


Asymptotic analysis of thin structures with point-dependent energy growth

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In this paper, 3D–2D-dimensional reduction for hyperelastic thin films modeled through energies with point-dependent growth, assuming that the sample is clamped on the lateral boundary, is performed in the framework of Γ -convergence. Integral representation results, with a more regular Lagrangian related to the original energy density, are provided for the lower dimensional limiting energy, in different contexts.

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1. Introduction

In solid mechanics, the equilibrium state of a body can be described by an energy minimization problem, over a suitable class of fields, of an integral functional of

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the type

$$\int_{\Omega} f(x, \nabla v(x)) dx,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, v is the deformation or the displacement, defined in a suitable function space $X(\Omega; \mathbb{R}^d)$, e.g. $W_{loc}^{1,1}(\Omega; \mathbb{R}^d)$, and $f : \Omega \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$, the hyperelastic energy density, is assumed to be a Carathéodory function satisfying suitable growth conditions.

In many applications, for instance dealing with multiphase materials, fine heterogeneities, shape optimization, thin structures, phase transitions, etc., other parameters may come into play and the equilibrium configurations arise as solutions of the following ε -parametrized problems

$$\min_{v \in \mathbb{X}_{\varepsilon}(\Omega_{\varepsilon}; \mathbb{R}^d)} \mathcal{F}_{\varepsilon}(v) := \min_{v \in \mathbb{X}_{\varepsilon}(\Omega_{\varepsilon}; \mathbb{R}^d)} \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} f(\varepsilon)(x, \nabla v(x)) dx, \tag{1.1}$$

where $\{\Omega_{\varepsilon}\}$ may represent an ε -dependent family of bounded open subsets in \mathbb{R}^N , $\mathbb{X}_{\varepsilon}(\Omega_{\varepsilon}; \mathbb{R}^d)$ is a suitable functional space, possibly encoding the anisotropy of the model, expressed through a point-dependent integrability condition on the deformation gradients, and $f(\varepsilon) : \Omega_{\varepsilon} \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$. Hence, one is interested in determining, as $\varepsilon \rightarrow 0^+$, the asymptotic behavior of such configurations.

Two main types of questions are addressed in this paper, one dealing with the detection of sufficient conditions on $\mathbb{X}_{\varepsilon}(\Omega_{\varepsilon}; \mathbb{R}^d)$, that, under special structure conditions on $f(\varepsilon)$, ensure existence of solutions of the problem (1.1) in the case $N = d = 3$, when the set Ω_{ε} is a thin domain of the form

$$\Omega_{\varepsilon} := \omega \times (-\varepsilon/2, \varepsilon/2), \tag{1.2}$$

with $\omega \subset \mathbb{R}^2$ a bounded, connected, open set with Lipschitz boundary, the other one concerning the asymptotic behavior of problems (1.1), under more general assumptions on $f(\varepsilon)$. In the latter case, by means of a suitable variational convergence, we derive a limit problem defined on ω with a structure analogous to the original one, which admits solutions. This approach rigorously justifies the fact that, in the applications, when dealing with very thin structures, it is convenient to work with a lower dimensional model. We refer to the pioneering papers^{1, 45} where this approach has been formally applied with the use of Γ -convergence, in the hyperelastic setting for strings and membranes, within the classical functional setting of Sobolev spaces.

The main novelty of this manuscript is that every energy density $f(\varepsilon)$ depends on the parameter ε also through its growth which involves a so-called *variable growth exponent* $p_{\varepsilon} = p_{\varepsilon}(x)$ or better, a family of variable growth exponents converging to a fixed one in some suitable topology. This leads to nontrivial mathematical questions, because it appears to be a challenging idea to control the dependence of the parameter ε both in the set and in the energy density.

In this context, when $N = d = 3$ it is convenient to rephrase the problem (1.1) on a fixed domain

$$\Omega := \Omega_1 = \omega \times (-1/2, 1/2), \tag{1.3}$$

through a re-scaling in the transverse direction x_3 , thus obtaining the equivalent minimization problem

$$\min_{u \in X_\varepsilon(\Omega; \mathbb{R}^3)} \mathcal{F}_\varepsilon(u) := \min_{u \in X_\varepsilon(\Omega; \mathbb{R}^3)} \int_\Omega f_\varepsilon(x, \nabla u(x)) dx, \tag{1.4}$$

where f_ε , $X_\varepsilon(\Omega; \mathbb{R}^3)$, and u have been obtained by $f(\varepsilon)$, $\mathbb{X}_\varepsilon(\Omega_\varepsilon; \mathbb{R}^3)$ and v , respectively, via the change of variables which maps x_3 from $(-\varepsilon/2, \varepsilon/2)$ into $(-1/2, 1/2)$. In the first case, we consider $p_\varepsilon : \Omega \rightarrow (1, +\infty)$, $X_\varepsilon(\omega; \mathbb{R}^3) = W^{1, p_\varepsilon(\cdot)}(\Omega; \mathbb{R}^3)$ and $f_\varepsilon : \Omega \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ of the following type:

$$f_\varepsilon(x, \xi) := \mathcal{W}_1^{p_\varepsilon(x)}(x, \xi) \quad \text{or} \quad f_\varepsilon(x, \xi) := a_\varepsilon(x) \mathcal{W}_2^{p_\varepsilon(x)}(\xi),$$

with suitable functions \mathcal{W}_i , $i \in \{1, 2\}$ and a_ε (which are made precise in the sequel). In particular, we provide lower semicontinuity with respect to the (strong $L^\infty \times L^1$) joint convergence of the sequences $\{p_\varepsilon\} \subseteq L^\infty(\Omega; (1, +\infty))$ and $\{u_\varepsilon\} \subseteq W^{1,1}(\Omega; \mathbb{R}^d)$ (see Theorems 1.1 and 1.2).

In the second case, we deal with the very dimensional reduction problem where the function f_ε in (1.4), representing the hyperelastic energy density of a 3D thin structure, is of the form

$$f_\varepsilon(x, \xi) := f \left(x_\alpha, \varepsilon x_3, p_\varepsilon(x), \xi_\alpha, \frac{1}{\varepsilon} \xi_3 \right), \tag{1.5}$$

for every

$$x \equiv (x_\alpha, x_3) \in \Omega, \quad \xi \equiv (\xi_\alpha, \xi_3) \in \mathbb{R}^{3 \times 3}, \quad \alpha = 1, 2.$$

Here, p_ε plays the role of a variable exponent defined in Ω , i.e.

$$p_\varepsilon(x) \equiv p_\varepsilon(x_\alpha, x_3) := p(x_\alpha, \varepsilon x_3), \tag{1.6}$$

for \mathcal{L}^3 -a.e. $x \in \Omega$, and the re-scaled variable $(\xi_\alpha, \frac{1}{\varepsilon} \xi_3) \in \mathbb{R}^{3 \times 3}$ appears to take into account the scaling of the gradient $\nabla v = (\nabla_\alpha v, \nabla_3 v)$ after the change of variable which maps it in the scaled deformation gradient of u in (1.4), cf. Sec. 4 for precise definition.

Dimensional reduction problems of the type (1.1) have attracted much attention in the past decades due to the many applications in engineering, materials science, conductors, micromagnetics, chemistry and biology. Indeed, a wide literature has been developed to rigorously deduce a simplified formulation, in a lower dimensional setting via a variational approach: we refer to Refs. 2 and 35 for micro-magnetic and ferromagnetic materials,^{4, 13} for brittle materials,¹⁵ in the case of materials which allow for fracture and plastic behavior,^{20, 43} in the case of magnetoelasticity,^{30, 31, 33, 34, 36, 38, 40} for the case of hyperelastic multistructures,³² to model delamination,^{29, 50} to detect bending effect, in particular, in shells as in

Ref. 39, or Refs. 7, 12 and 11 to describe thin structures made by heterogeneous materials with fine distribution, also in the discrete setting. Besides the above far from exhaustive bibliography, it is worth to point out that particular minimum problems like the one in (1.4) appear also in the modeling of conductors to detect the dielectric breakdown, see e.g. Ref. 8, or in the context of damaging and optimal design (cf. in Refs. 12, 16 and 42). It is worth pointing out that it has been analyzed the case of multicomponent materials which exhibit a very different behavior from one point to another, and imposing a perimeter penalization among the interfaces of the components, see Refs. 9 and 10; however, at the best of our knowledge, it has been always considered the case where the energetic pointwise-dependent behavior has a rough change from one point to another of the sample. The case where the position-dependent energetic behavior is suitably continuous has not yet been considered. This is indeed one of the topics of this paper, in the pure hyperelastic setting, neglecting damage or shape optimization. Indeed, despite the last mentioned results, the regularity of the energetic point-dependent behavior allows us to obtain an explicit representation of the limiting problems in any dimensional reduction setting.

We aim now to present the main results of the paper. To this purpose, referring to Ref. 23 for a comprehensive treatment of the topic, we recall that a (Lebesgue) measurable function $p : \Omega \rightarrow [1, +\infty]$, playing the role of an exponent, is called a *variable exponent* and we denote by $\mathcal{P}(\Omega)$ the class of variable exponents on Ω . Moreover, $\mathcal{P}_b(\Omega)$ and $\mathcal{P}^{\log}(\Omega)$ are the subsets of $\mathcal{P}(\Omega)$ whose elements satisfy the *boundedness condition*

$$1 < p^- := \operatorname{ess\,inf}_{\Omega} p(x) \leq \operatorname{ess\,sup}_{\Omega} p(x) =: p^+ < +\infty, \quad (1.7)$$

and the so-called *log-Hölder continuity property*:

$$\exists \gamma > 0 : |p(x) - p(y)| \leq \frac{\gamma}{|\log|x - y||} \quad \text{for every } x, y \in \Omega, \quad 0 < |x - y| < \frac{1}{2}, \quad (1.8)$$

respectively. Finally, we set $\mathcal{P}_b^{\log}(\Omega) := \mathcal{P}^{\log}(\Omega) \cap \mathcal{P}_b(\Omega)$ (according to Definitions 2.1 and 2.5). Condition (1.8) has been first introduced by Zhikov in Ref. 51; the failure of this condition is a possible cause of discontinuity of minimizers, while, by assuming it, it is possible to prove higher integrability of minimizers, which is the first step towards further regularity. From that moment onwards, a large number of papers devoted to the mathematical analysis of energy functionals involving variable exponents appeared in several and different directions, motivated by the fact that such types of energies describe models (also nonvariational) coming from Mathematical Physics that are built using a variable growth exponent. For instance, we refer to the recent contribution,³ where this theory is employed in the study of elastic problems,^{21, 48} where the variable exponent growth condition is considered in the framework of free discontinuity problems, i.e. for materials allowing for fractures and damage,³⁷ where a variable exponent model for image restoration has been

studied, from the existence point of view and the Γ -convergence one, in the case that the exponent attains the critical value one,²⁷ where power-law approximation of supremal functionals has been studied once more by means of the Γ -convergence tool.

In order to state our main results for the first class of problems, we recall that a Borel measurable function $g : \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ is said quasiconvex if

$$g(\xi) \leq \int_Q g(\xi + \nabla\varphi(x)) \, dx$$

holds, for every $\xi \in \mathbb{R}^{d \times N}$ and for every $\varphi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, where $Q := (-1/2, 1/2)^N$ and $W_0^{1,\infty}(Q; \mathbb{R}^d) = W^{1,\infty}(Q; \mathbb{R}^d) \cap W_0^{1,1}(Q; \mathbb{R}^d)$ (we recall that a finite valued quasiconvex function is locally Lipschitz, hence, continuous and locally bounded; thus the class of test functions can be replaced by $C_c^\infty(Q; \mathbb{R}^d)$ or $C_0^1(Q; \mathbb{R}^d)$, as in Ref. 46). We note that the quasiconvexity appears naturally in the relaxation processes (see Refs. 46 and 49). Moreover, the quasiconvex setting covers the polyconvex case, since any polyconvex function is also quasiconvex (cf. Ref. 18).

We are in a position to state the following lower semicontinuity results. We stress that the L^∞ -convergence on the variable exponents is a natural assumption in this context. Indeed, such uniformly converging sequences as in Theorems 1.1 and 1.2 can arise by means of a change of variables as in (1.6) from a single exponent p satisfying a uniform log-Hölder continuity assumption (see the comment after Theorem 1.5).

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $\mathcal{W} : \Omega \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$ be a function such that*

- $\mathcal{W}(\cdot, \xi)$ is $\mathcal{L}^N(\Omega)$ -measurable for every $\xi \in \mathbb{R}^{d \times N}$;
- $\mathcal{W}(x, \cdot)$ is quasiconvex for \mathcal{L}^N -a.e. $x \in \Omega$;
- there exists $C_2 > 0$ such that

$$\mathcal{W}(x, \xi) \leq C_2(|\xi| + 1) \quad \text{for } \mathcal{L}^N \text{ - a.e. } x \in \Omega, \text{ for every } \xi \in \mathbb{R}^{d \times N}. \quad (1.9)$$

Let $p_0 \in L^\infty(\Omega)$ such that $p_0 \in \mathcal{P}_b^{1,\text{log}}(A)$ for every open $A \subset\subset \Omega$. Let $\{p_k\} \subseteq L^\infty(\Omega)$ and $\{v_k\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ such that $p_k \rightarrow p_0$ in $L^\infty(\Omega)$ and $v_k \rightarrow v$ in $L^1(\Omega, \mathbb{R}^d)$ as $k \rightarrow +\infty$. If $\{v_k\}$ satisfies

$$\sup_{k \in \mathbb{N}} \int_\Omega |\nabla v_k(x)|^{p_k(x)} \, dx < +\infty, \quad (1.10)$$

then

$$v \in W_{\text{loc}}^{1,p_0(\cdot)}(\Omega; \mathbb{R}^d), \quad |\nabla v| \in L^{p_0(\cdot)}(\Omega), \quad (1.11)$$

and

$$\int_{\Omega} \mathcal{W}^{p_0(x)}(x, \nabla v(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx.$$

Taking into account, Theorem 4.12 of Ref. 46, the previous lower semicontinuity theorem can be extended (essentially with the same proof) to functionals of the form

$$F(u, p) = \int_{\Omega} \mathcal{W}^{p(x)}(x, u(x), \nabla u(x)) dx,$$

when \mathcal{W} is a Carathéodory integrand, quasiconvex with respect to the gradient variable, replacing assumption (1.9) by the mild condition

$$\mathcal{W}(x, u, \xi) \leq C_2(|u| + |\xi| + 1) \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \text{ for every } (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}.$$

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $\mathcal{W} : \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$ be a quasiconvex function. Assume that there exists a constant $C_2 > 0$ such that*

$$\mathcal{W}(\xi) \leq C_2(|\xi| + 1) \quad \text{for every } \xi \in \mathbb{R}^{d \times N}. \tag{1.12}$$

Let $p_0, \{p_k\}, v$ and $\{v_k\}$ be as in Theorem 1.1. Let $a_0 \in L^\infty(\Omega)$ be such that $a_0 \geq 0$ \mathcal{L}^N -a.e. in Ω and assume that $a_k \rightarrow a_0$ in $L^\infty(\Omega)$ as $k \rightarrow +\infty$. Then, v satisfies (1.11) and

$$\int_{\Omega} a_0(x) \mathcal{W}^{p_0(x)}(\nabla v(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} a_k(x) \mathcal{W}^{p_k(x)}(\nabla v_k(x)) dx.$$

In Sec. 3, with the aim of proving Theorems 1.1 and 1.2, we consider either convex or quasiconvex energies. The first choice was made with the aim of deducing information on the limiting fields of weakly convergent sequences $\{v_k\}$ when $v_k \in W^{1,p_k(\cdot)}$, i.e. we have in mind the case of modular-type energies. Concerning the lower semicontinuity results, we underline that, when $\mathcal{W}(x, \cdot)$ is convex, we can drop the regularity assumption on p_0 , i.e. $p_0 \in \mathcal{P}_b^{\text{log}}(\Omega)$ (thanks to the celebrated lower semicontinuity result obtained by De Giorgi and Ioffe). Moreover, we note that assumption (1.10) can be dropped when the energy density \mathcal{W} satisfies a suitable coercivity assumption, see Remark 3.4(2).

