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ORIGINAL PAPER

Geometry education at secondary level – a systematic literature review

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Abstract
This systematic literature review summarises research on geometry education published since 2013. It begins with a conceptualization of the geometry education using the didactic tetrahedron model. The PRISMA guidelines were used to search five databases, in addition to conference proceedings and journals. The keywords and abstracts from the articles were analyzed using the "Voyant Tool" that created a list of the frequencies of words and their possible relationships to each other. Based on this analysis, the articles were coded using the didactic tetrahedron model. This informed the development of codes that were analyzed to develop categories related to the process skills Arguing and Reasoning, Proving and Reasoning, Problem Solving, and Modeling; the content skills Plane Geometry, Spatial Geometry, and Measurement, including Area and Volume; concepts that include Assessment, Gamification and Computational Thinking; and finally Teaching Competencies and Teaching Theories. The results of this overview show trends and developments that should be considered in geometry research in the coming years.

Keywords Geometry · Geometry education · Literature review

1 Introduction

Geometry teaching has undergone several important changes and new approaches in the last 100 years (Sinclair, 2008). At the beginning of the last century, an axiomatic structure dominated school geometry based on Euclid's *Elements*. At the end of the 20th century, many countries offered students an intuitive approach to geometry in primary school with successive formalisation in high school (e.g, NCTM, 2000; Sinclair, 2008). There was a move away from classical Euclidean Geometry, from compass and ruler constructions and a systematic structure to the development of spatial reasoning through real-world applications. Nowadays, the use of dynamic geometry software (DGS) is obligatory, and dynamic spatial geometry software (3D-DGS) offers new ways to interact with 3D geometric objects.

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2 Research questions and theoretical framework

Building on these recent shifts in geometry education, several review studies have examined how geometry education continues to evolve These studies highlight recurring themes from earlier work while also describing emerging trends. In this context, our goal is to conduct a systematic literature review of research focused on the secondary level from 2013 to 2024. We were guided by the following two research questions:

1) What has been the focus of research studies conducted on the teaching and learning of geometry?
2) What new research directions have been pursued during that time period?

Our review is grounded in the Tetrahedron Model (Rezat & Sträßer, 2012; Fig. 1), which serves as a framework for conceptualizing the teaching and learning of geometry. This model extends the didactic triangle (Student, Teacher, Mathematics) with the addition of a vertex related to resources and artifacts. The model emphasizes the importance of the vertices and relationships among them, offering a holistic view of the teaching and learning of geometry. Its structure is especially useful for analyzing the role of resources, and

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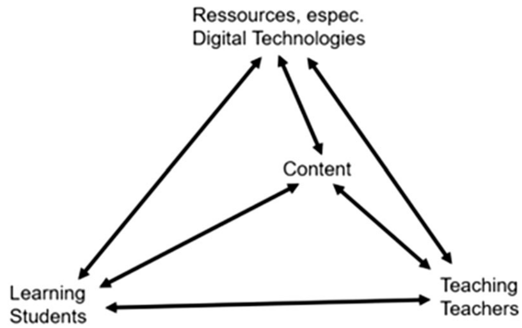


Fig. 1 The Tetrahedron Model (T-model)

it allows us to identify both well-studied and underexplored areas within the existing literature.

This model emphasizes how teaching and learning of mathematics content are mediated not only through teacher-student interactions, but also by the artifacts and resources present in instructional environments. Resources can include manipulatives like base-ten blocks, textbooks, or technology tools. Thus, it allows researchers to explore the pathways through which resources influence teaching practices and student learning. The Tetrahedron Model served as a guiding theoretical framework, shaping both the structure and analytical lens of our systematic literature review.

3 Review studies in geometry

A key focus across prior review studies has been the increasing role of digital technologies, particularly Dynamic Geometry Software (DGS), in enhancing students' understanding of geometry. For example, Sinclair et al.'s 2016 ICME survey report and Jones and Tzekaki's (2016) review emphasized the impact of technology on teaching and learning, particularly related to visual reasoning, supporting students with conjecturing, and assisting them with proving. Additionally, these reviews have noted that research has expanded to focus on the role of gestures and diagrams. There is also a trend to incorporate cognitive science perspectives, such as embodied cognition. The significance of spatial thinking and deductive reasoning, as well as the growing importance of instructional tools, materials development, and teacher education in shaping effective geometry instruction, has also been noted by Herbst et al. (2018).

Other review studies, such as Jablonski and Ludwig (2023) and Sinclair et al. (2017), have pointed to key gaps in geometry education research, including the need for broader methodological approaches and stronger connections between theoretical findings and classroom practice. These studies have also identified crucial areas for future exploration, such as the role of geometric transformations in introducing key concepts and the need for more robust theoretical frameworks to study geometry learning and teaching.

If we summarize these reviews, we see an increasing importance of the influence of digital technologies, especially DGS, on the learning of geometry concepts, and the emphasis on spatial thinking with 3D-DGS. Proofs and the act of proving remain an important reason for teaching geometry in schools. This review aims to identify important domains of geometry learning and teaching in secondary high school, starting with grade 5 or 6 up to grade 12 or 13, and in teacher education, if it is related to school mathematics, in the last ten years and to specify these by highlighting the contents, topics and themes in reviewed journals and conference proceedings.

4 Methods

For our review, we followed the Preferred Reporting Items for Systematic reviews and Meta-Analyses (PRISMA) guidelines, which include a search of databases and other sources (Page et al., 2021). Our process involved four main steps: identification, screening, analysis, and coding. Because we included both databases and other sources, the first step involved two different activities (Fig. 2).

4.1 The first step

For the identification step, we engaged in two different activities. The first activity was a hand search of peer-reviewed journals, and the second activity was a systematic review of the databases.

4.1.1 The first activity: identification of studies from other sources and databases

For the hand search of peer-reviewed journals, consistently identified as top-tier from 2015 to 2024 (Williams & Leatham, 2017), the keywords geometry, geometric, and geometry education were used. These journals included the *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *The Journal of Mathematical Behavior*, *Digital Experiences in Mathematics Education*, *ZDM – Mathematics Education*. We chose the proceedings of the conferences CERME 9 (2015) to CERME 13 (2023), and of PME 39 (2015) to PME 46 (2023). These conferences serve as meeting points for researchers from various countries, and also showcase the contributions from young researchers. They are intended to provide some insights into current research trends. We also evaluated the ICME 13 monograph, *International Perspectives on the Teaching* and *Learning of Geometry in Secondary Schools* (Herbst et al., 2018). We collected all articles we found in our journals and proceedings in an Excel file with the name(s) of the author(s), the title, the origin or source, the keywords, and the abstract. We listed nearly 350 papers.

