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THE STRETCHING ALONG THE PATHS: GENESIS, EVOLUTION AND APPLICATIONS

Abstract. A brief and very partial survey of the stretching along the paths method is given, outlining which were the initial ideas, where they originated from, and how they have been developed in the last twenty five years or so.

1. Introduction

During my Ph.D. studies at S.I.S.S.A. in Trieste between 1998 and 2000, Zanolin led my attention to Butler's paper [10] since he believed that it hid promising ideas that have yet to be tapped: the stretching along the path (SAP from now on) method developed as those ideas were investigated and refined. It looks that he was quite right, since he, together with his collaborators, published up to now almost fifty papers (which I tried to list in the bibliography: my apologies if I missed someone) on the SAP method: some included improvements or extensions of the method, but most of them grew out of specific mathematical problems in which the method turned out to be useful.

In Section 2 I consider Butler's papers with some details. The first half of that section is actually the real introduction of this survey. Its second half describes the first attempts to use Butler's ideas in a context still close to the same specific kind of equations. In Section 3 the setting of conical shells is considered. It is an intermediate step in the process of obtaining a more general and flexible setting in which to look for a rich chaotic behavior as close as possible to that of Bernoulli's shift maps. That is the subject of Section 4, where I describe the SAP method directly in the N-dimensional setting, specify the kind of chaotic behavior it implies and review its consequences in Butler's setting. The last Section 5 is dedicated to an application of the SAP in the context of linked twist maps.

I have to say that much of Zanolin's and his collaborators' work that is worth mentioning around the SAP has been left out of this survey. This is due mainly to the limits I have here. Almost all the nice applications to specific dynamical systems are not described in this survey, but can be recovered from the bibliography at the end of the paper.

Finally, this is a mathematical paper with no proofs at all. All definitions, lemmas, theorems and the like are precisely stated (hopefully), maybe in a simplified setting with respect to the original papers they are borrowed from. However, all remaining sentences are heuristics, trying to convey some ideas which are formalized in the corresponding references.

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Notation and glossary. $O = (0,0)$ is the origin of $\mathbb{R}^2$; $B(p,r)$ and $B[p,r]$ are, respectively, the open and closed disks with center $p \in \mathbb{R}^2$ and radius $r > 0$; $\mathbb{R}^+ =]0,+\infty[$; if $0 < r_1 < r_2$ we set $A[r_1,r_2] = B[O,r_2] \setminus B(O,r_1)$. Q_1, Q_2, Q_3 and Q_4 denote the four closed quadrants of $\mathbb{R}^2$, starting from the positive one and moving counter-clockwise. When γ is a continuous curve, I'll use the symbol γ to denote both the map and its image, with a little abuse of notation. I'll use *path* as a synonim of a continuous curve parametrized on a closed bounded interval. An *arc* will denote a path that is also injective.

2. Butler's papers

Butler was motivated by the paper [18] in which Jacobowitz proved that the equation

$$\ddot{x} + f(t,x) = 0 \qquad \text{in } [0,T]$$

has a T-periodic solution with exactly $2N$-zeros in $[0,T[$ for all N large enough, provided that f is a T-periodic C^1-function such that

1. $f(t,x)x > 0$ for all $x \neq 0$ and all t;
2. $f(t,x)/x \to +\infty$ as $x \to \pm\infty$, uniformly with respect to t;
3. $f(t,x)/x$ is bounded for x close to 0, uniformly with respect to t.

In [10] he obtained a similar result with a sign condition more general than 1. In order to get an idea of his arguments, let us focus on the simplified situation where $f(t,x) = q(t)|x|^{\beta-1}x$: assumption 1 means that q is a positive weight function while assumptions 2 and 3 are satisfied for $\beta > 1$. Butler's results applies when q changes sign and has some regularity: for the sake of simplicity let us assume that $q \in C^1(\mathbb{R})$ is a piecewise monotone T-periodic function such that

$$0 = t_0 < t_1 < t_2 \cdots < t_{2k-1} < t_{2k} = T$$

and $q(t) < 0$ in $]t_{2i-2}, t_{2i-1}[$ and $q(t) > 0$ in $]t_{2i-1}, t_{2i}[$, for $i = 1,2,\dots,k$.

The differential equation is equivalent to the first order system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(t)g(x) \end{cases} \tag{2.1}$$

in the phase plane $\mathbb{R}^2$. Let $z(t) = (x(t),y(t)) = z(t;s,p) = (x(t;s,p),y(t;s,p))$ be its solution satisfying the initial condition $z(s) = p$, for $(s,p) \in \mathbb{R} \times \mathbb{R}^2$, and $]\alpha_{s,p}, \omega_{s,p}[\ni s$ be the maximal interval for $z(\cdot;s,p)$. Let us also denote by $\operatorname{rot}(p;[a,b])$, the angle spanned by $z(t;a,p)$ as t ranges in $[a,b] \subset]\alpha_{a,p}, \omega_{a,p}[$ when $p \in \mathbb{R}^2 \setminus \{O\}$, namely

$$\operatorname{rot}(p;[a,b]) = \int_a^b \frac{q(t)g(x(t;a,p))x(t;a,p) + \dot{x}^2(t;a,p)}{x^2(t;a,p) + \dot{x}^2(t;a,p)}\,dt,$$

which is well defined, since $g(0) = 0$ and the uniqueness for initial value problems holds, and is positive in $[a,b] \subseteq [t_{2i-1}, t_{2i}]$ for $i = 1, \ldots, k$, thanks to the sign conditions. Butler's strategy consists in decomposing the T-periodic boundary condition into its radial and angular components. More precisely:

- prove the existence of a suitable closed and connected set $\Gamma \subset \mathbb{R}^2$ of initial conditions $p \in \mathbb{R}^2$ such that $[0,T] \subset]\alpha_{0,p}, \omega_{0,p}[$ and $|z(T;0,p)| = |p|$;
- prove that for each $n \in \mathbb{N}$ large enough there is $p \in \Gamma$ such that $\mathrm{rot}(p;[0,T]) = 2n\pi$.

