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Expanding vertices to triangles in cubic graphs

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Abstract

Contraction of triangles is a standard operation in the study of cubic graphs, as it reduces the order of the graph while typically preserving many of its properties. In this paper, we investigate the converse problem, wherein certain vertices of cubic graphs are expanded into triangles to achieve a desired property. We first focus on bridgeless cubic graphs and define the parameter $T(G)$ as the minimum number of vertices that need to be expanded into triangles so that the resulting cubic graph can be covered with four perfect matchings. We relate this parameter to the concept of shortest cycle cover. Furthermore, we show that if 5-Cycle Double Cover Conejcture holds true, then $T(G) \leq \frac{2}{5}|V(G)|$. We conjecture a tighter bound, $T(G) \leq \frac{1}{10}|V(G)|$, which is optimal for the Petersen graph, and show that this bound follows from major conjectures like the Petersen Coloring Conjecture. In the second part of the paper, we introduce the parameter $t(G)$ as the minimum number of vertex expansions needed for the graph to admit a perfect matching. We prove a Gallai type identity: $t(G) + \ell(G) = |V(G)|$, where $\ell(G)$ is the number of edges in a largest even subgraph of G. Then we prove the general upper bound $t(G) < \frac{1}{4}|V(G)|$ for cubic graphs, and $t(G) < \frac{1}{6}|V(G)|$ for cubic graphs without parallel edges. We provide examples showing that these bounds are asymptotically tight. The paper concludes with a discussion of the computational complexity of determining these parameters.

Keywords: Matching, cubic graph, perfect matching, 2-factor, triangle.

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1 Introduction

Many important conjectures in graph theory, such as the Cycle Double Cover Conjecture or 5-Flow Conjecture, can be reduced to the case of cubic graphs, graphs in which every vertex has degree three. We mean that proving a conjecture for cubic graphs is enough to establish it in its general formulation. When working with the conjectures mentioned above, triangles are often considered a trivial case, since their contraction reduces the size and complexity of the graph. In this paper, we consider the reverse approach: instead of contracting triangles, we study what happens when we expand certain vertices of a cubic graph into triangles. Our goal is to understand whether such an expansion can enforce a desired property that the original graph may not have satisfied.

In this paper, we consider finite, undirected graphs that do not contain loops. However, they may contain parallel edges. If they do not contain parallel edges, then we will refer to them as simple graphs.

For a bridgeless cubic graph G, we introduce the parameter $T(G)$ as the minimum number of vertices of G that we need to expand into triangles so that the edge-set of the resulting cubic graph can be covered with four perfect matchings (see the beginning of Section 3 for precise definition of this parameter). In some sense our parameter measures how far is our graph from being coverable with four perfect matchings. Similar measures for other properties of cubic graphs are presented in [24] and [4]. A strong motivation for studying the parameter $T(G)$ is that, if it can be shown that $T(G)$ is finite for every bridgeless cubic graph, this would imply the 5-CDC Conjecture (see [6] and Conjecture 3.2). We relate the parameter $T(G)$ to the length of a shortest cycle cover of the graph G. Moreover, we show that if 5-Cycle Double Cover Conjecture is verified, then $T(G) \leq \frac{2}{5}|V|$ holds. Anyway, this bound does not seem tight. In the paper we offer a conjecture that $T(G) \leq \frac{1}{10}|V|$. To evaluate the plausibility of our conjecture, observe that the bound is asymptotically tight, as evidenced by the Petersen graph, and we demonstrate that it follows as a consequence of several established conjectures.

In Section 4, we introduce another parameter $t(G)$, defined as the minimum number of vertices in an arbitrary cubic graph G that need to be expanded into triangles so that the resulting graph admits a perfect matching. In connection with this parameter, we preliminary show that for every cubic graph G the identity $t(G) + \ell(G) = |V(G)|$ holds, where $\ell(G)$ denotes the number of edges in a largest even subgraph of G. We then use this equality to show that $t(G) < \frac{1}{4}|V(G)|$ for an arbitrary cubic graph, and $t(G) < \frac{1}{6}|V(G)|$ for simple cubic graphs. These two bounds are complemented by examples demonstrating their tightness.

We conclude the paper in Section 5 by discussing the computational complexity of determining both parameters.

2 Notations and auxiliary results

This section is devoted to introducing the notation and definitions that will be used throughout the paper. We also recall all known results that will play a role in the forthcoming sections. Non-defined terms and concepts can be found, for instance, in [26].

If G is a graph, then let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. When the graph is clear from context, we simply write V and E.

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If G is a graph, then a *block* of G is an inclusion-wise maximal 2-connected subgraph of G. If B is a block in G and it contains at most one cut-vertex, then B will be called an *end-block*. Let G be a cubic graph and let B be an end-block of G. Then there is a unique cut-vertex y of G in B that is adjacent to a unique vertex x lying outside B. We will refer along the paper to x as the *root* of B.

In Section 4 a special role will be played by the graph with three vertices obtained from a triangle by duplicating one of its edges (see Figure 1). From now on, we will denote by W such a graph. Similarly, W' denotes the unique graph obtained from a complete graph K_4 by subdividing one of its edges exactly once (Figure 1).

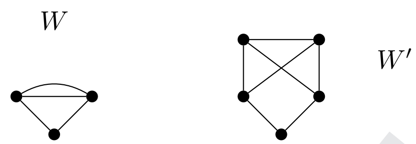

Figure 1: The graphs W and W'.

If G is a cubic graph and $e = uv$ is an edge in it, then the following operation will be relevant: subdivide the edge $e = uv$ with a new vertex w_e, add a copy of W and join the unique degree-two vertex of W to w_e (Figure 3). Note that the resulting graph is cubic, the added copy of W is an end-block in it whose root is w_e. We often say that *we subdivide e and attach a copy of W to it.* We will define a similar operation when instead of W we work with W'.

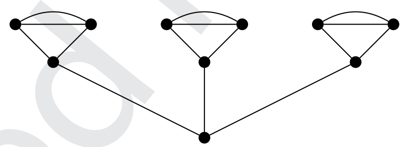

Figure 2: The graph S_{10}.

Figure 3: Subdividing e and attaching a copy of W to it.