Fig. 2 A description of our systematic review process following PRISMA guidelines that involves the identification of studies from other sources and databases

First Step: Identification of Studies
Activity 1: Identification of studies using a hand search process of journals and conference proceedings. A preliminary analysis of articles was conducted after this activity.
Activity 2: Identification of studies from databases
Second Step: Screening. Review of articles from Activity 2 to remove duplicates and exclude
those that did not meet inclusion criteria.
Third Step: Analysis. An analysis of abstracts to identify key concepts and themes.
Fourth Step: **Coding** of articles using the Tetrahedron model. Identification of themes used to organize the analysis of articles and organization of the literature review

1	students	952
2	geometry	779
3	study	509
4	mathematics	491
5	learning	480
6	mathematical	409
7	teachers	379
8	geometry	328
9	teaching	326
10	use	294
11	research	280
12	school	272
13	students	265
14	based	265
15	proof	244

Fig. 3 The most common words in the documents. An excerpt from the complete list

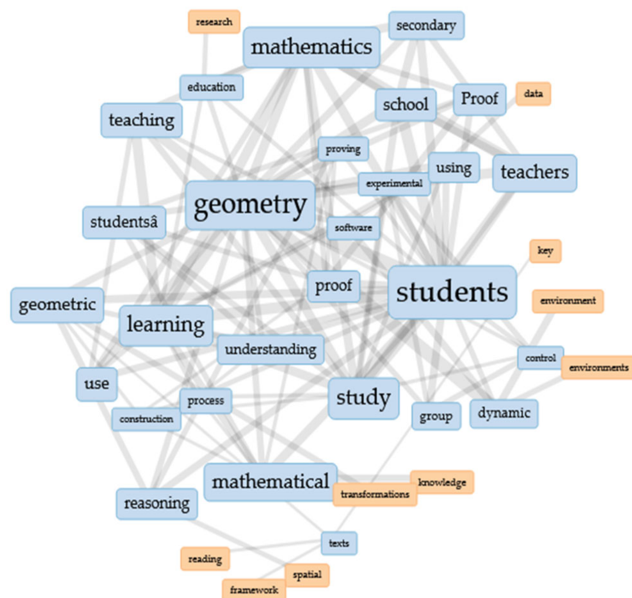


Fig. 4 The relations between keywords with the V-tool. Font size represents the frequency of a word, and lines represent the proximity of words

To gather more information about the keywords or concepts appearing in the documents and to learn more about the relation between these concepts, we used the Voyant Tool (V-tool). It is a web-based text reading and analysis environment. It lists the number of times words are used in the titles and abstracts of all articles and shows the relationship between these words. In Fig. 3 there is the top of the ranking list of the most common words in our documents.

Figure 4 shows the relations between these keywords or concepts with the V-tool. Keywords can also be chosen interactively in the "word-net" and put into the foreground.

The V-tool analysis confirmed that the T-model framework, which focuses on geometry (content), students, teachers, and resources, was a viable tool for analyzing the studies we identified from the hand search.

4.1.2 The second activity: a review of databases

The second activity in the first step of the process was a systematic review of the databases ERIC, JSTOR, SpringerLink, SCOPUS and Web of Science. We looked for research articles written in English, between 2013 and 2024 and used the search string "Geometry & Education & Secondary & Learning & Teaching". These keywords were informed by our research question and the keywords appearing in the V-tool analysis.

4.2 The second step: screening

Once the studies were identified from the databases they were screened. This process involved the use of Covidence tool, which is an online review tool. This platform identified duplicates (n = 26) that we reviewed and removed from our database. We then looked at the titles of studies to ensure that they were published in English and were related to geometry. Studies that did not were excluded (n = 795). After this screening process, we retrieved the abstracts of over 1300 studies. We reviewed the abstracts to determine whether they addressed geometry, reported results from an empirical study, were published in English, involved learning or teaching at the secondary level, and were

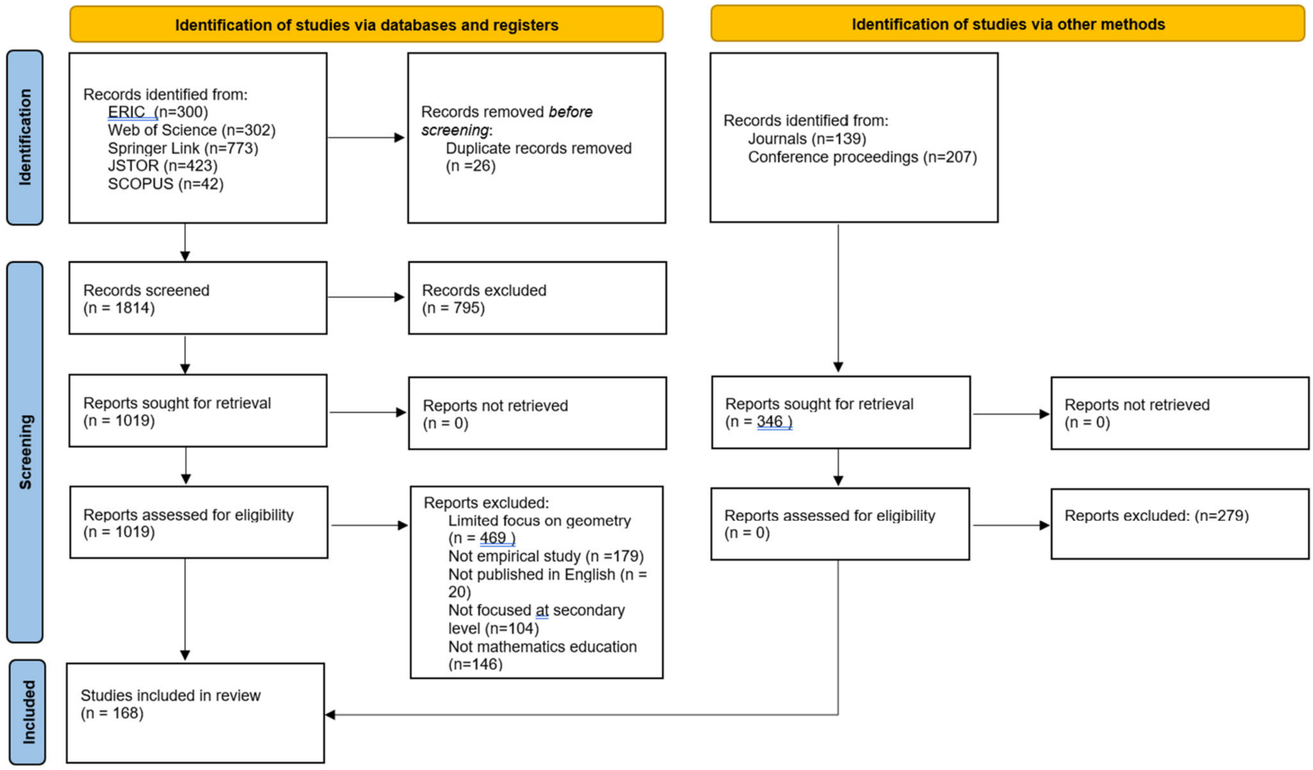


Fig. 5 The flow chart of the PRISMA search for the selected articles from both steps

related to mathematics education rather than pure mathematics. Covidence tracked the exclusion reasons and produced the PRISMA chart shown in Fig. 5.