Now, in the intervals where q is positive one has that

(+) thanks to the mild regularity imposed on q, blow up of solutions doesn't occur;

(+) solutions z that start large remain large;

(+) as a consequence of the superlinear growth of g, solutions turn around the origin as many times as desired provided that they stay sufficiently far from the origin, namely:
$$\lim_{|p| \to +\infty} \mathrm{rot}(p;[t_{2i-1}, t_{2i}]) = +\infty \qquad \text{for } i = 1, \ldots, k.$$

Hence, if one can find Γ such that the nodal behavior of $x(t;0,p)$ in $[0, t_{2k-1}]$ is somehow fixed for all $p \in \Gamma$ and $\sup_{p\in\Gamma} |z(t_{2k-1};a,p)| = +\infty$, then the existence of T-periodic solutions with arbitrary and large number of zeros in $[t_{2k-1}, T]$ follows from the continuity of $\mathrm{rot}(p;[t_{2k-1}, t_{2k}])$ with respect to $p \neq 0$ and the connectedness of Γ.

Butler was able to complete this program thanks to a topological lemma about continua in $\mathbb{R}^2$ and the exploitation of the behavior of the solutions in the intervals where q is negative:

(−) blow up of solutions *do occur*, no matter how much q is regular;

(−) large solutions pass uniformly close to the origin when they cross one of the axes of $\mathbb{R}^2$;

(−) each solution can cross only one of the axes and only once and, in particular, solutions do not turn around the origin.

A closer look to his argument show that some features were left unexplored. In particular, the interplay of the following two properties of the equation was not fully used:

(+) solutions can rotate around the origin an arbitrarily large number of times also in the *other* intervals of positivity of q (and not just the last one), provided that $|z(t_i)|$ has the right level of magnitude;

(−) any interval $[t_{2i}, t_{2i+1}]$ of negativity of q can be used to link different levels of magnitude of $|z(t_{2i})|$ and $|z(t_{2i+1})|$.

The following result was devised to try and take full advantage of that interplay.

LEMMA 1. *[31, Lemmas 3 and 4] There are $R > 0$ and $n^* \in \mathbb{N}$, depending only on q and g, such that, for every $n \geq n^*$, every $\delta \in \{0,1\}$ and every unbounded curve of initial conditions $\gamma\colon [0,+\infty[\, \to \mathbb{R}^2$, with $|\gamma(0)| \leq R$, that doesn't wind around the origin (e.g. it lies in a cone of $\mathbb{R}^2$), there is an interval $I_{n,\delta} \subset [0,+\infty[$ such that:*

1. *$x(\cdot;t_{2i-1},\gamma(s))$ has exactly n zeros in $[t_{2i-1},t_{2i}]$ for all $s \in I_{n,\delta}$;*
2. *$x(\cdot;t_{2i-1},\gamma(s))$ has exactly δ zeros and $\dot{x}(\cdot;t_{2i-1},\gamma(s))$ has exactly $1-\delta$ zeros in $[t_{2i},t_{2i+1}]$ for all $s \in I_{n,\delta}$.*
3. *the subpath $\gamma|_{I_{n,\delta}}$ is mapped by $p \mapsto z(t_{2i+1};t_{2i-1},p)$ into a curve with the same properties of γ;*

The zeros of x and $\dot{x}$ do not occur in t_{2i-1} , t_{2i} or t_{2i+1}.

Lemma 1 can be iteratively applied to $[t_1,t_3]$, $[t_3,t_5]\ldots$ to find curves of initial conditions at t_1 that carry prescribed nodal conditions on each subsequent intervals. What happens in in $[0,t_1]$ and in $[t_{2k-1},T]$ depends on the boundary conditions that are set on x. Lemma 1 was used to solve several boundary value problems for (2.1) (see [22, 27, 28, 31]). In particular, in [12] the chaotic features of (2.1) were outlined. In this respect, an inductive application of Lemma 1 provides a set of initial conditions $p \in \mathbb{R}^2$ such that $z(\cdot;0,p)$ is defined for all positive times (i.e. $\omega_{0,p} = +\infty$), which has the power of the continuum and a Cantor-Like structure.

In the paper [11], that can be considered a companion to [10], Butler studied the dual situation for (2.1) in which g is sublinear. In our simplified example, it means that $g(x) = |x|^{\beta-1}x$ with $0 < \beta < 1$ with the same kind of T-periodic weight function q that changes sign. Now we have that solutions $z = (x,y)$ rotate faster and faster around the origin in any interval where $q > 0$ as they become smaller and smaller. Moreover, the problem of blow up of solutions is replaced by the lack of uniqueness for the initial value problem of (2.1) with $z(s) = (0,0)$ if $q(s) < 0$. However, the formal similarities of the results of [10] and [11] pointed to an underlying common structure that was the subject of further investigations.

In order to get to that underlying structure, let us go back again to (2.1) in the case $\beta > 1$ and delve a little deeper into Butler's argument. In particular, I'd like to poit out how the first interval of negativity $[0,t_1]$ can be used to set up the stage for Lemma 1, even if Butler didn't have that lemma: in fact, the strategy I describe hereafter is detailed in [28]. We need to consider the *sets of continuability* $\Omega_0^{t_1}$ of all initial conditions $p \in \mathbb{R}^2$ such that $\omega_{0,p} > t_1$ or, more explicitly, such that the solution $z(\cdot;0,p)$ of (2.1) starting from p at time $t = 0$ is defined up to $t = t_1$. By [10, Lemma 2] and [31, Lemma 1], it is possible to show that:

- $\Omega_0^{t_1}$ is open and contains the origin;
- the intersections of $\Omega_0^{t_1}$ with the two axes are bounded, say, by some number $R_0 > 0$;

- if we let $\Omega_0^{t_1} \ni p \to p_0 \in \partial\Omega_0^{t_1}$, then $|z(t_1;0,p)| \to +\infty$.