The asymptotic analysis (via Γ -convergence) as $\varepsilon \rightarrow 0^+$ of the functionals appearing in (1.4), in the 3D–2D-dimensional reduction hyperelastic setting, besides our study can be carried out analogously for any nD – mD dimension reduction, ($m < n, m, n \in \mathbb{N}$). With this aim, here and in the sequel, we consider Ω_ε and Ω as in (1.2) and (1.3), respectively. The standard scaling argument in the x_3 direction introduced above, allows us to rephrase the problem in the fixed domain Ω , setting $v(x) \equiv v(x_\alpha, x_3) := u(x_\alpha, \varepsilon x_3)$, and taking p_ε as in (1.6). Moreover, assuming that the sample is clamped on its lateral boundary, we can make precise the functional space $X_\varepsilon(\Omega)$ appearing in (1.4). Indeed, denoting the lateral boundary of Ω by $\partial_L \Omega := \partial \omega \times (-1/2, 1/2)$, the functional space $X_\varepsilon(\Omega; \mathbb{R}^3)$ can be

specialized as

$$W_L^{1,p_\varepsilon(\cdot)}(\Omega; \mathbb{R}^3) := W_{0,\partial_L\Omega}^{1,p^-}(\Omega; \mathbb{R}^3) \cap W^{1,p_\varepsilon(\cdot)}(\Omega; \mathbb{R}^3),$$

where, according to Definition 2.3, $W_{0,\partial_L\Omega}^{1,p^-}(\Omega; \mathbb{R}^3)$ is the closure in $W^{1,p^-}(\Omega)$ of the subspace

$$\{v \in W_{\text{loc}}^{1,p^-}(\mathbb{R}^3; \mathbb{R}^3) : v(x) \equiv 0 \text{ in a neighborhood of } \partial_L\Omega\}.$$

Consequently, the functional $\mathcal{F}_\varepsilon : L^1(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ in (1.4) can be rewritten as

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_\Omega f\left(x_\alpha, \varepsilon x_3, p_\varepsilon(x), \nabla_\alpha u(x), \frac{1}{\varepsilon} \nabla_3 u(x)\right) dx & \text{if } u \in W_L^{1,p_\varepsilon(\cdot)}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3), \end{cases} \tag{1.13}$$

where the function f_ε , appearing in (1.4), is now defined by (1.5) and p_ε represents the growth condition of f_ε according to the re-scaling (1.6). In the following theorems, we aim to represent the Γ -limit of the family $\{\mathcal{F}_\varepsilon\}$ (with respect to the L^1 -convergence of the deformation fields u_ε) under a suitable convergence assumption of p_ε .

First, we consider the case when $f = f(x_\alpha, y, q, \xi)$ is a function defined on $\omega \times (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 3}$, convex in the gradient variable ξ and depending on the variable q (which assumes the value $p_\varepsilon(x)$).

The first result in the dimension reduction framework is the following.

Theorem 1.3. *Let $\omega \subset \mathbb{R}^2$ be a bounded, connected, open set with Lipschitz boundary. Let $f : \omega \times (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a function such that*

- $f(x_\alpha, \cdot, \cdot, \cdot)$ is continuous for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$;
- $f(\cdot, y, q, \xi)$ is measurable for every $(y, q, \xi) \in (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 3}$;
- $f(x_\alpha, y, q, \cdot)$ is convex for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ and for every $(y, q) \in (-1/2, 1/2) \times [1, +\infty)$.

Assume that there exist $0 < C_1 \leq C_2$ such that

$$C_1|\xi|^q - \frac{1}{C_1} \leq f(x_\alpha, y, q, \xi) \leq C_2(|\xi|^q + 1), \tag{1.14}$$

for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ and for every $(y, q, \xi) \in (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 3}$.

Let $p \in \mathcal{P}_b(\Omega)$ with $\Omega = \omega \times (-1/2, 1/2)$, let p_ε be as in (1.6) and let $\{\mathcal{F}_\varepsilon\}$ be the family of functionals in (1.13). If there exists $p_0 \in \mathcal{P}_b^{\text{log}}(\omega)$ such that $p_\varepsilon \rightarrow p_0$ in $L^1(\Omega)$ as $\varepsilon \rightarrow 0^+$, then

$$\Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon = \mathcal{F},$$

where $\mathcal{F} : L^1(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ is the functional given by

$$\mathcal{F}(u) := \begin{cases} \int_{\omega} f_0(x_{\alpha}, 0, p_0(x_{\alpha}), \nabla_{\alpha} u(x_{\alpha})) dx_{\alpha} & \text{if } u \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3), \end{cases} \tag{1.15}$$

with $f_0 : \omega \times (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ defined by

$$f_0(x_{\alpha}, y, q, \xi_{\alpha}) := \inf_{\xi_3 \in \mathbb{R}^3} f(x_{\alpha}, y, q, \xi_{\alpha}, \xi_3), \tag{1.16}$$

for \mathcal{L}^2 -a.e. $x_{\alpha} \in \omega$ and for every $(y, q, \xi_{\alpha}) \in (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 2}$.

In the following results, the convexity assumption on f with respect to the gradient variable is dropped, but a special structure is imposed on f in (1.5) and a stronger convergence assumption on $\{p_{\varepsilon}\}$ is required. More precisely, we neglect the dependence of f on the transverse variable and the dependence on the variable exponent p_{ε} is explicitly of power-law type. We denote this particular class of functionals as

$$\mathcal{I}_{\varepsilon}(u) := \begin{cases} \int_{\Omega} \mathcal{W}^{p_{\varepsilon}(x)} \left(x_{\alpha}, \nabla_{\alpha} u(x), \frac{1}{\varepsilon} \nabla_3 u(x) \right) dx & \text{if } u \in W_L^{1,p_{\varepsilon}(\cdot)}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3). \end{cases} \tag{1.17}$$

Theorem 1.4. *Let $\omega \subset \mathbb{R}^2$ be a bounded, connected, open set with Lipschitz boundary. Let $\mathcal{W} : \omega \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a function such that*

- $\mathcal{W}(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^{3 \times 3}$;
- $\mathcal{W}(x_{\alpha}, \cdot)$ is continuous for \mathcal{L}^2 -a.e. $x_{\alpha} \in \omega$;
- there exist $0 < C_1 \leq C_2$ such that

$$C_1 |\xi| - \frac{1}{C_1} \leq \mathcal{W}(x_{\alpha}, \xi) \leq C_2 (|\xi| + 1) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_{\alpha} \in \omega, \text{ for every } \xi \in \mathbb{R}^{3 \times 3}. \tag{1.18}$$

Assume that

$$|\mathcal{W}(x_{\alpha}, \xi) - \mathcal{W}(x_{\alpha}, \eta)| \leq w(x_{\alpha}, |\xi - \eta|) \quad \text{for every } \xi, \eta \in \mathbb{R}^{3 \times 3},$$

$$\text{for } \mathcal{L}^2\text{-a.e. } x_{\alpha} \in \omega, \tag{1.19}$$

where $w : \omega \times [0, +\infty) \rightarrow [0, +\infty)$ is such that

- $w(\cdot, t)$ is measurable for every $t \in [0, +\infty)$;
- $w(x_{\alpha}, \cdot)$ is a continuous, increasing function for \mathcal{L}^2 -a.e. $x_{\alpha} \in \omega$ satisfying $\lim_{t \rightarrow 0^+} w(x_{\alpha}, t) = 0$.

Let $p \in \mathcal{P}_b(\Omega)$ with $\Omega = \omega \times (-1/2, 1/2)$, let p_ε be as in (1.6) and let $\{\mathcal{I}_\varepsilon\}$ be the family of functionals in (1.17). If there exists $p_0 \in \mathcal{P}_b^{\log}(\omega)$ such that $p_\varepsilon \rightarrow p_0$ in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0^+$, then

$$\Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0^+} \mathcal{I}_\varepsilon = \mathcal{I},$$

where $\mathcal{I} : L^1(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ is the functional given by

$$\mathcal{I}(u) := \begin{cases} \int_\omega Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in \mathcal{W}_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3), \end{cases} \tag{1.20}$$

with $\mathcal{W}_0 : \omega \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ defined as

$$\mathcal{W}_0(x_\alpha, \xi_\alpha) := \inf_{\xi_3 \in \mathbb{R}^3} \mathcal{W}(x_\alpha, \xi_\alpha, \xi_3) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_\alpha \in \omega \text{ and for every } \xi_\alpha \in \mathbb{R}^{3 \times 2},$$

and $Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \cdot)$ denotes the quasiconvex envelope of $\mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \cdot)$, namely the greatest quasiconvex function below $\mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \cdot)$ for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$.

The third model considered in 3D–2D-dimensional reduction setting is linked to the class of functionals considered in Theorem 1.2, i.e. the function f in (1.5) still does not depend on the transverse variable, and the point dependence on the planar variable x_α is of product type, while the dependence on the variable exponents is of explicit power type. Its proof, relying on the technical Lemma 3.1, is not a direct application of Theorem 1.2, which, indeed, may provide only a possibly strict lower bound.

More precisely, the family of functionals we consider is the following:

$$\mathcal{J}_\varepsilon(u) := \begin{cases} \int_\Omega a(x_\alpha, \varepsilon x_3) \mathcal{W}^{p_\varepsilon(x)} \left(\nabla_\alpha u(x), \frac{1}{\varepsilon} \nabla_3 u(x) \right) dx & \\ \text{if } u \in W_L^{1,p_\varepsilon(\cdot)}(\Omega; \mathbb{R}^3), & \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3), \end{cases} \tag{1.21}$$

where $a \in L^\infty(\Omega)$.

Theorem 1.5. *Let $\mathcal{W} : \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a uniformly continuous function such that there exist $C_1 > 0$ and $C_2 \geq 1$ with*

$$C_1 |\xi| - \frac{1}{C_1} \leq \mathcal{W}(\xi) \leq C_2 (|\xi| + 1) \quad \text{for every } \xi \in \mathbb{R}^{3 \times 3}. \tag{1.22}$$

Let $a \in L^\infty(\Omega)$ be such that

$$a^- := \inf_{x \in \Omega} a(x) > 0, \tag{1.23}$$

and let

$$a_\varepsilon(x) := a(x_\alpha, \varepsilon x_3).$$

Assume that there exists $a_0 \in L^\infty(\omega)$ such that

$$\lim_{\varepsilon \rightarrow 0^+} a_\varepsilon(x) = \lim_{\varepsilon \rightarrow 0^+} a(x_\alpha, \varepsilon x_3) = a_0(x_\alpha) \quad \text{in } L^\infty(\Omega). \tag{1.24}$$

Let $p \in \mathcal{P}_b(\Omega)$, let p_ε be as in (1.6) and let $\{\mathcal{J}_\varepsilon\}$ be the family of functionals in (1.21). If there exists $p_0 \in \mathcal{P}_b^{\text{log}}(\omega)$ such that $p_\varepsilon \rightarrow p_0$ in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0^+$, then

$$\Gamma(L^1)\text{-}\lim_{\varepsilon \rightarrow 0^+} \mathcal{J}_\varepsilon = \mathcal{J},$$

where $\mathcal{J} : L^1(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ is the functional given by

$$\mathcal{J}(v) := \begin{cases} \int_\omega a_0(x_\alpha) Q(\mathcal{W}_0^{p_0(x_\alpha)})(\nabla_\alpha v(x_\alpha)) dx_\alpha & \text{if } v \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3), \end{cases} \tag{1.25}$$

with $\mathcal{W}_0 : \mathbb{R}^{3 \times 2} \rightarrow [0, +\infty)$ defined by

$$\mathcal{W}_0(\xi_\alpha) := \inf_{\xi_3 \in \mathbb{R}^3} \mathcal{W}(\xi_\alpha, \xi_3) \quad \text{for every } \xi_\alpha \in \mathbb{R}^{3 \times 2}, \tag{1.26}$$

and $Q(\mathcal{W}_0^{p_0(x_\alpha)})$ denoting the quasiconvex envelope of $\mathcal{W}_0^{p_0(x_\alpha)}$.

For what concerns the regularity imposed on p_0 in the previous results, we stress that it is a natural requirement, since relaxation in terms of a bounded Radon measure with subsequent integral representation may fail otherwise, as observed in a famous counterexample in Ref. 51, later expanded in Secs. 7 and 8 of Ref. 46. Furthermore, the condition $p_0^- > 1$ ensures that the target function v belongs to a Sobolev space.

We also observe that, in the previous dimensional reduction theorems, when $\varepsilon \rightarrow 0^+$, the uniform convergence of the family $\{p_\varepsilon\}$, defined by (1.6), is natural when $p \in \mathcal{P}_b^{\text{log}}(\Omega)$. Indeed, in this case, $\{p_\varepsilon\}$ is equicontinuous on $\bar{\Omega}$ and equibounded since, for every $\varepsilon > 0$, it holds

$$p_\varepsilon^+ := \text{ess sup}_{\omega \times (-\frac{1}{2}, \frac{1}{2})} p(x_\alpha, \varepsilon x_3) = \text{ess sup}_{\omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})} p(x_\alpha, x_3) \leq p^+ < +\infty, \tag{1.27}$$

and

$$p_\varepsilon^- := \text{ess inf}_{\omega \times (-\frac{1}{2}, \frac{1}{2})} p(x_\alpha, \varepsilon x_3) = \text{ess inf}_{\omega \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})} p(x_\alpha, x_3) \geq p^- > 1. \tag{1.28}$$

Hence, the Ascoli–Arzelà theorem applies and $p_\varepsilon \rightarrow p_0$ in $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0^+$ where $p_0(x_\alpha) = p(x_\alpha, 0) \in \mathcal{P}_b^{\text{log}}(\omega)$.

It is worth observing that the proofs of Theorems 1.4 and 1.5 do not follow as mere applications of Theorems 1.1 and 1.2. Indeed, the representation results obtained in (1.20) and (1.25) involve densities of the type $Q(\mathcal{W}_0^{p_0(x_\alpha)})(\cdot)$ which are, in general, greater than $(Q\mathcal{W}_0)^{p_0(x_\alpha)}(\cdot)$. This is easily seen in the scalar case (where the quasiconvex envelope coincides with the convex one): for instance, when

$p > 1$ is constant and $\mathcal{W} = \mathcal{W}_0$ is a homogeneous function defined as $\mathcal{W}(\xi) = \max\{|\xi|, |\xi|^{1/p}\}$, it results

$$Q(\mathcal{W}^p)(\xi) = \mathcal{W}^p(\xi) = \max\{|\xi|^p, |\xi|\} \geq |\xi|^p = (Q\mathcal{W})^p(\xi) \quad \text{for every } \xi \in \mathbb{R},$$

where the inequality is strict in $[-1, 1]$.

This paper is organized as follows. Section 2 contains notation and preliminary results regarding the functional spaces and Γ -convergence. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2 which follow as corollaries of the more general result Theorem 3.6. Finally, Sec. 4 contains the proofs of the dimensional reduction results stated above, namely Theorems 1.3–1.5, together with a compactness result for energy bounded sequences in $L^1(\Omega)$, namely Proposition 4.1, which motivates the choice of the topology for our results.

2. Notation and Preliminary Results

In the sequel Ω indicates a generic open set of $\mathbb{R}^N, N \geq 1$; by $\mathcal{A}(\Omega)$ we denote the class of open subsets of Ω and by $\mathcal{A}_0(\Omega)$ we define the subclass of $\mathcal{A}(\Omega)$ whose elements are well contained in Ω , i.e. $B \in \mathcal{A}_0(\Omega)$ if the open set $B \subset\subset \Omega$. Moreover, for every set $O \subset \mathbb{R}^N$, endowed with the natural topology, denote by $\mathcal{B}(O)$ the subclass of the Borel subsets of O . Finally, we denote by $\mathcal{L}^N(\Omega)$ the N -dimensional Lebesgue measure of the set Ω .