4.3 The third step: analysis

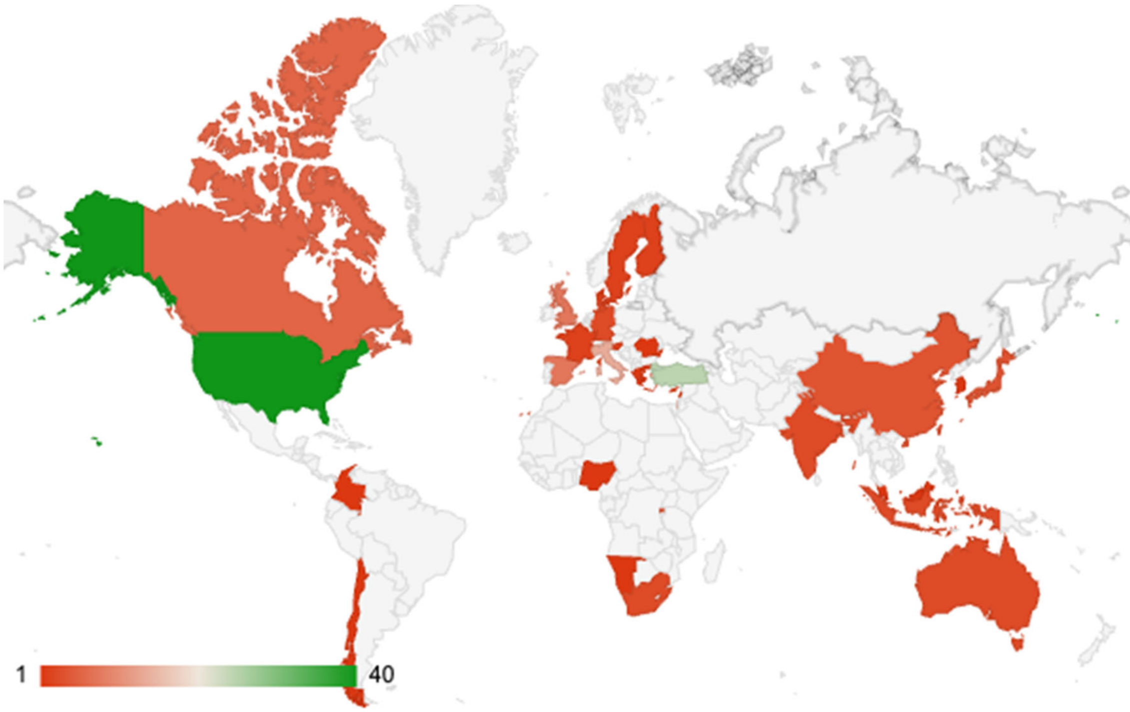
We used the V-Tool again with the new list of articles (titles and abstracts) to identify the most frequently used keywords. The results were quite similar to those from our manual search, which means the addition of new articles only changed the frequency of appearances but did not affect the ranking of the keywords or the relationships between them.

We also analyzed the first authors of our articles to determine the countries affiliated with their institutions to ensure a comprehensive review. The articles included 31 different countries depicted in the map shown in Fig. 6.

4.4 The fourth step: coding and the identification of categories

We coded the articles included in our database by reviewing the abstracts using the four vertices of the T-model (student, teacher, content, resources). If the codes could not be discerned from the abstract, the document was reviewed. For example, the article "Instrumental appropriation of a collaborative, dynamic-geometry environment and geometrical understanding" was coded student, content, and resources. We coded several together to gain an understanding of how we were applying the codes, coded individually, and then met to review our codes together to achieve consensus on the codes. Once the codes were applied, we analyzed the keywords that were used in the studies to identify categories. For example, with the studies that focused on content, we noticed many addressed reasoning, argumentation, or proof and summarized them using categories as shown in Fig. 7.

Reasoning, Argumentation, Proofs and *Proving* and *Problem solving* are usually described as "process competencies". Concerning the *Contents* we have distinguished between *plane*- and *space*-geometry. The category *Resources and Tools* is mainly about new developments in digital technologies. The category *Teaching* highlights teaching competencies as well as the preparation of teachers. Of course, it is not possible to clearly assign all articles to one of the four "vertices" of the T-model. There are many cross-relationships. Nevertheless, we have tried to do so and then shown cross-references to other "corners." We finally classified the articles of "one vertex" of the T-model concerning the frequency of their occurrence into sub-categories, based on our expert knowledge in geometry. Moreover, there are many topics like digital technologies, representations, gestures or assessment, which play a decisive role in all categories. We refer explicitly to these "all-over topics" in one category if we see the potential for new developments con-



Country	Count	Country	Count
US	58	Australia	3
Turkey	30	Germany	3
Italy	15	India	3
Israel	13	Sweden	2
UK	9	Cyprus	3
Spain	8	Finland	2
Canada	7	Denmark	2
Indonesia	3	Namibia	1
China	4	Nigeria	1
Japan	4	Greece	1
South Africa	2	Colombia	1
South Korea	1	Slovenia	2
Taiwan	3	Chile	1
Lebanon	1	Malaysia	1
France	2	Romania	1
		Rwanda	1

Fig. 6 Articles from the different countries

cerning a special topic. The full list of T-model codes and categories used to organize the review are shown in Fig. 8.

5 Learning: process competencies

Based on the results of the V-tool we highlight the geometry topics that were addressed in connection with the most frequently used process competencies, which include reasoning, argumentation, proofs and proving, problem solving, and modeling.

5.1 Reasoning and argumentation, proofs and proving

Various attempts have been made to summarise the findings, innovations and trends in reasoning and argumentation (e.g., Sommerhoff & Ufer, 2019; Mariotti et al., 2018). Stylianides et al. (2024) provided a systematic overview in school

Fig. 7 Sample categories and keywords used for the literature review

A sample of a Category	Examples of Keywords and Counts
Reasoning, Argumentation, poofs and proving	Reasoning (242), argumentation (52), understanding (207), process (127), processes (74), knowledge (170)
Resources	Digital (60), DGS (57), DGE (52), technology (98), dynamic (195), tools (90), representations (72)
Spatial geometry	Spatial (166), 3D (88),
Theories	Theory (72), understanding (207), knowledge (170), cognitive (63)

Fig. 8 Categories and their relationship to the T-model

T-model Code	Category
Learner and Learning	Reasoning, argumentation, proofs, and proving Solving problems mathematically Modeling
Contents	Plane geometry Definitions Geometric transformations Space geometry Measurement
Resources and Tools	Dynamic geometry systems 3D technologies
Teaching and Teacher Competencies	Teaching competencies Teacher education

and university mathematics. Similar to our process, they categorized articles using the didactic triangle and found that almost all articles were related to "Content." However, there are cross-relationships to the other "vertices". Mariotti et al. (2018) identified three emerging themes related to proofs and proving: the importance of textbooks, promoting design-based research, and specific aspects of logic and proof.

Argumentation, proof, and the use of digital technologies (DGS) Sinclair et al. (2016) emphasized "the importance of visual dynamic figures in DGS have been playing a vital epistemic role in studies that probed the process of generating geometrical conjectures" (p. 707). This was also highlighted by Dello Iacono (2021) and Komatsu and Jones (2020), who added to their studies the interaction with paper and pencil. Albano et al. (2019) investigated the transition from argumentation to (formal) proof through a computer-based didactic environment and that allows students to freely arrange language tiles to build arguments. In this context, it is particularly important to highlight and critically reflect on the relationship between experimental and empirical considerations with a DGS and deductive understanding.