It follows that, if we take any continuous path in the fourth quadrant, $\gamma_0 : [0,1] \to Q_4$, such that $|\gamma(0)|, |\gamma(1)| \geq R_0$ and $\gamma(0)$ belongs to the positive x-axis and $\gamma(1)$ to the negative y-axis, then γ must cross $\Omega_0^{t_1}$. More precisely, one can prove that there is an interval $I_1 \subset [0,1]$ such that the path $I_1 \ni s \mapsto \gamma_1(s) := z(t_1;0,\gamma_0(s))$ satisfies the assumption of Lemma 1. A similar situation holds for a path in the second quadrant, $\gamma_0 : [0,1] \to Q_2$. However, after $k-1$ successive applications of Lemma 1, we find an interval $I_k \subset [0,1]$ such that $\gamma_k(s) := z(t_{2k-1};0,\gamma_0(s))$ is well defined for $s \in I_k$, still satisfies the assumptions of the lemma, and $x(\cdot;0,\gamma_0(s))$ has prescribed nodal properties in all intervals $[t_1,t_3],\dots,[t_{2k-3},t_{2k-1}]$ (arbitrary and large number of zeros). The behavior of the solutions in the last interval of positivity $[t_{2k-1},T]$ implies that all solutions of (2.1) starting from points of $\gamma_0|_{I_k}$ at $t=0$ are defined up to $t=T$ and, moreover, that there exists at least one point $s_0 \in I_k$ such that $p = \gamma_0(s_0)$ satisfies the radial component of the T-periodic boundary condition: $|z(T;0,p)| = |z(0;0,p)| = |p|$.

By the topological lemma in [50, Lemma 3], the set of solutions $p \in \mathbb{R}^2$ of $|z(T;0,p)| = |p|$ must contain an unbounded continuum (i.e. closed and connected set) $\Gamma \subset Q_4$ such that $\Gamma \cap B[O,R_0] \neq \emptyset$ and all solutions of (2.1) starting at $t=0$ from any $p \in \Gamma$ have chosen nodal behavior in all intervals $[t_1,t_3],\dots,[t_{2k-3},t_{2k-1}]$. Moreover, it can be shown that $\Gamma_{2k-1} := z(t_{2k-1};0,\Gamma)$ also is unbounded, closed, connected and contained either in Q_1 or in Q_3 (which case holds depends on the chosen nodal behavior in $[0,t_{2k-1}]$). At this point, one can use the fact that $\mathrm{rot}(p;[t_{2k-1},T]) \to +\infty$ as $|p| \to +\infty$ to get the existence of infinitely many $p \in \Gamma$ that satisfy the angular component of the T-periodic boundary condition: $\mathrm{rot}(p[;0,T]) = 2n\pi$ for all n large enough.

3. Fixed points and chaos in conical shells

The quadrants of $\mathbb{R}^2$ are a special case of convex cones in $\mathbb{R}^2$ and a first attempt to find an abstract pattern from the argument described above can be found in [32], where the existence of fixed points and chaotic like dynamics for planar maps were discussed in conical shells, without assuming that maps are defined in the whole shells or that the shells are invariant. If $v,w \in \mathbb{R}^2$ are versors, let's denote by $\widehat{vOw}$ the angle spanned by the half lines $\mathbb{R}^+u$ as u ranges in $\partial B[0,1]$ from v to w in counter-clockwise sense, including $u=v$ and $u=w$. Moreover, we set $[\widehat{vOw}]_{r_1}^{r_2} = \widehat{vOw} \cap A[r_1,r_2]$, for $0<r_1<r_2$, and call it a *conical shell*, if $v \neq -w$ and $\widehat{vOw}$ is convex. The final part of the argument to obtain T-periodic solutions to (2.1) is contained in the following:

LEMMA 2. *[32, Corollary 2] Let $\psi : D \to \mathbb{R}^2$ be a continuous map with domain $D \subseteq \mathbb{R}^2$ and $W = [\widehat{vOw}]_{r_1}^{r_2}$ a conical shell. If there exists a connected set $\Gamma \subset \{p \in D : |\psi(p)| = |p|\}$ such that $\Gamma \subseteq W$, $\Gamma \cap \mathbb{R}^+v \neq \emptyset \neq \Gamma \cap \mathbb{R}^+w$ and $\psi(\Gamma) \subseteq W$, then ψ has at least one fixed point in Γ.*

The existence of the connected set Γ, where the radial component of the fixed point equation is satisfied, can be shown thanks to a topological lemma I already mentioned. It follows here with some definition that will shortly come in handy.

DEFINITION 1. *A set* $\mathcal{R} \subset \mathbb{R}^2$ *is a* 2D-cell *if there exists a homeomorphism* $h : [0,1] \times [0,1] \to \mathcal{R}$. *We set* $\mathcal{R}^l = h(\{0\} \times [0,1])$, $\mathcal{R}^r = h(\{1\} \times [0,1])$, $\mathcal{R}^t = h([0,1] \times \{1\})$ *and* $\mathcal{R}^b = h([0,1] \times \{0\})$ *its left, right, top and bottom sides, respectively. Moreover, a* path *is the image of a continuous curve* $\gamma : [a,b] \to \mathbb{R}^2$ *and a* subpath *is given by considering the restriction of* γ *to any closed sub interval of* $[a,b]$.

Any conical shell $[\widehat{vOw}]_{r_1}^{r_2}$ is a 2D-cell, its "left" and "right" sides being those lying on the sides $\mathbb{R}^+ v$ and $\mathbb{R}^+ w$ of the angle.

LEMMA 3. *[50, Lemma 3] Let* $\mathcal{R} \subset \mathbb{R}^2$ *be a 2D-cell and* $S \subset \mathcal{R}$ *a closed (hence, compact) set such that* $\sigma \cap S \neq \emptyset$ *for each path* $\sigma \subset \mathcal{R}$ *joining* $\mathcal{R}^t$ *and* $\mathcal{R}^b$. *Then, S contains a closed connected set (a continuum, hence)* Γ *that joins* $\mathcal{R}^l$ *and* $\mathcal{R}^r$.

