A PERTURBATION OF THE CAHN–HILLIARD EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. Our aim in this paper is to study a perturbation of the Cahn–Hilliard equation with nonlinear terms of logarithmic type. This new model is based on an unconstrained theory recently proposed in [5]. We prove the existence, regularity and uniqueness of solutions, as well as (strong) separation properties of the solutions from the pure states, also in three space dimensions. We finally prove the convergence to the Cahn–Hilliard equation, on finite time intervals.

1. Introduction

The Cahn–Hilliard equation,

$$\varphi_t + \Delta^2\varphi - \Delta\Psi'(\varphi) = 0,$$

is one of the most popular equations in the literature. Initially proposed by Cahn and Hilliard in [2] and [3] in order to describe phase separation processes in binary alloys, it was then employed, possibly with an additional source term, in many different situations, including astronomy and ecology. We refer the interested reader to, e.g., [15], [17] and references therein.

Very often, the potential Ψ is taken polynomial, typically,

$$\Psi(s) = \frac{1}{4}(s^2-1)^2,\ s \in \mathbb{R}.$$

In that case, the equation is very well understood from a mathematical point of view and one has, in particular, a complete and satisfactory picture concerning the well-posedness, regularity of solutions and asymptotic behavior of the associated dynamical system (existence of finite-dimensional attractors and convergence of single trajectories to steady states); see, e.g., [15] and the numerous references therein. However, as already mentioned in [2], the thermodynamically relevant potential Ψ should be logarithmic,

$$\Psi(s) = \frac{\theta}{2}\Big((1+s)\ln(1+s) + (1-s)\ln(1-s)\Big) - \frac{\theta_0}{2}s^2,\ 0<\theta<\theta_0,\ s\in(-1,1).$$

It is important to note that, in that case, the equation only makes sense for $\varphi \in (-1,1)$ a.e., meaning, roughly speaking, that, during the phase separation process, one never completely reaches the pure states, but always has at least one trace of the other component. One can indeed prove such a separation property from the pure states (see, e.g., [15] once more), allowing then to again prove the well-posedness, regularity of solutions,

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existence of finite dimensional attractors and convergence of solutions to steady states (see, e.g., [1], [15] and [18]). Another important property of the equation is known as the strict separation property (here on a time interval I),

$$|\varphi(x,t)| \leq 1-\delta, \quad \text{for all } (x,t) \in \overline{\Omega} \times I,$$

where Ω is the spatial domain occupied by the system, meaning, roughly speaking, that not only one never completely reaches the pure states, but there always remains some given quantity of the other component. This property was proved in one and two space dimensions in [18], for $I = (\sigma, +\infty)$, $\forall \sigma > 0$. In three space dimensions, such a property is not known, unless one makes growth assumptions on Ψ close to the pure states ± 1 that are not satisfied by the above logarithmic potentials. However, it follows from the results in [1], that this property holds asymptotically, in the sense that it holds for $I = (T, +\infty)$, where $T > 0$ depends on the initial datum and cannot be estimated, contrary to what we have in one and two space dimensions.

In [19], Novick–Cohen proposed the following variant of the Cahn–Hilliard equation, called viscous Cahn–Hilliard equation:

$$\varphi_t - \varepsilon \Delta \varphi_t + \Delta^2 \varphi - \Delta \Psi'(\varphi) = 0, \ \varepsilon > 0,$$

in order to account for viscosity effects in mixtures of polymers. It is proved in [18] that this perturbation of the Cahn–Hilliard equation enjoys the strict separation property in three space dimensions, although this property is uniform with respect to $\varepsilon \to 0$ only in one and two space dimensions.

The viscous Cahn–Hilliard equation is also a particular instance of the generalized Cahn–Hilliard equations proposed by Gurtin in [13] and based on a microforce balance (following [8] and [9]), i.e., a separate balance law for forces at a microscopic level, leading to a scalar phase field. This approach is further considered in [5], based also on internal constraints. More precisely, when considering a constrained theory, i.e., assuming that the concentration is constrained to be equal to the order parameter, one finds the Cahn–Hilliard equation. The unconstrained theory leads to the following system of equations:

$$\frac{\partial c}{\partial t} = \Delta(\Psi'(\varphi) - \Delta\varphi),$$

$$c = \varphi + \varepsilon(\Psi'(\varphi) - \Delta\varphi), \ \varepsilon > 0,$$

where c is the concentration. Introducing the chemical potential

$$\mu = \frac{1}{\varepsilon}(c - \varphi),$$

we end up with the system of equations

$$\varphi_t = \Delta\mu - \varepsilon\mu_t,$$

$$\mu = -\Delta\varphi + \Psi'(\varphi).$$

Note that, when $\varepsilon = 0$, we recover the Cahn–Hilliard equation. These equations were studied in [16], for a polynomial potential Ψ as above. In particular, there, the well-posedness, regularity of solutions and convergence to the Cahn–Hilliard equation on finite time intervals were obtained.

We also mention a related model proposed in [6] and based on microconstraints. This model was studied in [7], again for a polynomial potential.

Our aim in this paper is to study the model in [5], now with logarithmic (actually, slightly more general) potentials. For such potentials, we recover the same results as those obtained in [16]. Furthermore, a special attention is devoted to the strict separation property that also holds in three space dimensions, as it is the case for the viscous Cahn–Hilliard equation. However, again, in that case, this property is not uniform with respect to $\varepsilon \to 0$, contrary to what we have in two space dimensions.

Plan of the paper. In Section 2, after setting the problem in a rigorous way, we introduce the assumptions that are used throughout the paper and recall some mathematical tools that are needed for our analysis (see also Appendix 8 on a related elliptic problem). In Section 3, we perform several energy estimates, depending on ε. Then, we show the existence and uniqueness of weak and strong solutions in the subsequent Section 4. The proof of the separation property in three space dimensions is given in Section 5. Section 6 is devoted to the derivation of energy estimates that are independent of ε, therein applied to prove a uniform version of the separation property in two space dimensions. In the last Section 7, we prove the convergence of the model to the classical Cahn–Hilliard one when $\varepsilon \to 0$ and provide some quantitative error estimates.

2. Setting of the problem

We consider the following equations, in a bounded and regular domain Ω of $\mathbb{R}^d$, $d = 2$ or 3, with smooth boundary $\partial\Omega$:

$$\begin{cases} \varphi_t = \Delta\mu - \varepsilon\mu_t, \\ \mu = -\Delta\varphi + \Psi'(\varphi), \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{2.1}$$

completed with the boundary conditions

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{2.2}$$

$\mathbf{n}$ being the exterior normal on $\partial\Omega$, and the initial condition

$$\varphi(0) = \varphi_0, \quad \text{in } \Omega. \tag{2.3}$$

Setting

$$w = \varphi + \varepsilon\mu,$$

we rewrite the problem as

$$\begin{cases} \varepsilon w_t - \Delta w = -\Delta\varphi, \\ w = -\varepsilon\Delta\varphi + \varepsilon\Psi'(\varphi) + \varphi, \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{2.4}$$

completed with the boundary conditions

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}w = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{2.5}$$

and the initial condition

$$w(0) = \varphi_0 + \varepsilon\mu_0, \quad \text{in } \Omega, \tag{2.6}$$

having set $\mu_0 = -\Delta\varphi_0 + \Psi'(\varphi_0)$.

Remark 2.1. It is important to remark that, contrary to the Cahn-Hilliard equation corresponding to $\varepsilon = 0$, here for the ε-perturbed model the mass conservation property of φ does not hold, namely,

$$\langle \varphi(t) \rangle \neq \langle \varphi_0 \rangle, \quad \forall\, t \geq 0,$$

where we denote by $\langle f \rangle$ the average of f over Ω (see the subsequent (2.12)). Nonetheless, the mass of w in (2.4) is preserved by the evolution, that is,

$$\langle w(t) \rangle \equiv \langle w_0 \rangle, \quad \forall\, t \geq 0. \tag{2.7}$$

Preliminaries. We set $H = L^2(\Omega)$ with inner product denoted by $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$. For the standard Sobolev spaces, we use the notation $H^k = W^{k,2}(\Omega)$. We also set $V = H^1(\Omega)$ and denote by V' its dual space, with corresponding duality product $\langle \cdot,\cdot \rangle$.

Throughout this paper, with the same letter C, we denote a positive constant which may vary from line to line. Specific dependencies will be pointed out when necessary.

Assumptions. We formulate the standing assumptions on the potential Ψ and on the parameter ε that hold true throughout the paper.