A central aspect of working with DGS is the construction of tasks for argumentation and proofs. Komatsu and Jones (2017) developed principles for task design that fostered students' engagement in proofs and refutations. Haj et al. (2016) investigated the interrelationship between concept formation and proving. Cirillo (2014) developed a "scaffolding proof tool," a sample of competencies and their descriptions, to support teachers in understanding the different steps of a formal proof. Aaron and Herbst (2019) investigated the interrelationship of conjecturing and proving and made suggestions to "reunite" these activities, which are often separated. Overall, research on DGS has focused on developing strategies and tools that enhance students' engagement in proofs, concept formation, and conjecturing, with particular emphasis on verifying and refining their ideas.

Proof schemes and van Hiele levels Quite often the van Hiele levels are taken to evaluate how learners enhance their geometric thinking (e.g. Okazaki & Watanabe, 2021) and to classify students' or teachers' geometrical knowledge (e.g. Mariotti et al., 2023; Hassan et al., 2023). These are also used to classify the geometrical thinking in a DGS-environment (Adelabu et al., 2019), proofs by contradiction (Turiano and Boero, 2019). while proving in unfamiliar environments like non-Euclidean geometry or to show how difficulties of students (Haj et al., 2016) in the understanding of the concept definition affect their abilities to produce a proof. Beside the van Hiele levels, different taxonomies have been used to characterize and evaluate students' learning of proof and proving. For example, Keneloos et al. (2018) used a proof scheme taxonomy, Lee and Chen (2015) employed Polya's four-stage problem solving approach and Miyazaki et al. (2024) classified proof strategies using different levels of understanding. There are also some studies about the different elements of proof or proving, including conjectures (e.g. Harel et al., 2024), proof statements, and the formal proof (e.g. Winer & Battista, 2022). Finally, some studies examine the wide range of mistakes and misconceptions that occur during the process of proving (e.g. Poon & Leung, 2016).

The role of textbooks Textbooks play a major role in school practice (Mariotti et al., 2018). Some studies examined the influence of geometry textbooks and teachers' instructional decisions on students' opportunities to be engaged with proof and proving in school mathematics (Sears & Chávez, 2014). It is often noted that there are insufficient tasks and opportunities to develop reasoning and proof competencies (e.g. Wong, 2017). While some books include numerous tasks, proofs and proving is rarely an explicit object of reflection (e.g., Sinclair et al., 2016, p. 708; Otten et al., 2014; Nirode and Boyd, 2021). Some studies compare textbooks in different countries (e.g., Zhang & Qi, 2019) noting differences in the Western and Eastern textbooks on an intuitive versus a more formal approach to geometry.

Assessment While DGS has been incorporated into assessment systems, only a few studies addressing assessment with and without the use of technology were identified. Luz and Yerushalmy (2019) analyzed students' interactions with geometry interactive diagrams (GIDs) that were embedded in an online assessment platform, and they were able to look at students' interactions and categorized their conceptions, problem-solving approaches, and challenges they encountered in proving. In another study, Luz and Yerushalmy (2023) designed an activity for students in an online assessment system and provided evidence for how automated feedback to students early in their exploration can improve the quality of the conjectures they produce and can provide information to teachers about students' thinking (Yerushalmy & Osher, 2020).

5.2 Solving problems mathematically

There is an abundance of research on problem-solving in mathematics education (e.g. Liljedahl et al., 2016; Cáceres et al., 2018; Bokosmaty et al., 2015; DeJarnette & González, 2016; Koyuncu et al., 2015). For some years, studies about problem posing have increased as a method to engage students in problem solving (Andika et al., 2020). More recently, researchers have studied the use of digital media, including DGS, to support problem-solving and proving activities. Theories in this area have expanded, and there has been an increased focus on teachers and teaching.

The relationship between problem solving and proofs or proving is especially emphasised by Koichu and Leron (2015), whose article "proving as problem solving" relates the cognitive perspective of problem solving (e.g., hypothetical thinking, mental representation, and working memory capacity) to proving. Pedemonte and Balacheff (2016) analyzed students' conceptions in geometrical problem-solving and how they strongly impact the argumentation activities and the construction of proofs.

Theories Theories in mathematics education are often used to analyze problem-solving processes. Pankajkumar and Österling (2024) used commognition to examine how PSTs utilized procedures, constructs, and geometrical properties when solving a problem and developed an analytical tool for researching PSTs' mathematical discourse. Yao (2020) used the theory of instrumental genesis (Trouche, 2005) and the growth of understanding theory (based on the Pirie-Kieren theory, Pirie & Kieren, 1994) to describe learners' growth of understanding while solving geometrical problems (inscribing a square into a triangle). However, there are also some new ideas concerning theories in geometry. Miragliotta (2019) proposed a theoretical construct for students' thinking, especially the ability to make geometric predictions, while problem solving. Tůmová (2017) investigated the relationship between space structuring and volume and area problems and found a positive correlation. Overall, problem solving is a field for the specification of crucial theories in mathematics education.

Representations Finally, drawings and representations are important tools for supporting problem solving, especially in geometry. Krawitz and Schukajlow (2021) replicated findings of the surprising negative effect of drawing strategies and problem-solving. They emphasized the importance of the quality of the drawings for reducing this negative effect. Dongwi and Schäfer (2019) showed in a case study that visualizations influence mathematical reasoning and that they are integral components in the solving of geometrical problems.

5.3 Modeling

In geometry, modeling can be considered as a particular type of problem solving (see Sect. 5.2). Several studies examined how the use of modeling tasks could be used to support students' geometric thinking. Rellensmann et al. (2023) examined relationships between students' strategic knowledge and their mathematical modeling competencies. In their pre-post-test experimental study with 473 grade 9 students, they randomly assigned students to one of three 90-minute interventions that addressed: strategic knowledge about situational drawing, strategic knowledge about mathematical drawings, or strategic knowledge about situational and mathematical drawings. They found that strategic knowledge and drawing accuracy were mediating variables for students who demonstrated higher modeling competencies, and that students who drew accurate diagrams were more likely to be successful in solving the modeling problems. Simon and Cox (2019) found that using a design project and the lens of design and design thinking, they recommend thinking about prototyping as an act of mathematization and suggest broadening how researchers examine modeling to include design thinking, which could be helpful to both researchers and classroom teachers.

Other researchers examined how the implementation of real-world equity-focused social justice mathematics projects engaged students in geometric thinking (Harper & Kudaisi, 2023). They found that students expected real-world situations and social issues to be solved using mathematics and that the teacher embraced equity-directed pedagogical approaches. In the study of Ilgün et al. (2023), mathematical modeling activities using the history of mathematics resulted in a positive change in students' perceptions of geometry. Some studies show that DGS foster students' modeling competences (Cevikbas et al., 2022). Especially 3D computer modeling coincides with a lot of competences in design, spatial thinking, communication, creativity, and coding (Coşkun & Deniz, 2022, Šafhalter et al., 2022). However, modeling research often includes algebra or stochastics and is not especially related to geometry.