The closedness of S in $\mathcal{R}$ cannot be removed, generally speaking, as shown in [32, Example 1]. On the other hand, the typical maps $\psi : D \to \mathbb{R}^2$ we want to work with have an open domain D (namely, the Poincaré maps of the form $p \mapsto z(b; a, p)$ as a consequence of the continuous dependence on initial data for (2.1)). Therefore, we have that the set

$$S = \{p \in D \cap \mathcal{R} : |\psi(p)| = |p|\} \tag{3.1}$$

is relatively closed in D and in $D \cap \mathcal{R}$ but may not be closed in $\mathcal{R}$. An extra condition has to be added on ψ and its domain D which is usually satisfied by Poincaré maps, luckily (see [32, Examples 2-3]).

DEFINITION 2. *A continuous map* $\psi : D \to \mathbb{R}^2$ *is* proper *in D if* $\psi^{-1}(K)$ *is compact for each compact set* $K \subset \mathbb{R}^2$. *Moreover,* ψ *is* proper on compact sets *if it is proper on* $D \cap K$ *for each compact set K.*

Proper maps behave well with respect to composition and also to restriction to a relatively closed set (see [32, Appendix]). However, if ψ is proper on compact sets, then the set S in (3.1) is compact, indeed, and we get a first fixed point theorem in which we require neither the invariance of the domain nor the fact that the map is defined in the whole conical shell.

THEOREM 1. *[32, Theorem 1] Suppose that* $\psi : D \to \mathbb{R}^2$ *is continuous on* $D \subseteq \mathbb{R}^2$ *and proper on compact sets and consider a conical shell* $W = [\widehat{vOw}]_{r_1}^{r_2}$. *If every path* $\sigma \subset W$ *that intersects both* $\partial B[O, r_1]$ *and* $\partial B[O, r_2]$, *contains a subpath* $\sigma' \subset D$ *such that*

$$\psi(\sigma') \cap \partial B[O, r_1] \neq \emptyset \neq \psi(\sigma') \cap \partial B[O, r_2] \qquad \text{and} \qquad \psi(\sigma') \subset \widehat{vOw},$$

then ψ *has at least one fixed point in* $D \cap W$.

The key assumption is the one about the paths σ crossing the conical shell in a "radial" way: it is a stretching condition since it requires that a subpath of σ is mapped by ψ to a path that again crosses radially the conical shell, roughly speaking. As a consequence, it's not difficult to see that any such path σ intersects the set S: hence, Lemma 3 grants that S contains a continuum Γ joining the left and right sides of the conical shell and, finally, Lemma 2 applies.

A somewhat richer result is obtained when considering the presence of multiple disjoint conical shells such that ψ satisfies a stretching condition as in Theorem 1 between any two of them. Let us see the details of a situation in which there are two conical shells.

THEOREM 2. *[32, Theorem 2] Let $\psi : D \to \mathbb{R}^2$ be a continuous map which is proper on compact sets and*

$$W_0 = [\widehat{v_0 O w_0}]_{r_1^0}^{r_2^0} \qquad \textit{and} \qquad W_1 = [\widehat{v_1 O w_1}]_{r_1^1}^{r_2^1}$$

be two disjoint conical shells. Assume that, for each $i, j \in \{0,1\}$ and each path $\sigma_i \in W_i$ that intersects both $\partial B[O, r_1^i]$ and $\partial B[O, r_2^i]$, there is a subpath $\sigma_{ij} \subset D$ such that

$$\psi(\sigma_{ij}) \cap \partial B[O, r_1^j] \neq \emptyset \neq \psi(\sigma_{ij}) \cap \partial B[O, r_2^j] \qquad \textit{and} \qquad \psi(\sigma_{ij}) \subset \widehat{v_j O w_j}.$$

Then, the following statements hold:

1. *for each $k \in \mathbb{N}$, $k \geq 1$, and each $\delta = (\delta_1, \dots, \delta_k) \in \{0,1\}^k$ there is a fixed point z_δ of ψ^k such that*
$$\psi^j(z_\delta) \in D \cap W_{\delta_j} \qquad \forall j = 1, \dots, k;$$
2. *for each sequence $\delta = (\delta_0, \delta_1, \dots) \in \{0,1\}^{\mathbb{N}}$ there is a continuum $\Gamma_\delta \subset D$ that crosses $\mathbb{R}v_{\delta_0}$ and $\mathbb{R}w_{\delta_0}$ and such that*
$$\psi^j(z) \in D \cap W_{\delta_j} \qquad \forall z \in \Gamma_\delta \quad \forall j = 0, 1, 2, \dots$$
3. *for each doubly infinite sequence $\delta = (\delta_j)_{j \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ there is a doubly infinite sequence $(z_j)_{j \in \mathbb{Z}}$ such that*
$$z_j \in D \cap W_{\delta_j} \quad \textit{and} \quad \psi(z_j) = z_{j+1} \qquad \forall j \in \mathbb{Z}.$$

This result can be applied directly to (2.1) when q changes sign only once in a period, that is $q(t) > 0$ in $]0, \tau[$ and $q(t) < 0$ in $]\tau, T[$ for some $0 < \tau < T$: once we have fixed $n \in \mathbb{N}$ sufficiently large, it is possible to find $0 < r_1 < r_2$ such that the following twist condition holds:

$$|p| < r_1 \Rightarrow \mathrm{rot}(p; [0,\tau]) < 2(n-1)\pi \quad \text{and} \quad |p| > r_2 \Rightarrow \mathrm{rot}(p; [0,\tau]) > 2(n+1)\pi.$$