We assume that Ψ is a quadratic perturbation of a singular (strictly) convex function in $[-1, 1]$. Namely,

$$\Psi(s) = F(s) - \frac{\theta_0}{2} s^2,$$

where the convex part F belongs to $\mathcal{C}([-1,1]) \cap \mathcal{C}^2(-1,1)$, and fulfills

$$\lim_{s\to -1} F'(s) = -\infty, \quad \lim_{s\to 1} F'(s) = +\infty, \quad F''(s) \geq \theta, \quad \forall\, s \in (-1,1),$$

for some $\theta > 0$. Here, we study the physical case of double-well (singular) potentials, namely, we assume

$$\theta_0 > \theta.$$

We also extend $F(s) = +\infty$ for any $s \notin [-1,1]$. Note that the above assumptions imply that there exists $s_0 \in (-1,1)$ such that $F'(s_0) = 0$. Without loss of generality, we assume that $s_0 = 0$ and that $F(s_0) = 0$ as well, namely,

$$F(0) = F'(0) = 0.$$

In particular, this entails that $F(s) \geq 0$ for all $s \in [-1,1]$ and $F'(s)s \geq 0$.

Remark 2.2. The assumptions are satisfied and motivated by the logarithmic potential

$$\Psi(s) = \frac{\theta}{2}\Big((1+s)\ln(1+s) + (1-s)\ln(1-s)\Big) - \frac{\theta_0}{2} s^2, \quad \forall s \in (-1,1).$$

Concerning the parameter ε, we assume that $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 > 0$ satisfies

$$\varepsilon_0 < \frac{1}{\theta_0}. \tag{2.8}$$

Regularity results for the singular elliptic problem. An important role in the analysis of the model is played by the elliptic problem with the homogeneous Neumann boundary conditions,

$$\begin{cases} -\Delta u + F'(u) = f, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where F is a singular potential satisfying all the assumptions above and $f \in H$.

Let u be a solution to (2.9) such that $u \in H^2$ with $F'(u) \in H$, $\partial_{\mathbf{n}} u = 0$ on $\partial\Omega$ and u satisfies $-\Delta u + F'(u) = f$ for a.e. $x \in \Omega$. Note that, in particular, $\|u\|_{L^\infty(\Omega)} \le 1$ (see Section 8). We report here some useful results from [4, Appendix A], concerning further regularity properties of u and $F'(u)$.

Lemma 2.3. *Let $d = 2, 3$. Then the following hold true:*

(i) *If $f \in L^p(\Omega)$ with $2 \le p \le \infty$, then*

$$\|F'(u)\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)}.$$

(ii) *If $f \in V$, there exists $C = C(p) > 0$ such that*

$$\|u\|_{W^{2,p}(\Omega)} + \|F'(u)\|_{L^p(\Omega)} \le C\big(1 + \|f\|_V\big),$$

where $p = 6$ if $d = 3$ and for any finite $p \ge 2$ if $d = 2$.

In dimension $d = 2$, one usually obtains information on the regularity of F'' under the following additional condition on the potential: there exists a positive constant C such that

$$F''(s) \le \mathrm{e}^{C|F'(s)|+C}, \quad \forall s \in (-1, 1), \tag{2.10}$$

which is in particular satisfied by the logarithmic potentials. Indeed, based on the Trudinger-Moser inequality, the following result holds true (see [4, Lemma A.2]).

Lemma 2.4. *Let $d = 2$ and let F satisfy* (2.10)*. Then, for any $p \ge 2$, there exists a positive constant $C = C(p)$ such that*

$$\|F''(u)\|^p_{L^p(\Omega)} \le C\Big(1 + \mathrm{e}^{C\|f\|^2_V}\Big).$$

Moreover, we have the following very recent result [14, Lemma 3.2], see also [11].

Lemma 2.5. *Let $d = 2$ and* (2.10) *be in place. If $f \in V$, then*

$$\|F'(u)\|_{L^\infty(\Omega)} \le C,$$

for some $C > 0$ depending only on $\|f\|_V$.

Proof. By Lemma 2.3 we know that $u \in W^{2,p}(\Omega)$ and $F'(u) \in L^p(\Omega)$ for every $p \ge 2$. Besides, $F''(u) \in L^p(\Omega)$ in light of Lemma 2.4. Now observe that[1] $\nabla F'(u) = F''(u)\nabla u$, hence

$$\int_\Omega |\nabla F'(u)|^3 \le \left(\int_\Omega |F''(u)|^6\right)^{1/2} \left(\int_\Omega |\nabla u|^6\right)^{1/2} \le C.$$

This tells that $F'(u) \in W^{1,3}(\Omega)$. By the two dimensional Sobolev embedding $W^{1,3}(\Omega) \subset L^\infty(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the result follows. □

[1] This can be proven by a standard limiting procedure involving suitable cut-off functions for u.

Polynomial approximation of the singular potential. Let us recall some results in [10] concerning the existence of a sequence of regular functions F_λ which approximate the singular potential F. More precisely, there exists a family

$$F_\lambda : \mathbb{R} \to \mathbb{R}$$

($\lambda > 0$) such that $F_\lambda(0) = F_\lambda'(0) = 0$ and

(i) F_λ is convex with $F_\lambda''(s) \geq 0$ for all $s \in \mathbb{R}$,
(ii) F_λ' is Lipschitz continuous on $\mathbb{R}$ with constant $\frac{1}{\lambda}$,
(iii) there exist $0 < \overline{\lambda} \leq 1$ and $C > 0$ such that $F_\lambda(s) \geq \theta_0 s^2 - C$, $\forall\, s \in \mathbb{R}$, $\forall\, \lambda \in (0, \overline{\lambda}]$,
(iv) $F_\lambda(s) \nearrow F(s)$, for all $s \in \mathbb{R}$, $|F_\lambda'(s)| \nearrow |F'(s)|$ for $s \in (-1, 1)$ and F_λ' converges uniformly to F' on any set $[a, b] \subset (-1, 1)$.

For any $\lambda > 0$ we introduce the quadratic perturbation of F_λ by

$$\Psi_\lambda(s) = F_\lambda(s) - \frac{\theta_0}{2} s^2. \tag{2.11}$$

Note that

$$\Psi_\lambda''(s) \geq -\theta_0, \qquad \Psi_\lambda'(s)s \geq -\theta_0 s^2.$$

Zero-mean functions. We denote by $\langle f \rangle$ the average of f over Ω, that is,

$$\langle f \rangle = |\Omega|^{-1} \langle f, 1 \rangle, \tag{2.12}$$

for all f in the dual space V'. Then, we introduce the space of zero-mean functions $V_0 = \{f \in V : \langle f \rangle = 0\}$ and its dual space $V_0' = \{g \in V' : \langle g, 1 \rangle = 0\}$. We consider the operator $A \in \mathcal{L}(V, V')$ defined by

$$\langle Af, v \rangle = \int_\Omega \nabla f \cdot \nabla v \,\mathrm{d}x, \quad \forall\, f, v \in V.$$

Since the restriction of A to V_0 is an isomorphism from V_0 onto V_0', we define the inverse map $\mathcal{N} : V_0' \to V_0$. It is well-known that for all $g \in V_0'$, $\mathcal{N}g$ is the unique $f \in V_0$ such that $\langle Af, v \rangle = \langle g, v \rangle$, for all $v \in V$. On account of the above definitions, we have $\langle Af, \mathcal{N}g \rangle = \langle g, f \rangle$, for all $f \in V$ and $g \in V_0'$. It turns out that $f \to \|f\|_\sharp = \|\nabla \mathcal{N} f\|$ and $f \to \|f\|_* = (\|f - \langle f \rangle\|_\sharp^2 + |\langle f \rangle|^2)^{\frac{1}{2}}$ are norms on V_0' and V', respectively, that are equivalent to the standard ones. We also recall the Poincaré type inequality

$$\|f - \langle f \rangle\| \leq C \|\nabla f\|, \qquad \forall f \in V$$

and the following chain rule:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f(t)\|_\sharp^2 = \langle \partial_t f(t), \mathcal{N} f(t) \rangle, \quad \text{for a.e. } t \in (0,T),\ \forall\, f \in H^1(0,T; V_0').$$

3. Energy estimates (depending on ε)

Let $\varepsilon > 0$ complying with (2.8) be fixed. We consider the λ-family of regularized systems corresponding to (2.4) obtained by replacing the singular potential Ψ with the regular approximations Ψ_λ as in (2.11), namely,

$$\begin{cases} \varepsilon w_t - \Delta w = -\Delta \varphi, \\ w = -\varepsilon \Delta \varphi + \varepsilon \Psi_\lambda'(\varphi) + \varphi, \end{cases}$$

with the same boundary conditions and initial datum

$$w_0 \in H$$

arbitrarily chosen. Let us denote this problem by $(2.4)_\lambda$. In this section we perform several energy estimates[2] that are *independent of* λ. To this aim, let us simply denote by (w, φ) the solution to $(2.4)_\lambda$, neglecting dependence on λ.