2.1. Variable exponents Lebesgue spaces

In this section, we collect some basic results concerning variable exponent Lebesgue spaces. For more details we refer to the monograph,²³ see also Refs. 41 and 24–26.

Let $\Omega \subset \mathbb{R}^N$ be an open set (where $N \geq 1$).

Definition 2.1. For any (Lebesgue) measurable function $p : \Omega \rightarrow [1, +\infty]$ we define

$$p^- := \operatorname{ess\,inf}_{x \in \Omega} p(x) \quad p^+ := \operatorname{ess\,sup}_{x \in \Omega} p(x).$$

Such function p is called *variable exponent* on Ω . If $p^+ < +\infty$ then we call p a *bounded variable exponent*.

We denote by $\mathcal{P}(\Omega)$ the class of variable exponents and with $\mathcal{P}_b(\Omega)$ the class of variable exponents satisfying (1.7).

In the sequel we consider the case $p^+ < +\infty$. In this case, the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ can be defined as

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable such that } \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\}.$$

Let us note that in the case $p^+ = +\infty$ the space above defined may even fail to be a vector space (see Sec. 2 of Ref. 17) and a different definition of the variable Lebesgue spaces has been given in order to preserve the vectorial structure of the space (we refer to Definition 3.2.1 of Ref. 23). In addition, if $p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

(see Theorem 3.2.7 of Ref. 23). Moreover, if $p^+ < +\infty$, the space $L^{p(\cdot)}(\Omega)$ is separable and $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$, while, if $1 < p^- \leq p^+ < +\infty$, the space $L^{p(\cdot)}(\Omega)$ is reflexive and uniformly convex (see Theorems 3.4.7, 3.4.9 and 3.4.12 of Ref. 23). For any variable exponent p , we define p' by setting

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention that, if $p(x) = +\infty$, then $p'(x) = 1$. The function p' is called *the dual variable exponent of p* .

The following result holds (for more details, see Lemma 3.2.20 of Ref. 23, in the case $\varphi(t) = |t|^p$).

Theorem 2.2. (Hölder’s inequality) *Let p, q, s be measurable exponents such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega.$$

Then, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, it holds

$$\|fg\|_{s(\cdot)} \leq \left(\left(\frac{s}{p}\right)^+ + \left(\frac{s}{q}\right)^+ \right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where, in the case $s = p = q = \infty$, we use the convention $\frac{s}{p} = \frac{s}{q} = 1$. In particular, in the case $s = 1$, it holds

$$\left| \int_{\Omega} fg \, dx \right| \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

We introduce the *modular* of the space $L^{p(\cdot)}(\Omega)$ which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u(x)|^{p(x)} \, dx.$$

Thanks to Lemma 3.2.4 in Ref. 23, for every $u \in L^{p(\cdot)}(\Omega)$

$$\begin{aligned} \|u\|_{p(\cdot)} \leq 1 &\Rightarrow \rho_{p(\cdot)}(u) \leq 1, \\ \|u\|_{p(\cdot)} \leq 1 &\Rightarrow \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)} \quad \text{and} \quad \|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u). \end{aligned} \tag{2.1}$$

The following further results hold in the special case $p^+ < +\infty$. By Lemma 3.2.5 of Ref. 23, for every $u \in L^{p(\cdot)}(\Omega)$ it holds

$$\min\{(\rho_{p(\cdot)}(u))^{\frac{1}{p^-}}, (\rho_{p(\cdot)}(u))^{\frac{1}{p^+}}\} \leq \|u\|_{p(\cdot)} \leq \max\{(\rho_{p(\cdot)}(u))^{\frac{1}{p^-}}, (\rho_{p(\cdot)}(u))^{\frac{1}{p^+}}\}. \tag{2.2}$$

In particular, if $\mathcal{L}^N(\Omega) < +\infty$, then

$$\min\{(\mathcal{L}^N(\Omega))^{\frac{1}{p^-}}, (\mathcal{L}^N(\Omega))^{\frac{1}{p^+}}\} \leq \|1\|_{p(\cdot)} \leq \max\{(\mathcal{L}^N(\Omega))^{\frac{1}{p^-}}, (\mathcal{L}^N(\Omega))^{\frac{1}{p^+}}\}. \tag{2.3}$$

Moreover, from (2.2), taking into account (2.1), it follows that for every $u \in L^{p(\cdot)}(\Omega)$

$$\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+},$$

and

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}.$$

Finally, by Corollary 3.3.4 in Ref. 23, if $0 < \mathcal{L}^N(\Omega) < +\infty$ and p and q are variable exponents such that $q \leq p$ \mathcal{L}^N -a.e. in Ω , then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous. In view of (2.3) and of Theorem 3.3.1, Part (a) in Ref. 23, the embedding constant $C_{p,q,\Omega}$ satisfies

$$\begin{aligned} C_{p,q,\Omega} &\leq 2 \min\{\mathcal{L}^N(\Omega) + 1, \max\{\mathcal{L}^N(\Omega)^{(\frac{1}{q} - \frac{1}{p})^+}, \mathcal{L}^N(\Omega)^{(\frac{1}{q} - \frac{1}{p})^-}\}\} \\ &\leq 2(\mathcal{L}^N(\Omega) + 1). \end{aligned} \tag{2.4}$$

In particular, if $q(x) \equiv p^-$

$$C_{p,p^-, \Omega} \leq 2 \min\{\mathcal{L}^N(\Omega), \mathcal{L}^N(\Omega)^{\frac{1}{p^-} - \frac{1}{p^+}}\} + 2.$$

2.2. Trace operator in Sobolev spaces

In view of introducing some important results concerning variable exponents Sobolev spaces, in this section, we recall some useful remarks concerning the trace operator.

Let Ω be a bounded, connected open set with Lipschitz boundary and $1 \leq r \leq \infty$. Hence, set

$$s = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ q \in [1, +\infty) & \text{if } r = N, \\ +\infty & \text{if } r > N, \end{cases}$$

it is well defined the linear and continuous “trace operator” $T_{\partial\Omega} : W^{1,r}(\Omega) \rightarrow L^s(\partial\Omega)$ (where the measure on $\partial\Omega$ is the Hausdorff measure \mathcal{H}^{N-1}) such that

(i) when $1 \leq r \leq N$, it holds

$$T_{\partial\Omega}(u)(x) = u(x) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega, \text{ for every } u \in W^{1,r}(\Omega) \cap C(\overline{\Omega});$$

(ii) when $r > N$, it holds

$$T_{\partial\Omega}(u)(x) = u(x) \quad \text{for every } x \in \partial\Omega, \text{ for every } u \in W^{1,r}(\Omega),$$

(see Theorem 4.3.12 of Ref. 14).

Definition 2.3. For every set $\Gamma \in \mathcal{B}(\partial\Omega)$ and $1 \leq r < +\infty$, we define $W_{0,\Gamma}^{1,r}(\Omega)$ to be the closure in $W^{1,r}(\Omega)$ of the subspace

$$\{u \in W_{\text{loc}}^{1,r}(\mathbb{R}^N) : u \equiv 0 \text{ on a neighborhood of } \Gamma\}.$$

Hence, $W_{0,\Gamma}^{1,r}(\Omega)$ is closed once we endow it with the weak- $W^{1,r}(\Omega)$ topology and

$$W_{0,\Gamma}^{1,r}(\Omega) \subseteq \{u \in W^{1,r}(\Omega) : T_{\partial\Omega}u \equiv 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma\}.$$

Thanks to Proposition 4.3.13 of Ref. 14, it holds

$$W_{0,\partial\Omega}^{1,r}(\Omega) = W_0^{1,r}(\Omega) = \{u \in W^{1,r}(\Omega) : T_{\partial\Omega}u \equiv 0\}.$$

Finally, by Theorem 4.3.18 of Ref. 14, for $1 \leq r < +\infty$, the following Poincaré inequality holds:

$$\|u\|_{L^{r^*}(\Omega)} \leq C(\Omega, r, N, \Gamma) \|\nabla u\|_{L^r(\Omega)} \quad \forall u \in W_{0,\Gamma}^{1,r}(\Omega), \tag{2.5}$$

where

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r \in [1, N), \\ +\infty & \text{otherwise.} \end{cases}$$

2.3. Variable exponents Sobolev spaces

In this section, we recall the definition of variable exponents Sobolev spaces. For more details, we refer to Ref. 17 (see also Definition 8.1.2 of Ref. 23).

Definition 2.4. Let $k, d \in \mathbb{N}$, $k \geq 0$, and let $p \in \mathcal{P}(\Omega)$. We define

$$W^{k,p(\cdot)}(\Omega; \mathbb{R}^d) := \{u : \Omega \rightarrow \mathbb{R}^d : u, \partial_\alpha u \in L^{p(\cdot)}(\Omega; \mathbb{R}^d) \forall \alpha \text{ multi-index such that } |\alpha| \leq k\},$$

where

$$L^{p(\cdot)}(\Omega; \mathbb{R}^d) := \{u : \Omega \rightarrow \mathbb{R}^d : |u| \in L^{p(\cdot)}(\Omega)\}.$$

We define the semimodular on $W^{k,p(\cdot)}(\Omega)$ by

$$\rho_{W^{k,p(\cdot)}(\Omega)}(u) := \sum_{0 \leq |\alpha| \leq k} \rho_{L^{p(\cdot)}(\Omega)}(|\partial_\alpha u|),$$

which induces a norm by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{W^{k,p(\cdot)}(\Omega)} \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

For $k \in \mathbb{N} \setminus \{0\}$, the space $W^{k,p(\cdot)}(\Omega)$ is called *Sobolev space* and its elements are called *Sobolev functions*. Clearly, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$. The space $W^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if $1 \leq p^- \leq p^+ < +\infty$, and reflexive and uniformly convex if $1 < p^- \leq p^+ < +\infty$ (see Theorem 8.1.6 of Ref. 23). We recall that the Sobolev conjugate exponent $p^* : \Omega \rightarrow (1, +\infty]$ is defined as

$$p^*(x) := \begin{cases} \frac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ +\infty & \text{otherwise.} \end{cases}$$

Finally, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the set of $W^{1,p(\cdot)}(\Omega)$ -functions with compact support.

Now, we complement Definition 2.1 with the following one.

Definition 2.5. We denote by $\mathcal{P}^{\log}(\Omega)$ the class of variable exponents p satisfying (1.8) and by $\mathcal{P}_b^{\log}(\Omega) = \mathcal{P}^{\log}(\Omega) \cap \mathcal{P}_b(\Omega)$.

Remark 2.6. We recall that, when Ω is a bounded open set, (1.8) implies the so-called *decay condition*

$$\exists p_\infty \in \mathbb{R}, \gamma_\infty > 0 : |p(x) - p_\infty| \leq \frac{\gamma_\infty}{\log(e + |x|)} \quad \text{for every } x \in \Omega,$$

(see Remark 2.4 of Ref. 22). This ensures that our class $\mathcal{P}^{\log}(\Omega)$ coincides with the class of variable exponents denoted in the same way in Definition 4.1.4 of Ref. 23.

In the sequel, we frequently use the following results.

Remark 2.7. Let Ω be a bounded open set. Hence

- (1) if $p \in \mathcal{P}^{\log}(\Omega)$ satisfies $p^+ < +\infty$, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ (see Corollary 11.2.4 of Ref. 23);
- (2) if $p \in \mathcal{P}^{\log}(\Omega)$, then the following Poincaré inequality holds:

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c \operatorname{diam}(\Omega) \|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{2.6}$$

Finally, we report here the proof of the compact embedding $W^{1,p(\cdot)}(\Omega; \mathbb{R}^d) \hookrightarrow L^{p(\cdot)}(\Omega; \mathbb{R}^d)$ by using arguments similar to those employed in Proposition 3.3 in Ref. 47 (see also Theorem 1.3 of Ref. 28).

Proposition 2.8. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded open set and let $p \in C(\overline{\Omega})$ such that $p \geq 1$. Then*

$$\{u \in L^1(\Omega; \mathbb{R}^d) : \nabla u \in L^{p(\cdot)}(\Omega; \mathbb{R}^{d \times N})\} = W^{1,p(\cdot)}(\Omega; \mathbb{R}^d). \tag{2.7}$$

Moreover, the embedding $W^{1,p(\cdot)}(\Omega; \mathbb{R}^d) \hookrightarrow L^{p(\cdot)}(\Omega; \mathbb{R}^d)$ is compact and on $W^{1,p(\cdot)}(\Omega; \mathbb{R}^d)$ the norm

$$\| \|u\| \| = \|u\|_{L^1(\Omega; \mathbb{R}^d)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega; \mathbb{R}^{d \times N})},$$

is equivalent to the norm $\| \cdot \|_{W^{1,p(\cdot)}(\Omega; \mathbb{R}^d)}$.

Proof. Without loss of generality, we can assume $d = 1$. Identity (2.7) follows from Lemma 2.4 of Ref. 17. In order to show that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact, first of all we construct a suitable covering of $\overline{\Omega}$. Since p is uniformly continuous on $\overline{\Omega}$, for every fixed $0 < \sigma < \frac{(p^-)^2}{N+p^-}$ there exists $\rho > 0$ such that

$$\sup_{B_r \cap \Omega} p - \inf_{B_r \cap \Omega} p < \sigma, \tag{2.8}$$

for every ball B_r di radius $r < \rho$. Being $\overline{\Omega}$ a compact set with Lipschitz boundary, we can find a finite number of open balls $(B_j)_{j \in F}$ and $(\tilde{B}_j)_{j \in F}$, such that $\overline{B_j} \subseteq \tilde{B}_j$, B_j and \tilde{B}_j have the same centers and

$$\begin{aligned} \overline{\Omega} &\subseteq \bigcup_{j \in F} B_j, \\ \Omega \cap \tilde{B}_j &\text{ has Lipschitz continuous boundary,} \end{aligned} \tag{2.9}$$

for every j the radius r_j of \tilde{B}_j satisfies $r_j < \rho$.

For every $j \in F$, we set

$$p_j^+ = \sup_{\tilde{B}_j \cap \Omega} p, \quad p_j^- = \inf_{\tilde{B}_j \cap \Omega} p. \tag{2.10}$$

We claim that, for every $j \in F$, there exists $\delta_j > 0$ such that the embedding

$$W^{1,p_{0,j}^- - \delta_j}(\Omega) \hookrightarrow L^{p_{0,j}^+}(\Omega) \quad \text{holds and is compact.} \tag{2.11}$$

Indeed, fix $j \in F$. We distinguish two cases:

- if $p_j^- > N$, then there exists $\delta_j > 0$ such that $p_j^- - \delta_j > N$. Hence, the claim follows thanks to the Rellich–Kondrachov theorem;
- if $p_j^- \leq N$, since $p^- \leq p_j^- \leq p_j^+$ and the function $x \rightarrow \frac{x^2}{N+x}$ is increasing on $[0, +\infty)$, we get that

$$\sigma < \frac{(p^-)^2}{N+p^-} \leq \frac{(p_j^+)^2}{N+p_j^+}.$$

Thanks to the choice of σ and by applying (2.8) with $r = r_j < \rho$, the above inequality implies

$$p_j^- > p_j^+ - \sigma > p_j^+ - \frac{(p_j^+)^2}{N + p_j^+} = \frac{Np_j^+}{N + p_j^+}.$$

Then, there exists $\delta_j > 0$ such that

$$p_j^- - \delta_j > \frac{Np_j^+}{N + p_j^+}.$$

Hence, taking into account that $p_j^- - \delta_j < p_j^- \leq N$, we obtain that

$$p_j^+ < \frac{(p_j^- - \delta_j)N}{N - p_j^- + \delta_j} = (p_j^- - \delta_j)^*,$$

and, by applying again the Rellich–Kondrachov theorem, we get that $W^{1,p_j^- - \delta_j}(\Omega)$ is compactly embedded into $L^{p_j^+}(\Omega)$.