As usual, a *circuit* in a graph G is a connected 2-regular subgraph. If C is a circuit in G, we refer to $|E(C)|$ as the *length* of C. If C is a circuit of length three, then we will call it a *triangle*. Whereas, in this context, we use the term *cycle* in a broader sense: a subgraph in which every vertex has positive even degree. In the case of cubic graphs, this definition implies that a cycle is a union of circuits, and the length of a cycle is simply the sum of the lengths of its circuits. A *cycle cover* $\mathcal{C}$ of G is a list of cycles in G such that every edge of G belongs to at least one element of $\mathcal{C}$. The *length* of a cycle cover $\mathcal{C}$ of G is the sum of length of all cycles in $\mathcal{C}$. For a bridgeless graph G, $scc(G)$ denotes the length of a shortest cycle cover of G. We define an *even subgraph* of a graph G as a spanning subgraph in which every vertex has even degree. Under this terminology, even subgraphs may include isolated vertices (i.e., vertices of degree zero), in contrast to cycles, which by definition do not. The complement of an even subgraph is referred to as a *parity subgraph* of G.

In this paper we also need to consider the Petersen Coloring Conjecture of Jaeger and its classical consequences. This conjecture asserts that for every bridgeless cubic graph G its edge-set $E(G)$ can be colored by using as set of colors $E(P_{10})$, where P_{10} is the Petersen graph (Figure 4), such that adjacent edges of G receive as colors adjacent edges of P_{10}.

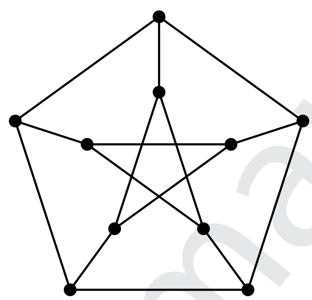

Figure 4: The Petersen graph P_{10}.

Here, we introduce such a conjecture in a more formal way. Let G and H be two cubic graphs. If there is a mapping $\phi\colon E(G) \to E(H)$, such that for each $v \in V(G)$ there is $w \in V(H)$ such that $\phi(\partial_G(v)) = \partial_H(w)$, then ϕ is called an *H-coloring* of G. Here $\partial_G(u)$ denotes the set of edges of G incident to the vertex u. If G admits an H-coloring, then we will write $H \prec G$. It can be easily seen that if $H \prec G$ and $K \prec H$, then $K \prec G$. In other words, $\prec$ is a transitive relation defined on the set of cubic graphs.

Example 2.1. If G is the complete bipartite graph $K_{3,3}$ and H is the complete graph K_4, then Figure 5 shows an example of an H-coloring of G. Here $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Figure 5 shows the colors of edges of G with the edges of H.

Conjecture 2.2 (Jaeger, 1988 [9, 10, 11])**.** *For any bridgeless cubic graph G, one has $P_{10} \prec G$.*

The conjecture is well-known in graph theory. It is considered hard to prove since it implies some other classical conjectures in the field such as Berge-Fulkerson Conjecture (Conjecture 2.3 below), Cycle Double Cover Conjecture, 5-Cycle Double Cover Conjecture (Conjecture 2.4 below) and the Shortest Cycle Cover Conjecture (Conjecture 2.5 below) (see [5, 10, 27]).

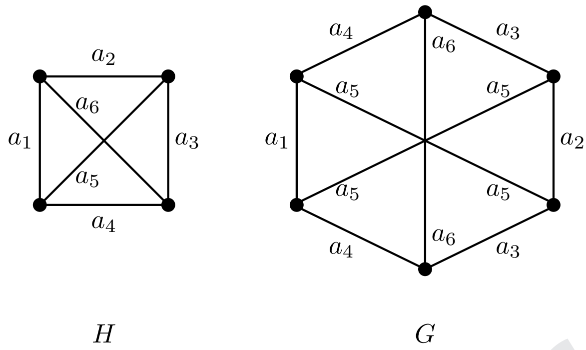

Figure 5: An example of an H-coloring of G.

Conjecture 2.3 (Berge-Fulkerson, 1972 [5, 22]). *Any bridgeless cubic graph G contains six (not necessarily distinct) perfect matchings $F_1, \dots, F_6$ such that any edge of G belongs to exactly two of them.*

Conjecture 2.4 ([1, 20, 5-Cycle Double Cover Conjecture]). *Any bridgeless graph G contains five cycles such that any edge of G belongs to exactly two of them.*

Conjecture 2.5 ([12, 27, 28, Shortest Cycle Cover Conjecture]). *Let G be a bridgeless graph. Then, $scc(G) \leq \frac{7}{5}|E(G)|$ holds.*

More conjectures similar to Conjecture 2.2 can be found in [6, 17]. Recent results about H-colorings can be found in [14] and [16], where the graphs under consideration are regular, but not necessarily cubic.

We conclude this section with a list of some results that would be used later in the paper.

Proposition 2.6. *Let T be a tree in which every vertex is of degree 1 or 3. Assume T has n vertices. Let k_1 and k_3 be the number of vertices with degrees 1 and 3, respectively. Then, $k_1 = \frac{n}{2} + 1$ and $k_3 = \frac{n}{2} - 1$ hold.*

The trees discussed above have played an important role in [18, 19]. A classical theorem by Petersen can be stated as follows.

Theorem 2.7 (Petersen, see [21] or Section 3.4 of [13]). *(a) Let G be a bridgeless cubic graph. Then $G - e_1 - e_2$ has a perfect matching for every $e_1, e_2 \in E(G)$;*

(b) Let G be a cubic graph with at most two bridges. Then G has a perfect matching.

Corollary 2.8. *Let G be a bridgeless graph in which all degrees of vertices are three, except one which is of degree two. Then G contains a 2-factor. In particular, every end-block of a cubic graph contains a 2-factor.*

Theorem 2.9 ([25]). *Let G be a bridgeless cubic graph. The edge-set of G can be covered with four perfect matchings if and only if*

$$scc(G) = \frac{4}{3}|E(G)|.$$

Theorem 2.10 ([8, 25]). *Let G be a bridgeless graph. Then G admits a cover with five cycles such that each edge is covered twice, if and only if G can be covered with four parity subgraphs such that every edge is covered at most twice.*

3 Expanding vertices to reduce the perfect matching index

In this section, we focus on bridgeless cubic graphs. Let G be a cubic graph and let $U \subseteq V(G)$. The cubic graph obtained from G by expanding the vertices of U to triangles will be denoted by G_U. For a bridgeless cubic graph G, let $T(G)$ be the size of a smallest set U such that the edge-set of the resulting cubic graph G_U can be covered with four perfect matchings. In this section we obtain some bounds on $T(G)$ and we link its value to some well-known conjectures. The following remark is an immediate consequence of the definition of a parity subgraph.

Remark 3.1. Let G be a cubic graph, J be a parity subgraph of G and U be the set of degree-three vertices in J. Then J is a perfect matching in G_U.

A priori it is unclear why starting with any bridgeless cubic graph and expanding some vertices to triangles, we will obtain a bridgeless cubic graph that can be covered with four perfect matchings. However, we proved that 5-Cycle Double Cover Conjecture is equivalent to the following statement.