Setting $W_0 = Q_1 \cap A[r_1, r_2]$ and $W_1 = Q_3 \cap A[r_1, r_2]$, Theorem 2 can be applied to $\psi(p) = z(T; 0, p)$ in order to find:

1. kT-periodic solutions of (2.1) which have exactly $2n$ zeros in $[(j-1)T, jT[$ and either 0 or 1 zero in $[(j-1)T+\tau, jT[$, for each $j = 1,\dots,k$ and all $k \in \mathbb{N}$, $k \geq 1$;

2. continua of initial conditions p such that $x(\cdot;0,p)$ is defined on $[0,+\infty[$ (i.e. $\omega_{0,p} = +\infty$) and has exactly $2n$ zeros in $[(j-1)T, jT[$ and either 0 or 1 zero in $[(j-1)T+\tau, jT[$, for each $j = 1,2,\dots$;

3. initial conditions p such that $x(\cdot;0,p)$ is globally defined and has exactly $2n$ zeros in $[(j-1)T, jT[$ and either 0 or 1 zero in $[(j-1)T+\tau, jT[$, for each $j \in \mathbb{Z}$.

In order to obtain a result that allows q to change sign more times in a single period or to consider solutions with different numbers of zeros in different periods some refinement of Theorem 2 is needed. I refer to the discussion in [32] for the detalis. Here I prefer to point out that the results on conical shells are yet too influenced by the behavior of (2.1). More precisely, the shape of conical shells reminds the fact that solutions of (2.1) wind around the origin when $q > 0$ and are stretched in a radial fashion when $q < 0$. A somewhat more abstract result should ignore the particular shapes of trajectory and rely only on the basic topological properties of the dynamical system generated by the iterations of ψ.

4. The SAP for oriented ND-rectangles

As already observed, a conical shell is just a 2D-cell in $\mathbb{R}^2$ whose boundary is homeomorphic to the boundary of a square $[0,1]^2$. Moreover, in view of the stretching condition required in Theorems 1 and 2, a specific couple of opposite sides in the boundary of the conical shell is selected: those sides are transversal to the direction of the stretching in some general and rough sense, since in our context there is no smoothness assumption to make transversality rigorous. The 2-dimensional case has been set up in [32,34,35]. However, I prefer to discuss the more general N-dimensional case which is established by Pireddu and Zanolin in [46]. In that paper there are many interesting results, especially about fixed points for continuous maps that are defined on topological N-dimensional rectangles and are expanding along some directions and contracting along the remaining ones. I'll focus here only on the stretching along the paths with its consequences, skipping also many interesting tools that are needed to get to the results that follow.

DEFINITION 3. *A* ND-rectangle $\widehat{X} = (X,h)$ *is couple given by a subset* X *of a metric space* Z *and a homeomorphism* $h : [0,1]^N \to X$. *The* contour *of* $\widehat{X}$ *is the set* $\vartheta X = h(\partial[0,1]^N)$. *For each* $i = 1,\dots,N$ *we let* $[x_i = s] = \{x = (x_1,\dots,x_N) \in [0,1]^N : x_i = s\}$. *Moreover,* $X_\ell = [x_N = 0]$, $X_r = [x_N = 1]$ *and* $X^- = X_\ell \cup X_r$ *are the "vertical" sides of* X *and we call the couple* $\widetilde{X} = (X, X^-)$ *an* oriented ND-rectangle.

The choice of X^- is purely conventional: one could use any of the other couples of opposite "faces" of ϑX.

DEFINITION 4. *Let* $\widetilde{X} = (X, X^-)$ *and* $\widetilde{Y} = (Y, Y^-)$ *be two oriented ND-rectangels of a metric space Z,* $\psi : Z \supseteq D_\psi \to Z$ *be a (not necessarily continuous) map and consider a set* $D \subseteq D_\psi$. *We say that* (D, ψ) stretches $\widetilde{X}$ to $\widetilde{Y}$ along the paths, *and write*

$$(D, \psi) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{Y},$$

if there exists a compact set $K \subseteq D$ *such that* ψ *is continuous on K and each path* $\gamma \subseteq X$, *with* $\gamma \cap X_\ell \neq \emptyset \neq \gamma \cap X_r$, *has a subpath* σ *such that*

$$\sigma \subseteq K, \quad \psi(\sigma) \subseteq Y \quad \textit{and} \quad \psi(\sigma) \cap Y_\ell \neq \emptyset \neq \psi(\sigma) \cap Y_r.$$

In this case, we also write

$$(D, K, \psi) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{Y} \qquad \textit{or} \qquad (K, \psi) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{Y},$$

when we want to stress the role of K versus the one of D. If there are m pairwise disjoint compact sets $K_0, K_1, \ldots, K_{m-1} \subset D$ *such that* $(D, K_i, \psi) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{Y}$ *for* $i = 0, 1, \ldots, m-1$, *for some* $m \geq 1$, *we write* $(D, \psi) : \widetilde{X} \overset{m}{\Leftrightarrow\!\!\!\rightarrow} \widetilde{Y}$ *and say that* ψ stretches $\widetilde{X}$ to $\widetilde{Y}$ m times (with respect to $K_0, \ldots, K_{m-1}$).

With this definition one gets the following:

THEOREM 3. *[46, Theorem 5.1] If* $(K, \psi) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{X}$, *then* ψ *has a fixed point in K.*

A comparison with Theorem 1 shows immediately that the requirement of ψ being proper on compact sets is replaced by the compactness of the set K. This last assumption again cannot be removed, generally speaking (see [35, §3.3]). The SAP property is preserved by composition of maps.

PROPOSITION 1. *(See [46, Theorem 5.2]) Assume that:*

$$(D_1, K_1, \psi_1) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{Y} \qquad \textit{and} \qquad (D_2, K_2, \psi_2) : \widetilde{Y} \Leftrightarrow\!\!\!\rightarrow \widetilde{Z}$$

and set:

$$D = \{z \in D_1 : \psi_1(z) \in D_2\} \quad \textit{and} \quad K = \{z \in K_1 : \psi_1(z) \in K_2\}.$$

Then $(D, K, \psi_2 \circ \psi_1) : \widetilde{X} \Leftrightarrow\!\!\!\rightarrow \widetilde{Z}$.