Multiplying the second equation of $(2.4)_\lambda$ by φ we get

$$(w, \varphi) = \varepsilon\|\nabla\varphi\|^2 + \varepsilon(\Psi'_\lambda(\varphi), \varphi) + \|\varphi\|^2.$$

Exploiting the inequality

$$(\Psi'_\lambda(\varphi), \varphi) \geq -\theta_0\|\varphi\|^2,$$

it turns out that

$$\varepsilon(\Psi'_\lambda(\varphi)\varphi, \varphi) + \|\varphi\|^2 \geq (1 - \varepsilon\theta_0)\|\varphi\|^2 = 2c_1\|\varphi\|^2,$$

where

$$c_1 := \frac{1 - \varepsilon\theta_0}{2} > 0,$$

owing to the constraint (2.8). By the estimate $|(w, \varphi)| \leq c_1\|\varphi\|^2 + \frac{1}{4c_1}\|w\|^2$, we end up with

$$\varepsilon\|\nabla\varphi\|^2 + c_1\|\varphi\|^2 \leq \frac{1}{4c_1}\|w\|^2. \tag{3.1}$$

Now test the first equation of $(2.4)_\lambda$ by w, so obtaining

$$\frac{\varepsilon}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 = (\nabla\varphi, \nabla w) \leq \frac{1}{2}\|\nabla\varphi\|^2 + \frac{1}{2}\|\nabla w\|^2,$$

hence

$$\varepsilon\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \leq \|\nabla\varphi\|^2.$$

Exploiting (3.1), we arrive at

$$\varepsilon\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \leq \frac{1}{4c_1\varepsilon}\|w\|^2,$$

hence the Gronwall lemma, together with $w_0 \in H$, gives

$$w \in L^\infty(0, T; H) \cap L^2(0, T; V),$$

with bounds of the norms that do not depend on λ.

Finally, multiplying the second equation of $(2.4)_\lambda$ by $-\Delta\varphi$ we have

$$-(w, \Delta\varphi) = \varepsilon\|\Delta\varphi\|^2 + \varepsilon(\Psi''_\lambda(\varphi)\nabla\varphi, \nabla\varphi) + \|\nabla\varphi\|^2.$$

Using the property

$$(\Psi''_\lambda(\varphi)\nabla\varphi, \nabla\varphi) \geq -\theta_0\|\nabla\varphi\|^2,$$

it turns out that

$$\varepsilon(\Psi''_\lambda(\varphi)\nabla\varphi, \nabla\varphi) + \|\nabla\varphi\|^2 \geq 2c_1\|\nabla\varphi\|^2.$$

[2]that are properly justified within a suitable Galerkin scheme.

Since $|(w,\Delta\varphi)| \le \frac{\varepsilon}{2}\|\Delta\varphi\|^2 + \frac{1}{2\varepsilon}\|w\|^2$, we obtain

$$\varepsilon\|\Delta\varphi\|^2 + 4c_1\|\nabla\varphi\|^2 \le \frac{1}{\varepsilon}\|w\|^2. \tag{3.2}$$

In conclusion, we have proved that

$$\varphi \in L^\infty(0,T;H^2),$$

with an estimate that is independent of λ.

We proceed by looking for estimates on the time-derivatives of φ and w. We immediately get by comparison in the first equation that

$$w_t \in L^2(0,T;V'). \tag{3.3}$$

Then, differentiating the second equation with respect to time and multiplying by φ_t we find

$$\langle w_t, \varphi_t\rangle = \varepsilon\|\nabla\varphi_t\|^2 + (\varepsilon\Psi''_\lambda(\varphi)\varphi_t, \varphi_t) + \|\varphi_t\|^2 \ge \varepsilon\|\nabla\varphi_t\|^2 + 2c_1\|\varphi_t\|^2.$$

Note that

$$\langle w_t, \varphi_t\rangle = \langle A\varphi_t, \mathcal{N}w_t\rangle = (\nabla\mathcal{N}w_t, \nabla\varphi_t) \le \|w_t\|_\sharp\|\nabla\varphi_t\| \le \frac{\varepsilon}{2}\|\nabla\varphi_t\|^2 + \frac{1}{2\varepsilon}\|w_t\|_\sharp^2.$$

We thus obtain

$$\varepsilon\|\nabla\varphi_t\|^2 + 4c_1\|\varphi_t\|^2 \le \frac{1}{\varepsilon}\|w_t\|_{V'}^2,$$

giving the bound

$$\varphi_t \in L^2(0,T;V),$$

uniformly in λ, in light of (3.3).

Based on the above a priori estimates and a proper Galerkin scheme, exploiting the global Lipschitz continuity of Ψ'_λ, it is possible to prove the existence of a (unique) weak solution $(w_\lambda, \varphi_\lambda)$ to each approximating problem $(2.4)_\lambda$, satisfying

$$\langle\varepsilon\partial_t w_\lambda, v\rangle + (\nabla w_\lambda, \nabla v) - (\nabla\varphi_\lambda, \nabla v) = 0, \quad \forall\, v \in V,$$

where $w_\lambda = -\varepsilon\Delta\varphi_\lambda + \varepsilon\Psi'_\lambda(\varphi_\lambda) + \varphi_\lambda$ a.e. $(x,t) \in \Omega\times(0,T)$, and

$$\begin{aligned}
&\|w_\lambda\|_{L^\infty(0,T;H)} + \|w_\lambda\|_{L^2(0,T;V)} \le C,\\
&\|\varphi_\lambda\|_{L^\infty(0,T;H^2)} \le C,\\
&\|\partial_t w_\lambda\|_{L^2(0,T;V')} + \|\partial_t\varphi_\lambda\|_{L^2(0,T;V)} \le C,
\end{aligned}$$

for some $C > 0$ independent of λ. It is nonetheless apparent from the computations above that C depends on ε, and $C \to \infty$ as $\varepsilon \to 0$. Furthermore, by comparison in the second equation of $(2.4)_\lambda$ we also have the uniform estimate

$$\|F'_\lambda(\varphi_\lambda)\|_{L^\infty(0,T;H)} \le C. \tag{3.4}$$

We omit the proof of the existence, that is essentially contained in [16]; see also Section 8 for the Galerkin scheme and the next section for the passage to the limit.

4. Well-posedness

Our aim in this section is to show the well-posedness of the model, for any given $\varepsilon > 0$ complying with (2.8). We start from equation (2.4).

Theorem 4.1. *Let $\varepsilon \in (0, \varepsilon_0)$ and let $w_0 \in H$. Given $T > 0$, there exists a unique weak solution (w, φ) on $[0, T]$ to (2.4)-(2.5) with initial condition $w(0) = w_0$ in the following sense:*

$$w \in L^\infty(0,T;H) \cap L^2(0,T;V), \qquad w_t \in L^2(0,T;V'),$$
$$\varphi \in L^\infty(0,T;H^2), \qquad \varphi_t \in L^2(0,T;V),$$
$$F'(\varphi) \in L^\infty(0,T;H),$$

with

$$|\varphi(x,t)| < 1 \ \text{a.e.}\ (x,t) \in \Omega \times (0,T).$$

Besides,

$$\langle \varepsilon w_t, v\rangle + (\nabla w, \nabla v) - (\nabla \varphi, \nabla v) = 0, \qquad \forall\, v \in V, \tag{4.1}$$

for almost every $t \in (0,T)$, and

$$w = -\varepsilon\Delta\varphi + \varepsilon\Psi'(\varphi) + \varphi$$

holds for a.a. $(x,t) \in \Omega \times (0,T)$. Moreover, $\partial_{\mathbf{n}}\varphi = 0$ a.e. on $\partial\Omega \times (0,T)$ and $w(0) = w_0$ a.e. in Ω.

Remark 4.2 (Continuity of the solution)**.** By a classical Lions–Magenes result we see that $w \in \mathcal{C}([0,T],H)$. Besides, from the Aubin-Simon theorem, $\varphi \in \mathcal{C}([0,T],H^{2-\delta})$ for every $\delta > 0$, and by the Strauss lemma, $\varphi \in \mathcal{C}([0,T],H^2_w)$. In particular, it turns out that $\varphi \in \mathcal{C}(\overline{\Omega} \times [0,T])$.

Remark 4.3 (Initial conditions)**.** Note that, besides $w(0) = w_0$ for a.a. $x \in \Omega$, we also have $\varphi(0) = \varphi_0$, where φ_0 is the unique solution to the Neumann problem

$$-\varepsilon\Delta u + \varepsilon\Psi'(u) + u = f, \qquad \partial_{\mathbf{n}} u|_{\partial\Omega} = 0,$$

in the Appendix with $f = w_0 \in H$, satisfying all the properties therein. In particular, $\varphi_0 \in H^2$ and $|\varphi_0(x)| < 1$ a.e. $x \in \Omega$, or, equivalently, $|\langle\varphi_0\rangle| < 1$.