Conjecture 3.2 ([6]). *Any claw-free bridgeless cubic graph can be covered with four perfect matchings.*

See [6] for the proof of this equivalence. The proof given there relies on Theorem 2.10 from [8]. This means that if 5-Cycle Double Cover Conjecture holds true then $T(G)$ is well-defined and $T(G) \leq |V(G)|$.

By definition, $T(G) = 0$ if and only if the graph G itself admits a cover of the edge-set with at most four perfect matchings. In particular, if G admits a 3-edge-coloring then $T(G) = 0$.

3.1 Relations between $T(G)$ and $scc(G)$

Theorem 2.9 suggests a link between $scc(G)$, the length of a shortest cycle cover of G and the parameter $T(G)$. In what follows, we aim to strengthen this connection. In it, we assume that 5-Cycle Double Cover Conjecture (see Conjecture 2.4) is true, hence $T(G)$ is well defined.

Proposition 3.3. *Let G be a bridgeless cubic graph. Then*

$$scc(G) \leq \frac{4}{3}|E(G)| + T(G).$$

Proof. Let G be any bridgeless cubic graph. If we replace one vertex of G with a triangle, then for the resulting cubic graph G' we will have

$$scc(G') \geq scc(G) + 3.$$

Now, if U is a smallest subset of $V(G)$, such that G_U can be covered with four perfect matchings, then we will have

$$scc(G_U) \geq scc(G) + 3|U| = scc(G) + 3T(G).$$

By Theorem 2.9, we have

$$scc(G_U) \leq \frac{4}{3}|E(G_U)|,$$

hence

$$\begin{aligned} scc(G) \leq scc(G_U) - 3T(G) &\leq \frac{4}{3}|E(G_U)| - 3T(G) \\ &= \frac{4}{3}|E(G)| + 4T(G) - 3T(G) = \frac{4}{3}|E(G)| + T(G). \end{aligned}$$

The proof is complete. □

We are unable to prove that, in the previous statement, the inequality can be replaced by an equality. However, we suspect this may be the case, at least under the additional assumption that G is 3-edge-connected, and we propose it as a conjecture.

Conjecture 3.4. *Let G be a 3-edge-connected cubic graph. Then*

$$scc(G) = \frac{4}{3}|E(G)| + T(G).$$

To support our conjecture, we prove that it follows from another well-known conjecture by C.-Q. Zhang (see [27]). To state this conjecture, we first define the *depth* of an edge e in a cycle cover as the number of cycles containing e. Accordingly, we define the *depth* of a cycle cover as the maximum depth among all edges of G.

Conjecture 3.5 ([27, Conjecture 8.11.6])**.** *Let G be a 3-edge-connected graph. Then, G admits a shortest cycle cover of depth* 2*.*

Proposition 3.6. Conjecture 3.5 *implies* Conjecture 3.4*.*

Proof. By Proposition 3.3, it suffices to prove that for any 3-edge-connected cubic graph G, we have

$$scc(G) \geq \frac{4}{3}|E(G)| + T(G),$$

or equivalently,

$$T(G) \leq scc(G) - \frac{4}{3}|E(G)|.$$

Consider a shortest cycle cover $\mathcal{C}$ of depth 2 in G, which means that each edge of G belongs to one or two cycles in $\mathcal{C}$. The vertex-set of G is partitioned into two subsets: the set U of vertices incident to three edges of depth 2 and the set of vertices incident to exactly one edge of depth 2. Clearly, the number of edges covered twice in $\mathcal{C}$ is equal to $\frac{3|U|+|V(G)\setminus U|}{2}$ and therefore:

$$scc(G) = |E(G)| + \frac{3|U| + (|V(G) \setminus U|)}{2} = \frac{4}{3}|E(G)| + |U|.$$

Next, we expand every vertex u in U into a triangle, thus obtaining a bridgeless cubic graph G_U. We then extend every cycle of $\mathcal{C}$ passing through u to a cycle in G_U by adding exactly one edge of the corresponding triangle. The resulting set of cycles forms a cycle cover of G_U with length $\frac{4}{3}|E(G_U)|$, since the set of edges covered twice forms a perfect matching of G_U, while every other edge is covered once. Since G_U admits a cycle cover of length

$\frac{4}{3}|E(G_U)|$, it follows from Theorem 2.9 that its edge-set can be covered with four perfect matchings. Hence,

$$T(G) \leq |U| = scc(G) - \frac{4}{3}|E(G)|.$$

The proof is complete. □

We leave as an open problem determining whether there exists a bridgeless cubic graph such that $scc(G) < \frac{4}{3}|E(G)| + T(G)$.

3.2 Some upper bounds for $T(G)$

We have already observed that 5-Cycle Double Cover Conjecture implies $T(G) \leq |V(G)|$ for bridgeless cubic graphs. In this section, we propose some stronger upper bounds for the parameter $T(G)$ in terms of the order of the graph.

Theorem 3.7. *If* Conjecture 2.4 *holds true, then* $T(G) \leq \frac{2}{5}|V(G)|$ *for every bridgeless cubic graph* G.

Proof. We follow the approach of [8] and [25]. Let G be a bridgeless cubic graph. Let $\{C_0, ..., C_4\}$ be a 5-CDC of G. Observe that

$$|C_0| + ... + |C_4| = 2|E|,$$

where $|C_j|$ is the length of C_j. We can assume that

$$|C_0| \geq \frac{2}{5}|E|.$$

Consider the set

$$\mathcal{C}_0 = \{C_0 \triangle C_1, ..., C_0 \triangle C_4\}.$$

Note that all elements in it are even subgraphs of G. As in [8], we have if $e \in C_0$, then $\mathcal{C}_0$ covers e three times, and if $e \notin C_0$, then $\mathcal{C}_0$ covers e two times. Let

$$\mathcal{J}_0 = \{\overline{C_0 \triangle C_1}, ..., \overline{C_0 \triangle C_4}\}$$

be the set of complements of the subgraphs $C_0 \triangle C_1, ..., C_0 \triangle C_4$. Observe that all of them are parity subgraphs. Moreover, if $e \in C_0$, then $\mathcal{J}_0$ covers e once, and if $e \notin C_0$, then $\mathcal{J}_0$ covers e twice. Observe that if an edge e is covered once in $\mathcal{J}_0$, then no parity subgraph of $\mathcal{J}_0$ has degree three on endpoints of e. This means that the number of vertices of G which have degree three in one of parity subgraphs of $\mathcal{J}_0$ is at most

$$|V| - |C_0| \leq |V| - \frac{2}{5}|E| = |V| - \frac{3}{5}|V| = \frac{2}{5}|V|.$$

By Remark 3.1 we have

$$T(G) \leq |V| - |C_0| \leq \frac{2}{5}|V|.$$

The proof is complete. □

One may wonder what could be the best upper bound for $T(G)$. The following conjecture tries to answer this question.