5.4 Results

Regarding new developments in reasoning and proving in plane geometry, the use of DGS still plays the most important role. DGS are now a standard component in plane geometry curricula and textbooks, and many ideas developed in mathematics education research have become common in secondary education and teacher training materials. This includes the design of new tasks, content ideas, and strategic approaches to DGS-supported argumentation and proof. While there are no far-reaching new findings, findings from previous research have been confirmed. Furthermore, plane geometry is a suitable field for analyzing and explaining actions and activities on a theoretical basis, but also for demonstrating the relevance of familiar theories in mathematics lessons. In particular, modeling in geometry offers rich opportunities to observe problem-solving processes and gain insights into the geometric thinking of learners. An emerging area is the development of computerized systems for formative and summative assessments.

6 Contents

This section focuses on the learning of geometry content, which is considered both curricular content in secondary schools and content knowledge in teacher training.

6.1 Shapes and conceptualization of figures in plane geometry

Parallel and perpendicular line segments is a topic quite often used from the perspective of studying students' example spaces (Watson and Mason, 2005) and as prototypical examples for proofs and argumentation. Ulusoy (2016) and Srinivas et al. (2019) examined grade 6-9 students' answers and showed that students generally provided prototypical examples, did not recognize segments of different lengths as parallel, and confused the notions of verticality and perpendicularity.

Angle Misconceptions about angles have been discussed in the literature at different school levels. Tanguay and Venant (2016) studied the types of mathematical work that support grade 6 students' conceptualizations of angle and confirmed that the students shared the textbooks' conception of angle as a magnitude. By analyzing tasks related to defining, recognising, and comparing angles, Mullins (2020) found the conceptualisations of angles provided by three different school level students depended on the students' approaches to geometry. From the perspective of the development of geometric language, Chesnais (2021) analyzed students' responses to questions focusing on the distinction between empirical and theoretical aspects of angle measures.

Circle Some research focused on the concept of circle and its properties and theorems. For example, Akyuz (2016) examined students' misconceptions about angles of inscribed triangles and quadrilaterals using DGS activities. Other researchers have proposed using circle activities that use material artefacts to be manipulated (Fitri & Prahmana, 2020).

Heights of a triangle Krajcevski and Sears (2019) discussed how the use of prototypical images influence the conceptualisation of height for parallelograms and triangles. Soldano and Sabena (2022) developed a DGS-based inquiry-based activity conducted with grade 7 students with examples addressing non-prototypical figures.

Based on these findings, it would be worthwhile to continue to pose tasks that challenge prototypical figures and support the construction of example spaces. Overall, research on students' learning of particular geometric topics should remain a focus for further research.

6.2 Definitions and classifications in plane geometry

Sinclair et al. (2016) argued that recent studies underscore the critical role of definitions, highlighting two key aspects: the process and necessity of defining concepts and specific definitions related to triangles and quadrilaterals. Building on these findings, current research examines factors influencing students' conceptualizations and definitions, shifting attention from secondary students to PSTs.

Non-Euclidean contexts Some studies explore the process of defining in non-Euclidean contexts, under the assumption that learners can deepen their understanding of new concepts if they step somewhat apart from the acquired and internalised habits developed in Euclidean geometry. Dibbs and Beach (2017) have done this in geometry courses for PSTs.

Arnal-Bailera and Manero (2023) focused on the definition of the circumference of a circle in Taxicab geometry, suitable for assessing geometric thinking of PSTs according to van Hiele's levels. Kemp and Vidakovic (2023) explored the interaction between Taxicab and Euclidean geometry when students enrolled in a college geometry course on the circle concept; their analysis showed that one obstacle is the tendency to neglect the definition.

Quadrilaterals Research focused on the definitions and classification of quadrilaterals has yielded significant insights, particularly in understanding how students and PSTs consider relationships within convex quadrilaterals.

Ulusoy (2015) highlighted two main points for middle school students: students often applied exclusive criteria in defining and classifying figures; prototypical images emerge and influence their answers. Bernabeu et al. (2022) confirmed that preservice primary teachers did not use hierarchical definitions or used partial hierarchical definitions, mainly focusing on exclusive classification.

Fujita et al. (2019) analyzed how tasks promoting dialogic interactions in small group work support the transformation of students' definitions toward formal ones.

The research reported here emphasizes that students' learning does not seem to evolve during secondary school, as PSTs provide similar responses to students, and underlines that the results of the studies discussed in the previous section have implications related to the process of defining and classifying. Hence, the need to develop challenging learning paths to support the development of geometric thinking and to meet instructional needs in teaching geometry.

6.3 Geometric transformations in plane geometry

Geometric transformations are another topic that students encounter in primary school and revisit throughout secondary school and at the university. This subject can play a fundamental role in the development of mathematical conceptualization and encourage various forms of generalization (Yao & Manouchehri, 2019). However, research highlights that students, including PSTs, encounter difficulties with these concepts. Research related to this topic at the secondary school is mainly carried out in a DGS.

Transformations and functions Hollebrands et al. (2021) explored applications of geometric transformations to mediate the concept of function within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008). They engaged students aged 15–16 in tasks using DGS to manipulate points to explore dynamic relationships between independent and dependent variables. Mariotti and Montone (2020) designed activities on reflection for students aged 9 to 14 years, combining pins and paper and a GeoGebra file; this integrated approach seeks to enrich students' comprehension by offering diverse tactile and digital learning experiences.

Isometries Drawing on Balacheff's cK¢ model of students' conceptions, DeJarnette et al. (2016) identified five distinct conceptions of reflection evident in students' work: the paper-folding conception, flipping conception, rotating conception, perpendicular bisector conception, and visual conception. Yanik's study (2014) explored middle school students' concept images and definition of geometric translation: their prototypical examples show that not all geometric shapes can undergo translations, such as circles, due to the lack of angles and straight edges.

These studies underscore the versatility of geometric transformations in educational settings, showing their potential to deepen students' mathematical understanding across different age groups. However, research on this topic remains relatively underexplored, representing a niche area ripe for further investigation.

6.4 Space geometry

Space geometry and spatial thinking play a pivotal role, heavily influenced by how geometric objects are represented. For instance, Pittalis and Christou (2013) identified middle school students' representational abilities (e.g., construction of nets, orthogonal drawing of 3D shapes) in tasks involving 3D shapes formulated with 2D representations, while Fujita et al. (2017) explored methods to assess students' geometric thinking when solving 3D geometry problems. Teachers' definitions of 3D shapes have been investigated by Erdem and Man (2023). Widder et al. (2018) investigated students' visual difficulties in 3D-problems and found out (however, only in a small qualitative study) that 3D-DGS could reduce these visual difficulties.

Sinclair et al. (2016) and Mithalal & Balacheff (2019) emphasized that using 3D-DGS supports the transition between different modes of visualization, which is crucial for developing geometric thinking and approaching proofs. Sua et al. (2022, 2023) analyzed strategies carried out by students (aged 11 to 14) in solving problems employing "soft constructions" (which are not completely robust under dragging).