Proof of Theorem 4.1.

Existence. For $\lambda > 0$ let $(w_\lambda, \varphi_\lambda)$ be a solution to the approximated problem $(2.4)_\lambda$. In light of the uniform (with respect to λ) estimates in Section 3, we can pass to the limit $\lambda \to 0$ on any interval $[0,T]$, $T > 0$, with the following convergences (up to subsequences):

$\varphi_\lambda \to \varphi$ weakly star in $L^\infty(0,T;H^2)$,
$w_\lambda \to w$ weakly star in $L^\infty(0,T;H)$ and weakly in $L^2(0,T;V)$,
$\partial_t\varphi_\lambda \to \partial_t\varphi$ weakly in $L^2(0,T;V)$,
$\partial_t w_\lambda \to \partial_t w$ weakly in $L^2(0,T;V')$.

By Aubin-Lions theorem, we also deduce that $\varphi_\lambda \to \varphi$ in $L^2(0,T;V) \cap \mathcal{C}([0,T],H)$, and $\varphi_\lambda(x,t) \to \varphi(x,t)$ almost everywhere in $\Omega \times (0,T)$.

We claim that the limit (w, φ) is a weak solution according to Theorem 4.1. Indeed, the required regularity immediately follows by the above convergences. The boundedness of φ can be proved by a standard argument as follows. For any fixed $\eta \in (0,1/2)$ we introduce the set

$$E_\eta^\lambda = \{(x,t) \in \Omega \times [0,T] \,:\, |\varphi_\lambda(x,t)| > 1-\eta\}.$$

It is easy to see from (3.4) that

$$|E_\eta^\lambda| \leq \frac{C}{\min\{F_\lambda'(1-\eta), |F_\lambda'(-1+\eta)|\}}.$$

Passing to the limit as $\lambda \to 0$ and $\eta \to 0$, we have $|\,\{(x,t) \in \Omega \times (0,T) : |\varphi(x,t)| \geq 1\}\,| = 0$, meaning that $\varphi \in L^\infty(\Omega \times (0,T))$, with $|\varphi(x,t)| < 1$ for a.a. $(x,t) \in \Omega \times (0,T)$.

This allows to handle the nonlinear potential. Indeed, using the pointwise convergence of φ_λ and the uniform convergence of F_λ' to F' on any compact set in $(-1,1)$, we infer that $F_\lambda'(\varphi_\lambda) \to F'(\varphi)$, for almost every $(x,t) \in \Omega \times (0,T)$. Then, in light of (3.4), a weak form of the Lebesgue convergence theorem implies that $F_\lambda'(\varphi_\lambda) \to F'(\varphi)$ weakly in $L^2(0,T;H)$, which allows us to identify w in $L^\infty(0,\infty;H)$ as $w = -\varepsilon\Delta\varphi + \varepsilon\Psi'(\varphi) + \varphi$. In a standard way, we now pass to the limit in the weak formulation of $(2.4)_\lambda$, proving the validity of (4.1).

Uniqueness. Set $w = w_1 - w_2$ and $\varphi = \varphi_1 - \varphi_2$, where (w_i, φ_i) is a solution departing from the initial data w_{i0}, $i = 1,2$. Then, we have

$$\varepsilon\langle w_t, v\rangle + (\nabla w, \nabla v) = (\nabla\varphi, \nabla v), \quad \forall v \in V,$$

where

$$w = -\varepsilon\Delta\varphi + \varepsilon[\Psi'(\varphi_1) - \Psi'(\varphi_2)] + \varphi, \quad \text{a.e. in } \Omega \times (0,T).$$

Testing the second equation by φ yields

$$(w,\varphi) = \varepsilon\|\nabla\varphi\|^2 + \varepsilon(\Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi) + \|\varphi\|^2.$$

Noting that

$$\varepsilon(\Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi) + \|\varphi\|^2 \geq 2c_1\|\varphi\|^2,$$

we end up with

$$\varepsilon\|\nabla\varphi\|^2 + c_1\|\varphi\|^2 \leq \frac{1}{4c_1}\|w\|^2. \tag{4.2}$$

Testing the first equation by w gives

$$\frac{\varepsilon}{2}\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 = (\nabla\varphi, \nabla w) \leq \frac{1}{2}\|\nabla\varphi\|^2 + \frac{1}{2}\|\nabla w\|^2,$$

so that

$$\varepsilon\frac{d}{dt}\|w\|^2 + \|\nabla w\|^2 \leq \|\nabla\varphi\|^2 \leq \frac{1}{4c_1\varepsilon}\|w\|^2.$$

Finally,

$$\frac{d}{dt}\|w\|^2 \leq \frac{1}{4c_1\varepsilon^2}\|w\|^2,$$

and the Gronwall lemma, together with (4.2), yields the continuous dependence estimate

$$\|w_1(t) - w_2(t)\| + \sqrt{\varepsilon}\|\varphi_1(t) - \varphi_2(t)\|_V \leq C\|w_{10} - w_{20}\|, \qquad \forall t \in [0,T],$$

where C depends on ε and T. In particular, if $w_{10} = w_{20}$, then $w_1 = w_2$ and $\varphi_1 = \varphi_2$, proving the uniqueness. $\square$

4.1. **Strong solutions.** If the initial datum is more regular, namely $w_0 \in V$, we obtain further regularity on the solution. Indeed, having in mind the regularization scheme of Section 3, testing the first equation of (4.1) by $-\Delta w$ we get

$$\frac{\varepsilon}{2}\frac{d}{dt}\|\nabla w\|^2 + \|\Delta w\|^2 = (\Delta\varphi, \Delta w) \leq \frac{1}{2}\|\Delta\varphi\|^2 + \frac{1}{2}\|\Delta w\|^2,$$

which, in light of (3.2), tells us that

$$\varepsilon\frac{d}{dt}\|\nabla w\|^2 + \|\Delta w\|^2 \leq \frac{1}{\varepsilon^2}\|w\|^2.$$

Therefore, the Gronwall lemma yields

$$w \in L^\infty(0,T;V) \cap L^2(0,T;H^2),$$

and by comparison,

$$\varepsilon w_t = \Delta w - \Delta\varphi \in L^2(0,T;H).$$

At this point, we can deduce further regularity properties of φ and $F'(\varphi)$ in light of Lemma 2.3, observing that φ solves the homogeneous Neumann problem

$$-\Delta\varphi + F'(\varphi) = f, \qquad \partial_{\mathbf{n}}\varphi|_{\partial\Omega} = 0,$$

with

$$f = \frac{1}{\varepsilon}[w - \varphi + \varepsilon\theta_0\varphi].$$

Since $f \in L^\infty(0,T;V)$, we immediately learn from Lemma 2.3 (ii) that

$$\|\varphi\|_{W^{2,p}(\Omega)} + \|F'(\varphi)\|_{L^p(\Omega)} \leq C,$$

where $p = 6$ if $d = 3$ and for any finite $p \geq 2$ if $d = 2$.

Summing up, we have

Theorem 4.4. *Let $\varepsilon \in (0,\varepsilon_0)$ and let $w_0 \in V$. Given $T > 0$, the unique solution (w,φ) on $[0,T]$ to (2.4)-(2.5) with initial condition $w(0) = w_0$ is strong in the following sense:*

$$\begin{aligned} &w \in L^\infty(0,T;V) \cap L^2(0,T;H^2), \quad w_t \in L^2(0,T;H), \\ &\varphi \in L^\infty(0,T;W^{2,p}(\Omega)), \quad \varphi_t \in L^2(0,T;V), \\ &F'(\varphi) \in L^\infty(0,T;L^p(\Omega)), \end{aligned}$$

where $p = 6$ if $d = 3$ and any $2 \leq p < \infty$ if $d = 2$. Moreover,

$$\varepsilon w_t - \Delta w + \Delta\varphi = 0, \quad a.e.\ in\ \Omega \times (0,T),$$

and $\partial_{\mathbf{n}} w = 0$ a.e. on $\partial\Omega \times (0,T)$.

4.2. **Back to the original problem.** Let us now go back to the original ε-model (2.1)-(2.3). Regarding the initial condition $\varphi(0) = \varphi_0$, we assume that φ_0 is an *admissible datum*. Namely, as it is done for the classical Cahn-Hilliard equation with singular potential, we assume that

$$|\varphi_0(x)| < 1 \quad \text{for a.a. } x \in \Omega.$$

Equivalently, $\varphi_0 \in L^\infty(\Omega)$ with $\|\varphi_0\|_{L^\infty(\Omega)} \le 1$ and $|\langle\varphi_0\rangle| < 1$. Notice that such an assumption prevents the admissibility of the pure states (i.e., $\varphi \equiv 1$ or $\varphi \equiv -1$) as initial conditions.

The following well-posedness results hold.