Conjecture 3.8. *Let G be a bridgeless cubic graph. Then*

$$T(G) \leq \frac{|V(G)|}{10}.$$

Note that the upper bound in Conjecture 3.8 is going to be tight: indeed, there exist infinitely many cubic graphs reaching such a bound (see for instance [15] for more details), the smallest of them is the Petersen graph (Figure 4), where $|V| = 10$ and $T(G) = 1$ since P_{12} can be covered with four perfect matchings (Figure 6).

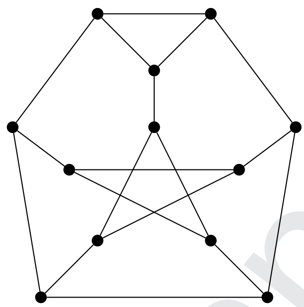

Figure 6: The graph P_{12}.

Also, observe that if Conjecture 3.8 is true, then by Proposition 3.3 for any bridgeless cubic graph G we will have

$$scc(G) \leq \frac{4}{3}|E(G)| + T(G) \leq \frac{4}{3}|E(G)| + \frac{|V(G)|}{10} = \frac{7}{5}|E(G)|,$$

which matches $\frac{7}{5}|E|$-bound from Conjecture 2.5 for cubic graphs. Thus, if we can show that Conjecture 3.8 follows from Conjecture 2.4, then it would mean that Conjecture 2.4 implies the restriction of Conjecture 2.5 to cubic graphs, something that was not previously known.

Unfortunately, we are not able to derive Conjecture 3.8 directly from Conjecture 2.4. However, this becomes possible if we start from a stronger version of the 5-CDC Conjecture. First of all, note that Conjecture 2.4 can be reformulated in the following equivalent way.

Conjecture 3.9. *Let G be any bridgeless graph. Then G admits a 5-CDC $\mathcal{C} = (E_0, ..., E_4)$ such that $|E_0| \geq \frac{2}{5}|E(G)|$.*

We offer the following seemingly stronger version of the 5-Cycle Double Cover Cover Conjecture.

Conjecture 3.10. *Let G be any bridgeless graph. Then G admits a 5-CDC $\mathcal{C} = (E_0, ..., E_4)$ such that $|E_0| \geq \frac{3}{5}|E(G)|$.*

Now, we show that this strengthened version of Conjecture 2.4 is enough to derive Conjecture 3.8.

Theorem 3.11. Conjecture 3.10 *implies* Conjecture 3.8.

Proof. Let G be a bridgeless cubic graph. Let $(C_0, ..., C_4)$ be a 5-CDC of G with $|C_0| \geq \frac{3|E|}{5}$. Consider the set

$$\mathcal{C}_0 = \{C_0 \triangle C_1, ..., C_0 \triangle C_4\}.$$

Note that every element in it is an even subgraph of G. As in [8], we have if $e \in C_0$, then $\mathcal{C}_0$ covers e three times, and if $e \notin C_0$, then $\mathcal{C}_0$ covers e two times. Let

$$\mathcal{J}_0 = \{\overline{C_0 \triangle C_1}, ..., \overline{C_0 \triangle C_4}\}$$

be the set of complements of the subgraphs $C_0 \triangle C_1, ..., C_0 \triangle C_4$. Observe that all of them are parity subgraphs. Moreover, if $e \in C_0$, then $\mathcal{J}_0$ covers e once, and if $e \notin C_0$, then $\mathcal{J}_0$ covers e twice. If an edge e is covered once in $\mathcal{J}_0$, then no parity subgraph of $\mathcal{J}_0$ has degree three on endpoints of e. This means that the number of vertices of G which have degree three in one of parity subgraphs of $\mathcal{J}_0$ is at most

$$|V| - |C_0| \leq |V| - \frac{3}{5}|E| = |V| - \frac{9}{10}|V| = \frac{1}{10}|V|.$$

Thus, by Remark 3.1,

$$T(G) \leq |V| - |C_0| \leq \frac{1}{10}|V|.$$

The proof is complete. □

One may wonder how realistic is Conjecture 3.10: it is not difficult to show that it is a consequence of Petersen Coloring Conjecture.

Theorem 3.12. *Petersen Coloring Conjecture implies the restriction of* Conjecture 3.10 *to cubic graphs.*

Proof. Assume that G is a bridgeless cubic graph. Then by Petersen Coloring Conjecture, G admits a Petersen coloring f. Since P_{10} has ten vertices, one of its vertices, say z, is an image of at most $\frac{|V|}{10}$ vertices under f. Let C be a 9-cycle of the Petersen graph that does not pass through z. P_{10} admits a 5-CDC such that one of the even subgraphs in it is C. Observe that by definition of z, at least $\frac{9|V|}{10}$ vertices of G map to vertices of C under f. Thus, $f^{-1}(C)$ is an even subgraph of G such that it is part of a 5-CDC of G and its size is at least $\frac{9|V|}{10} = \frac{3}{5}|E|$. The proof is complete. □

Corollary 3.13. *Petersen Coloring Conjecture implies* Conjecture 3.8.

Proof. Let G be a bridgeless cubic graph. By Theorem 3.12, we have that Petersen Coloring Conjecture implies the existence of a 5-CDC in G, such that one of the even subgraphs in it has at least $\frac{3}{5} \cdot |E|$ edges. As we have shown in Theorem 3.11, this implies that $T(G) \leq \frac{|V|}{10}$. The proof is complete. □

We have derived the restriction of Conjecture 3.10 to cubic graphs as a consequence of Conjecture 2.2. The authors suspect that one should be able to derive Conjecture 3.10 as a consequence of Conjecture 3.9. In other words, they would like to offer:

Conjecture 3.14. Conjecture 3.10 *is equivalent to* Conjecture 3.9.

4 Expanding vertices to obtain a graph with a perfect matching

In this section, we consider cubic graphs that may contain bridges. For a cubic graph G let $t(G)$ be the size of a smallest subset U of $V(G)$, such that G_U has a perfect matching. Note that $t(G)$ is well-defined and $t(G) \leq |V|$, since if we replace all vertices of G with a triangle then the set of edges not included in the introduced triangles forms a perfect matching in G_V.

We start by obtaining a Gallai type equality for $t(G)$. If G is an arbitrary cubic graph, then let $\ell(G)$ be the number of edges in a cycle of G having maximum length.