Researchers are attending more to gestures since touch screens and tablets can be controlled with finger swiping, and geometrical objects can be manipulated in virtual reality via tactile movements (e.g., Bairral & Arzarello, 2015). Researchers have studied the relation between gestures and eye movements (Hannula & Gaye, 2016) and the meaning of collaborative gestures (Walkington et al., 2019; Buchbinder & Zaslavsky, 2019). Okumus and Hollebrands (2019) focused on gestures made by 12–14-year-old students engaged in tasks involving solid figures with methods such as extrusion and rotation to explore volume, using physical artifacts and Cabri 3D sketches. Gestures are helpful to support the communication about 3D objects and to assist students in anticipating their 3D-DGS constructions. However, spatial thinking and reasoning are not absolute competencies and have to be seen in relation to domain-specific knowledge (Fujita et al., 2020).

Approaching 3D geometry from an enactive perspective, Palatnik and Abrahamson (2022) and Rosenski and Palatnik (2022) emphasized the cognitive role of sensory perception in understanding spatial shapes. The use of 3D-DGS and 3D manipulatives are helpful tools and resources for the development of this competence as well as a "network of spatial visual representations" (Mamolo et al., 2015). Even "Origami-based instruction" might support spatial thinking. Arıcı and Aslan-Tutak (2015) showed its significant effect on spatial visualization and geometric reasoning. Other research focused on the use of AR for studying polyhedra (e.g., Fernández-Enríquez & Delgado-Martín, 2020).

However, proofs and proving in 3D geometry are much more complex than in 2D geometry (Hollebrands & Okumus, 2017). It needs modifications if a figure, such as an equilateral triangle, is constructed in a 3D environment (e.g. Okumus & Hollebrands, 2017; Sua et al., 2023). Additionally, the relationships between visualisation and reasoning, e.g., proving relationships between or arguing about properties of geometrical objects, become even more important when engaging in spatial thinking (Dongwi & Schäfer, 2019; Tůmová, 2017).

6.5 Measurement

Research found that many students and teachers have a superficial understanding of many measurement concepts: e.g., they confuse the concept of perimeter with the concept of area and volume with surface area (Seah & Horne, 2020). According to this, de Freitas and Sinclair (2020) highlighted that the topic of measurement is undervalued in the mathematics curriculum and reduced to a set of formulae. Moreover, Harris et al. (2023) claimed that understanding area can be considered a foundation for more advanced geometric concepts. Ng and Sinclair (2015) analyzed two tasks designed in DGS to shift students' focus from a formula-driven conception of area to the concept of surface equivalence by comparing and constructing equivalent figures, while Proulx (2023) investigated problem solving processes when grade 7 students were engaging in solving tasks about area and perimeter simultaneously.

The studies on measurement highlight some points to be developed in future research: the first concerns the process of measuring a geometric quantity before learning the formulas, also related to spatial reasoning; the second concerns the relationships between the variations of geometric measures concerning the same figure, such as area and perimeter for 2D figures. A third concern is about the well-known difficulties of developing and maybe proving formulas for the volume of solids like pyramids, cones, and spheres.

6.6 Results

There are some quantitative and qualitative studies and, as a follow-up, learning materials, both pen and paper and computer-based, to support understanding and overcome misconceptions. A topic that has been pushed into the background in recent decades, working with transformations, seems to be experiencing a revival with the increasing use of dynamic software. However, there is still a lack of empirical research in this area: does the understanding of traditional geometric content differ when it is based on this dynamic view?

The importance of spatial geometry, but also the difficulties in this area, has long been emphasized. Nowadays, we see that digital technologies offer new opportunities for accessing 3D geometry. Empirical studies show the importance of combining 3D digital activities with 2D representations, paper and pencil drawings, and enactive activities. However, good strategies for developing spatial thinking will remain a new field.

7 Resources and tools

There are a variety of resources and tools available to support the teaching and learning of geometry, including DGS, computer programming languages, virtual manipulatives, and augmented or virtual reality systems. The use of one's body, which can include movements and gestures, can also be considered a resource that can be used to communicate geometric ideas and express one's thinking. The foci of the research studies related to the use of resources and tools focused on reasoning, gamification, task design, problem-solving, proving, and spatial visualization.

7.1 The crucial role of DGS

DGS and geometric reasoning and achievement Several studies examined the effect of incorporating DGS on students' achievement (Bhagat & Chang, 2015; Birgin & Topuz, 2021; Ganaesan & Eu, 2020) and geometric reasoning (e.g., Adelabu et al., 2019; Adelabu et al., 2022; Ng & Sinclair, 2015; Okumus & Bozkurt, 2022), showing positive results. For example, Özçakir and Çakiroglu (2019) found an increase in students' van Hiele levels after using the technology to study quadrilaterals. Other studies examined students' use of particular features of DGS and their impact on students' reasoning (e.g., Alqahtani & Powell, 2016; Anwar et al., 2024; Canogullari & Erbas, 2024). For example, Nagar et al. (2022a) explored students' discernment of invariant properties in a DGS setting and found that they mostly associated dragging with generating examples, rather than looking at an object as it varies continuously. They also identified four types of invariants: shapes, measurements, locations, and calculations. They noted that sometimes the use of paper and pencil assisted students in identifying invariant properties, particularly when there are conflicts between conceptual and figural aspects (Nagar et al., 2022b).

DGS, gamification, and computational thinking Several researchers reported on the design of activities that incorporate gamification and computational thinking (Soldano & Arzarello, 2016; Sinclair & Patterson, 2018; Su, 2017). For example, Soldano and Arzarello (2016) designed five game-based activities centred on geometric theorems, where pairs of players use a touchscreen DGS. The goal of the activities was to improve students' geometric thinking through game-strategic thinking by focusing on the construction of logical links between concepts during the argumentation phase of the proving process. Sinclair and Patterson (2018) explored how DGSs can be presented to students as a type of computer programming language and found relationships between the use of propositions in programming languages with the spatial, temporal language used while students are exploring geometric objects in a DGS.

DGS, task engagement, and problem solving Students' interactions with DGS can support their engagement in tasks and problem-solving. Trocki and Hollebrands (2018) introduced a framework designed to assess task quality, highlighting its role in shaping students' mathematical activity and engagement in problem-solving. This framework has been used by other researchers to examine DGS tasks developed by teachers and included in textbooks (Ulusoy & Girit-Yildiz, 2024; Ulusoy & Turuş, 2022; Zengin, 2023). Komatsu & Jones (2019) studied how tasks could be designed to engage students in the refutation of heuristics and Olsson and Granberg (2022) looked at the effect of including guidelines in DGS tasks on students' success in solving them. Studies also addressed language shifts and instructional strategies observed when students interact with a DGS, emphasizing the transformational impact of digital tools on classroom discourse and pedagogical practices (Akyuz, 2016; Chang et al., 2022; Gulkilik, 2023; Lai & White, 2014; Oner, 2016; Alqahtani and Powell, 2017b; Schacht, 2018).