Inductively, the last result can be easily extended to the composition of a finite number of maps and that is precisely what is needed to study (2.1) when q and g are as in Section 2 and, in particular, q is allowed several changes of sign as follows:

$$0 = t_0 < t_1 < t_2 \cdots < t_{2k-1} < t_{2k} = T$$

and $q(t) < 0$ in $]t_{2i-2}, t_{2i-1}[$ and $q(t) > 0$ in $]t_{2i-1}, t_{2i}[$, for $i = 1, 2, \ldots, k$. For each couple of adjacent intervals $[t_{2j-2}, t_{2j-1}]$ and $[t_{2j-1}, t_{2j}]$ $(1 \leq j \leq k)$, the first of negativity for q and the second of positivity, it is possible to show that the Poincaré map

$\psi_j : p \mapsto z(t_{2j}; t_{2j-2}, p)$ stretches two suitably defined conical shells, one on the other one, multiple times. More precisely, for every $N \in \mathbb{N}$, $N \geq 1$, it is possible to find radii $0 < r_j < R_j$ and numbers $n_j^* \in \mathbb{N}$, for $j = 1, \ldots, k$, such that:

1. we set $\widetilde{X}_j = (X_j, X_j^-)$ and $\widetilde{Y}_j = (Y_j, Y_j^-)$ where:

$$X_j = A[r_j, R_j] \cap Q_4 \qquad X_j^- = X_j \cap ([x=0] \cup [y=0])$$
$$Y_j = A[r_j, R_j] \cap Q_2 \qquad Y_j^- = Y_j \cap ([x=0] \cup [y=0])$$

   with $\widetilde{X}_0 = \widetilde{X}_k$ and $\widetilde{Y}_0 = \widetilde{Y}_k$;

2. for every path γ in X_{j-1} joining the two sides of X_{j-1}^- and every $n \in n_j^* + \{1, \ldots, N\}$ there are *four* subpaths $\sigma_{1,n}, \sigma_{2,n}, \sigma_{3,n}, \sigma_{4,n}$ of γ such that $\psi_j(\sigma_{1,n})$ and $\psi_j(\sigma_{2,n})$ are contained in X_j and joins the two components of X_j^-, while $\psi_j(\sigma_{3,n})$ and $\psi_j(\sigma_{4,n})$ are contained in Y_j and joins the two components of Y_j^-; moreover for $x(t) := x(t; t_{2j-2}, p)$ we have that

$$p \in \sigma_{1,n} \Longrightarrow x \text{ has no zeros in } [t_{2j-2}, t_{2j-1}] \text{ and } 2n \text{ zeros in } [t_{2j-1}, t_{2j}]$$
$$p \in \sigma_{2,n} \Longrightarrow x \text{ has 1 zero in } [t_{2j-2}, t_{2j-1}] \text{ and } 2n-1 \text{ zeros in } [t_{2j-1}, t_{2j}]$$
$$p \in \sigma_{3,n} \Longrightarrow x \text{ has no zeros in } [t_{2j-2}, t_{2j-1}] \text{ and } 2n+1 \text{ zeros in } [t_{2j-1}, t_{2j}]$$
$$p \in \sigma_{4,n} \Longrightarrow x \text{ has 1 zero in } [t_{2j-2}, t_{2j-1}] \text{ and } 2n \text{ zeros in } [t_{2j-1}, t_{2j}]$$

   and a similar statement holds if γ is any path in Y_{j-1} joining the two components of Y_{j-1}^-;

3. as a consequence, we have that

$$(D_j, \psi_j) : \widetilde{X}_{j-1} \overset{2N}{\Longleftrightarrow\!\!\!\!\to} \widetilde{X}_j \qquad (D_j, \psi_j) : \widetilde{X}_{j-1} \overset{2N}{\Longleftrightarrow\!\!\!\!\to} \widetilde{Y}_j$$
$$(D_j, \psi_j) : \widetilde{Y}_{j-1} \overset{2N}{\Longleftrightarrow\!\!\!\!\to} \widetilde{X}_j \qquad (D_j, \psi_j) : \widetilde{Y}_{j-1} \overset{2N}{\Longleftrightarrow\!\!\!\!\to} \widetilde{Y}_j$$

   where $D_j = \{p \in \mathbb{R}^2 : t_{2j} < \omega_{p, t_{2j-1}}\}$ for all $j = 1, \ldots, k$;

4. using Proposition 1, we deduce that:

$$(D, \psi) : \widetilde{X}_0 \overset{M}{\Longleftrightarrow\!\!\!\!\to} \widetilde{X}_k = \widetilde{X}_0 \qquad \text{and} \qquad (D, \psi) : \widetilde{Y}_0 \overset{M}{\Longleftrightarrow\!\!\!\!\to} \widetilde{Y}_k = \widetilde{Y}_0$$

   where $D = \{p \in \mathbb{R}^2 : T < \omega_{0,p}\}$, $\psi(p) = z(T; 0, p)$ and $M = (4N)^{k-1} 2N$, which already gives the existence of $(4N)^k$ T-periodic solutions of (2.1) that are distinguished by their nodal behavior as in step 2, thanks to Theorem 3.

I'm not giving more details here, beside commenting that the first two steps follows from the qualitative analysis performed, for instance, in [28]. On the other hand, it's time to make more precise the kind of chaotic dynamics a result like that outlines.