Theorem 4.1. *Let G be a cubic graph. Then,*

$$|V(G)| = t(G) + \ell(G).$$

Proof. We use an argument similar to the one used to prove standard Gallai equalities, see [7]. Let C be a longest cycle of G. Let us expand the vertices of G in $V(G) \setminus V(C)$ to triangles. Then C together with these new triangles will form a 2-factor in G. The complement of this 2-factor will be a perfect matching in the expanded graph. Thus:

$$t(G) \leq |V(G)| - |V(C)| = |V(G)| - \ell(G),$$

or

$$\ell(G) + t(G) \leq |V(G)|.$$

For the proof of the converse inequality, let $U \subseteq V(G)$ be a smallest subset of vertices whose expansion to triangles leads to a cubic graph G_U with a perfect matching. Since G_U is cubic, this is equivalent to G_U having a 2-factor. Since U is minimum, hence $|U| = t(G)$ we have these new triangles will be a part of any 2-factor of the cubic graph G_U. Let C be a cycle of G resulting from the 2-factor by removing the new triangles. Then it contains

$$|V(G)| - t(G)$$

vertices and that many edges. Hence,

$$\ell(G) + t(G) \geq |V(G)|.$$

The proof is complete. □

Next, we prove a lemma that will be used later in order to obtain the main result of this section.

Lemma 4.2. *Let G be a bridgeless cubic graph and let $E_0 \subseteq E(G)$. Consider the cubic graph H obtained from G by subdividing every edge $e \in E_0$ and adding a copy of W to it. Then,*

$$t(H) \leq \min_M |E_0 \cap M|$$

where M is a perfect matching of G.

Proof. Let M be a perfect matching of G (see Theorem 2.7) such that $|E_0 \cap M|$ is minimum. Let U be the subset of $V(H)$ consisting of all roots of a copy of W which subdivide an edge of G in $E_0 \cap M$. It suffices to prove the existence of a perfect matching in H_U. We construct a perfect matching N of H_U arising from the perfect matching M of G as follows: if an edge $e \notin E_0$, then the corresponding edge in H_U belongs to N only if e belongs to M. If an edge $e = xy \in E_0$, then the perfect matching N of H_U contains one of the two parallel edges of the copy of W corresponding to e, and the bridge joining such a copy of W to the rest of the graph H_U: moreover, if $xy \in M$ then N contains also the two edges incident with x and y and with the triangle obtained by the expansion of the root of W in H. The assertion follows. □

Next lemma is a direct consequence of previous one and the observation that for every $h, g \in E_0$, $G - h - g$ has a perfect matching ((a) of Theorem 2.7).

Lemma 4.3. *Let G be a bridgeless cubic graph and let $E_0 \subseteq E(G)$ consisting of at least two edges. Consider the cubic graph H obtained from G by subdividing every edge $e \in E_0$ and adding a copy of W to it. Then,*

$$t(H) \leq |E_0| - 2.$$

In our next theorem, we obtain asymptotically tight upper bounds for $t(G)$ in the classes of cubic graphs and simple cubic graphs.

Theorem 4.4. *Let G be a cubic graph. Then, the following holds:*

(i) $t(G) < \frac{|V|}{4}$;

(ii) *if G is simple, then* $t(G) < \frac{|V|}{6}$;

Moreover, previous bounds are asymptotically tight.

Proof. For the proof of statement (i), assume that G is a counterexample of minimum order. Observe that $t(G) \geq \frac{1}{4}|V(G)|$ is equivalent to $\frac{\ell(G)}{|V(G)|} \leq \frac{3}{4}$ by Theorem 4.1. Clearly, G is connected. By Theorem 2.7, we can assume that G contains at least three bridges. We proceed by proving a series of intermediate statements that ultimately yield a contradiction, disproving the existence of such a counterexample.

Claim 4.5. *Every end-block of G is isomorphic to W.*

Proof. Suppose some end-block B has $h \geq 5$ vertices. Define a smaller cubic graph G' obtained from G by removing B from G and attaching a copy of W (Figure 7).

Figure 7: Obtaining the cubic graph G' from G.

We have:

$$|V(G')| = |V(G)| - (h - 3),$$

and

$$\ell(G') = \ell(G) - (h - 3).$$

The latter follows from observation that the end-block has a 2-factor (see Corollary 2.8), hence every maximum even subgraph covers all these h vertices. Since $h \geq 5$, we have $|V(G')| < |V(G)|$. Thus, by minimality of G, we have

$$\frac{\ell(G')}{|V(G')|} > \frac{3}{4},$$

therefore

$$\frac{\ell(G)}{|V(G)|} = \frac{\ell(G') + (h-3)}{|V(G')| + (h-3)} \geq \frac{\ell(G')}{|V(G')|} > \frac{3}{4}$$

contradicting that G is a counterexample. Here we also used the fact that $\ell(G') \leq |V(G')|$. The proof of Claim 4.5 is complete. □

Claim 4.6. *There is no vertex w of G, such that two end-blocks of G are joined to w with two bridges. In other words, different end-blocks have different roots.*

Proof. On the opposite assumption, assume that w is incident to vertices x and y such that x and y lie in end-blocks (Figure 8). By Claim 4.5, these end-blocks have three vertices. Consider the cubic graph H obtained from G by removing these two end-blocks and attaching a copy of W to w (Figure 8).

Figure 8: Obtaining the cubic graph H from G.

We have

$$|V(G)| = |V(H)| + 4,$$

and

$$\ell(G) = \ell(H) + 3.$$

Since $|V(H)| < |V(G)|$, by minimality of G, we have

$$\frac{\ell(H)}{|V(H)|} > \frac{3}{4}.$$

Hence,

$$\frac{\ell(G)}{|V(G)|} = \frac{\ell(H) + 3}{|V(H)| + 4} > \frac{3}{4},$$

which contradicts our choice of G as a counterexample. The proof of Claim 4.6 is complete. □

Our next claim states that roots of end-blocks form an independent set.

Claim 4.7. *There are no two roots w_1 and w_2 of end-blocks in G, such that $w_1 w_2 \in E(G)$.*

Proof. Suppose two roots w_1 and w_2 are adjacent in G (Figure 9). First of all, let us observe that w_1w_2 cannot be a double edge since G has more than two bridges. Moreover, by previous claim there are no two end-blocks of G joined in w_2. Then, we denote by w_3 the neighbour of w_2 distinct from w_1 and such that w_2w_3 is not the bridge in an end-block.

Consider the cubic graph H obtained from G by removing the end-block having w_2 as a root and adding a new edge w_1w_3 (Figure 9). Note that w_1w_3 could be a double edge in H.

