, \ldots, r_n)$ is a split Skolem sequence.

In both cases we have $\partial\mathcal{F}' = \mathbb{Z}_{10n+5} \setminus H$.

Let $\overline{B}_1 = (0, m, 2m, 3m, 4m)$ and $\overline{B}_2 = (0, 2m, 4m, m, 3m)$ and set $\mathcal{F} = \mathcal{F}' \cup \{\overline{B}_1, \overline{B}_2\}$. The set $\mathcal{F}$ is a $(K_{10n+5}, C_5)$-difference system, the two cycles $\overline{B}_1, \overline{B}_2$ are of type 5, while all the cycles of $\mathcal{F}'$ are full.

**Proposition 5.3.** *We have:*

- $NC(K_{10n+5}, C_5) \geq 38.4^n$ *if* $n \equiv 3, 5, 24, 26 \ (mod\ 28)$ *is sufficiently large;*
- $NC(K_{10n+5}, C_5) \geq 29.7^n$ *if* $n \not\equiv 3, 5, 24, 26 \ (mod\ 28)$ *is sufficiently large.*

*Proof.* Let $\mathcal{F}$ be the $(K_{10n+5}, C_5)$-difference system described above. For each cycle $C_i$ there are 4! simple orderings of $\vec{\Delta}^* C_i$ and, following the notation of Theorem 4.4, the set $\{\mathcal{F}_\sigma \mid \sigma \in O\}$ contains $24^n$ difference systems that give rise to $24^n$ distinct cyclic $(K_{10n+5}, C_5)$-designs. Now observe that if $S$ and $S'$ are two different split Skolem sequences of the same order $n$, and if $\mathcal{F}$ and $\mathcal{F}$' are the related difference systems obtained as described above, then $\mathcal{D}(\mathcal{F}) \neq \mathcal{D}(\mathcal{F}')$ since $\Lambda(\mathcal{F}) \neq \Lambda(\mathcal{F}')$. This also implies that the set of $24^n$ designs arising from $\mathcal{F}$ is different from the set of $24^n$ distinct designs arising from $\mathcal{F}'$. From Proposition 1.3 we can conclude that the number of non isomorphic cyclic $(K_{10n+5}, C_5)$-designs is at least $\frac{24^n \cdot Ro(n)}{\phi(10n+5)}$.

We can use the known bounds for $Ro(n)$ given in [15] to obtain the result. That is, if $n \equiv 0, 3 \pmod 4$ and $n = 7t + 3$, then for sufficiently large $t \equiv 0, 3 \pmod 4$ we have $n \equiv 3, 24 \pmod{28}$, $2t+1 = (2n+1)/7$ and $Ro(n) > 6.492^{(2n+1)/7}$. If $n \equiv 1, 2 \pmod 4$, $n \neq 1$ and $n = 7t + 5$, then for sufficiently large $t \equiv 0, 3 \pmod 4$ we have $n \equiv 5, 26 \pmod{28}$, $2t + 1 = (2n - 3)/7$ and $Ro(n) > 6.492^{(2n-3)/7}$. Therefore, if $n$ is sufficiently large and $n \equiv 3, 24 \pmod{28}$ or $n \equiv 5, 26 \pmod{28}$, we have, respectively, $NC(K_{10n+5}, C_5) \geq \frac{24^n \cdot 6.492^{\frac{2n+1}{7}}}{\phi(10n+5)}$ or $NC(K10n + 5, C_5) \geq \frac{24^n \cdot 6.492^{\frac{2n-3}{7}}}{\phi(10n+5)}$. An easy calculation shows that both of these numbers are greater than $31.35^n$ if $n \geq 25$ and greater than $38.4^n$ if $n \geq 136$. If $n \not\equiv 3, 5, 24, 26 \pmod{28}$ is sufficiently large, then $NC(K_{10n+5}, C_5) \geq \frac{24^n \cdot 2^{\lfloor n/3 \rfloor}}{\phi(10n+5)}$ and this number is greater than $24^n$ when $n \geq 27$ and greater than $29.76^n$ if $n \geq 570$. □

### 5.3 Odd $k$-cycle systems with $k > 5$

In this section we obtain lower bounds for the number of non isomorphic cyclic $(K_v, C_k)$-designs when $k > 5$, $k$ odd. For this purpose we use the examples of cyclic $(K_v, C_k)$-designs, with either $v = 2nk + 1$ or $v = 2nk + k$, which are obtained in [9, 10, 26] via Skolem-type sequences. We will use the tecnicque of flipping the signs illustrated in Section 2. The lower bounds depend on both $n$ and $k$ and they become significant when $n$ is sufficiently large and, for instance, $k$ is linear in $n$.