Theorem 4.5. *Let φ_0 be an admissible datum. We assume that $\varphi_0 \in H^2$ with $\partial_{\mathbf{n}}\varphi_0 = 0$ a.e. on $\partial\Omega$ and*

$$\mu_0 := -\Delta\varphi_0 + \Psi'(\varphi_0) \in H.$$

Given $\varepsilon \in (0, \varepsilon_0)$ and $T > 0$, there exists a unique weak solution (φ, μ) to (2.1)-(2.3) on $[0, T]$ with

$$|\varphi(x,t)| < 1 \text{ for a.e. } (x,t) \in \Omega \times (0,T),$$

such that

$$\begin{aligned}
&\varphi \in L^\infty(0,T;H^2), \quad \varphi_t \in L^2(0,T;V),\\
&F'(\varphi) \in L^\infty(0,T;H),\\
&\mu \in L^\infty(0,T;H) \cap L^2(0,T;V), \quad \mu_t \in L^2(0,T;V'),
\end{aligned}$$

and

$$(\varphi_t, v) + (\nabla\mu, \nabla v) + \langle\varepsilon\mu_t, v\rangle = 0, \quad \forall\, v \in V,$$

for almost every $t \in (0,T)$, where

$$\mu = -\Delta\varphi + \Psi'(\varphi)$$

holds a.e. $(x,t) \in \Omega \times (0,T)$. Besides, $\partial_{\mathbf{n}}\varphi = 0$ a.e. on $\partial\Omega \times (0,T)$ and $\varphi(0) = \varphi_0$ a.e. in Ω.

When the datum is more regular, namely, $\mu_0 \in V$, then the solution is strong according to the next theorem.

Theorem 4.6. *Let the assumptions of Theorem 4.5 be in place, and assume that $\mu_0 \in V$. Then, the solution (φ, μ) to (2.1)-(2.3) on $[0,T]$ is strong, in the sense that*

$$\begin{aligned}
&\varphi \in L^\infty(0,T;W^{2,p}(\Omega)), \quad \varphi_t \in L^2(0,T;V),\\
&F'(\varphi) \in L^\infty(0,T;L^p(\Omega)),\\
&\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^2), \quad \mu_t \in L^2(0,T;H),
\end{aligned}$$

where $p = 6$ if $d = 3$ and any $2 \le p < \infty$ if $d = 2$. Besides, the solution satisfies

$$\varphi_t - \Delta\mu + \varepsilon\mu_t = 0, \quad \text{a.e. in } \Omega \times (0,T),$$

and $\partial_{\mathbf{n}}\mu = 0$ a.e. on $\partial\Omega \times (0,T)$.

Indeed, it is enough to take the unique solution (w, φ) to the boundary value problem (2.4)-(2.5) with initial datum $w(0) = w_0$ given by

$$w_0 = \varphi_0 + \varepsilon\mu_0,$$

as in (2.6). Then, in light of Theorem 4.1 and Theorem 4.4, the pair (φ, μ) with

$$\mu = \frac{1}{\varepsilon}(w - \varphi)$$

has all the required properties, depending on whether $w_0 \in H$ or $w_0 \in V$. As a matter of fact, in light of Remark 4.3, the boundary value problems (2.1)-(2.3) and (2.4)-(2.6), when endowed with a suitable initial condition as above, are completely equivalent.

We conclude the section by stressing once again that all the norm-bounds obtained so far *depend on* ε, growing indefinitely as $\varepsilon \to 0$.

5. Separation property

We recall the following definition.

Strict separation property in the time interval I. There exists $\delta > 0$ such that

$$|\varphi(x,t)| \leq 1 - \delta, \quad \text{for all } (x,t) \in \overline{\Omega} \times I. \tag{5.1}$$

For the classical Cahn–Hilliard equation (7.1) (see, e.g., [15]), such a property is well-known to be true in dimension $d = 2$ under the additional condition (2.10) on the growth of the singular potential at ± 1. Nonetheless, in dimension three, the validity of (5.1) is still an important open issue.

One of the main results of the paper is that for the ε-perturbation (2.1)-(2.3) of the classical Cahn–Hilliard system, the strict separation property from the pure state holds true even when $d = 3$, and without any extra assumption on the singular potential. More precisely, we have the following.

Theorem 5.1. *Let $d = 3$, $\varepsilon \in (0, \varepsilon_0)$ be fixed and let (φ, μ) be a strong solution to* (2.1)-(2.3) *according to Theorem 4.6. Then, φ satisfies property* (5.1) *on any time interval $I = [\sigma, T]$, $0 < \sigma < T$.*

Proof. Take any initial datum (φ_0, μ_0) complying with the assumptions of Theorem 4.6 and consider the corresponding strong solution (φ, μ) to (2.1)-(2.3) on $[0, T]$.

Setting $w = \varphi + \varepsilon\mu$, then (w, φ) is a strong solution to (2.4)-(2.6), satisfying in particular $\varepsilon w_t - \Delta w = -\Delta\varphi$, almost everywhere. Differentiating this equality with respect to t, we find[3]

$$\varepsilon w_{tt} - \Delta w_t = -\Delta\varphi_t,$$

that we test by w_t to get

$$\frac{\varepsilon}{2}\frac{d}{dt}\|w_t\|^2 + \|\nabla w_t\|^2 = (\nabla\varphi_t, \nabla w_t).$$

[3]Actually, the following computations are only formal, but one can make them rigorous by working with the finite differences $\partial_t^h w(t) := \frac{1}{h}[w(t+h) - w(t)]$ for any $h \neq 0$, see, e.g., [4] and [12].

This yields the differential inequality

$$\varepsilon\frac{d}{dt}\|w_t\|^2 + \|\nabla w_t\|^2 \leq \|\nabla\varphi_t\|^2.$$

It is apparent from Theorem 4.4 that there exists $C > 0$, depending on T and ε, such that

$$\|w_t\|_{L^2(t,t+\sigma;H)} \leq C, \quad \|\nabla\varphi_t\|_{L^2(t,t+\sigma;H)} \leq C, \tag{5.2}$$

for all $t \in [0,T]$, and any $\sigma \in (0,T)$. Therefore, we can apply the uniform Gronwall lemma, yielding

$$\|w_t\|_{L^\infty(\sigma,T;H)} \leq C, \tag{5.3}$$

for some $C > 0$ depending on σ, T and ε via the estimates (5.2). By comparison in the first equation (written as $w_t = \Delta\mu$), we learn that $\mu \in L^\infty(\sigma,T;H^2)$, implying in turn that $\mu \in L^\infty(\Omega\times(\sigma,T))$.

Now recall that φ solves the Neumann problem (2.9) with $f = \mu + \theta_0\varphi$. Having proved that $f \in L^\infty(\Omega\times(\sigma,T))$, part (i) of Lemma 2.3 with $p = \infty$ ensures that

$$\|F'(\varphi)\|_{L^\infty(\Omega\times(\sigma,T))} \leq C.$$

Since F' diverges at ±1 and $\varphi \in \mathcal{C}(\overline{\Omega}\times[0,T])$, we immediately deduce the existence of $\delta > 0$ complying with (5.1) in $I = [\sigma, T]$. □

It is apparent form the proof that the separation parameter δ in (5.1) depends on ε through (5.2) and (5.3), hence it deteriorates to 0 as $\varepsilon \to 0$. Besides, it depends on the size of the initial data and on the specific form of F.

Remark 5.2. In the light of the strict separation property, one can say more about the regularity of φ for strong solutions. Indeed, the singular potential F becomes regular on the interval $[-1+\delta, 1-\delta]$. As a consequence, reasoning as in [16], for $F \in \mathcal{C}^3(-1,1)$ one obtains $\varphi \in L^\infty(0,\infty;H^3)$.

6. Uniform (in ε) energy estimates

As already observed, the norm-bounds obtained so far *depend on* ε. Having in mind to study the limit $\varepsilon \to 0$, a crucial goal is to derive a number of estimates that are independent of ε. To this aim, let $\varepsilon \in (0,\varepsilon_0)$ complying with (2.8). According to Theorem 4.6, we consider problem (2.1) with homogeneous Neumann boundary conditions for φ and μ and initial datum $\varphi(0) = \varphi_0$ satisfying the assumption therein. In particular,

$$\mu_0 \in V.$$

Along the section, we denote by (φ,μ) the corresponding unique strong solution, omitting for simplicity the superscript ε. The generic positive constant C appearing in the following computations may depend on ε_0 but is independent of ε.