Now, for every $j \in F$, let $\varphi_j \in C^1(\Omega)$ be such that

$$0 \leq \varphi_j \leq 1, \quad \varphi_j = 1 \quad \text{on } B_j \cap \Omega, \quad \varphi_j = 0 \quad \text{on } \Omega \setminus \tilde{B}_j,$$

and let $\sup_{j \in F} \|\nabla \varphi_j\|_{L^\infty(\Omega)} = C$. Since $p_j^- - \delta_j < p(\cdot)$ on $\Omega \cap \tilde{B}_j$, using the continuity of the embedding $L^{p(\cdot)}(\Omega \cap \tilde{B}_j) \hookrightarrow L^{p_j^- - \delta_j}(\Omega \cap \tilde{B}_j)$, we get that there exists a constant $C_1 = C_1(\Omega) > 0$ such that for all $u \in W^{1,p(\cdot)}(\Omega)$ it holds

$$\|u\|_{L^{p(\cdot)}(\Omega \cap B_j)} \leq \|\varphi_j u\|_{L^{p(\cdot)}(\Omega \cap \tilde{B}_j)} \leq C_1 \|\varphi_j u\|_{L^{p_j^+}(\Omega \cap \tilde{B}_j)} = C_1 \|\varphi_j u\|_{L^{p_j^+}(\Omega)}.$$

Hence, set $E_j := \{v \in L^{p_j^+}(\Omega) : v = \varphi_j u, u \in W^{1,p(\cdot)}(\Omega)\}$ endowed with the norm $\|\cdot\|_{L^{p_j^+}(\Omega)}$, for every $j \in F$ we have that

$$E_j \ni \varphi_j u \mapsto u \in L^{p(\cdot)}(\Omega) \quad \text{holds and is continuous.} \tag{2.12}$$

Moreover, for every $j \in F$ and for every $u \in W^{1,p(\cdot)}(\Omega)$, it holds

$$\|\varphi_j u\|_{L^{p_j^- - \delta_j}(\Omega)} = \|\varphi_j u\|_{L^{p_j^- - \delta_j}(\Omega \cap \tilde{B}_j)} \leq C_1 \|\varphi_j u\|_{L^{p(\cdot)}(\Omega \cap \tilde{B}_j)} \leq C_1 \|u\|_{L^{p(\cdot)}(\Omega)},$$

and

$$\begin{aligned} \|D(\varphi_j u)\|_{L^{p_j^- - \delta_j}(\Omega)} &= \|D(\varphi_j u)\|_{L^{p_j^- - \delta_j}(\Omega \cap \tilde{B}_j)} \\ &\leq C_1 (\|\varphi_j Du\|_{L^{p(\cdot)}(\Omega \cap \tilde{B}_j)} + \|uD\varphi_j\|_{L^{p(\cdot)}(\Omega \cap \tilde{B}_j)}) \\ &\leq C_2 \|u\|_{W^{1,p(\cdot)}(\Omega)}, \end{aligned}$$

i.e. for every $j \in F$ we have that

$$W^{1,p(\cdot)}(\Omega) \ni u \mapsto \varphi_j u \in W^{1,p_j^-(\cdot)-\delta_j}(\Omega) \quad \text{holds and is continuous.} \quad (2.13)$$

By combining (2.11)–(2.13), it easily follows that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact.

Finally, it is easy to show that $W^{1,p(\cdot)}(\Omega; \mathbb{R}^d)$ is complete with respect to the norm $\|\cdot\|$. Since $\|\cdot\| \leq C\|\cdot\|_{W^{1,p(\cdot)}(\Omega; \mathbb{R}^d)}$, thanks to a classical result of Functional Analysis, we get that the two norms are equivalent on $W^{1,p(\cdot)}(\Omega; \mathbb{R}^d)$. \square

We conclude this section by recalling the following theorem which is of crucial importance for the results contained in Sec. 4. It is a consequence of Theorem 11.2.7 of Ref. 23, and properties of sets with Lipschitz boundary in Sec. 9.5 of Ref. 23.

Theorem 2.9. *Let Ω be a bounded open set with Lipschitz boundary and let $p \in \mathcal{P}_b^{\log}(\Omega)$. If $u \in W_0^{1,1}(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ then $u \in W_0^{1,p(\cdot)}(\Omega)$.*

2.4. Γ -convergence

For an introduction to Γ -convergence, we refer to Ref. 19. We recall the sequential characterization of the Γ -limit when X is a metric space.

Proposition 2.10. (Proposition 8.1 in Ref. 19) *Let X be a metric space and let $\varphi_k : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ for every $k \in \mathbb{N}$. Then, $\{\varphi_k\}$ Γ -converges to φ with respect to the strong topology of X (and we write $\Gamma(X)\text{-}\lim_{k \rightarrow +\infty} \varphi_k = \varphi$) if and only if*

- (i) (*Γ -liminf inequality*) for every $x \in X$ and for every sequence $\{x_k\}$ converging to x , it is

$$\varphi(x) \leq \liminf_{k \rightarrow +\infty} \varphi_k(x_k);$$

- (ii) (*Γ -limsup inequality*) for every $x \in X$, there exists a sequence $\{x_k\}$ converging to $x \in X$ such that

$$\varphi(x) = \lim_{k \rightarrow +\infty} \varphi_k(x_k).$$

We recall that the $\Gamma\text{-}\lim_{k \rightarrow +\infty} \varphi_k$ is lower semicontinuous on X (see Proposition 6.8 of Ref. 19).

Definition 2.11. We say that a family $\{\varphi_\varepsilon\}$ Γ -converges to φ , with respect to the topology considered on X as $\varepsilon \rightarrow 0^+$, if $\{\varphi_{\varepsilon_k}\}$ Γ -converges to φ for all sequences $\{\varepsilon_k\}$ of positive numbers converging to 0 as $k \rightarrow +\infty$.

3. Some Lower Semicontinuity Results

In this section, we provide some lower semicontinuity results for different classes of integral functionals of the type

$$F(u, p) = \int_{\Omega} \varphi(x, p(x), \nabla u(x)) \, dx, \quad (3.1)$$

where the variable p may stand for a variable growth exponent and $\Omega \subset \mathbb{R}^N$ is an open set. We discuss the two cases: convexity and quasiconvexity of $\varphi(x, p, \cdot)$, in order to point out the different sets of assumptions.

Before proceeding, we recall the following definition.

Definition 3.1. Let O be a bounded open subset of \mathbb{R}^N . A function $\varphi : O \times \mathbb{R}^m \times \mathbb{R}^s \rightarrow (-\infty, \infty]$ is said to be a normal integrand if

- φ is $\mathcal{L}^N(O) \otimes \mathcal{B}(\mathbb{R}^m \times \mathbb{R}^s)$ -measurable;
- $\varphi(x, \cdot, \cdot)$ is lower semicontinuous for \mathcal{L}^N -a.e. $x \in O$.

Given a closed set $M \subseteq \mathbb{R}^m$, a function $\varphi : O \times M \times \mathbb{R}^s \rightarrow \mathbb{R}$ is said to be a Carathéodory integrand if

- $\varphi(\cdot, z, \xi)$ is $\mathcal{L}^N(O)$ -measurable for every $(z, \xi) \in M \times \mathbb{R}^m$;
- $\varphi(x, \cdot, \cdot)$ is continuous for \mathcal{L}^N -a.e. $x \in O$.

3.1. The convex case

If the function φ in (3.1) is convex in the last variable, then the functional F (in (3.1)) is sequentially lower semicontinuous along sequences $u_k \rightharpoonup u$ in $W^{1,1}(\Omega, \mathbb{R}^d)$ and $p_k \rightarrow p_0$ in $L^1(\Omega; \mathbb{R}^m)$, in view of the following result, shown by De Giorgi and Ioffe (see Theorem 5.8 of Ref. 6).

Theorem 3.2. Let O be a bounded open subset of \mathbb{R}^N and let $\varphi : O \times \mathbb{R}^m \times \mathbb{R}^s \rightarrow [0, +\infty]$ be a normal integrand such that the map $\xi \in \mathbb{R}^s \rightarrow \varphi(x, z, \xi)$ is convex for \mathcal{L}^N -a.e. $x \in O$ and every $z \in \mathbb{R}^m$. Let $\{z_k\} \subset L^1(O, \mathbb{R}^m)$ and $\{U_k\} \subset L^1(O; \mathbb{R}^s)$. If $z_k \rightarrow z_0$ in $L^1(O; \mathbb{R}^m)$ and $U_k \rightharpoonup U_0$ in $L^1(O; \mathbb{R}^s)$ then it holds

$$\int_O \varphi(x, z_0(x), U_0(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_O \varphi(x, z_k(x), U_k(x)) dx.$$

As an application of Theorem 3.2 with $m = 1$ and $s = d \times N$, we give the following result, that is crucial in the proof of Theorem 1.1.

Corollary 3.3. Let $\{p_k\} \subset \mathcal{P}(\Omega) \cap L^1(\Omega)$ and $\{v_k\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$. If $p_k \rightarrow p_0$ in $L^1(\Omega)$ and $v_k \rightharpoonup v$ in $W^{1,1}_{loc}(\Omega; \mathbb{R}^d)$, then

$$\int_{\Omega} |\nabla v(x)|^{p_0(x)} dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} |\nabla v_k(x)|^{p_k(x)} dx. \tag{3.2}$$

In particular, if $\{v_k\}$ satisfies the equiboundedness assumption (1.10), then $\nabla v \in L^{p_0(\cdot)}(\Omega; \mathbb{R}^{d \times N})$. Finally, if $\{v_k\}$ satisfies (1.10), Ω is a Lipschitz bounded open set, $p_0 \in C(\overline{\Omega})$ and $v \in L^1(\Omega; \mathbb{R}^d)$, then $v \in W^{1,p_0(\cdot)}(\Omega; \mathbb{R}^d)$.

Proof. In order to show (3.2), it is sufficient to apply Theorem 3.2 on a sequence of open sets $\{O_n\}$ such that $O_n \subset\subset \Omega$ and $\cup_n O_n = \Omega$, with $\varphi : \Omega \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty]$

given by

$$\varphi(x, z, U) = \begin{cases} |U|^z & \text{if } z \in [1, +\infty), \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$z_k = p_k, \quad U_k = \nabla u_k.$$

For every $n \in \mathbb{N}$, we obtain that

$$\int_{O_n} |\nabla v(x)|^{p_0(x)} dx \leq \liminf_{k \rightarrow +\infty} \int_{O_n} |\nabla v_k(x)|^{p_k(x)} dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} |\nabla v_k(x)|^{p_k(x)} dx,$$

and (3.2) follows sending $n \rightarrow +\infty$. Finally, the additional assumption (1.10) combined with inequality (3.2) implies that $\nabla v \in L^{p_0(\cdot)}(\Omega; \mathbb{R}^{d \times N})$. The last part of the theorem follows from Proposition 2.8. \square

Remark 3.4. Let Ω be an open set with finite Lebesgue measure. Let $\{p_k\} \subset \mathcal{P}_b(\Omega) \cap L^1(\Omega)$ be such that, for some $k_0 \in \mathbb{N}$, it holds

$$1 < \inf_{k \geq k_0} p_k^-. \tag{3.3}$$

(1) If $\{v_k\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ is equibounded in the sense of (1.10) and $v_k \rightharpoonup v$ in $L^1(\Omega; \mathbb{R}^d)$ then $v_k \rightharpoonup v$ in $W^{1,1}(\Omega; \mathbb{R}^d)$.

Indeed, exploiting (3.3), we can take $1 < r < \inf_{k \geq k_0} p_k^-$. Hence, by (2.4), it follows that

$$\begin{aligned} & \sup_{k \geq k_0} \|\nabla v_k\|_{L^r(\Omega; \mathbb{R}^{d \times N})} \\ & \leq 2(\mathcal{L}^N(\Omega) + 1) \sup_{k \geq k_0} \|\nabla v_k\|_{L^{p_k(\cdot)}(\Omega; \mathbb{R}^{d \times N})} \\ & \leq 2(\mathcal{L}^N(\Omega) + 1) \sup_{k \geq k_0} \{(\rho_{p_k(\cdot)}(\nabla v_k))^{\frac{1}{p_k^-}}, (\rho_{p_k(\cdot)}(\nabla v_k))^{\frac{1}{p_k^+}}\} < +\infty. \end{aligned}$$

Since $r > 1$, we obtain that there exists a not relabeled subsequence $\{v_k\}$ such that $\nabla v_k \rightharpoonup V$ in $L^r(\Omega; \mathbb{R}^{d \times N})$. This easily implies that $V = \nabla v$ and that $v_k \rightharpoonup v$ in $W^{1,1}(\Omega; \mathbb{R}^d)$. This fact will be used in the proof of Theorem 1.1.

(2) Let $\varphi : \Omega \times \mathbb{R} \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ be a $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^{d \times N})$ -measurable function satisfying the growth condition

$$\varphi(x, p, \xi) \geq b(x) + C|\xi|^p \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^{d \times N}, \forall p \geq 1, \tag{3.4}$$

where $C > 0$ and $b \in L^1(\Omega)$. Assume that $p_k \rightarrow p_0$ in $L^1(\Omega)$ with $p_0 \in \mathcal{P}(\Omega) \cap L^1(\Omega)$ and let $\{v_k\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ be such that

$$\sup_k F(v_k, p_k) < +\infty, \tag{3.5}$$

where F is the functional defined by (3.1). If $v_k \rightharpoonup v$ in $L^1(\Omega; \mathbb{R}^d)$ then $v_k \rightharpoonup v$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ and $\nabla v \in L^{p_0(\cdot)}(\Omega; \mathbb{R}^{d \times N})$. Indeed, (3.4) and (3.5) imply (1.10)

and then it is sufficient to use the Part (1) of this remark, combined with Corollary 3.3.

In particular, the coercivity assumption (3.4) is satisfied by integrands of the form $\varphi(x, p, \xi) = \mathcal{W}^p(x, \xi)$ with $p \geq 1$ and

$$\mathcal{W}(x, \xi) \geq b(x) + C|\xi| > 0 \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^{d \times N},$$

where $C > 0$ and $b(x) \in L^\infty(\Omega)$.

3.2. The quasiconvex case

The following special case of Theorem 4.12 in Ref. 46 holds (see also Theorem 7.4 in Ref. 5). It is important in order to show Theorems 1.1 and 1.2. For the reader's convenience, after the statement, we provide an outline of the main steps necessary to deduce this application from the results contained in Ref. 46.