Alqahtani and Powell (2017b) investigated how teachers used GeoGebra, and found its use supported geometrical problem-solving and fostered mathematical knowledge construction over time. Several researchers have explored how students interact with touch-based DGS (Ng, 2019; Pittalis & Drijvers, 2023). For example, Sua et al. (2022) and Chartouny et al. (2019) employed theories of semiotic mediation and instrumentation processes, respectively, to examine students' problem-solving strategies and conceptual schemes in DGSs, highlighting the dynamics between tool use and mathematical reasoning. Koyuncu et al. (2015) looked at differences in problem-solving strategies in DGS and paper-and-pencil environments, noting that prospective teachers used geometric approaches in the former and algebraic strategies in the latter environment.

DGS and proving Several studies examined students' deductive reasoning processes while using DGS. For example, Lachmy and Koichu (2014) studied the interplay between empirical and deductive reasoning as middle grades students engaged in discovering and justifying "if" and "only if" statements related to the properties of quadrilaterals. They found that the DGS constructions supported students in exploring and proving "if" claims, but not the "only if" claims. Komatsu and Jones (2019) found that students were successful in heuristic refutation when they engaged in DGS tasks created using three design principles. Lingefjard and Ghosh (2019) investigated the impact of DGS on grade nine students' geometrical reasoning, finding positive outcomes in conjecture-making and proof development.

Other researchers have studied the use of technology to support students' development of proofs. Baccaglini-Frank et al. (2018) discussed the transition from dynamic explorations to proofs by contradiction, highlighting the role of a pseudo-object, which is a geometric object related to another object but contains contradictory properties. In this study, students were challenged to use DGS to create a non-constructible object, a triangle with angle bisectors that are perpendicular to each other. They found that students who used pseudo-objects were more successful in producing proofs by contradiction than those who did not. Komatsu and Jones (2020) examined the interplay between paper-and-pencil activities and DGS use in generalization and proving tasks. They found that the technology assisted students in developing a generalization, but that they were better able to prove statements using paper and pencil. Miragliotta (2023) found that when students first solved a task in paper and pencil and then in a DGS, they revised their predictions about the solution of the task when they were provided opportunities to examine whether their prediction matched or did not match the behavior of the DGS sketch.

7.2 Technology and spatial visualization

Researchers have also studied students' uses of tools designed to support their visualization abilities and learning of 2-D and 3-D geometry concepts (Chang et al., 2016; Gülburnu, 2022; Uwurukundo et al., 2022; Uwurukundo et al., 2024; Žakelj & Klancar, 2022). Okumus and Hollebrands (2019) focused on middle school students' gestures alongside Cabri 3D, highlighting how gestures supplement visual representations. Widder et al. (2018) analyzed high school students' problem-solving strategies with DGS, noting how the software helped to mitigate visual difficulty and supported dynamic exploration, particularly in predicting geometric changes and interpreting feedback. Hollebrands and Okumus (2017) investigated prospective teachers solving optimization problems with Cabri 3D, revealing their evolving reasoning processes and the strategic use of tools like the "Length tool" to derive generalized conjectures. Cohen et al. (2017) investigated in-service mathematics teachers' constructions of rhombi and highlighted the diverse conceptualizations that they had. Uygan and Bozkurt (2021) focused on a PST's development of construction strategies for cyclic quadrilaterals in DGS and reported the evolution of both instrumental techniques and conceptual understanding. Uygun (2020) explored inquiry-based learning sequences and demonstrated how structured activities fostered deeper understanding and operational definitions of geometric transformations. There have also been research studies that have examined the impact of augmented reality on students' achievement and spatial visualization skills (Cetintav & Yilmaz, 2023; Koparan et al., 2023) and its use as a resource for teaching 3D concepts (Fernández-Enríquez & Delgado-Martín, 2020).

7.3 Results

Dynamic geometry is still an important tool to develop geometric thinking. Many empirical studies with DGS show how the use of this tool influences methods of drawing, drawing schemes, and problem solving and logical reasoning, e.g. making assumptions and conjectures and supporting deductive reasoning as well as understanding concepts, e.g. categorizing quadrilaterals.

The last decade has also seen an increasing number of empirical studies looking at the relationship between the use of tools and the explanation of user actions using theories such as instrumental genesis or semiotic mediation. These theories seem to be an integral part of mathematics education, allowing us to interpret many empirical results on geometric outcomes and geometric reasoning.

There are also several studies examining teaching with DGS that show a positive impact on inquiry-based approaches, deeper student engagement with geometric content, and a positive impact on prospective teachers' knowledge. Overall, they show a positive impact on teachers' professional development.

8 Teaching and teacher competencies

In this section, we examine studies on teachers' knowledge and competencies for teaching geometry. While we do not delve into teachers' professional development in geometry, we highlight some points that could suggest new perspectives for research.

8.1 Teaching competencies

We consider research paying attention to cultural aspects, proof, and the use of technology (Niss & Højgaard, 2011, 2019). Other studies focus on noticing students' understanding (Ulusoy & Çakiroglu, 2021; Haj-Yahya, 2022).

Cultural contexts Although literature promotes ethnomathematical practices for geometry education, the methodological basis for their integration in schools has not been well established. For instance, Verner et al. (2019) analyzed specific competencies teachers need to teach geometry in a cultural context. Schwarts and Karsenty (2020) reported that activities culturally "far" from the teachers' reality (e.g., Israeli teachers' discussions on a lesson from Japan) fostered the teachers to re-see everyday aspects of their practice and promote reflective actions.

Teaching of proof In addition to learning proof, research considers the teaching of proof from teachers' perspectives. For instance, Sears and Chávez (2014) studied the features of two USA textbooks and their influence on geometry teachers' facilitation of proof and students' engagement with proof tasks. Similarly, Otten et al. (2017) argued that the use of textbooks and the authority the teacher manifests in whole-class proving discussions can limit students' opportunities to collectively engage in proving. Finally, Dimmel and Herbst (2020) investigated what teachers expect from students when presenting their geometric proofs on the board, with attention to the two-column proof format characterizing proof teaching in USA secondary schools.

Competences on noticing Ulusoy and Çakiroglu (2021) claimed that middle school mathematics prospective teachers (PST) enhance their professional noticing skills through their engagement in different kinds of tasks (i.e., individual video analysis, reflection papers, and group discussions). Haj-Yahya (2022) showed that in-service teachers improve their professional noticing performance by working on theoretical and empirical resources on different aspects of geometrical thinking.

Competencies and theories in technology Nagar et al. (2022a) investigated how high-school mathematics teachers understood draggable geometric objects and invariant properties in a DGS. The teachers seemed to prefer the view of draggable objects as generating numerous examples of the object, mainly related to a static view rather than a dynamic one. Some works focus on teachers' competencies and knowledge in managing rich technology environments. For instance, Bretscher (2023) used the TPACK model to analyze the processes to introduce students to a circle theorem using a DGS. Other research refers to theoretical approaches to investigate how mathematics teachers engaged with DGS: Hollebrands and Okumuş (2018) focused on secondary mathematics teachers' processes of instrumental integration, while Alqahtani and Powell (2017a) claimed that understanding how teachers use technological tools can inform the design of professional development programs that engage them with those tools to extend their specialized content knowledge. Studies highlighted that DGS can support inquiry-based approaches that can facilitate deeper student engagement with geometric concepts (Gómez-Chacón & Kuzniak, 2015; Hollebrands & Lee, 2016), though teachers often require support in effectively integrating technology into their pedagogy (Bozkurt & Uygan, 2020).