Basic estimate. We introduce the energy of the solution as

$$\mathcal{E}_{\varphi_0}(t) = \|\nabla\varphi(t)\|^2 + \varepsilon\|\mu(t)\|^2 + 2\int_\Omega \Psi(\varphi(t))dx.$$

Multiplying the first equation in (2.1) by μ and the second one by φ_t, we see that

$$\frac{d}{dt}\mathcal{E}_{\varphi_0}(t) + 2\|\nabla\mu(t)\|^2 = 0.$$

Recalling that $|\varphi| < 1$ almost everywhere, we easily deduce from this differential equality that

$$\nabla\varphi \in L^\infty(0,\infty;H), \quad \sqrt{\varepsilon}\mu \in L^\infty(0,\infty;H), \quad \nabla\mu \in L^2(0,\infty;H). \tag{6.1}$$

In turn, since $\|\varphi\|_{L^\infty(\Omega)} \leq 1$, this tells us that

$$\varphi \in L^\infty(0,\infty;V), \tag{6.2}$$

with a bound on the norm that is independent of ε.

Uniform control of $\langle\varphi\rangle$. This is a key point, since the mean of φ is not conserved during the evolution. Nonetheless, we have the conservation law (see (2.7))

$$\langle\varphi(t) + \varepsilon\mu(t)\rangle = \langle\varphi_0 + \varepsilon\mu_0\rangle, \qquad \forall t \geq 0.$$

By the uniform estimate $\sqrt{\varepsilon}\mu \in L^\infty(0,\infty;H)$, we obtain

$$|\langle\sqrt{\varepsilon}\mu\rangle| \leq C\int_\Omega \sqrt{\varepsilon}|\mu| \leq C\left(\int_\Omega \varepsilon|\mu|^2\right)^{1/2} \leq C.$$

Therefore, writing

$$\langle\varphi(t)\rangle = -\sqrt{\varepsilon}\langle\sqrt{\varepsilon}\mu(t)\rangle + \langle\varphi_0 + \varepsilon\mu_0\rangle, \tag{6.3}$$

we easily see that

$$\langle\varphi(t)\rangle \to \langle\varphi_0\rangle, \qquad \varepsilon \to 0$$

uniformly in $[0,\infty)$.

Now observe that, since $|\langle\varphi_0\rangle| < 1$, then $\langle\varphi_0\rangle \in (-1+2\delta, 1-2\delta)$ for some $\delta > 0$. Therefore, there exists $\tilde{\varepsilon} > 0$ (we assume that $\tilde{\varepsilon} < \varepsilon_0$), depending on δ, such that

$$\langle\varphi(t)\rangle \in (-1+\delta, 1-\delta), \qquad \forall t \geq 0, \, \forall\varepsilon \in (0,\tilde{\varepsilon}). \tag{6.4}$$

Uniform estimates for φ_t and μ_t. We test the first equation in (2.1) by μ_t, finding

$$-(\varphi_t, \mu_t) = \frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \varepsilon\|\mu_t\|^2.$$

Then, differentiating the second equation, we have[4]

$$\mu_t = -\Delta\varphi_t + \Psi''(\varphi)\varphi_t,$$

and multiplying by φ_t, we get

$$(\mu_t, \varphi_t) = \|\nabla\varphi_t\|^2 + \langle\Psi''(\varphi)\varphi_t, \varphi_t\rangle.$$

This yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \varepsilon\|\mu_t\|^2 + \|\nabla\varphi_t\|^2 = -\langle\Psi''(\varphi)\varphi_t, \varphi_t\rangle \leq \theta_0\|\varphi_t\|^2.$$

[4]Recall that φ is separated from the pure states, hence $\Psi'(\varphi)$ is regular and can be differentiated with respect to time; besides $\Psi''(\varphi) \in L^\infty(\Omega)$.

Adding and subtracting $\|\varphi_t\|^2$, by the interpolation inequality $\|\varphi_t\|^2 \leq C\|\varphi_t\|_V\|\varphi_t\|_{V'}$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \varepsilon\|\mu_t\|^2 + \|\varphi_t\|_V^2 \leq \frac{1}{2}\|\varphi_t\|_V^2 + C\|\varphi_t\|_{V'}^2.$$

Now observe that

$$\varphi_t + \varepsilon\mu_t = \Delta\mu,$$

hence

$$\|\varphi_t + \varepsilon\mu_t\|_{V'} = \|\Delta\mu\|_{V'} = C\|\nabla\mu\|.$$

We thus learn that

$$\|\varphi_t\|_{V'} \leq \varepsilon\|\mu_t\|_{V'} + C\|\nabla\mu\| \leq \varepsilon C_1\|\mu_t\| + C\|\nabla\mu\|,$$

for some $C_1 > 0$ given by the embedding $H \subset V'$. This provides the differential inequality

$$\frac{d}{dt}\|\nabla\mu\|^2 + \varepsilon(2 - C_2\varepsilon)\|\mu_t\|^2 + \|\varphi_t\|_V^2 \leq C\|\nabla\mu\|^2,$$

for some structural parameter $C_2 > 0$. Assuming without loss of generality that $\varepsilon \in (0, 2/C_2)$, an integration in time in light of (6.1), recalling that $\mu_0 \in V$, yields

$$\|\nabla\mu\|_{L^\infty(0,\infty;H)} \leq C, \qquad \|\varphi_t\|_{L^2(0,\infty;V)} + \|\sqrt{\varepsilon}\mu_t\|_{L^2(0,\infty;H)} \leq C. \tag{6.5}$$

At this point, we recall an important inequality satisfied by the singular potential, firstly envisaged in [18]; see also [15, Ch. 4]:

$$\int_\Omega |F'(u)|\,\mathrm{d}x \leq \tilde{C}\left|\int_\Omega F'(u)(u - \langle u\rangle)\,\mathrm{d}x\right| + \tilde{C},$$

where $\tilde{C}$ goes to $+\infty$ as $|\langle u\rangle| \to 1$. According to (6.4), we obtain the existence of $\tilde{C} > 0$ (depending on δ) such that, for all $t \geq 0$,

$$\int_\Omega |F'(\varphi)|\,\mathrm{d}x \leq \tilde{C}\left|\int_\Omega F'(\varphi)(\varphi - \langle\varphi\rangle)\,\mathrm{d}x\right| + \tilde{C}. \tag{6.6}$$

This is the key ingredient for the next step.

Uniform control for $\langle\mu\rangle$. By definition of μ written as $\mu = -\Delta\varphi + F'(\varphi) - \theta_0\varphi$, we have $\langle\mu\rangle = |\Omega|^{-1}\int_\Omega F'(\varphi) - \theta_0\langle\varphi\rangle$, hence

$$|\langle\mu\rangle| \leq C(1 + \|F'(\varphi)\|_{L^1(\Omega)}). \tag{6.7}$$

In order to estimate the L^1-norm of $F'(\varphi)$, we use a classical argument, starting by testing the equation for μ by $\overline{\varphi} = \varphi - \langle\varphi\rangle$:

$$(\mu, \overline{\varphi}) = (-\Delta\varphi, \overline{\varphi}) + (\Psi'(\varphi), \overline{\varphi}).$$

Since

$$(\mu, \overline{\varphi}) = (\overline{\mu}, \overline{\varphi}), \qquad (-\Delta\varphi, \overline{\varphi}) = \|\nabla\varphi\|^2,$$

we obtain

$$\|\nabla\varphi\|^2 + (F'(\varphi), \overline{\varphi}) = \theta_0(\varphi, \overline{\varphi}) + (\overline{\mu}, \overline{\varphi}).$$

As a result, recalling that $\|\varphi\|_{L^\infty(\Omega)} \leq 1$ and that $\nabla\varphi \in L^\infty(0,\infty;H)$ uniformly in ε, we get

$$|(F'(\varphi), \overline{\varphi})| \leq \|\nabla\varphi\|^2 + \theta_0|(\varphi, \overline{\varphi})| + |(\overline{\mu}, \overline{\varphi})| \leq C(1 + \|\nabla\mu\|).$$

Exploiting (6.6), we thus obtain

$$\int_\Omega |F'(\varphi)|\,\mathrm{d}x \leq C(1+\|\nabla\mu\|),$$

and inserting into (6.7), we have

$$|\langle\mu\rangle| \leq C(1+\|\nabla\mu\|).$$

In light of (6.5), we thus conclude that

$$\|\mu\|_{L^\infty(0,\infty,V)} \leq C. \tag{6.8}$$

Finally, recalling that φ solves the elliptic problem (2.9) with $f = \mu + \theta_0\varphi$ uniformly bounded in $L^\infty(0,\infty;V)$ by (6.2) and (6.8), Lemma 2.3 (ii) yields

$$\|\varphi\|_{L^\infty(0,\infty;W^{2,p}(\Omega))} + \|F'(\varphi)\|_{L^\infty(0,\infty;L^p(\Omega))} \leq C,$$

where $p = 6$ if $d = 3$ and for any finite $p \geq 2$ if $d = 2$. In both cases,

$$\|\varphi\|_{L^\infty(0,\infty;H^2)} + \|F'(\varphi)\|_{L^\infty(0,\infty;H)} \leq C.$$

Collecting all the estimates above, we can state the following.