Theorem 3.5. *Let $p_0 \in C(\Omega)$ be such that $p_0 \in \mathcal{P}_b^{\log}(A)$ for every open set $A \subset \subset \Omega$. Let $\varphi : \Omega \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$ be a function such that*

- $\varphi(\cdot, \xi)$ is $\mathcal{L}^N(\Omega)$ -measurable for every $\xi \in \mathbb{R}^{d \times N}$;
- $\varphi(x, \cdot)$ is quasiconvex for \mathcal{L}^N -a.e. $x \in \Omega$;
- there exists $C > 1$ and $b(x) \in L^1_{\text{loc}}(\Omega)$ with $b(x) \geq 0$ such that

$$0 \leq \varphi(x, \xi) \leq b(x) + C|\xi|^{p_0(x)}.$$

Then, for every sequence $\{v_k\} \subset W^{1,1}(\Omega; \mathbb{R}^d)$ with $v_k \rightarrow v$ in $L^1(\Omega; \mathbb{R}^d)$ and such that

$$\sup_k \int_{\Omega} |\nabla v_k(x)|^{p_0(x)} dx < +\infty, \tag{3.6}$$

we have that $v \in W^{1,p_0(\cdot)}_{\text{loc}}(\Omega; \mathbb{R}^d)$, $|\nabla v|^{p_0(\cdot)} \in L^1(\Omega)$ and

$$\int_{\Omega} \varphi(x, \nabla v(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \varphi(x, \nabla v_k(x)) dx. \tag{3.7}$$

Sketch of the proof. We observe that, by applying Remark 3.4(1) with $p_k \equiv p_0$, we have that the sequence $\{v_k\}$ weakly converges to v in $W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^d)$. Hence, by Corollary 3.3, taking into account (3.6), we get that $v \in W^{1,p_0(\cdot)}_{\text{loc}}(\Omega; \mathbb{R}^d)$ and $|\nabla v|^{p_0(\cdot)} \in L^1(\Omega)$. Then, the lower semicontinuity result given by (3.7) follows as an application of Theorem 4.12 in Ref. 46, recalling that, for every open set $A \Subset \Omega$,

- if $p_0 \in C(A)$ and $p_0 > 1$ on A , then the function $\psi(x, \xi) = |\xi|^{p_0(x)}$ enjoys the Sobolev-type property and the Rellich-type property on A required by Theorem 4.12 of Ref. 46 (see Definition 3.2 and Example 3.4 in Ref. 46 and Lemmas 2.4 and 2.8 in Ref. 17);

- if $p_0 \in \mathcal{P}^{\text{log}}(A)$ then ψ verifies the maximal property on A , see Proposition 5.2 and Definition 4.4 in Ref. 46. Hence, ψ satisfies also the density property on A , see Definition 4.5 and Proposition 4.6 in Ref. 46.

Hence, Theorem 4.12 in Ref. 46 entails that for every open set $A \subset\subset \Omega$

$$\int_A \varphi(x, \nabla v(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_A \varphi(x, \nabla v_k(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \varphi(x, \nabla v_k(x)) dx.$$

By passing to the supremum on $A \subset\subset \Omega$, we get the desired conclusion (3.7). \square

By using the arguments of Lemma 4.1 in Ref. 24 (see also Lemma 5.1 in Ref. 44), we show the following key result.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$ an open set with finite Lebesgue measure. Let $\mathcal{W} : \Omega \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty)$ be a Borel function. Let $a_0 \in L^\infty(\Omega)$ be such that $a_0 \geq 0$ \mathcal{L}^N -a.e. in Ω . Let $p_0 \in \mathcal{P}_b(\Omega)$ and let $\{p_k\} \subset L^\infty(\Omega)$ be such that $p_k \rightarrow p_0$ in $L^\infty(\Omega)$. Then, for every $0 < \delta < \min\{1/2, p_0^- - 1\}$, there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$ and for every $U \in L^1(\Omega, \mathbb{R}^{d \times N})$, it holds*

$$\begin{aligned} & \int_{\Omega} a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, U(x)) dx \\ & \leq 2\delta \mathcal{L}^N(\Omega) + (2\delta)^{-2\delta} \|a_0\|_{L^\infty(\Omega)} \int_{\Omega} a_0(x) \mathcal{W}^{p_k(x)}(x, U(x)) dx. \end{aligned} \quad (3.8)$$

Proof. Let $0 < \delta < \min\{1/2, p_0^- - 1\}$. Thanks to the uniform convergence of $\{p_k\}$ to p_0 , there exists $k_0 \in \mathbb{N}$ and a negligible measurable set $N \subseteq \Omega$ such that

$$1 < p_0(x) - \delta < p_k(x) < p_0(x) + \delta \quad \text{for every } x \in \Omega \setminus N,$$

for every $k \geq k_0$. Hence, for every $x \in \Omega \setminus N$ and for every $U \in L^1(\Omega, \mathbb{R}^{d \times N})$, if $2\delta < \mathcal{W}(x, U(x)) < 1$, then

$$(2\delta)^{2\delta} < (2\delta)^{p_k(x)-p_0(x)+\delta} < \mathcal{W}^{p_k(x)-p_0(x)+\delta}(x, U(x)) < 1,$$

that implies

$$1 < \mathcal{W}^{p_0(x)-p_k(x)-\delta}(x, U(x)) < (2\delta)^{-2\delta},$$

while, if $\mathcal{W}(x, U(x)) \leq 2\delta$, then

$$\mathcal{W}^{p_0(x)-\delta}(x, U(x)) \leq (2\delta)^{p_0(x)-\delta} < 2\delta,$$

being $2\delta < 1$ and $p_0(x) - \delta > 1$. Hence, for every $k \geq k_0$ and for every $U \in L^1(\Omega, \mathbb{R}^{d \times N})$, it holds

$$\begin{aligned} & \int_{\Omega} a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, U(x)) dx \\ &= \int_{\{\mathcal{W}(x, U(x)) \leq 2\delta\}} a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, U(x)) dx \\ &+ \int_{\{2\delta < \mathcal{W}(x, U(x)) < 1\}} a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, U(x)) dx \\ &+ \int_{\{\mathcal{W}(x, U(x)) \geq 1\}} a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, U(x)) dx \\ &\leq 2\delta \|a_0\|_{L^\infty(\Omega)} \mathcal{L}^N(\Omega) + (2\delta)^{-2\delta} \left(\int_{\{2\delta < \mathcal{W}(x, U(x)) < 1\}} \mathcal{W}^{p_k(x)}(x, U(x)) dx \right. \\ &\quad \left. + \int_{\{\mathcal{W}(x, U(x)) \geq 1\}} \mathcal{W}^{p_k(x)}(x, U(x)) dx \right) \\ &\leq 2\delta \|a_0\|_{L^\infty(\Omega)} \mathcal{L}^N(\Omega) + (2\delta)^{-2\delta} \int_{\Omega} \mathcal{W}^{p_k(x)}(x, U(x)) dx, \end{aligned}$$

which gives the desired conclusion. □

Now, we are in a position to show the following general result which allows us to deduce Theorems 1.1 and 1.2 as immediate corollaries.

Theorem 3.6. *Let $\Omega, \mathcal{W}, p_0, \{p_k\}, v, \{v_k\}$ be as in Theorem 1.1 and let $a_0, \{a_k\}$ be as in Theorem 1.2. Then, v satisfies (1.11) and*

$$\int_{\Omega} a_0(x) \mathcal{W}^{p_0(x)}(x, \nabla v(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} a_k(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx.$$

Proof. First of all, we note that, for every open set $A \subset\subset \Omega$, since $\inf_A p_0 > 1$ and p_k uniformly converges to p_0 as $k \rightarrow \infty$, we have that there exists $k_0 = k_0(A) \in \mathbb{N}$ such that

$$\inf_{k \geq k_0} \operatorname{ess\,inf}_A p_k(x) > 1.$$

In view of Remark 3.4(1), this implies that $v_k \rightharpoonup v$ in $W^{1,1}(A; \mathbb{R}^d)$ for every open set $A \subset\subset \Omega$. By applying Corollary 3.3 in view of (1.10), we get that $v \in W_{loc}^{1, p_0(\cdot)}(\Omega; \mathbb{R}^d)$ and $\nabla v \in L^{p_0(\cdot)}(\Omega)$. Now, we claim that for every open $A \subset\subset \Omega$ it holds

$$\int_A a_0(x) \mathcal{W}^{p_0(x)}(x, \nabla v(x)) dx \leq \liminf_{k \rightarrow +\infty} \int_A a_k(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx. \quad (3.9)$$

Hence, by passing to the supremum in (3.9) with respect to $A \subset\subset \Omega$, we get the desired conclusion.

In order to show (3.9), for fixed $A \subset\subset \Omega$, we note that

$$\begin{aligned} & \liminf_{k \rightarrow +\infty} \int_A a_k(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \\ &= \liminf_{k \rightarrow +\infty} \left(\int_A (a_k(x) - a_0(x)) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \right. \\ & \quad \left. + \int_A a_0(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \right) \\ & \geq \liminf_{k \rightarrow +\infty} \left(-\|a_k - a_0\|_{L^\infty(A)} \int_A \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \right. \\ & \quad \left. + \int_A a_0(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \right). \end{aligned}$$

On the other hand, (1.10) and (1.12) ensure

$$\liminf_{k \rightarrow +\infty} \int_A \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx < +\infty,$$

and, since $a_k \rightarrow a_0$ in $L^\infty(\Omega)$, this implies

$$\lim_{k \rightarrow +\infty} \|a_k - a_0\|_{L^\infty(A)} \int_A \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx = 0.$$

Hence

$$\liminf_{k \rightarrow +\infty} \int_A a_k(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \geq \liminf_{k \rightarrow +\infty} \int_A a_0(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx. \tag{3.10}$$

Since $\{p_k\}$ uniformly converges to p_0 on A and $p_0 > 1$ on A , by applying Lemma 3.1, there exists $k_0 = k_0(A) \in \mathbb{N}$ such that, for every $k \geq k_0$, it holds

$$\begin{aligned} & \int_A \mathcal{W}^{p_0(x)-\delta}(x, U(x)) dx \\ & \leq 2\delta \mathcal{L}^N(A) + (2\delta)^{-2\delta} \int_A \mathcal{W}^{p_k(x)}(x, U(x)) dx \quad \forall U \in L^1(A, \mathbb{R}^{d \times N}). \end{aligned} \tag{3.11}$$

By applying (3.11) with $U = \nabla v_k$ and then passing to the limit when $k \rightarrow \infty$, we obtain that

$$\begin{aligned} & \liminf_{k \rightarrow +\infty} \int_A a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, \nabla v_k(x)) dx \\ & \leq (2\delta)^{-2\delta} \liminf_{k \rightarrow +\infty} \int_\Omega a_0(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx + 2\delta \|a\|_{L^\infty(A)} \mathcal{L}^N(A). \end{aligned}$$

Taking into account the growth condition (1.12), we can apply Theorem 3.5 with the variable exponent $p_0(\cdot) - \delta \in P_b^{\text{log}}(A)$ to obtain that

$$\begin{aligned} & \int_A a_0(x) \mathcal{W}^{p_0(x)-\delta}(x, \nabla v(x)) dx \\ & \leq (2\delta)^{-2\delta} \liminf_{k \rightarrow +\infty} \int_A a_0(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx + 2\delta \|a_0\|_{L^\infty(A)} \mathcal{L}^N(A). \end{aligned}$$

By sending $\delta \rightarrow 0$ and by using Fatou’s lemma in the above inequality, we finally get the desired conclusion

$$\begin{aligned} \int_A a_0(x) \mathcal{W}^{p_0(x)}(x, \nabla v(x)) dx & \leq \liminf_{k \rightarrow \infty} \int_A a_0(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx \\ & \leq \liminf_{k \rightarrow \infty} \int_A a_k(x) \mathcal{W}^{p_k(x)}(x, \nabla v_k(x)) dx, \end{aligned}$$

where the last inequality follows from (3.10). □

Proof of Theorem 1.1. It follows from Theorem 3.6 with $a_k(x) = a(x) = 1$ for every $x \in \Omega$. □

Proof of Theorem 1.2. It is a straightforward consequence of Theorem 3.6. □

4. Dimensional Reduction

The aim of this section consists in proving the Γ -convergence results stated in Theorems 1.3–1.5 concerning the dimension reduction problems defined in $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$ where $\omega \subseteq \mathbb{R}^2$ is a bounded, Lipschitz open set. To this end, in view of Definition 2.11, in this section, $\{\varepsilon_k\} \subseteq (0, 1/2)$ is any sequence converging to 0.

We start proving a compactness result for energy bounded sequences, which, indeed, motivates the choice of the topology in our Γ -convergence results. Furthermore, according to the regularity of the variable exponent, modeling the point-dependent growth of the hyperelastic energy, we obtain different results for the limiting deformation fields. Finally, the proofs of Theorems 1.3–1.5 are provided showing that, for every infinitesimal sequence $\{\varepsilon_k\}$, the Γ -limit of $\{\mathcal{F}_{\varepsilon_k}\}$ exists and admits the same integral representation, when $p \in \mathcal{P}_b^{\text{log}}(\Omega)$.

4.1. Compactness of bounded energy sequences

Let $\{\mathcal{F}_\varepsilon\}$ be the family of integral energies in (1.13). The following compactness result for energy bounded sequences, relies on Poincaré inequality (2.5), Corollary 3.3 and Proposition 2.8.

Proposition 4.1. *Let $f : \omega \times (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be such that*

- *f is $\mathcal{L}^2(\omega) \otimes \mathcal{B}((-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3})$ -measurable;*
- *f satisfies (1.14).*

Let $p \in \mathcal{P}_b(\Omega)$, let $\varepsilon_k \rightarrow 0^+$ and let $\{p_{\varepsilon_k}\}$ be as in (1.6). Suppose that $\{u_{\varepsilon_k}\} \subset L^1(\Omega; \mathbb{R}^3)$ is a sequence such that

$$\sup_{k \in \mathbb{N}} \mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty. \tag{4.1}$$

Then

- (1) up to a not relabeled subsequence, $\{u_{\varepsilon_k}\}$ converges weakly in $W^{1,p^-}(\Omega; \mathbb{R}^3)$, (where p^- is given by (1.7)), to a function $u_0 \in W_{0,\partial_L\Omega}^{1,p^-}(\Omega; \mathbb{R}^3)$ (cf. Definition 2.3), which, with an abuse of notation, we can identify with a function $u_0 \in W_0^{1,p^-}(\omega; \mathbb{R}^3)$;
- (2) if $p_0 \in C(\overline{\omega})$ and $p_{\varepsilon_k} \rightarrow p_0$ in $L^1_{\text{loc}}(\Omega)$, then $u_0 \in W^{1,p_0(\cdot)}(\Omega; \mathbb{R}^3) \cap W_{0,\partial_L\Omega}^{1,p^-}(\Omega; \mathbb{R}^3)$ and it can be identified with a function $u_0 \in W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3) \cap W_0^{1,p^-}(\omega; \mathbb{R}^3)$;
- (3) if $p_0 \in \mathcal{P}_b^{\text{log}}(\omega)$ and $p_{\varepsilon_k} \rightarrow p_0$ in $L^1_{\text{loc}}(\Omega)$, then u_0 can be identified with a function $u_0 \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$;
- (4) if $p_0 \in C(\overline{\omega})$ and $p_{\varepsilon_k} \rightarrow p_0$ in $L^\infty(\Omega)$, then $u_{\varepsilon_k} \rightarrow u_0$ in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$.

Remark 4.2. Note that, if p satisfies the assumptions (1.7) and (1.8) on Ω , then $\{p_{\varepsilon_k}\}$ is equicontinuous on $\overline{\Omega}$ and equibounded (thanks to (1.27) and (1.28)). By applying Ascoli–Arzelà theorem, it follows that $\|p_{\varepsilon_k} - p_0\|_{L^\infty(\Omega)} \rightarrow 0$ with $p_0(x_\alpha, x_3) = p(x_\alpha, 0)$.