First of all we recall the contructions of [9]. More precisely, suppose $n \equiv 0, 1 \pmod 4$ and fix a Skolem sequence $(s_1, \ldots, s_n)$ of order $n$. For $i = 1, \ldots, n$, consider the $k$-cycle $C_k$ defined as follows:

$$c_{i1} = 0, \quad c_{i2} = -s_i$$

$$c_{ij} : \begin{cases} \frac{-jn}{2} & \text{for } j \text{ even}, \ j \neq 2 \\ i + \frac{(j-3)n}{2} & \text{for } j \text{ odd}, \ j \leq \frac{k+1}{2} \\ i + \frac{(k+j+\varepsilon)n}{2} & \text{for } j \text{ odd}, \ j > \frac{k+1}{2} \end{cases}$$

where $\varepsilon = -2$ or $0$ according to whether $k \equiv 1$ or $3 \pmod 4$, respectively.

The set $\mathcal{F} = \{C_1, \ldots, C_n\}$ is a $(K_{2nk+1}, C_k)$-difference system.

If $n \equiv 2, 3 \pmod 4$, we fix a hooked Skolem sequence $(s_1, \ldots, s_n)$ of order $n$. In this case, a difference system is obtained by replacing the cycle $C_1$ of $\mathcal{F}$ with the either the cycle $A$, or the cycle $A'$, according to whether $k \equiv 1 \pmod 4$ or $3 < k \equiv 3 \pmod 4$. Here $A$ is defined as follows:

$$a_1 = 1 \quad a_3 = 0 \quad a_{k-2} = 5n \quad a_{k-1} = (4-k)n \quad a_k = (k+3)n/2 + 1$$
$$a_j = c_{1j} \quad \text{for } j \notin \{1, 3, k-2, k-1, k\}.$$

While $A'$ is defined as follows:

$$a_1 = 1 \quad a_3 = 0$$
$$a_j = c_{1j} \quad \text{for } 1 \neq j \neq 3.$$

We can prove the following

**Proposition 5.4.** *We have:*

- $NC(K_{2nk+1}, C_k) \geq \dfrac{2.49^n}{k}$ *for any sufficiently large* $n \not\equiv 4$ *(mod* 7*);*
- $NC(K_{2nk+1}, C_k) \geq \dfrac{3.35^n}{k}$ *fore any sufficiently large* $n \equiv 4$ *(mod* 7*).*

*Proof.* Observe that if $S$ and $S'$ are two different Skolem sequences of the same order $n$, and if $\mathcal{F}$ and $\mathcal{F}$' are the $(K_v, C_k)$-difference systems obtained from $S$ and $S'$ respectively as above described, then $\mathcal{D}(\mathcal{F}) \neq \mathcal{D}(\mathcal{F}')$ since $\Lambda(\mathcal{F}) \neq \Lambda(\mathcal{F}')$. We know that all the cycles of a difference system are full, and then recalling Theorem 2.2, we conclude that $NC(K_{2nk+1}, C_k) \geq 2^n \times Sk(n)$. We known that $\mathrm{Sk}(n) \geq 2^{\lfloor n/3 \rfloor}$ whenever $n \not\equiv 4 \pmod 7$, while $\mathrm{Sk}(n) \geq 6.492^{(2n-1)/7}$ whenever $n \equiv 4 \pmod 7$. In the first case, we

obtain: $NC(K_{2nk+1}, C_k) \geq \frac{2^n \cdot 2^{\lfloor n/3 \rfloor - 1}}{kn}$ and a numeric calculation shows that this number is bigger than $\frac{2.49^n}{k}$ for $n \geq 640$.

In the second case, we obtain $NC(K_{2nk+1}, C_k) \geq \frac{2^n \cdot 6.492^{(2n-1)/7}}{2nk}$ and a numeric calculation shows that this number is bigger than $\frac{3.35^n}{k}$ for $n \geq 375$. □

Now we consider the case of a cyclic $(K_{2nk+k}, C_k)$-design, $k$ odd, $k > 5$.

**Proposition 5.5.** *We have:*

- $NC(K_{2nk+k}, C_k) \geq \frac{3.35^n}{k}$ *for any sufficiently large* $n$, $n \equiv 3, 5, 24, 26 \ (mod\ 28)$*;*
- $NC(K_{2nk+k}, C_k) \geq \frac{2.49^n}{k}$ *for any sufficiently large* $n$, $n \not\equiv 3, 5, 24, 26 \ (mod\ 28)$.

*Proof.* If $k \neq 15$ and $k \neq q$, with $q$ a prime power that is not a prime, the way to construct a $(K_{2nk+k}, C_k)$-difference system $\mathcal{F}'$ is given in [9], as already described for the case $k = 5$ in the previous section 5.2.2.