Theorem 6.1. *Let* $(\varphi_0,\mu_0) \in H^2 \times V$ *comply with the assumptions of Theorem 4.6. Then, there exists* $\varepsilon_1 > 0$ *such that, for every* $\varepsilon \in (0,\varepsilon_1)$*, the corresponding strong solution* $(\varphi^\varepsilon,\mu^\varepsilon)$ *to* (2.1)-(2.3) *has the following properties:*

$$\begin{aligned}
&\varphi^\varepsilon \in L^\infty(0,\infty;H^2), \quad \varphi_t^\varepsilon \in L^2(0,\infty;V),\\
&F'(\varphi^\varepsilon) \in L^\infty(0,\infty;H),\\
&\mu^\varepsilon \in L^\infty(0,\infty;V), \quad \sqrt{\varepsilon}\mu_t^\varepsilon \in L^2(0,\infty;H),
\end{aligned}$$

where all the norms are bounded independently of ε*.*

Uniform strict separation in dimension two. Owing to the uniform estimates provided by Theorem 6.1, we are able to prove the validity of a strict separation property in dimension 2 which is uniform with respect to ε.

Theorem 6.2. *Let* $d = 2$ *and* (2.10) *be in place. For* $\varepsilon \in (0,\varepsilon_1)$*, let* $(\varphi^\varepsilon,\mu^\varepsilon)$ *be a strong solution to* (2.1)-(2.3) *according to Theorem 4.6. Then, there exists* $\delta > 0$ *such that*

$$|\varphi^\varepsilon(x,t)| \leq 1-\delta, \quad \forall (x,t) \in \overline{\Omega}\times[0,\infty),\ \forall\varepsilon\in(0,\varepsilon_1).$$

Proof. Observe that φ^ε solves the elliptic problem (2.9) with

$$f = \mu^\varepsilon + \theta_0\varphi^\varepsilon.$$

By Theorem 4.6, we know in particular that

$$\|\varphi^\varepsilon\|_{L^\infty(0,\infty;V)} + \|\mu^\varepsilon\|_{L^\infty(0,\infty;V)} \leq C,$$

for some $C > 0$ (which is independent of ε). Accordingly,

$$\|f\|_{L^\infty(0,\infty;V)} \leq C.$$

This allows the application of Lemma 2.5, yielding

$$\|F'(\varphi^\varepsilon)\|_{L^\infty(\Omega\times(0,\infty))} \leq C.$$

Since F' diverges at ± 1 and $\varphi^\varepsilon \in \mathcal{C}(\overline{\Omega} \times [0,\infty))$ (uniformly in ε), we immediately deduce the existence of $\delta > 0$ complying with (5.1) in $I = [0,\infty)$. Since all the estimates are independent of T, the proof is completed. □

Let us remark that this short and elegant proof of the two-dimensional strict separation property can be extended to the classical Cahn–Hilliard equation. An analogous argument can be found in [14] for a singular Navier–Stokes–Cahn–Hilliard system.

7. Convergence to the Cahn–Hilliard system

Let us consider the classical Cahn–Hilliard equation formally obtained by setting $\varepsilon = 0$, namely,

$$\begin{cases} \varphi_t = \Delta\mu, \\ \mu = -\Delta\varphi + \Psi'(\varphi), \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{7.1}$$

completed with the same boundary conditions

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0, \qquad \text{on } \partial\Omega \times (0,T), \tag{7.2}$$

and the initial condition

$$\varphi(0) = \varphi_0, \quad \text{in } \Omega. \tag{7.3}$$

After the uniform estimates of Section 6 we are now ready to prove that the convergence of the ε-model (2.1)-(2.3) to the Cahn-Hilliard one is not only formal, according to the next result.

Theorem 7.1. *Let the initial datum (φ_0, μ_0) comply with the assumptions of Theorem 4.6. Then, on every time interval $[0,T]$, $T > 0$, the strong solution $(\varphi^\varepsilon, \mu^\varepsilon)$ to (2.1)-(2.3) converges* [5] *as $\varepsilon \to 0$ to a strong solution (φ^0, μ^0) to (7.1)-(7.3).*

Proof. In light of the uniform with respect to ε estimates proven above, we can pass to the limit $\varepsilon \to 0$ with the following convergences (up to subsequences):

$$\begin{aligned} &\varphi^\varepsilon \to \varphi^0 \text{ weakly star in } L^\infty(0,T;H^2), \\ &\mu^\varepsilon \to \mu^0 \text{ weakly star in } L^\infty(0,T;V) \text{ and weakly in } L^2(0,T;V), \\ &\varphi^\varepsilon_t \to \varphi^0_t \text{ weakly in } L^2(0,T;V), \\ &\varepsilon\mu^\varepsilon_t \to 0 \text{ weakly in } L^2(0,T;H). \end{aligned}$$

By Aubin-Lions compactness theorems, we also deduce that $\varphi^\varepsilon \to \varphi^0$ in $L^2(0,T;V) \cap \mathcal{C}([0,T],H)$, and $\varphi^\varepsilon(x,t) \to \varphi^0(x,t)$ almost everywhere in $\Omega \times (0,T)$.

We can now pass to the limit in a standard way in the weak formulation of (2.1)-(2.3) to show that the limit (φ^0, μ^0) is indeed a weak solution to the Cahn–Hilliard system. Just note that the weak convergence $F'(\varphi^\varepsilon) \to F'(\varphi^0)$ in $L^2(0,T;H)$ is ensured (as above) by a weak form of the Lebesgue convergence theorem in light of the uniform bound $F'(\varphi^\varepsilon) \in L^\infty(0,\infty;H)$ in Theorem 6.1 and the pointwise convergence $F'(\varphi^\varepsilon) \to F'(\varphi^0)$ (recall that φ^ε and φ^0 belong a.e. to $(-1,1)$). Then, since the initial datum $\varphi_0 \in H^2$ with $\partial_{\mathbf{n}}\varphi_0 = 0$ on $\partial\Omega$ and $\mu_0 \in V$, it is a standard matter to prove that actually (φ^0, μ^0) is a strong solution to the Cahn–Hilliard system. □

[5] In the sense specified along the proof.

We conclude the section with an estimate of the error between the solutions of the ε-model and the solutions of the unperturbed one.

Theorem 7.2. *Let $T > 0$ be given. Under the assumptions of Theorem 7.1, we have*

$$\|\varphi^\varepsilon(t) - \varphi^0(t)\|_{V'}^2 + \int_0^T \|\nabla\varphi^\varepsilon(\tau) - \nabla\varphi^0(\tau)\|^2 d\tau \leq C\varepsilon,$$

for any $t \in [0, T]$, where $C > 0$ depends on T and on the size of the initial datum.

Proof. Let $\varphi = \varphi^\varepsilon - \varphi^0$ denote the difference between the solutions of the two problems departing from the same initial datum (φ_0, μ_0). First, note that $\langle\varphi(t)\rangle = \langle\varphi^\varepsilon(t)\rangle - \langle\varphi_0\rangle$ for every $t \geq 0$, due to the mass conservation property for the Cahn–Hilliard equation (7.1). Hence, by (6.3),

$$\langle\varphi(t)\rangle = \langle\varphi^\varepsilon(t)\rangle - \langle\varphi_0\rangle = -\varepsilon\langle\mu^\varepsilon(t)\rangle + \varepsilon\langle\mu_0\rangle.$$

Recalling that $\mu^\varepsilon \in L^\infty(0, \infty; V)$ by Theorem 6.1, we infer that

$$|\langle\varphi(t)\rangle| \leq C\varepsilon, \qquad \forall t \geq 0. \tag{7.4}$$

Next, observe that φ solves

$$\langle\varphi_t, v\rangle + (\nabla\mu, \nabla v) = -\varepsilon(\mu_t^\varepsilon, v), \quad \forall\, v \in V, \tag{7.5}$$

for almost every $t \in (0, T)$, where

$$\mu = -\Delta\varphi + \Psi'(\varphi^\varepsilon) - \Psi'(\varphi^0),$$

with

$$\varphi(0) = 0.$$

Besides,

$$\|\varphi^i(t)\|_V \leq C \quad \text{and} \quad \|\varphi^i(t)\|_{L^\infty(\Omega)} \leq 1, \quad \forall\, t \geq 0, \quad i = 0, \varepsilon,$$

where here and in what follows C is independent of ε. Taking $v = \mathcal{N}(\varphi - \langle\varphi\rangle)$ in (7.5), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \langle\varphi\rangle\|_\sharp^2 + (\mu, \varphi - \langle\varphi\rangle) = -\varepsilon(\mu_t^\varepsilon, \mathcal{N}(\varphi - \langle\varphi\rangle)).$$