Proof. (1) Thanks to the growth condition (1.14), the assumption (4.1) implies that $u_{\varepsilon_k} \in W_{0,\partial_L\Omega}^{1,p^-}(\Omega; \mathbb{R}^3) \cap W^{1,p_{\varepsilon_k}(\cdot)}(\Omega; \mathbb{R}^3)$ for every $k \in \mathbb{N}$ and that there exists a constant $C \geq 1$ such that

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \left| \left(\nabla_\alpha u_{\varepsilon_k}, \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k} \right) \right|^{p_{\varepsilon_k}(x)} dx \leq C. \tag{4.2}$$

By exploiting (1.28), both the sequences $\{|\nabla u_{\varepsilon_k}|\}$ and $\{|\frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}|\}$ are bounded in $L^{p^-}(\Omega)$ and, by using the Poincaré inequality (2.5) with $\Gamma := \partial\omega \times (-1/2, 1/2)$, it turns out that $\{u_{\varepsilon_k}\}$ is bounded in $W^{1,p^-}(\Omega; \mathbb{R}^3)$. Hence, $\{u_{\varepsilon_k}\}$ admits a not relabeled subsequence which is weakly converging in $W^{1,p^-}(\Omega; \mathbb{R}^3)$ to $u_0 \in W_{0,\partial_L\Omega}^{1,p^-}(\Omega; \mathbb{R}^3)$, being the latter space weakly closed. On the other hand, $\frac{\partial u_0}{\partial x_3} = 0$ \mathcal{L}^3 -a.e. in Ω , hence, with an abuse of notation, we can identify u_0 with a function in $W_0^{1,p^-}(\omega; \mathbb{R}^3)$;

(2) the statement follows from Part (1) and Corollary 3.3;

(3) in view of Theorem 2.9, under the additional assumption, $p_0 \in \mathcal{P}_b^{\text{log}}(\omega)$, the space $W_0^{1,p^-}(\omega; \mathbb{R}^3) \cap W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ coincides with $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$;

(4) under the additional hypothesis that

$$p_{\varepsilon_k} \rightarrow p_0 \quad \text{in } L^\infty(\Omega), \tag{4.3}$$

we show that $u_{\varepsilon_k} \rightarrow u_0$ in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$. Since $p_0 \in C(\overline{\Omega})$ and $\partial\Omega$ is Lipschitz continuous, we can find a finite number of open balls $(B_i)_{i \in F}$ and $(\tilde{B}_i)_{i \in F}$, such that $\tilde{B}_i \subset\subset B_i$ and (2.9) holds. By following the same argument in Proposition 2.8, for every $j \in F$ there exists $\delta_j > 0$ such that (2.11) holds with $p_{0,j}^+, p_{0,j}^-$ given by (2.10).

Let $\varphi_j \in C^1(\Omega)$ be such that $0 \leq \varphi_j \leq 1$, $\varphi_j = 1$ on B_j , $\text{supp } \varphi_j \subset \tilde{B}_j$ and $|\nabla \varphi_j| \leq C$. Now, we choose $0 < \delta < \min_{j \in F} \delta_j$. Since $\|p_{\varepsilon_k} - p_0\|_{L^\infty(\Omega)} \rightarrow 0$, there exists $k_0 \in \mathbb{N}$ such that $p_{\varepsilon_k}(x) > p_0(x) - \delta$ for $k \geq k_0$ and for \mathcal{L}^3 -a.e. $x \in \Omega$. This, together with Hölder’s inequality, implies that for $k \geq k_0$ the embedding

$$W^{1,p_{\varepsilon_k}(\cdot)}(\Omega; \mathbb{R}^3) \hookrightarrow W^{1,p_0(\cdot)-\delta_j}(\Omega; \mathbb{R}^3) \quad \text{is continuous } \forall j \in F. \tag{4.4}$$

Moreover, by reasoning as in the proof of (2.13), the embedding

$$W^{1,p_0(\cdot)-\delta_j}(\Omega; \mathbb{R}^3) \ni u \mapsto \varphi_j u \in W^{1,p_{0,j}^-(\cdot)-\delta_j}(\Omega; \mathbb{R}^3) \tag{4.5}$$

holds and is continuous $\forall j \in F$.

By (4.4), (4.5) and (2.12), it easily follows that, $\forall j \in F$ and for $k \geq k_0$, the embedding

$$W^{1,p_{\varepsilon_k}(\cdot)}(\Omega; \mathbb{R}^3) \ni u \hookrightarrow \varphi_j u \in L^{p_0(\cdot)}(\Omega; \mathbb{R}^3) \quad \text{is compact.}$$

Hence, for every $j \in F$, the sequence $\{u_{\varepsilon_k}\}$ admits a subsequence $\{u_{\varepsilon_k^j}\}$ such that $\{\varphi_j u_{\varepsilon_k^j}\}$ converges strongly in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$ to a function u_0^j for every $j \in F$. Since $p^- \leq p_0(\cdot)$ in Ω then $\varphi_j u_{\varepsilon_k^j} \rightarrow u_0^j$ as $k \rightarrow +\infty$ in $L^{p^-}(\Omega; \mathbb{R}^3)$. Then, $u_0^j = u_0$ for every $j \in F$ that implies $u_{\varepsilon_k^j} \rightarrow u_0$ in $L^{p_0(\cdot)}(\Omega \cap B_j; \mathbb{R}^3)$. Passing to subsequences repeatedly, we obtain that $u_{\varepsilon_k} \rightarrow u_0$ in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$. □

With the aim at proving Theorems 1.3–1.5, we start with a preliminary lemma.

Lemma 4.1. *Let $f : \omega \times (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be such that*

(1_f) *f is $\mathcal{L}^2(\omega) \otimes \mathcal{B}((-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3})$ -measurable.*

Then

(1_{f₀}) *the function $f_0 : \omega \times (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$, defined by (1.16), is $\mathcal{L}^2(\omega) \otimes \mathcal{B}((-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2})$ -measurable.*

Furthermore, assuming that

(2_f) *$f(x_\alpha, \cdot, \cdot, \cdot)$ is lower semicontinuous for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$;*

(3_f) there exists $C_1 > 0$ such that for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ and for every $(y, q, \xi) \in (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3}$, it holds

$$f(x_\alpha, y, q, \xi) \geq C_1 |\xi|^q - \frac{1}{C_1},$$

then

(2_{f₀}) $f_0(x_\alpha, \cdot, \cdot, \cdot)$ is lower semicontinuous on $(-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2}$ for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$;

(3_{f₀}) for every $(y, q, \xi_\alpha) \in (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2}$ it holds

$$f_0(x_\alpha, y, q, \xi_\alpha) \geq C_1 |\xi_\alpha|^q - \frac{1}{C_1}. \tag{4.6}$$

Moreover, if

(4_f) for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ the function $f(x_\alpha, y, q, \cdot)$ is convex on $\mathbb{R}^3 \times \mathbb{R}^3$ for every $(y, q) \in (-1/2, 1/2) \times [1, \infty)$;

then

(4_{f₀}) for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ the function $f_0(x_\alpha, y, q, \cdot)$ is convex on $\mathbb{R}^3 \times \mathbb{R}^2$ for every $(y, q) \in (-1/2, 1/2) \times [1, \infty)$.

Finally, if f satisfies (1.14), then

$$C_1 |\xi_\alpha|^q - \frac{1}{C_1} \leq f_0(x_\alpha, y, q, \xi_\alpha) \leq C_2 (|\xi_\alpha|^q + 1), \tag{4.7}$$

for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$, and for every $(y, q, \xi_\alpha) \in (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2}$.

Proof. For simplicity, in the sequel we use $f(x, y, q, \xi_\alpha, \xi_3)$ in place of $f(x, y, q, (\xi_\alpha, \xi_3))$. (1_{f₀}) follows from the fact that, by definition, f_0 is the infimum of the family of functions $f(\cdot, \cdot, \cdot, \cdot, \xi_3)$ which are $\mathcal{L}^2(\omega) \otimes \mathcal{B}((-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2})$ -measurable.

The proofs of (2_{f₀})–(3_{f₀}) and (4.7) follow along the lines the arguments of Proposition 1 of Ref. 45. The property (4_{f₀}) is an easy consequence of the very definition (1.16) of f_0 . □

4.2. The case of convex densities

Proposition 4.3. *Let $f : \omega \times (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ be a function satisfying (1_f), (2_f) and (4_f) of Lemma 4.1. Let $p \in \mathcal{P}_b(\Omega)$, let $\{p_{\varepsilon_k}\}$ be as in (1.6) and let $\mathcal{F}_{\varepsilon_k}$ be as in (1.13) with $\varepsilon_k \rightarrow 0^+$. Assume that $p_{\varepsilon_k} \rightarrow p_0$ in $L^1(\Omega)$ with $p_0 = p_0(x_\alpha) \in C(\overline{\Omega})$. Let $\{u_{\varepsilon_k}\} \subseteq L^1(\Omega; \mathbb{R}^3)$ be a sequence satisfying (4.2) and such that $u_{\varepsilon_k} \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$. Then*

$$\mathcal{F}_{p^-}(u) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}),$$

where $\mathcal{F}_{p^-} : L^1(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ is the functional defined by

$$\mathcal{F}_{p^-}(v) := \begin{cases} \int_{\omega} f_0(x_\alpha, 0, p_0(x_\alpha), \nabla_\alpha v(x_\alpha)) dx_\alpha & \text{if } v \in W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3) \cap W_0^{1,p^-}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3), \end{cases} \tag{4.8}$$

with f_0 given by (1.16).

Proof. Without loss of generality, we can assume that $\liminf_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty$, and that, up to a not relabeled subsequence, such a liminf is a limit. Then, arguing as in Proposition 4.1 (see Parts (1) and (3)) in view of (4.2), u_{ε_k} weakly converges to u in $W^{1,p^-}(\Omega; \mathbb{R}^3)$, and u can be identified with an element in $W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3) \cap W_0^{1,p^-}(\omega; \mathbb{R}^3)$. Moreover, up to a subsequence

$$\frac{1}{\varepsilon_{k_h}} \nabla_3 u_{\varepsilon_{k_h}} \rightharpoonup \bar{w} \in L^{p^-}(\Omega; \mathbb{R}^3), \tag{4.9}$$

where the latter function might depend on the selected subsequence. Now, we observe that the hypotheses on f guarantee that the function $\varphi : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty]$, given by

$$\varphi(x, y, q, \xi) = \begin{cases} f(x_\alpha, y, q, \xi) & \text{if } (y, q) \in (-1/2, 1/2) \times [1, +\infty), \\ +\infty & \text{otherwise,} \end{cases}$$

satisfies all the assumptions of Theorem 3.2 with $O = \Omega$, $m = 2$ and $d = 3 \times 3$. Since, as $k \rightarrow \infty$, we have that

$$z_k(x) = (\varepsilon_k x_3, p_{\varepsilon_k}(x)) \rightarrow (0, p_0(x_\alpha)) \quad \text{in } L^1(\Omega; \mathbb{R}^2),$$

and, thanks to (4.9) (up to a not relabeled subsequence)

$$U_k(x) = \left(\nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) \rightharpoonup (\nabla u(x_\alpha), \bar{w}(x)) \quad \text{in } L^1(\Omega, \mathbb{R}^{3 \times 3}),$$

it follows that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{\Omega} f \left(x_\alpha, \varepsilon_k x_3, p_{\varepsilon_k}(x), \nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ & \geq \int_{\Omega} f(x_\alpha, 0, p_0(x_\alpha), \nabla_\alpha u(x_\alpha), \bar{w}(x)) dx \\ & \geq \int_{\Omega} f_0(x_\alpha, 0, p_0(x_\alpha), \nabla_\alpha u(x_\alpha)) dx_\alpha dx_3 \\ & = \int_{\omega} f_0(x_\alpha, 0, p_0(x_\alpha), \nabla_\alpha u(x_\alpha)) dx_\alpha, \end{aligned}$$

which concludes the proof, in view of the measurability property (1_{f_0}) satisfied by f_0 (thanks to Lemma 4.1). \square

Proposition 4.4. *Let $f : \omega \times (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty)$ be a function such that*

- $f(x_\alpha, \cdot, \cdot, \cdot)$ is continuous for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$;
- $f(\cdot, y, q, \xi)$ is measurable for every $(y, q, \xi) \in (-1/2, 1/2) \times [1, +\infty) \times \mathbb{R}^{3 \times 3}$.

Moreover, assume that (1.14) holds. Let $p \in \mathcal{P}_b(\Omega)$, let $\{p_{\varepsilon_k}\}$ be as in (1.6) with $\varepsilon_k \rightarrow 0^+$. Assume that $p_{\varepsilon_k} \rightarrow p_0$ in $L^1(\Omega)$ with $p_0 \in \mathcal{P}_b^{\text{log}}(\omega)$. Then, it holds

$$\Gamma(L^{p_0(\cdot)})\text{-lim sup}_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_k} \leq \mathcal{F}, \tag{4.10}$$

where $\mathcal{F}_{\varepsilon_k}$ and \mathcal{F} are defined, respectively, by (1.13) and (1.15).

Proof. Define

$$F := \Gamma(L^{p_0(\cdot)})\text{-lim sup}_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_k}. \tag{4.11}$$

By the very definition of \mathcal{F} , we can restrict to prove that $F \leq \mathcal{F}$ on $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$. First, we consider the case $u, w \in \mathcal{C}_0^\infty(\omega; \mathbb{R}^3)$. Set

$$u_{\varepsilon_k}(x) := u(x_\alpha) + \varepsilon_k x_3 w(x_\alpha),$$

we have that, as $k \rightarrow +\infty$

$$u_{\varepsilon_k}(x_\alpha, x_3) \rightarrow u(x_\alpha) \quad \text{in } L^\infty(\Omega; \mathbb{R}^3),$$

and

$$\begin{aligned} & \left(\nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) \\ &= (\nabla u(x_\alpha) + \varepsilon_k x_3 \nabla w(x_\alpha), w(x_\alpha)) \\ &\rightarrow (\nabla u(x_\alpha), w(x_\alpha)) \quad \text{in } L^\infty(\Omega; \mathbb{R}^{3 \times 3}). \end{aligned}$$

Thus, in particular, there exists $M > 0$ such that

$$\sup_{k \in \mathbb{N}} \|(\nabla u + \varepsilon_k x_3 \nabla w, w)\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})} \leq M.$$

Being $f(x_\alpha, \cdot, \cdot, \cdot)$ continuous for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$, by the above convergences we get that, as $k \rightarrow +\infty$

$$f \left(x_\alpha, \varepsilon_k x_3, p_{\varepsilon_k}(x), \nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) \rightarrow f(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha), w(x_\alpha)),$$

for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ and for every $x_3 \in (-1/2, 1/2)$. Moreover, by (1.14) and (1.27), it holds

$$\begin{aligned} & \operatorname{ess\,sup}_{x \in \Omega} f \left(x_\alpha, \varepsilon_k x_3, p_{\varepsilon_k}(x), \nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) \\ & \leq C_2 \left(\operatorname{ess\,sup}_{x \in \Omega} \left| \left(\nabla_\alpha u_{\varepsilon_k}, \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k} \right) \right|^{p_{\varepsilon_k}(x)} + 1 \right) \\ & = C_2 (\operatorname{ess\,sup}_{x \in \Omega} |(\nabla u + \varepsilon_k x_3 \nabla w, w)|^{p_{\varepsilon_k}(x)} + 1) \\ & \leq C_2 ((1 + M)^{p^+} + 1). \end{aligned}$$

Hence, by applying the Dominated Convergence Theorem, by (1.14) it follows that

$$\begin{aligned} F(u) & \leq \limsup_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}) \\ & = \limsup_{k \rightarrow +\infty} \int_\Omega f \left(x_\alpha, \varepsilon_k x_3, p_{\varepsilon_k}(x), \nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ & = \int_\omega f(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha), w(x_\alpha)) dx_\alpha < +\infty. \end{aligned}$$

Being $w \in C_0^\infty(\omega; \mathbb{R}^3)$ an arbitrary function, we get

$$F(u) \leq \inf_{w \in C_0^\infty(\omega; \mathbb{R}^3)} \int_\omega f(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha), w(x_\alpha)) dx_\alpha \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}^3).$$

Since $C_0^\infty(\omega; \mathbb{R}^3)$ is dense in $L^{p_0(\cdot)}(\omega; \mathbb{R}^3)$, arguing in components and taking into account the growth condition (1.14), by applying Vitali–Lebesgue Dominated Convergence Theorem, we get

$$F(u) \leq \inf_{w \in L^{p_0(\cdot)}(\omega; \mathbb{R}^3)} \int_\omega f(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha), w(x_\alpha)) dx_\alpha \quad \forall u \in C_0^\infty(\Omega; \mathbb{R}^3). \tag{4.12}$$