Let $m = 2n + 1$, $k = 2h + 1$ and let $H = \{0, m, 2m, \ldots, (k-1)m\}$. Let $\mathcal{F}' = \{C_1, \ldots, C_n\}$ with $C_i$ defined by following rules:

$$c_{ij} : \begin{cases} \dfrac{m(j-1)}{2} & \text{for } j \text{ odd, } j \neq k \\ m(h - \dfrac{j}{2}) - i & \text{for } j \text{ even,} \end{cases}$$

$$c_{ik} = r_i + (hm - n - 1),$$

where $(r_1, \ldots, r_n)$ is a split Skolem sequence of order $n$. We have $\partial\mathcal{F}' = \mathbb{Z}_{2kn+k} \setminus H$ (Theorem 3.2 of [9]), and the cycles of $\mathcal{F}'$ are full. Now to obtain a $(K_{2nk+k}, C_k)$-difference system we need a $(K_k, C_k)$-difference system $\mathcal{B}$ with vertex set given by the additive cyclic group $H$. The difference system $\mathcal{B}$ is given in [9] if $k$ is a prime, and in [10] if $k$ is not a prime. Since $\mathcal{D}(\mathcal{B})$ is Hamiltonian, each cycle of $\mathcal{B}$ is short. Applying Theorem 4.1 of [9] we know that $\mathcal{F} = \mathcal{F}' \cup \mathcal{B}$ is a $(K_{2nk+k}, C_k)$-difference system. Moreover, if we change the split Skolem sequence which defines $\mathcal{F}'$ we obtain a new set $\mathcal{F}$" and $\bar{\mathcal{F}} = \mathcal{F}" \cup \mathcal{B}$ is still a $(K_{2n+k}, C_k)$-difference system. We have $\mathcal{D}(\mathcal{F}) \neq \mathcal{D}(\bar{\mathcal{F}})$, since $\Lambda\mathcal{F} \neq \Lambda\bar{\mathcal{F}}$.

If $k = q$, with $q$ a prime power that is not a prime, a $(K_{2nk+k}, C_k)$-difference system $\mathcal{F}_q$ was constructed in [26]. It is $\mathcal{F}_q = \mathcal{F}'_q \cup \mathcal{B}'$, where all the cycles of $\mathcal{B}'$ are short (see the description, page 303 of [26]), while the cycles of $\mathcal{F}'_q$ are full cycles. This set of full cycles is obtained by modifying one cycle of the family $\mathcal{F}'$ constructed in [10] and decribed above, (see p.305 of [26]). Namely, $\mathcal{F}'_q = (\mathcal{F}' - \{C_1\}) \cup \{C'_1\}$. Here $C'_1 = (c'_{11}, \ldots, c'_{1k})$ is obtained by replacing certain vertices of $C_1 = (c_{11}, \ldots, c_{1k})$, but $c'_{1k}$ remains equal to $c_{1k}$. This assures that if we change the split Skolem sequence which defines $\mathcal{F}'$, we obtain a new family $\mathcal{F}''_q$ instead of $\mathcal{F}'_q$ and $\bar{\mathcal{F}}_q = \mathcal{F}''_q \cup \mathcal{B}'$ is still a $(K_{2nk+k}, C_k)$-difference system. Since $\Lambda\mathcal{F}_q \neq \Lambda\bar{\mathcal{F}}_q$, we have $\mathcal{D}(\mathcal{F}_q) \neq \mathcal{D}(\bar{\mathcal{F}}_q)$.

If $k = 15$, a $(K_{30n+15}, C_{15})$-difference system $\mathcal{F} = \mathcal{F}' \cup \mathcal{B}$ was constructed in Theorem 2.1 of [26]. Here the cycles of $\mathcal{B}$ are short, namely $\mathcal{B}$ contains one cycle of type 3 and two cycles of type 15. The cycles of $\mathcal{F}'$ are full and are constructed via a split Skolem sequence $(r_1, \dots, r_n)$ of order $n$. Namely $\mathcal{F}' = \{C_1^*, C_2, \dots C_n\}$ with $C_i = (0, 6m-i, m, 5m-i, 2m, 4m-i, \dots, 6m, -i, r_i + \frac{13m-1}{2})$ for each $1 \leq i \leq n$ and $C_1^*$ is obtained from $C_1$ by repleacing the vertex $6m-1$ with $-5m+1$ and the vertex $6m$ with $6m-1$. Obviously, also in this case if we change the split Skolem sequence we obtain a new difference system giving rise to a different design.

Now, we can achieve a lower bound for $NC(K_{2nk+k}, C_k)$ by using the method of flipping the signs to the set of $n$ full cycles.

We obtain $NC(K_{2nk+k}, C_k) \geq \frac{2^n \times Ro(n)}{2nk+k}$. As already observed in Proposition 5.1, if $n$ is sufficiently large, we have either $Ro(n) \geq 6.492^{(2n+1)/7}$ or $Ro(n) \geq 6.492^{(2n-3)/7}$ according to whether $n \equiv 3, 24$ or $n \equiv 5, 26 \pmod{28}$. Numeric calculations show that both the fractions $\frac{2^n \times 6.492^{(2n+1)/7}}{(2n+1)k}$ and $\frac{2^n \times 6.492^{(2n-3)/7}}{(2n+1)k}$ are bigger than $\frac{3.35^n}{k}$ starting from $n \geq 403$.

If $n \not\equiv 3, 5, 24, 26 \ (mod\ 28)$ is sufficiently large, we have $Ro(n) \geq 2^{\lfloor n/3 \rfloor}$ and numeric calculations show that the fraction $\frac{2^n \times 2^{\lfloor n/3 \rfloor}}{(2n+1)k}$ is bigger than $\frac{2.49^n}{k}$ starting from $n \geq 597$. □

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