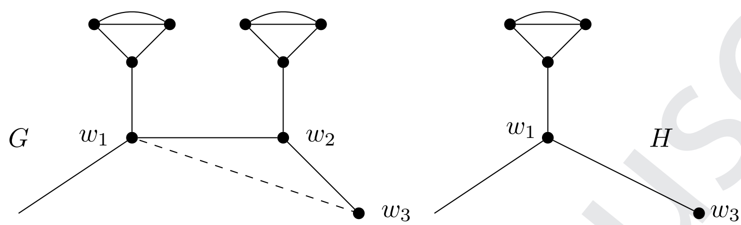

Figure 9: Obtaining the cubic graph H from G.

We have:

$$|V(G)| = |V(H)| + 4,$$

and

$$\ell(G) \geq \ell(H) + 3.$$

Since $|V(H)| < |V(G)|$, we have

$$\frac{\ell(H)}{|V(H)|} > \frac{3}{4}$$

by minimality of G. Hence

$$\frac{\ell(G)}{|V(G)|} \geq \frac{\ell(H)+3}{|V(H)|+4} > \frac{3}{4},$$

which contradicts our choice of G as a counterexample. The proof of Claim 4.7 is complete. □

The proved claims allow us to view our counterexample G as one obtained from a cubic graph G_0 by taking a subset $E_0 \subseteq E(G_0)$, subdividing edges of E_0 once and attaching a copy of W to them.

Let us say that a bridge e of G is trivial if

$$\min\{|V(G_1)|, |V(G_2)|\} = 3,$$

where G_1 and G_2 denote the components of $G - e$.

Claim 4.8. *G contains a non-trivial bridge.*

Proof. Assume that all bridges in G are trivial. Then the graph G_0 discussed above is a bridgeless cubic graph. By Lemma 4.2,

$$t(G) \leq |E_0| \leq \frac{|V(G)| - |V(G_0)|}{4} < \frac{|V(G)|}{4},$$

since for every edge $e \in E_0$ we have four vertices in G. This contradicts the assumption that G is a counterexample. The proof of Claim 4.8 is complete. □

We are ready to complete the proof of statement (i). Let f be a non-trivial bridge in G. Let G_1 and G_2 be the components of $G - f$. Consider two cubic graphs H_1 and H_2 obtained from G_1 and G_2, respectively, by attaching copies of W to the end-vertices of f (Figure 10).

Figure 10: Obtaining the cubic graphs H_1 and H_2 from G.

Since f is non-trivial, we have $|V(H_1)|, |V(H_2)| < |V(G)|$. Note that we can always choose the non-trivial bridge f such that

(a) H_1 contains at least two end-blocks,

(b) all bridges in H_1 are trivial.

We have:

$$|V(H_1)| = |V(G_1)| + 3, |V(H_2)| = |V(G_2)| + 3,$$

and since f is a bridge

$$\ell(H_1) = \ell(G_1) + 3, \ell(H_2) = \ell(G_2) + 3.$$

Since $|V(H_2)| < |V(G)|$, we have

$$\ell(H_2) > \frac{3}{4}|V(H_2)|,$$

hence

$$\ell(G_2) + 3 = \ell(H_2) > \frac{3}{4}|V(H_2)| = \frac{3}{4}\left(|V(G_2)| + 3\right)$$

by minimality of G. This is equivalent to

$$\ell(G_2) \geq \frac{3}{4}|V(G_2)| - \frac{3}{4}. \tag{4.1}$$

On the other hand, since in H_1 all bridges are trivial, by Lemma 4.3 and minimality of G

$$t(H_1) \leq |E_0(H_1)| - 2 \leq \frac{|V(H_1)|}{4} - 2 < \frac{|V(H_1)|}{4} - \frac{3}{2}.$$

By Theorem 4.1, the latter means

$$\ell(H_1) > \frac{3|V(H_1)|}{4} + \frac{3}{2}.$$

Since $\ell(H_1) = \ell(G_1) + 3$ and $|V(H_1)| = |V(G_1)| + 3$, the last inequality is equivalent to

$$\ell(G_1) > \frac{3}{4}|V(G_1)| + \frac{3}{4}. \tag{4.2}$$

(4.1) and (4.2) together imply

$$\ell(G) = \ell(G_1) + \ell(G_2) > \frac{3}{4}|V(G_1)| + \frac{3}{4} + \frac{3}{4}|V(G_2)| - \frac{3}{4} = \frac{3}{4}|V(G)|$$

which contradicts our assumption that G is a counterexample. In the last equation, we used equalities

$$|V(G)| = |V(G_1)| + |V(G_2)|,$$

and

$$\ell(G) = \ell(G_1) + \ell(G_2).$$

The latter follows from our choice of f as a bridge. Proof of (i) is complete.

Now, we prove that the bound in (i) is asymptotically tight. Let T be a tree on n vertices in which every vertex is of degree three or one. Let k_1 be the number of vertices with degree one, and let k_3 be the number of vertices of degree three. Attach a copy of W to every vertex of degree one in T so that we get a cubic graph G (if $T = K_{1,3}$ is the claw, then $G = S_{10}$, see Figure 2).

By Proposition 2.6,

$$|V(G)| = k_3 + 3k_1 = 3k_1 + (k_1 - 2) = 4k_1 - 2 = 4\left(\frac{n}{2} + 1\right) - 2 = 2n + 2.$$

Since T is a tree, all its edges are bridges in G. Thus, these edges cannot lie on a cycle, hence on even subgraphs of G. Thus, in order to get an even subgraph, we can take just triangles in copies of W attached to degree one vertices. Thus:

$$\ell(G) = 3k_1.$$

Hence, by Theorem 4.1

$$t(G) = k_1 - 2 = \left(\frac{n}{2} + 1\right) - 2 = \frac{n}{2} - 1.$$

Thus:

$$\lim_{n\to+\infty} \frac{t(G)}{|V(G)|} = \lim_{n\to+\infty} \frac{\frac{n}{2} - 1}{2n + 2} = \lim_{n\to+\infty} \frac{n-2}{4n+4} = \frac{1}{4}.$$

The proof of point (ii) proceeds in exactly the same way as in point i), replacing W with W'. The only difference lies in Claim 6.3, which corresponds to Claim 4.7, where an additional argument is required to ensure that no parallel edges are created. For this reason, we defer the detailed proof of this point to the Appendix.

In order to prove that the bound in (ii) is the best possible, again, we start with a tree T in which all degrees are either one or three. Then we attach to every vertex of degree one a copy of W'. For example, if $T = K_{1,3}$ is the claw, then we get the cubic graph from Figure 11.