DEFINITION 5. *Let $\psi : D \to Z$ be a map, where $D \subseteq Z$ and Z is a metric space, and $m \geq 2$ be an integer. We say that ψ* induces chaotic dynamics on m symols (in the set D) *if there are m compact sets $K_0, K_1, \dots, K_{m-1} \subset D$ which are non-empty and pairwise disjoint and such that, for each 2-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \Sigma_m = \{0, 1, \dots, m-1\}^{\mathbb{Z}}$, there exists a corresponding 2-sided sequence $(w_i)_{i\in\mathbb{Z}} \in D^{\mathbb{Z}}$ satisfying:*

1. *$w_{i+1} = w_i$ for all $i \in \mathbb{Z}$;*

2. *$w_i \in K_{s_i}$ for all $i \in \mathbb{Z}$;*

3. *if $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence for some $k \in \mathbb{N}$, then $(w_i)_{i\in\mathbb{Z}}$ is k-periodic, too.*

Requirements 1 and 2 in the case $m = 2$ are equivalent to say that the discrete dynamical system generated by the iterates of ψ shows chaos in the sense of coin tossing according to [19]. All three properties 1–3 are inherited by maps $\psi : D \to D$ which are *topologically conjugated* to the Bernoulli shift $\sigma : \Sigma_m \to \Sigma_m$, meaning that there exists a homeomorphism $h : D \to \Sigma_m$ such that the following diagram

$$\begin{array}{ccc} D & \xrightarrow{\;h\;} & \Sigma_m \\ \psi\big\downarrow & & \big\downarrow\sigma \\ D & \xrightarrow{\;h\;} & \Sigma_m \end{array}$$

commutes. In that case ψ has the same topological entropy as σ, that is $\log m$. On the other hand, when h is just continuous and surjective, then ψ is *topologically semi-conjugated* to the Bernouilli shift and the topological entropy of σ is still a lower bound for that of ψ, provided that D is an invariant set for ψ. However, the only surjectivity of h doesn't grant condition 3 about periodic points, while, if ψ induces chaos in D on m symbols then there exists a non-empty compact set $\Lambda \subset K_0 \cup \dots \cup K_{m-1}$ which is invariant with respect to ψ and such that $\psi|_\Lambda$ is topologically semi-conjugated to σ and the set of periodic points of $\psi|_\Lambda$ is dense in Λ (see [46, Lemma 5.1]).

THEOREM 4. *[46, Theorem 5.3] If $(D, \psi) : \widetilde{X} \stackrel{m}{\Bumpeq\!\!\!\!\!\!\longrightarrow} \widetilde{X}$, then ψ induces chaos on m symbols.*

[46, Theorem 5.3] gives more information that I'm neglecting here for the sake of brevity. However, it's enough to be applied to (2.1) in the specific case of q and g I considered here. As far as application of the SAP in more than two dimensions are concerned, I just point to the papers [51, 52].

5. Linked twist maps

The theory of linked twist maps is a useful tool to show the presence of chaotic dynamics generated by the switching of two different maps that satisfy a twist condition on two different annuli that are somehow linked (see [1, 13, 55]). Margheri, Rebelo and Zanolin used the SAP in [23] to extend that framework to the case of topological annuli. A topological annulus may be constructed starting from two Jordan curves that determines its boundary. If γ is a Jordan curve in $\mathbb{R}^2$, we denote by $\mathcal{I}(\gamma)$ its interior, i.e. the open and bounded connected component of $\mathbb{R}^2 \setminus \gamma$, and by $\mathcal{E}(\gamma)$ its exterior, which is the open and unbounded connected component of $\mathbb{R}^2 \setminus \gamma$. If γ_i, γ_e are two Jordan curves such that $\gamma_i \subset \mathcal{I}(\gamma_e)$ or, equivalently, $\gamma_e \subset \mathcal{E}(\gamma_i)$, then they define an annulus $A = \overline{\mathcal{I}(\gamma_e)} \setminus \mathcal{I}(\gamma_i)$ and we set $\partial^i A = \gamma_i$ and $\partial^e A = \gamma_e$, so that $\partial A = \partial^i A \cup \partial^e A$.

It is possible to associate covering projections

$$\Pi_i : \widehat{A}_i = \mathbb{R} \times [a_i, b_i] \to A_i \qquad i = 1,2$$

to two topological annuli $A_1, A_2 \subset \mathbb{R}^2$, in such a way that:

1. $\Pi_i(\theta + 1, \alpha) = \Pi_i(\theta, \alpha)$ for all $\theta \in \mathbb{R}$ and $\alpha \in [a_i, b_i]$;
2. $C_i(\alpha) = \Pi_i(\mathbb{R}, \alpha)$ is a Jordan curve for all $\alpha \in [a_i, b_i]$ and, in particular, $C_i(a_i) = \partial^i A_i$ and $C_i(b_i) = \partial^e A_i$;
3. $L_i(\theta) := \Pi_i(\theta, [a_i, b_i])$ is an arc in A_i that connects the inner and the outer boundaries of A_i.

Two annuli A_1 and A_2 are *linked through a topological rectangle* $\mathcal{R}$ if there is a 2D-cell $\mathcal{R} \subset A_1 \cap A_2$ whose boundary is the concatenation of four arcs $\sigma_1^i * \sigma_2^i * \sigma_1^e * \sigma_2^e$, where $*$ stands for the concatenation of curves and $\sigma_1^i \in \partial^i A_1$, $\sigma_2^i \in \partial^i A_2$, $\sigma_1^e \in \partial^e A_1$ and $\sigma_2^e \in \partial^e A_2$. It is possible to give two different orientations $\widetilde{\mathcal{R}}_1 = (\mathcal{R}, \mathcal{R}_1^-)$ and $\widetilde{\mathcal{R}}_2 = (\mathcal{R}, \mathcal{R}_2^-)$to $\mathcal{R}$ by setting:

$$\mathcal{R}_1^- := \sigma_1^i \cup \sigma_1^e \qquad \text{and} \qquad \mathcal{R}_2^- := \sigma_2^i \cup \sigma_2^e.$$

In particular, the Jordan curve $C_2(\alpha)$ crosses $\mathcal{R}$ through $\mathcal{R}_1^-$ for all $\alpha \in [a_2, b_2]$, while $C_1(\alpha)$ crosses $\mathcal{R}$ through $\mathcal{R}_2^-$ for all $\alpha \in [a_1, b_1]$.