8.2 Teacher education and theories for teaching geometry

Several researchers have investigated PSTs' knowledge on the different contents, processes, and aspects of geometrical thinking presented in the previous sections. İmamoğlu et al. (2023) designed a geometry course for PSTs based on the Geometric Working Spaces to improve mathematical and pedagogical content knowledge related to geometry, while Miller (2018) analyzed PSTs' ability to define quadrilaterals. Herbst et al. (2020) reported how they measured teachers' growth in their mathematical knowledge for teaching geometry after engaging in online practice-based professional development.

Many articles on problem-solving offer suggestions for teacher education. Researchers (e.g. Camacho-Machin et al., 2016; Singer et al., 2017) suggest long-standing courses are needed to get teachers familiar with different kinds of problem solving and to develop their abilities to teach problem solving.

PSTs also need to develop strategies that they can use to support their students with the proving process. Crucial are the understanding of the logical structure of proofs, the knowledge of different types of proofs, and the knowledge of proof strategies; they can be supported by special courses (Buchbinder & McCrone, 2023). Also, the transformation perspective of geometry, using reflections, translations and rotations in Euclidean geometry, might be a possibility to have a different, compared to the congruence perspective, access to proofs (St. Goar & Lay, 2022). Finally, PTSs should develop skills that allow them to facilitate whole-class discussions focused on proving tasks while teaching (Conner et al., 2014).

Researchers conducted studies to examine how the use of DGS impacted prospective teachers' knowledge (Yildiz & Gokcek, 2018; Yilmaz, 2015; Zambak & Tyminski, 2020). Some studies examined teachers' beliefs about and acceptance of using DGS in teaching (Chan, 2015; Pittalis, 2021) and studies that described the benefits of teacher collaboration in the adoption of technology (Mavani et al., 2018).

To support the connection between undergraduate content coursework and secondary teaching, Lai et al. (2023) analyzed the connection between the mathematics coursework in PST programs and secondary teaching practice, in terms of the quality of mathematical knowledge in geometry teaching. Kumar et al. (2019) claimed that challenging situations are useful to support the development of SMK and PCK to structure the classroom discourse. In order to implement the Common Core State Standards (NGAC, 2010) around geometric transformations, Seago et al. (2014) created a sequence of video cases on similarity. In their works, Martinovic and Manizade (2018) discussed the challenge of designing assessment instruments to measure data related to teachers' PCK concerning the topic of the area of a trapezoid.

In general, research that focuses on teachers from the perspective of the knowledge required for teaching remains, appropriately, at a general level. The studies cited here focus on the specific content and skills related to geometry; some other research on teachers' knowledge has been reported in the section on content. These studies show that PSTs' geometric knowledge is partial and sometimes weak and, consequently, highlight the need to emerge weak and lacking on one hand, and work on the characteristics of geometric thinking on the other hand.

8.3 Results

Four findings can be highlighted. First, the development of teachers' perceptions of students' understanding and skills seems to be a crucial element of teachers' PCK and should be more emphasised. Second, the importance of teachers' familiarity with teaching theories, especially in relation to digital technologies, such as TPACK or instrumental genesis had to be developed in closer relationship to specific content. Third, a still important issue is the construction of appropriate tasks, especially while using digital technologies and adapting textbook problems to individual classroom situations. Fourth and finally, the development of computer-based tasks is a major challenge because, compared to algebra or calculus, geometric problems generally tend to stress more the conceptual than procedural knowledge. And these are more difficult to implement in a digital development.

9 Conclusion and perspectives

If we compare our results with the review studies of the last decades, we can confirm the continuing interest in traditional topics in learning and teaching of geometry: the use of digital technologies, geometric reasoning and proof, spatial reasoning, problem solving, and teacher training. We also see the continued interest in developments that have already emerged in recent decades: gesture and embodiment, application of well-accepted theories in mathematics, and the use of evolving resources such as VR and AR. However, judging by the number of publications, this is far from being a flourishing trend. We summarize our conclusions and show some perspectives along the four main themes, the "vertices" of the T-model.

Learning While there are no significant new developments in 2D geometry, the situation is different with spatial geometry and the role of 3D DGS. They offer new opportunities for research in terms of visualization, developing the relevance of this software for spatial thinking and reasoning and for real-world applications. In this respect, it is still an open question how new developments such as VR and AR and the continued availability of tablets will influence these spatial thinking skills. This will be a promising area of research in the upcoming years.

Content The (miss)understanding of classical geometric 2D-objects, problem-solving strategies, proofs and proving, and the possibilities offered in dynamic environments will be a continuing area of interest. In addition, 3D geometry and the development of spatial thinking open up a wide range of future research questions. Nowadays, digital technologies offer the possibility to visualize spatial situations and make them more accessible, and, for example, make formulas for the volume of solids more insightful. However, spatial geometry is much more difficult than plane geometry, and the verification of the effect of the computer-supported learning is and remains an important empirical task.

Resources and tools By far the most important new area in the last ten years is the use of 3D software, 3D DGS, VR and AR. The research focuses mainly on teachers' knowledge before and during professional training. However, the effectiveness of these technologies in promoting students' geometric thinking, especially in the transition from 2D to 3D representations, is a field for upcoming research. There is a new quality of interaction between learners, teachers and the tool when students have continuous access to tablets and the internet. The well-known questions e.g., concerning understanding of concepts or geometrical and spatial thinking, will be posed in relation to tool use.

Teaching The technical foundations of (Euclidean) geometry are no longer taught in high school in such a way that they are sufficient as a basis for teacher training. Both the content and the tool-based teaching of this content must therefore be a significant focus of training. The competence of teachers to use digital technologies with regard to understanding-oriented geometry teaching is of central importance. The numerous teaching theories can provide orientation for this.

Many other specific topics are open for further research. The inclusion of gestures and embodiment, particularly in relation to digital and VR technologies, is an important area of research. Theoretical frameworks remain essential for understanding student learning and classroom dynamics, helping to identify student conceptions and develop effective teaching strategies. Digital textbooks are playing an increasingly important role, providing multimedia resources such as explanatory videos and interactive learning modules that promote student engagement and understanding. Assessment methods are also evolving with the advent of digital feedback systems, enhancing opportunities for formative assessment (Weigand et al., 2024). The role of artificial intelligence (AI) in supporting teachers and students in the teaching and learning of geometry is a fruitful area for future research. Ultimately, the evolving landscape of geometry education underscores the interdisciplinary nature of the subject, which integrates advances in technology, theoretical insights, and innovative pedagogical practices to enhance teaching and learning experiences in geometry.

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Declarations

Competing Interests The authors declare no competing interests.

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