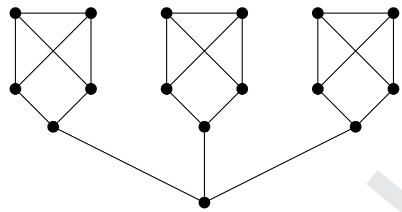

Figure 11: The graph S_{16}.

We will have:

$$|V(G)| = k_1 - 2 + 5k_1 = 6k_1 - 2 = 6\left(\frac{n}{2} + 1\right) - 2 = 3n + 4,$$

and

$$\ell(G) = 5k_1.$$

Thus, by Theorem 4.1,

$$t(G) = k_1 - 2 = \frac{n}{2} - 1,$$

and

$$\lim_{n\to+\infty} \frac{t(G)}{|V(G)|} = \lim_{n\to+\infty} \frac{\frac{n}{2} - 1}{3n+4} = \lim_{n\to+\infty} \frac{n-2}{6n+8} = \frac{1}{6}.$$

□

5 Computational complexity of introduced parameters

In this section, we discuss the computational complexity of computing our parameters when on the input one is given a cubic graph. We start with the discussion of $t(G)$.

If J is a parity subgraph of a cubic graph G, then let $V_1(J)$ and $V_3(J)$ be the sets of vertices of degree one and three in J, respectively. Note that by Remark 3.1 and Theorem 4.1,

$$t(G) = \min_J |V_3(J)|. \tag{5.1}$$

By counting the sum of degrees in J, we have

$$|V_1(J)| + 3|V_3(J)| = 2|E(J)|.$$

Taking into account that J is a spanning subgraph, we have

$$|V_1(J)| + |V_3(J)| = |V(G)|.$$

Thus,

$$2|E(J)| = |V_1(J)| + 3|V_3(J)| = |V(G)| + 2|V_3(J)|,$$

or

$$2|E(J)| = |V(G)| + 2|V_3(J)|. \tag{5.2}$$

The problem of computing $t(G)$, by (5.1) is equivalent to minimization of $|V_3(J)|$. The latter is equivalent to minimization of $|E(J)|$ by (5.2). Thus, we need to understand the computational complexity of finding a parity subgraph with minimum number of edges in a given cubic graph G. The latter is polynomial time solvable as it is stated on pages 233–234 of [13]. The authors of [13] refer to [2] as a source for the polynomial time solvability of the problem of finding a smallest parity subgraph in arbitrary graphs which may not be cubic. Let us note that in [13], the authors use the word "join" in order to refer to parity subgraphs.

Finally, let us turn to the parameter $T(G)$ that we defined for all bridgeless cubic graphs. Note that $T(G) = 0$ is equivalent to the statement that $E(G)$ can be covered with four perfect matchings. The problem of checking this property is NP-complete in the class of bridgeless cubic graphs as it is proved in [3]. Note that the problem remains NP-complete in the class of cyclically 4-edge-connected cubic graphs as [23] demonstrates. Recall that a connected cubic graph is cyclically 4-edge-connected if it does not contain 2-edge-cuts and all 3-edge-cuts in it are trivial.

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6 Appendix

For the proof of statement (ii) of Theorem 4.4, assume that G is a counterexample of minimum order. Observe that $t(G) \geq \frac{1}{6}|V(G)|$ is equivalent to $\frac{\ell(G)}{|V(G)|} \leq \frac{5}{6}$ by Theorem 4.1. Clearly, G is connected. By Theorem 2.7, we can assume that G contains at least three bridges. We proceed by proving a series of intermediate statements that ultimately yield a contradiction, disproving the existence of such a counterexample.

Claim 6.1. *Every end-block of G is isomorphic to W'.*

Proof. Suppose some end-block B has $h \geq 7$ vertices. Define a smaller simple cubic graph G' obtained from G by removing B from G and attaching a copy of W' (Figure 12).

Figure 12: Obtaining the cubic graph G' from G.

We have:

$$|V(G')| = |V(G)| - (h - 5),$$

and

$$\ell(G') = \ell(G) - (h - 5).$$

The latter follows from observation that the end-block has a 2-factor (see Corollary 2.8), hence every maximum even subgraph covers all these h vertices. Since $h \geq 7$, we have $|V(G')| < |V(G)|$. Thus, by minimality of G, we have

$$\frac{\ell(G')}{|V(G')|} > \frac{5}{6},$$

hence

$$\frac{\ell(G)}{|V(G)|} = \frac{\ell(G') + (h-5)}{|V(G')| + (h-5)} \geq \frac{\ell(G')}{|V(G')|} > \frac{5}{6}$$

contradicting that G is a counterexample. Here we also used the fact that $\ell(G') \leq |V(G')|$. The proof of Claim 6.1 is complete. □

Claim 6.2. *There is no vertex w of G, such that two end-blocks of G are joined to w with two bridges. In other words, different end-blocks have different roots.*

Proof. On the opposite assumption, assume that w is incident to vertices x and y such that x and y lie in end-blocks (Figure 13). By Claim 6.1, these end-blocks have five vertices. Consider the simple cubic graph H obtained from G by removing these two end-blocks and attaching a copy of W' to w (Figure 13).

We have

$$|V(G)| = |V(H)| + 6,$$

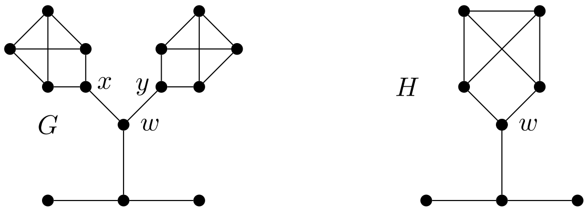

Figure 13: Obtaining the cubic graph H from G.

and

$$\ell(G) = \ell(H) + 5.$$

Since $|V(H)| < |V(G)|$, by minimality of G, we have

$$\frac{\ell(H)}{|V(H)|} > \frac{5}{6}.$$

Hence,

$$\frac{\ell(G)}{|V(G)|} = \frac{\ell(H)+5}{|V(H)|+6} > \frac{5}{6},$$

which contradicts our choice of G as a counterexample. The proof of Claim 6.2 is complete. □

Our next claim states that roots of end-blocks form an independent set.

Claim 6.3. *There are no two roots w_1 and w_2 of end-blocks in G, such that $w_1w_2 \in E(G)$.*

Proof. Suppose two roots w_1 and w_2 are adjacent in G (Figure 14). First of all, let us observe that w_1w_2 cannot be a double edge since G has more than two bridges. Moreover, by previous claim there are no two end-blocks of G joined in w_2. Then, we denote by w_3 the neighbour of w_2 distinct from w_1 and such that w_2w_3 is not the bridge in an end-block (Figure 14).