Let us consider, now, two continuous maps $\psi_i : A_i \to A_i$, $i = 1,2$, and their liftings $\widehat{\psi}_i : \widehat{A}_i \to \widehat{A}_i$ such that

$$\Pi_i \circ \widehat{\psi}_i = \psi_i \circ \Pi_i \qquad \text{and} \qquad \widehat{\psi}_i(\theta, \alpha) = (\theta + g_i(\theta, \alpha), R_i(\theta, \alpha)),$$

where $g_i, R_i : \widehat{A}_i \to \mathbb{R}$ are continuous and 1-periodic with respect to θ. We consider the following two assumptions:

(BI): $R_i(\theta, a_i) = a_i$ and $R_i(\theta, b_i) = b_i$ for all $\theta \in \mathbb{R}$ and $i = 1,2$.

(TC): For each $i = 1,2$ there exists $m_i \in \mathbb{N}$ such that either

$$\max_{\theta \in [0,1]} g_i(\theta, a_i) \le -1 \qquad \text{and} \qquad \min_{\theta \in [0,1]} g_i(\theta, b_i) \ge 1 + m_i$$

or

$$\min_{\theta \in [0,1]} g_i(\theta, a_i) \ge 1 + m_1 \qquad \text{and} \qquad \max_{\theta \in [0,1]} g_i(\theta, b_i) \le -1$$

(BI) is a boundary invariance condition since it prescribes that $\psi_i(\partial^i A_i) \subseteq \partial^i A_i$ and $\psi_i(\partial^e A_i) \subseteq \partial^e A_i$. **(TC)** is a twist condition, since it implies that ψ_i moves the points of the inner and of the outer boundaries of A_i in opposite directions: the larger is m_i the larger is the gap of the angular displacement of points on the two components of the boundary under the action of ψ_i.

Now, if **(BI)** and **(TC)** hold and we consider any path γ in $\mathcal{R}$ joining, for example, σ_1^i and σ_1^e, the twist condition implies that $\psi_1(\gamma)$ crosses $\mathcal{R}$ through $\mathcal{R}_2^-$ at least m_1+1 times (in fact, if $m_1 \geq 1$, $\psi_1(\gamma)$ makes more than m_1 turns inside A_1). Similarly, if γ joins σ_2^i and σ_2^e, then $\psi_2(\gamma)$ crosses $\mathcal{R}$ through $\mathcal{R}_1^-$ at least m_2+1 times. Therefore, it can be proved that

$$(\mathcal{R}, \psi_1) : \widetilde{\mathcal{R}}_1 \overset{m_1+1}{\Longleftrightarrow\!\!\!\!\rightarrow} \widetilde{\mathcal{R}}_2 \qquad \text{and} \qquad (\mathcal{R}, \psi_2) : \widetilde{\mathcal{R}}_2 \overset{m_2+1}{\Longleftrightarrow\!\!\!\!\rightarrow} \widetilde{\mathcal{R}}_1$$

and, thus the following result holds.

THEOREM 5. *[23, Theorem 3.1] If the annuli A_1 and A_2 are linked through $\mathcal{R}$ and ψ_1, ψ_2 satisfy* **(BI)** *and* **(TC)***, then $\psi_2 \circ \psi_1$ has at least a fix point in $\mathcal{R}$. If, moreover, $m_1 \geq 1$ or $m_2 \geq 1$, then $\psi_2 \circ \psi_1$ induces chaotic dynamics on $(m_1+1)(m_2+1)$ symbols.*

In [23] a second result is proved in which the boundary invariance is removed at the cost of a strengthened twist condition (for the details, see [23, Theorem 3.2]). Those kind of results can be applied, for instance, to planar Hamiltonian systems of the form

$$\dot{z} = J\nabla H(z) + e(t)$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function, $J \in \mathbb{R}^{2\times 2}$ is the symplectic matrix and $e : \mathbb{R} \to \mathbb{R}^2$ is a T-periodic piecewise constant function such that

$$e(t) = \begin{cases} e_1 & \text{if } 0 \leq t < \tau_1 \\ e_2 & \text{if } \tau \leq t < \tau_1 + \tau_2 = T \end{cases}$$

for some $e_1, e_2 \in \mathbb{R}^2$ and $\tau_1, \tau_2 > 0$. If each of the two switching systems:

$$\text{(H1) } \dot{z}_1 = J\nabla H(z_1) + e_1 \qquad \text{and} \qquad \text{(H2) } \dot{z}_2 = J\nabla H(z_2) + e_2$$

possesses an annulus A_i ($i = 1, 2$) filled by periodic orbits and A_1 and A_2 are linked through a topological rectangle $\mathcal{R}$, then the stage is ready for a direct application of Theorem 5 to the Poincaré maps $\psi_i(p) = z_i(\tau_i; 0, p)$. The twist condition is granted provided that the periods of the orbits running the interior and the exterior boundaries of each annulus are different and τ_1, τ_2 are large enough.

As a final remark, I point out that results like Theorem 5 can be applied also to systems that are not Hamiltonian. Some examples can be found in [29, 30], where a different notion of topologically linked annuli is given, which is easier to check in applications and may cover situations in which two annuli are not linked through a topological rectangle.

6. Final remark

I've been a lucky person up to now, very lucky. A great piece of my luck was to be in the right place at the right moment and to meet Fabio. However, being supervised by Fabio is an experience that may be also misleading, in some sense. You've got to work with a boss who is gentle, meek, kind, beside having an entire and continuously updating mathematical library stored in his head and ready at his fingertips; a kind of "no-stress" and caring presence, quick to help as well as to discuss about science fiction novels, videogames, or the last instalment of a Star Trek tale. You might even be induced to believe that the academic world is that way. The world is as we make it and Fabio surely makes the piece of the world he walks in a far better place.

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