Following the same argument in p. 558 of Ref. 45, there exists a measurable function $w_0 = w_0(x_\alpha)$ such that

$$\begin{aligned} f_0(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha)) & = \inf_{\xi_3 \in \mathbb{R}^3} [f(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha), \xi_3)] \\ & = f(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha), w_0(x_\alpha)), \end{aligned}$$

for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$. Thanks to (1.14), such a function w_0 belongs to $L^{p_0(\cdot)}(\omega; \mathbb{R}^3)$. Hence, by using (4.12) and by recalling the definition of \mathcal{F} , we get that for every $u \in C_0^\infty(\omega; \mathbb{R}^3)$ it holds

$$\begin{aligned} F(u) & \leq \int_\omega f(x_\alpha, 0, p_0(x_\alpha), \nabla_\alpha u(x_\alpha), w_0(x_\alpha)) dx_\alpha \\ & = \int_\omega f_0(x_\alpha, 0, p_0(x_\alpha), \nabla u(x_\alpha)) dx_\alpha = \mathcal{F}(u). \end{aligned} \tag{4.13}$$

In view of the coercivity assumption on f (see (1.14)), we can apply Lemma 4.1 to obtain that the function $f_0(x_\alpha, \cdot, \cdot, \cdot)$ is lower semicontinuous on $(-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2}$ for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$; being also the supremum of continuous function, we obtain that $f_0(x_\alpha, \cdot, \cdot, \cdot)$ is continuous on $(-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 2}$ for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$. Moreover, taking into account (4.7)

$$f_0(x_\alpha, 0, p_0(x_\alpha), \xi_\alpha) \leq C_2(|\xi_\alpha|^{p_0(x_\alpha)} + 1) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_\alpha \in \omega, \forall \xi_\alpha \in \mathbb{R}^{3 \times 2}.$$

Therefore, \mathcal{F} is strongly continuous in $W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$. Moreover, by the very definition (see (4.11)), the functional F is lower semicontinuous with respect to the strong convergence in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$. Hence, by applying the density of $C_0^\infty(\omega; \mathbb{R}^3)$ in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ (see Remark 2.7, Part (1)), (4.13) implies that

$$F(u) \leq \mathcal{F}(u) \quad \forall u \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3),$$

which gives the desired conclusion (4.10). □

Proof of Theorem 1.3. By Theorem 2.9, when $p_0 \in \mathcal{P}_b^{\log}(\omega)$, the functionals \mathcal{F} in (1.15) and \mathcal{F}_{p^-} in (4.8) coincide. Hence, the result follows from applying on one hand, Proposition 4.3, noticing that for every $\{u_{\varepsilon_k}\} \subseteq L^1(\Omega; \mathbb{R}^3)$ satisfying $\liminf_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty$, the growth condition in (1.14) implies (4.2), and on the other hand, using Proposition 4.4. □

Corollary 4.5. *Under the same assumptions and with the same notation of Theorem 1.3, it results that*

$$\Gamma(L^{p_0(\cdot)})\text{-}\lim_{k \rightarrow +\infty} \mathcal{F}_{\varepsilon_k} = \mathcal{F}.$$

Proof. The result is a direct consequence of Theorem 1.3 together with Proposition 4.4, since the first ensures that \mathcal{F} is a lower bound for the \liminf of $\mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k})$ for every $u_{\varepsilon_k} \rightarrow u$ in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$ while the upper bound is given by Proposition 4.4. □

Remark 4.6. (i) A careful inspection of the result of Corollary 4.5 shows that it holds also if we replace, in the definition of the functional $\mathcal{F}_{\varepsilon_k}$, the null boundary datum on $\partial_L \Omega$ by any other function $u_0 \in \cap_{\varepsilon_k > 0} W^{1,p_{\varepsilon_k}(\cdot)}(\Omega; \mathbb{R}^3) \cap W^{1,p_0(\cdot)}(\Omega; \mathbb{R}^3)$ satisfying $u_0 \equiv u_0(x_\alpha)$.
 (ii) The Γ -convergence result with respect to the $L^{p_0(\cdot)}$ -strong convergence, stated in Corollary 4.5, appears as a natural convergence under the assumption (4.3) on $\{p_{\varepsilon_k}\}$.

4.3. The nonconvex case

This section is devoted to the proofs of Theorems 1.4 and 1.5. With this aim, we recall a relaxation result which extends, in the case of variable exponents, to integral functionals with trace constraints on the boundary the more general results proven

in Ref. 46 in the unconstrained case. For the readers' convenience we state it in our framework, which deals with functionals with growth of the type $|\xi|^{p(x)}$, referring to Theorem 5.1 of Ref. 49, for the result stated in full generality.

Theorem 4.7. *Let U be a bounded open subset of \mathbb{R}^N with Lipschitz boundary, let $p \in \mathcal{P}_b(U)$ and consider a Carathéodory integrand $\varphi : U \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$ satisfying the growth condition*

$$C_1 + C_2|\xi|^{p(x)} \leq \varphi(x, \xi) \leq C_3 + C_4|\xi|^{p(x)}, \tag{4.14}$$

for a.e. $x \in U$ and $\xi \in \mathbb{R}^{d \times N}$, with $0 < C_2 < C_4$. Let $J : L^1(U; \mathbb{R}^d) \rightarrow [0, +\infty]$ be the functional defined by

$$J(u) := \begin{cases} \int_{\Omega} \varphi(x, \nabla u(x)) dx & \text{if } \varphi(\cdot, \nabla u(\cdot)) \in L^1(U), \\ +\infty & \text{otherwise in } L^1(U; \mathbb{R}^d). \end{cases}$$

Assume that $u_0 \in W^{1,p^-}(U; \mathbb{R}^d)$ is such that $J(u_0) < \infty$. If there exists a sequence $\{u_k\}$ such that $u_k \in W^{1,p^-}(U; \mathbb{R}^d) \cap C^1(U_k; \mathbb{R}^d)$, where

$$U_k := \{x \in U : \text{dist}(x, \partial U) > 1/k\}, \quad k \in \mathbb{N},$$

with $u_k|_{\partial U} = u_0|_{\partial U}$, $u_k \rightarrow u_0$ in $W^{1,p^-}(U; \mathbb{R}^d)$ and $J(u_k) \rightarrow J(u_0)$, as $k \rightarrow +\infty$, then there exists a sequence $\{\tilde{u}_k\}$ such that $u_k \in W^{1,p^-}(\Omega; \mathbb{R}^d) \cap W^{1,\infty}(U_k; \mathbb{R}^d)$, $\tilde{u}_k|_{\partial U} = u_0|_{\partial U}$, $\tilde{u}_k \rightarrow u_0$ in $W^{1,p^-}(U; \mathbb{R}^d)$ and

$$\lim_{k \rightarrow \infty} J(\tilde{u}_k) = \int_U Q\varphi(x, \nabla u_0(x)) dx,$$

where $Q\varphi$ denotes the quasiconvex envelope of $\varphi(x, \cdot)$.

Now, exploiting the same techniques of Theorem 1.1, we are in a position to show the lower bound inequality.

Proposition 4.8. *Let $\{\mathcal{I}_{\varepsilon_k}\}$ and \mathcal{I} be the functionals defined by (1.17) and (1.20), respectively. Under the assumptions of Theorem 1.4*

$$\mathcal{I}(u) \leq \liminf_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_k}(u_{\varepsilon_k}),$$

for every $L^1(\Omega; \mathbb{R}^3) \ni u_{\varepsilon_k} \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$ as $k \rightarrow +\infty$.

Proof. Without loss of generality, assume that $\liminf_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty$. By exploiting the assumption $p_0 \in \mathcal{P}_b^{\text{log}}(\omega)$, Proposition 4.1 ensures that u can be identified with a function in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ and $u_{\varepsilon_k} \rightarrow u$ in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$ and $u \rightharpoonup u$ in $W^{1,p^-}(\Omega; \mathbb{R}^3)$. Taking into account that $p_0^- > 1$, let $0 < \delta < \min\{1/2, p_0^- - 1\}$ and let $k_0 \in \mathbb{N}$ be such that

$$1 < p_0(x) - \delta < p_{\varepsilon_k}(x) < p_0(x) + \delta \quad \text{for } \mathcal{L}^3\text{-a.e. } x \in \Omega,$$

for every $k \geq k_0$. By using (3.8), we have that

$$\begin{aligned} & 2\delta|\Omega| + (2\delta)^{-2\delta} \int_{\Omega} \mathcal{W}^{p_{\varepsilon_k}(x)} \left(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ & \geq \int_{\Omega} \mathcal{W}^{p_0(x_{\alpha})-\delta} \left(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx. \end{aligned}$$

Hence, by passing to the liminf as $k \rightarrow +\infty$ and by applying the very definition of \mathcal{W}_0 and $Q(\mathcal{W}_0^{p_0(\cdot)})$, we get that

$$\begin{aligned} & 2\delta|\Omega| + (2\delta)^{-2\delta} \liminf_{k \rightarrow +\infty} \int_{\Omega} \mathcal{W}^{p_{\varepsilon_k}(x)} \left(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ & \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} \mathcal{W}^{p_0(x_{\alpha})-\delta} \left(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ & \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} \mathcal{W}_0^{p_0(x_{\alpha})-\delta} (x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x)) dx \\ & \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} Q(\mathcal{W}_0^{p_0(x_{\alpha})-\delta})(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x)) dx. \end{aligned} \tag{4.15}$$

Then, by Theorem 3.5, we have

$$\begin{aligned} & \liminf_{k \rightarrow +\infty} \int_{\Omega} Q(\mathcal{W}_0^{p_0(x_{\alpha})-\delta})(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x)) dx \\ & \geq \int_{\Omega} Q(\mathcal{W}_0^{p_0(x_{\alpha})-\delta})(x_{\alpha}, \nabla_{\alpha} u(x_{\alpha})) dx_{\alpha} dy \\ & = \int_{\omega} Q(\mathcal{W}_0^{p_0(x_{\alpha})-\delta})(x_{\alpha}, \nabla_{\alpha} u(x_{\alpha})) dx_{\alpha}. \end{aligned}$$

Hence, by combining (4.15) and the above inequality, as $\delta \rightarrow 0^+$, we obtain

$$\begin{aligned} & \liminf_{k \rightarrow +\infty} \int_{\Omega} \mathcal{W}^{p_{\varepsilon_k}(x)} \left(x_{\alpha}, \nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ & \geq \liminf_{\delta \rightarrow 0^+} \int_{\omega} Q(\mathcal{W}_0^{p_0(x_{\alpha})-\delta})(x_{\alpha}, \nabla_{\alpha} u(x_{\alpha})) dx_{\alpha}. \end{aligned}$$

Now, we claim that

$$\begin{aligned} & |\mathcal{W}_0(x_{\alpha}, \xi_{\alpha}) - \mathcal{W}_0(x_{\alpha}, \eta_{\alpha})| \leq w(x_{\alpha}, |\xi_{\alpha} - \eta_{\alpha}|) \\ & \text{for every } \xi_{\alpha}, \eta_{\alpha} \in \mathbb{R}^{3 \times 2}, \text{ for } \mathcal{L}^2\text{-a.e. } x_{\alpha} \in \omega. \end{aligned} \tag{4.16}$$

Indeed, let $N \subseteq \omega$ be such that $\mathcal{L}^2(N) = 0$ and (1.18), (1.9) and (1.19) hold for every $x_{\alpha} \in \omega \setminus N$ and for every $\xi, \eta \in \mathbb{R}^{3 \times 3}$. Hence, thanks to the coercivity assumption (1.18), for a.e. $x_{\alpha} \in \omega \setminus N$ and for every $\xi_{\alpha} \in \mathbb{R}^{3 \times 2}$ there exists $\bar{\xi}_3 \in \mathbb{R}^3$ such that

$\mathcal{W}_0(x_\alpha, \xi_\alpha) = \mathcal{W}(x_\alpha, \xi_\alpha, \bar{\xi}_3)$. Therefore, for every $\eta_\alpha \in \mathbb{R}^{3 \times 2}$

$$\begin{aligned} \mathcal{W}_0(x_\alpha, \eta_\alpha) &\leq \mathcal{W}(x_\alpha, \eta_\alpha, \bar{\xi}_3) \leq w(x_\alpha, |(\xi_\alpha, \bar{\xi}_3) - (\eta_\alpha, \bar{\xi}_3)|) + \mathcal{W}(x_\alpha, \xi_\alpha, \bar{\xi}_3) \\ &= w(x_\alpha, |\xi_\alpha - \eta_\alpha|) + \mathcal{W}(x_\alpha, \xi_\alpha, \bar{\xi}_3) \\ &\leq w(x_\alpha, |\xi_\alpha - \eta_\alpha|) + \mathcal{W}_0(x_\alpha, \xi_\alpha). \end{aligned}$$

By changing the role of ξ_α and η_α we get (4.16), which implies that the map $(x_\alpha, q, \xi) \mapsto \mathcal{W}_0^q(x_\alpha, \xi)$ is a Carathéodory integrand. In particular, this in turn yields that for every $q \geq 1$ and \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ the function $\mathcal{W}_0^q(x_\alpha, \cdot)$ is Borel measurable and locally bounded; moreover, it is bounded from below by the right-hand side of (1.18), hence, its quasiconvex envelope $Q(\mathcal{W}_0^q)(x_\alpha, \cdot)$ satisfies the assumptions of Theorem 6.9 of Ref. 18, and can be represented as

$$Q(\mathcal{W}_0^q)(x_\alpha, \xi_\alpha) = \inf \left\{ \frac{1}{\mathcal{L}^2(D)} \int_D \mathcal{W}_0^q(x_\alpha, \xi_\alpha + \nabla_\alpha \psi(y_\alpha)) dy_\alpha : \psi \in W_0^{1,\infty}(D; \mathbb{R}^3) \right\},$$

with D any bounded open subset of \mathbb{R}^2 .

Hence, by using Part 3(b) in Proposition 9.5 of Ref. 18, the map $(x_\alpha, q, \xi) \mapsto Q(\mathcal{W}_0^q)(x_\alpha, \xi)$ is a Carathéodory integrand too. Therefore, for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ and for every $\xi_\alpha \in \mathbb{R}^{3 \times 2}$ it holds

$$\lim_{\delta \rightarrow 0^+} Q(\mathcal{W}_0^{p_0(x_\alpha) - \delta})(x_\alpha, \xi_\alpha) = Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \xi_\alpha), \tag{4.17}$$

and, by using Fatou’s lemma, we finally get

$$\begin{aligned} \liminf_{\varepsilon_k \rightarrow 0^+} \int_\Omega \mathcal{W}^{p_{\varepsilon_k}(x)} \left(x_\alpha, \nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ \geq \int_\omega Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \nabla_\alpha u(x_\alpha)) dx_\alpha. \end{aligned} \quad \square$$

Now, we are in a position to prove the upper bound inequality for Theorem 1.4.

Proposition 4.9. *Under the assumptions of Theorem 1.4, it results that*

$$\Gamma(L^1)\text{-}\limsup_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_k} \leq \mathcal{I},$$

where \mathcal{I}_ε and \mathcal{I} are defined by (1.17) and (1.20), respectively.

Proof. First of all, we observe that the functional $\mathcal{I}_{\varepsilon_k}$ coincides with the functional $\mathcal{F}_{\varepsilon_k}$ in (1.13) when $f : \omega \times (-1/2, 1/2) \times [1, \infty) \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ is given by $f(x_\alpha, y, q, \xi) = \mathcal{W}^q(x_\alpha, \xi)$. Moreover, the assumptions on \mathcal{W} ensure that for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$ and $\xi_\alpha \in \mathbb{R}^{3 \times 2}$, it holds

$$\inf_{\xi_3 \in \mathbb{R}^3} (\mathcal{W}^{p_0(x_\alpha)}(x_\alpha, \xi_\alpha, \xi_3)) = \mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \xi_\alpha), \tag{4.18}$$

and the infimum above is attained.