By the assumptions on Ψ, we have

$$\begin{aligned}(\mu, \varphi - \langle\varphi\rangle) &= \|\nabla\varphi\|^2 + (\Psi'(\varphi^\varepsilon) - \Psi'(\varphi^0), \varphi^\varepsilon - \varphi^0) - (\Psi'(\varphi^\varepsilon) - \Psi'(\varphi^0), \langle\varphi\rangle)\\ &\geq \|\nabla\varphi\|^2 - \theta_0\|\varphi\|^2 - |(\Psi'(\varphi^\varepsilon) - \Psi'(\varphi^0), \langle\varphi\rangle)|\\ &\geq \|\varphi\|_V^2 - (\theta_0 + 1)\|\varphi\|^2 - \big(\|\Psi'(\varphi^\varepsilon)\|_{L^1(\Omega)} + \|\Psi'(\varphi^0)\|_{L^1(\Omega)}\big)|\langle\varphi\rangle|.\end{aligned}$$

Besides, we control $\|\varphi\|^2$ by interpolation,

$$(\theta_0 + 1)\|\varphi\|^2 \leq \frac{1}{4}\|\varphi\|_V^2 + C\|\varphi\|_*^2.$$

By the Poincaré type inequality $\|\mathcal{N}(\varphi - \langle\varphi\rangle)\| \leq C\|\nabla\varphi\|$, we obtain the estimate

$$-\varepsilon(\mu_t^\varepsilon, \mathcal{N}(\varphi - \langle\varphi\rangle)) \leq C\sqrt{\varepsilon}\|\sqrt{\varepsilon}\mu_t^\varepsilon\|\|\nabla\varphi\| \leq C\varepsilon\|\sqrt{\varepsilon}\mu_t^\varepsilon\|^2 + \frac{1}{4}\|\varphi\|_V^2.$$

Since by definition $\|\varphi\|_*^2 = \|\varphi - \langle\varphi\rangle\|_\sharp^2 + |\langle\varphi\rangle|^2$, we can write $\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi - \langle\varphi\rangle\|_\sharp^2 = \frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 - \frac{\mathrm{d}}{\mathrm{d}t}|\langle\varphi\rangle|^2$. Collecting everything, we arrive at the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \|\varphi\|_V^2 \leq C\|\varphi\|_*^2 + g|\langle\varphi\rangle| + h\varepsilon + \frac{\mathrm{d}}{\mathrm{d}t}|\langle\varphi\rangle|^2,$$

having defined

$$g(t) = C\big(\|\Psi'(\varphi^\varepsilon(t))\|_{L^1(\Omega)} + \|\Psi'(\varphi^0(t))\|_{L^1(\Omega)}\big), \qquad h(t) = C\|\sqrt{\varepsilon}\mu_t^\varepsilon(t)\|^2.$$

Note that g and h belong to $L^1(0,T)$, due to Theorem 6.1. In light of (7.4), we finally end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \|\varphi\|_V^2 \leq C\|\varphi\|_*^2 + Cg\varepsilon + h\varepsilon + \frac{\mathrm{d}}{\mathrm{d}t}|\langle\varphi\rangle|^2.$$

Therefore, an application of the Gronwall lemma gives

$$\|\varphi(t)\|_*^2 \leq C_T\|\varphi(0)\|_*^2 + C_g\varepsilon + C_h\varepsilon + |\langle\varphi(t)\rangle|^2 - |\langle\varphi(0)\rangle|^2, \quad \forall\, t \in [0,T].$$

Since $\varphi(0) = 0$, invoking once again (7.4), we conclude that

$$\|\varphi(t)\|_*^2 \leq C\varepsilon.$$

A subsequent integration on $[0,T]$ of the differential inequality concludes the proof, recalling that $\|\cdot\|_*$ is an equivalent norm on V'. ☐

8. Appendix. The associated Neumann problem

For the sake of completeness and for the reader's convenience, in this appendix we consider the elliptic problem associated to the model (2.1)-(2.2). We have the following existence result.

Theorem 8.1. *Let $\varepsilon \in (0,\varepsilon_0)$ and $f \in H$. Then, the problem*

$$\begin{cases} -\varepsilon\Delta u + \varepsilon\Psi'(u) + u = f, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega, \end{cases} \tag{8.1}$$

admits a unique solution $u \in H^2$ that satisfies the equation a.e. in Ω and the boundary conditions a.e. on $\partial\Omega$. Besides, $F'(u) \in H$. In particular, $\|u\|_{L^\infty(\Omega)} \leq 1$ and $|u(x)| < 1$ for almost every $x \in \Omega$.

We report here the main steps of the proof, which is obtained by classical tools.

Proof. Let $\{b_n\}$ be a sequence of eigenvectors of the operator $-\Delta$ with Neumann boundary conditions ($-\Delta b_i = \lambda_i b_i$, $\partial_{\mathbf{n}} b_i = 0$), forming a complete orthonormal basis in H and a complete orthogonal one in V.

For any $\lambda > 0$, consider the regular potential Ψ_λ defined in (2.11) instead of the singular potential Ψ in (8.1). Accordingly, the weak formulation of the problem in the Galerkin scheme reads: find $u_m \in V_m = \mathrm{Span}\{b_1, ..., b_m\}$ such that

$$\varepsilon(\nabla u_m, \nabla v) + \varepsilon(\Psi_\lambda'(u_m), v) + (u_m, v) = (f, v), \qquad \forall v \in V_m.$$

Use $v = u_m$ and $v = -\Delta u_m$ to get

$$\varepsilon\|\nabla u_m\|^2 + \varepsilon(\Psi_\lambda'(u_m), u_m) + \|u_m\|^2 = (u_m, f) \quad \Longrightarrow \quad \varepsilon\|\nabla u_m\|^2 + c_1\|u_m\|^2 \leq \frac{1}{4c_1}\|f\|^2,$$

where c_1 is as in Section 3, and

$$\varepsilon\|\Delta u_m\|^2 + \varepsilon(\Psi''_\lambda(u_m)\nabla u_m, \nabla u_m) + \|\nabla u_m\|^2 = -(\Delta u_m, f),$$

providing

$$\varepsilon\|\Delta u_m\|^2 + 4c_1\|\nabla u_m\|^2 \le \frac{1}{\varepsilon}\|f\|^2.$$

We thus have

$$\|u_m\|_V \le C, \qquad \|u_m\|_{H^2} \le C,$$

with constants independent of m and λ. We can now pass to the weak limit $u_m \rightharpoonup u$ in $V \cap H^2$: by the compact embedding $H^2 \subset\subset \mathcal{C}(\overline{\Omega})$, the convergence is pointwise and uniform too. Besides, since F'_λ is Lipschitz continuous (with constant $1/\lambda$), we immediately have $\Psi'_\lambda(u_m) \to \Psi'_\lambda(u)$ uniformly (in particular in H). Hence we can pass to the limit in the weak formulation, proving that $u \in H^2$ satisfies

$$\varepsilon(\nabla u, \nabla v) + \varepsilon(\Psi'_\lambda(u), v) + (u, v) = (f, u), \qquad \forall v \in V.$$

Integrating by parts, we see that the equation is satisfied a.e. in Ω. Finally, by comparison in the equation, we get $\varepsilon\Psi'_\lambda(u) \in H$ with

$$\|\varepsilon\Psi'_\lambda(u)\| \le C, \tag{8.2}$$

independently of λ. Now consider the sequence of solutions u_λ and let $\lambda \to 0$. By the above estimates, $u_\lambda \rightharpoonup u$ in $V \cap H^2$, hence pointwise and uniformly in $\overline{\Omega}$. The boundedness of u can be proved by a standard argument as follows. For any fixed $\eta \in (0, 1/2)$, we introduce the set $E^\lambda_\eta = \{x \in \Omega : |u_\lambda(x)| > 1 - \eta\}$. It is easy to see from (8.2) that

$$|E^\lambda_\eta| \le \frac{C_\varepsilon}{\min\{F'_\lambda(1-\eta), |F'_\lambda(-1+\eta)|\}}.$$

Passing to the limits $\lambda \to 0$ and $\eta \to 0$, we have $|\{x \in \Omega : |u(x)| \ge 1\}| = 0$, meaning that $u \in L^\infty(\Omega)$ with $|u(x)| < 1$ for almost every $x \in \Omega$.

Using now the pointwise convergence of u_λ and the uniform convergence of F'_λ to F' on any compact set in $(-1, 1)$, we infer that $F'_\lambda(u_\lambda) \to F'(u)$, for almost every $x \in \Omega$. Then, in light (8.2), a weak form of the Lebesgue convergence theorem implies that $F'_\lambda(u_\lambda) \to F'(u)$ weakly in H. Therefore, u is a weak solution to the problem (8.1). Then, it is a standard matter to verify that both the equation and the boundary conditions are satisfied almost everywhere. □

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