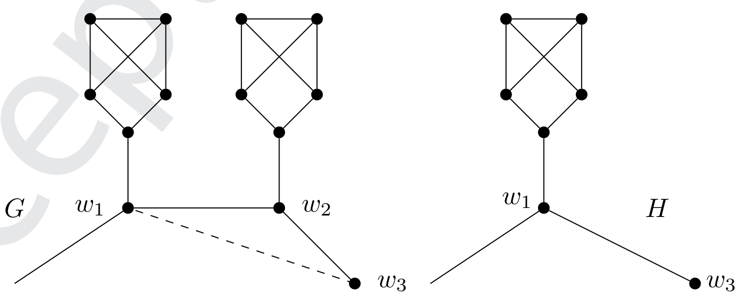

Figure 14: Obtaining the cubic graph H from G.

We consider two cases.

Case 1: $w_1w_3 \notin E(G)$. Consider the cubic graph H obtained from G by removing the end-block having w_2 as a root and adding a new edge w_1w_3 (Figure 14). Note that w_1w_3 is not a double edge in H. Hence, H is a simple cubic graph.

We have:

$$|V(G)| = |V(H)| + 6,$$

and

$$\ell(G) \geq \ell(H) + 5.$$

Since $|V(H)| < |V(G)|$, we have

$$\frac{\ell(H)}{|V(H)|} > \frac{5}{6}$$

by minimality of G. Hence

$$\frac{\ell(G)}{|V(G)|} \geq \frac{\ell(H)+5}{|V(H)|+6} > \frac{5}{6},$$

which contradicts our choice of G as a counterexample.

Case 2: $w_1w_3 \in E(G)$. Note that w_1, w_2, w_3 form a triangle K in G. Moreover, $H = G/K$ is a simple cubic graph with $|V(H)| < |V(G)|$. Thus,

$$\frac{\ell(H)}{|V(H)|} > \frac{5}{6}.$$

Since $|V(G)| = |V(H)| + 2$ and $\ell(G) = \ell(H) + 3$, we get

$$\frac{\ell(G)}{|V(G)|} = \frac{\ell(H)+3}{|V(H)|+2} > \frac{\ell(H)+2}{|V(H)|+2} \geq \frac{\ell(H)}{|V(H)|} > \frac{5}{6}.$$

We used the trivial inequality $\ell(H) \leq |V(H)|$ above. The proof of Claim 6.3 is complete. □

The proved claims allow us to view our counterexample G as one obtained from a cubic graph G_0 by taking a subset $E_0 \subseteq E(G_0)$, subdividing edges of E_0 once and attaching a copy of W' to them.

Let us say that a bridge e of G is trivial if

$$\min\{|V(G_1)|, |V(G_2)|\} = 5,$$

where G_1 and G_2 denote the components of $G - e$.

Claim 6.4. *G contains a non-trivial bridge.*

Proof. Assume that all bridges in G are trivial. Then the graph G_0 discussed above is a bridgeless cubic graph. By Lemma 4.2,

$$t(G) \leq |E_0| \leq \frac{|V(G)| - |V(G_0)|}{6} < \frac{|V(G)|}{6},$$

since for every edge $e \in E_0$ we have six vertices in G. This contradicts the assumption that G is a counterexample. The proof of Claim 6.4 is complete. □

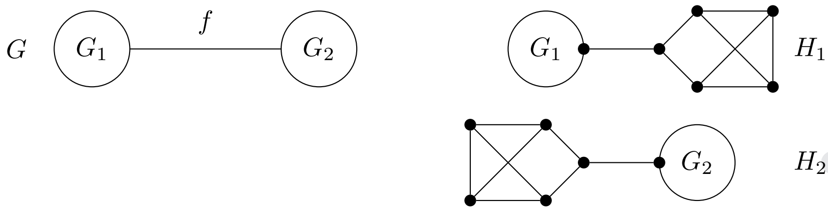

Figure 15: Obtaining the cubic graphs H_1 and H_2 from G.

We are ready to complete the proof of statement (ii). Let f be a non-trivial bridge in G. Let G_1 and G_2 be the components of $G - f$. Consider two simple cubic graphs H_1 and H_2 obtained from G_1 and G_2, respectively, by attaching copies of W' to the end-vertices of f (Figure 15).

Since f is non-trivial, we have $|V(H_1)|, |V(H_2)| < |V(G)|$. Note that we can always choose the non-trivial bridge f such that

(a) H_1 contains at least two end-blocks,

(b) all bridges in H_1 are trivial.

We have:

$$|V(H_1)| = |V(G_1)| + 5, |V(H_2)| = |V(G_2)| + 5,$$

and since f is a bridge

$$\ell(H_1) = \ell(G_1) + 5, \ell(H_2) = \ell(G_2) + 5.$$

Since $|V(H_2)| < |V(G)|$, we have

$$\ell(H_2) > \frac{5}{6}|V(H_2)|,$$

therefore

$$\ell(G_2) + 5 = \ell(H_2) > \frac{5}{6}|V(H_2)| = \frac{5}{6}(|V(G_2)| + 5)$$

This is equivalent to

$$\ell(G_2) > \frac{5}{6}|V(G_2)| - \frac{5}{6}. \tag{6.1}$$

On the other hand, since in H_1 all bridges are trivial, by Lemma 4.3

$$t(H_1) \leq |E_0(H_1)| - 2 \leq \frac{|V(H_1)|}{6} - 2 < \frac{|V(H_1)|}{6} - \frac{5}{3}.$$

By Theorem 4.1, the latter means

$$\ell(H_1) > \frac{5|V(H_1)|}{6} + \frac{5}{3}.$$

Since $\ell(H_1) = \ell(G_1) + 5$ and $|V(H_1)| = |V(G_1)| + 5$, the last inequality is equivalent to

$$\ell(G_1) > \frac{5}{6}|V(G_1)| + \frac{5}{6}. \tag{6.2}$$

(6.1) and (6.2) together imply

$$\ell(G) = \ell(G_1) + \ell(G_2) > \frac{5}{6}|V(G_1)| + \frac{5}{6} + \frac{5}{6}|V(G_2)| - \frac{5}{6} = \frac{5}{6}|V(G)|$$

which contradicts our assumption that G is a counterexample. In the last equation, we used equalities

$$|V(G)| = |V(G_1)| + |V(G_2)|,$$

and

$$\ell(G) = \ell(G_1) + \ell(G_2).$$

The latter follows from our choice of f as a bridge. Proof of (ii) is complete.