Hence, by applying Proposition 4.4, we have that

$$I := \Gamma(L^{p_0(\cdot)})\text{-}\limsup_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_k} \leq \mathcal{I}_1, \tag{4.19}$$

where

$$\mathcal{I}_1(u) := \begin{cases} \int_{\omega} \mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } u \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3). \end{cases}$$

We introduce

$$J(u) := \begin{cases} \int_{\omega} \mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \nabla_\alpha u(x_\alpha)) dx_\alpha & \text{if } \mathcal{W}_0^{p_0(\cdot)}(\cdot, \nabla_\alpha u(\cdot)) \in L^1(\omega), \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^3). \end{cases}$$

Thanks to (1.9) and by applying Lemma 4.1, $\mathcal{W}_0 : \omega \times \mathbb{R}^{3 \times 2} \rightarrow \mathbb{R}$ is a Carathéodory integrand satisfying

$$C_1 |\xi_\alpha| - \frac{1}{C_1} \leq \mathcal{W}_0(x_\alpha, \xi_\alpha) \leq C_2 (|\xi_\alpha| + 1) \tag{4.20}$$

for \mathcal{L}^2 -a.e. $x_\alpha \in \omega$, for every $\xi \in \mathbb{R}^{3 \times 3}$.

This, clearly, implies that also the map $(x_\alpha, \xi_\alpha) \rightarrow \mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \xi_\alpha)$ is a Carathéodory integrand satisfying the growth condition (4.14).

In order to show that $I \leq \mathcal{I}$, without loss of generality, we can consider the case when $\mathcal{I}(u) < +\infty$. Then, $u \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ and, thanks to (4.14), $J(u) < +\infty$. Moreover, being $p_0 \in \mathcal{P}_b^{\log}(\omega)$, Part (1) in Remark 2.7 entails that $C_0^\infty(\omega; \mathbb{R}^3)$ is dense in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$. Since J is continuous with respect to the convergence in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$, we get that J satisfies the assumptions of Theorem 4.7 on $U = \omega$. Hence, we can infer the existence of a sequence $\tilde{u}_{\varepsilon_k} \in W_0^{1,p_0^-}(\omega; \mathbb{R}^3) \cap W^{1,\infty}(\omega_{\varepsilon_k}; \mathbb{R}^3)$, such that $\tilde{u}_{\varepsilon_k} \rightharpoonup u$ in $W^{1,p_0^-}(\omega; \mathbb{R}^3)$ and

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \int_{\omega} \mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \nabla \tilde{u}_{\varepsilon_k}(x_\alpha)) dx_\alpha \\ &= \int_{\omega} Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \nabla u(x_\alpha)) dx_\alpha = \mathcal{I}(u) < +\infty. \end{aligned} \tag{4.21}$$

In particular, thanks to the coercivity of $\mathcal{W}_0^{p_0(\cdot)}$, (4.21) implies that there exists $k_0 \in \mathbb{N}$ such that $\nabla \tilde{u}_{\varepsilon_k} \in L^{p_0(\cdot)}(\omega; \mathbb{R}^{3 \times 2})$ for every $k \geq k_0$. By exploiting Proposition 2.8, this implies that $u_{\varepsilon_k} \in W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$. Finally, by Theorem 2.9, we obtain that $u_{\varepsilon_k} \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ for $k \geq k_0$. Hence, $\mathcal{I}_1(\tilde{u}_{\varepsilon_k}) = J(\tilde{u}_{\varepsilon_k})$ for every $k \geq k_0$. Moreover, by using (4.20) and the Poincaré inequality in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ (see (2.6)), it is easy to obtain that $\tilde{u}_{\varepsilon_k} \rightharpoonup u$ in $W^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$. Therefore, by Proposition 2.8, $\tilde{u}_{\varepsilon_k} \rightarrow u$ in $L^{p_0(\cdot)}(\omega; \mathbb{R}^3)$ and then $\tilde{u}_{\varepsilon_k} \rightarrow u$ in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$. Since I is lower

semicontinuous with respect to such a convergence, by (4.19) and (4.21), we get that

$$\begin{aligned} I(u) &\leq \liminf_{k \rightarrow +\infty} I(\tilde{u}_{\varepsilon_k}) \leq \lim_{k \rightarrow +\infty} \mathcal{I}_1(\tilde{u}_{\varepsilon_k}) = \lim_{k \rightarrow +\infty} J(\tilde{u}_{\varepsilon_k}) \\ &= \int_{\omega} Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \nabla \tilde{u}(x_\alpha)) dx_\alpha = \mathcal{I}(u). \end{aligned} \quad \square$$

Proof of Theorem 1.4. It follows from Propositions 4.8 and 4.9. □

Finally, we prove Theorem 1.5, splitting the proof in a double inequality. We underline that the adopted arguments are very similar to those exploited in the proof of Theorem 1.4, relying for the lower bound on techniques stemmed from Theorem 1.2. Indeed, we provide for the reader’s convenience only the proof of the main differences. Arguing as in Theorem 3.6, it is possible to show that the result generalizes to the case when W depends also on x_α ; however we prefer to focus just on the model behavior of an energy density of product-type between the spatial variable and a gradient dependence.

Proposition 4.10. *Let $\{\mathcal{J}_{\varepsilon_k}\}$ and \mathcal{J} be the functionals defined by in (1.21) and (1.25), respectively. Under the assumptions of Theorem 1.5*

$$\mathcal{J}(u) \leq \liminf_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}),$$

for every $L^1(\Omega; \mathbb{R}^3) \ni u_{\varepsilon_k} \rightarrow u$ in $L^1(\Omega; \mathbb{R}^3)$ as $\varepsilon_k \rightarrow 0^+$.

Proof. Without loss of generality, we can assume that $\liminf_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty$. Thus, by (1.23)

$$\liminf_{k \rightarrow +\infty} \int_{\Omega} \mathcal{W}^{p_{\varepsilon_k}(x)} \left(\nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \leq \frac{1}{a^-} \liminf_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_k}(u_{\varepsilon_k}) < +\infty.$$

In turn, thanks to (1.22), (4.2) holds. Hence, in view of our assumptions on p_{ε_k} and p_0 , by (1), (3) and (4) in Proposition 4.1, we have that, up to a not relabeled subsequence $u_{\varepsilon_k} \rightharpoonup u$ in $W^{1,p^-}(\Omega; \mathbb{R}^3)$ and strongly in $L^{p_0(\cdot)}(\Omega; \mathbb{R}^3)$ with $u \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$.

Furthermore, a_{ε_k} , p_{ε_k} and u_{ε_k} satisfy the assumptions of Theorem 1.2, hence, we can mimick the arguments therein. More precisely, by reasoning as in the first part of Theorem 1.2 (see (3.10)) we have that

$$\begin{aligned} &\liminf_{k \rightarrow +\infty} \int_{\Omega} a_{\varepsilon_k}(x) \mathcal{W}^{p_{\varepsilon_k}(x)} \left(\nabla_{\alpha} u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\ &\geq \liminf_{k \rightarrow +\infty} \int_{\Omega} a_0(x_\alpha) \mathcal{W}_0^{p_{\varepsilon_k}(x)}(\nabla_{\alpha} u_{\varepsilon_k}(x)) dx, \end{aligned}$$

where \mathcal{W}_0 is the function appearing in (1.26). Moreover, by applying the same argument of Lemma 3.1 and Theorem 3.5, for every $0 < \delta < \min\{1/2, p_0^- - 1\}$ it holds

$$\begin{aligned}
 & (2\delta)^{-2\delta} \liminf_{k \rightarrow +\infty} \int_{\Omega} a_0(x_\alpha) \mathcal{W}_0^{p_{\varepsilon_k}(x)} (\nabla_\alpha u_{\varepsilon_k}(x)) dx + 2\delta \|a\|_{L^\infty(\Omega)} \mathcal{L}^3(\Omega) \\
 & \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} a_0(x_\alpha) \mathcal{W}_0^{p_0(x_\alpha) - \delta} (\nabla_\alpha u_{\varepsilon_k}(x)) dx \\
 & \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} a_0(x_\alpha) Q(\mathcal{W}_0^{p_0(x_\alpha) - \delta}) (\nabla_\alpha u_{\varepsilon_k}(x)) dx \\
 & \geq \int_{\Omega} a(x_\alpha) Q(\mathcal{W}_0^{p_0(x_\alpha) - \delta}) (\nabla_\alpha u(x_\alpha)) dx.
 \end{aligned} \tag{4.22}$$

Now, arguing as in Proposition 4.8 (see (4.17)) i.e. exploiting the fact that quasisconvex envelope $Q(\mathcal{W}_0^q)$ satisfies the assumptions of Theorem 6.9 of Ref. 18, it follows that

$$\lim_{\delta \rightarrow 0} Q(\mathcal{W}_0^{p_0(x_\alpha) - \delta})(\xi_\alpha) = Q(\mathcal{W}_0^{p_0(x_\alpha)})(\xi_\alpha). \tag{4.23}$$

Hence, combining (4.22) and (4.23) and sending $\delta \rightarrow 0$, Fatou’s lemma gives

$$\begin{aligned}
 & \liminf_{k \rightarrow +\infty} \int_{\Omega} a_{\varepsilon_k}(x) \mathcal{W}^{p_{\varepsilon_k}(x)} \left(\nabla_\alpha u_{\varepsilon_k}(x), \frac{1}{\varepsilon_k} \nabla_3 u_{\varepsilon_k}(x) \right) dx \\
 & \geq \int_{\omega} a_0(x_\alpha) Q(\mathcal{W}_0^{p_0(x_\alpha)}) (\nabla_\alpha u(x_\alpha)) dx_\alpha,
 \end{aligned}$$

for every $u \in \mathcal{W}_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$, which, in view of the definition of \mathcal{J} in (1.25), concludes the proof. □

Proposition 4.11. *Under the assumptions of Theorem 1.5, it results that*

$$\Gamma(L^1)\text{-}\limsup_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_k} \leq \mathcal{J},$$

where $\{\mathcal{J}_{\varepsilon_k}\}$ and \mathcal{J} are the functionals defined by (1.21) and (1.25), respectively.

Proof. Without loss of generality, we can restrict to consider the case when $\mathcal{J}(u) < +\infty$. Taking into account (1.24), up to some minor modifications, we can reason as in the first part of Proposition 4.4 and obtain that

$$I(u) \leq \inf_{w \in L^{p_0(\cdot)}(\omega; \mathbb{R}^3)} \int_{\omega} a_0(x_\alpha) \mathcal{W}^{p_0(x_\alpha)} (\nabla u(x_\alpha), w(x_\alpha)) dx_\alpha \quad \forall u \in C_0^\infty(\omega; \mathbb{R}^3), \tag{4.24}$$

where

$$I := \Gamma(L^{p_0(\cdot)})\text{-}\limsup_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_k}.$$

Following the same argument in p. 558 of Ref. 45, there exists a measurable function $w_0 = w_0(x_\alpha)$ such that

$$\mathcal{W}_0(\nabla u(x_\alpha)) = \inf_{\xi_3 \in \mathbb{R}^3} [\mathcal{W}(\nabla u(x_\alpha), \xi_3)] = \mathcal{W}(\nabla u(x_\alpha), w_0(x_\alpha)) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_\alpha \in \omega.$$

In particular, by using (4.18), we deduce that

$$\begin{aligned} a_0(x_\alpha) \mathcal{W}_0^{p_0(x_\alpha)}(\nabla_\alpha u(x_\alpha)) &= a_0(x_\alpha) \mathcal{W}^{p_0(x_\alpha)}(\nabla_\alpha u(x_\alpha), w_0(x_\alpha)) \\ &= \inf_{\xi_3 \in \mathbb{R}^3} [a_0(x_\alpha) \mathcal{W}^{p_0(x_\alpha)}(\nabla_\alpha u(x_\alpha), \xi_3)]. \end{aligned}$$

Moreover, thanks to (1.22), such a function w_0 belongs to $L^{p_0(\cdot)}(\omega; \mathbb{R}^3)$. Hence, by using (4.24), we get that

$$I(u) \leq \mathcal{J}_1(u) \quad \forall u \in C_0^\infty(\omega; \mathbb{R}^3),$$

where $\mathcal{J}_1 : L^1(\omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ is the functional defined by

$$\mathcal{J}_1(u) := \begin{cases} \int_\omega a_0(x_\alpha) \mathcal{W}_0^{p_0(x_\alpha)}(\nabla_\alpha u(x_\alpha)) \, dx_\alpha & \text{if } u \in W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3), \\ +\infty & \text{otherwise in } L^1(\omega; \mathbb{R}^3). \end{cases}$$

Since $a_0 \in L^\infty(\omega)$ and thanks to the growth condition

$$\mathcal{W}_0(\xi_\alpha) \leq C_2(|\xi_\alpha| + 1) \quad \text{for every } \xi_\alpha \in \mathbb{R}^{3 \times 2},$$

inherited by (1.12), \mathcal{J}_1 results strongly continuous on $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$.

By using the density of $C_0^\infty(\omega; \mathbb{R}^3)$ in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ (see Part (1) in Remark 2.7) and the lower semicontinuity of I with respect to the strong convergence in $W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ we obtain that

$$I(u) \leq \mathcal{J}_1(u) \quad \forall u \in L^1(\omega; \mathbb{R}^3). \tag{4.25}$$

Now, by reasoning as in the proof of Proposition 4.9, we can show that there exists a sequence $\{\tilde{u}_{\varepsilon_k}\} \subseteq W_0^{1,p_0(\cdot)}(\omega; \mathbb{R}^3)$ such that $\tilde{u}_{\varepsilon_k} \rightharpoonup u$ in $L^{p_0(\cdot)}(\omega; \mathbb{R}^3)$ and

$$\begin{aligned} \lim_{k \rightarrow +\infty} \mathcal{J}_1(\tilde{u}_{\varepsilon_k}) &= \lim_{k \rightarrow +\infty} \int_\omega a_0(x_\alpha) \mathcal{W}_0^{p_0(x_\alpha)}(x_\alpha, \nabla \tilde{u}_{\varepsilon_k}(x_\alpha)) \, dx_\alpha \\ &= \int_\omega Q(a_0(x_\alpha) \mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \nabla u(x_\alpha)) \, dx_\alpha. \end{aligned} \tag{4.26}$$

Since I is lower semicontinuous with respect to the $L^{p_0(\cdot)}$ -convergence and

$$Q(a_0(x_\alpha) \mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \cdot) = a_0(x_\alpha) Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \cdot), \quad \text{for } \mathcal{L}^2\text{-a.e. } x_\alpha \in \omega,$$

(4.26) combined with (4.25) implies

$$\begin{aligned} I(u) &\leq \liminf_{k \rightarrow +\infty} I(\tilde{u}_{\varepsilon_k}) \leq \lim_{k \rightarrow +\infty} \mathcal{J}_1(\tilde{u}_{\varepsilon_k}) = \lim_{k \rightarrow +\infty} J(\tilde{u}_{\varepsilon_k}) \\ &= \int_{\omega} Q(\mathcal{W}_0^{p_0(x_\alpha)})(x_\alpha, \nabla \tilde{u}(x_\alpha)) dx_\alpha = \mathcal{J}(u), \end{aligned}$$

and the proof is over. □

Proof of Theorem 1.5. It follows from Propositions 4.10 and 4.11. □

Analogously to Corollary 4.5, the following results hold. The proof is omitted, being very similar to the one presented in the convex case.

Corollary 4.12. *Let $\mathcal{I}_{\varepsilon_k}, \mathcal{I}, \mathcal{J}_{\varepsilon_k}$ and \mathcal{J} be the functionals in (1.17), (1.20), (1.21) and (1.25), respectively. Under the same assumptions and with the same notation of Theorems 1.4 and 1.5 it results that*

$$\Gamma(L^{p_0(\cdot)})\text{-}\lim_{k \rightarrow +\infty} \mathcal{I}_{\varepsilon_k} = \mathcal{I},$$

and

$$\Gamma(L^{p_0(\cdot)})\text{-}\lim_{k \rightarrow +\infty} \mathcal{J}_{\varepsilon_k} = \mathcal{J}.$$

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