Nonlinear Anal. RWA 85 (2025) 104361

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

# A Hyperbolic–parabolic framework to manage traffic generated pollution

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ARTICLE INFO

*MSC:*
35L65
76A30
35Q93
35K15

*Keywords:*
Hyperbolic traffic models
Parabolic pollution models
Optimal traffic management
Stop & go waves

ABSTRACT

Vehicular traffic flows through a merge regulated by traffic lights and produces pollutant that diffuses in the surrounding region. This situation motivates a general hyperbolic - parabolic system, whose well-posedness and stability are here proved in $\mathbf{L}^1$. Roads are allowed to be also 2-dimensional. The effects of stop & go waves are comprised, leading to measure source terms in the parabolic equation. The traffic lights, as well as inflows and outflows, can be regulated to minimize the presence of pollutant in given regions.

## 1. Introduction

We study a macroscopic model describing how vehicular traffic along a road network produces pollution and how this pollution propagates. The well-posedness and stability properties obtained allow to tackle the problem of optimal management of traffic lights in order to minimize the presence of pollution in given regions.

The network we consider consists of a merge with 2 incoming roads $\mathcal{R}^1$ and $\mathcal{R}^2$ and one outgoing road $\mathcal{R}^3$. All roads may also have a 2-dimensional geometry, so that $\mathcal{R}^i \subset \mathbb{R}^2$, and the traffic along them is described by a 2-dimensional Lighthill–Whitham [1] and Richards [2] model

$$\begin{cases} \partial_t \rho^i + \nabla \cdot \left(\vec{q}(x,\rho^i)\right) = 0 & x \in \mathcal{R}^i & i = 1,2,3 \\ \rho^i(0,x) = \rho^i_o(x) & x \in \mathcal{R}^i & i = 1,2,3 \\ \vec{q}\left(\xi,\rho^i(t,\xi)\right) = f^i_{in}(t,\xi) & \xi \in \text{entry to } \mathcal{R}^i & i = 1,2 \\ \vec{q}\left(\xi,\rho^i(t,\xi)\right) = f^i_{out}(t,\xi) & \xi \in \text{exit to } \mathcal{R}^i & i = 3\,. \end{cases} \tag{1.1}$$

As usual, $\rho = \rho(t,x)$ is the traffic density and $\vec{q} = \vec{q}\,(x,\rho(t,x))$ the corresponding traffic flow. A precise statement of the whole model is presented below.

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https://doi.org/10.1016/j.nonrwa.2025.104361
Received 12 February 2024; Received in revised form 24 February 2025; Accepted 25 February 2025
Available online 13 March 2025

Traffic is controlled through the inflows $f^1_{in}$, $f^2_{in}$, the outflow $f^3_{out}$ and through traffic lights at the merge. The latter consists in selecting time intervals $J_j$ where the light is green for $\mathcal{R}^1$, while in the complement $\mathbb{R}^+ \setminus \bigcup J_j$ the light is green for $\mathcal{R}^2$, i.e.,

$$\begin{array}{llll} t\in \bigcup_j J_j & f^1_{out}=f^3_{in} & \text{and} & f^2_{out}=0\,; \\ t\in \mathbb{R}^+\setminus \bigcup_j J_j & f^1_{out}=0 & \text{and} & f^2_{out}=f^3_{in}\,. \end{array}$$

Traffic flow produces pollutant whose density $u = u(t, x)$ propagates according to

$$\begin{cases} \partial_t u + \nabla \cdot (u\,\beta(t,x) - \mu(t,x)\,\nabla u) + \kappa(t,x)\,u = G \\ u(0,x) = u_o(x) \end{cases} \qquad x \in \mathbb{R}^2 \tag{1.2}$$

where $\mu = \mu(t, x)$ is the diffusion coefficient, $\beta = \beta(t, x)$ represents the wind velocity field, $\kappa = \kappa(t, x)$ is the pollutant decay rate and $u_o = u_o(x)$ is the initial datum. The source term $G$ models the pollutant production and is a function possibly depending locally or non locally on $\rho$, $\vec q$. We underline that $G$ may well depend also on the spatial derivative of the flow $\vec q$, so that it accounts also for accelerations, well known to be a major source of pollutant.

We remark that, differently from other results in the literature, we carefully adopt a conservative form for (1.2) so that the production of pollutant is exclusively due to vehicular traffic, while its decay is that explicitly prescribed in the equation through the term $\kappa\, u$.

Analytically, we develop an $\mathbf{L}^1$ well-posedness theory for solutions, understood in the weak sense, to the system consisting of the nonlinear hyperbolic equation (1.1) on the merge coupled to the parabolic equation (1.2). The coupling, nonlinear and possibly nonlocal, may also contain in measure terms.

At the hyperbolic level, this is achieved extending [3] to comprise the presence of the boundary and [4] to account for flux constraints, all this first in the 1d case corresponding to a 1–dimensional version of (1.1), i.e., an initial boundary value problem with flux constraints. Then, we introduce a framework for a 2–dimensional road, further extending [3,4]. At last, we also include the 2–dimensional merge. In this part, the techniques developed below rely on the definitions of solution from [5–7], notably stable with respect to $\mathbf{L}^1$ convergence. Moreover, precise $\mathbf{BV}$ estimates are obtained to allow that the distributional spatial derivatives of $\rho$ and $\vec q$ appear in $G$, as required by the physical application considered. Note that the roads' geometry is essentially arbitrary and the extension to more general networks is a matter of iterating the techniques below to more roads or junctions.

The parabolic part is settled in $\mathbf{L}^1$, as in [8], a norm whose physical meaning in (1.2) is evident. This choice requires our introduction of an *ad hoc* definition of weak solution and of the related uniqueness and continuous dependence theorems, obtained essentially extending the classical results limited to strong solutions in [9]. The case of the source term $G$ in (1.2) being a Radon measure falls within the general well-posedness framework here developed and allows to account for stop & go waves.

Once stability estimates are available, we optimize the timing of the traffic lights, the inflows $f^1_{in}$, $f^2_{in}$ and the outflow $f^3_{out}$, so that an integral functional measuring the presence of pollutant in a given region is minimized. For completeness, we refer to [10] for a numerical approach to the control of a conservative equation like (1.2).

Thus, we provide a rigorous analytical framework to various results in the literature motivated by pollution production and propagation. We refer for instance to [11,12], consisting mainly in numerical simulations devoted to general networks. Second order models along a junction connecting several 1–dimensional roads are considered in [13,14]. The minimization of pollution by means of *ad hoc* speed limits is presented in the recent work [15].

The use of mixed hyperbolic–parabolic systems is frequent in various mathematical models: see for instance [16] motivated by tumor growth and analytically limited to radially symmetric solutions, or [17] devoted to the circulation of blood on a network, analytically settled in $\mathbf{L}^2$ and considering strong solutions on a bounded domain with Robin boundary conditions. A related mixed ODE – parabolic PDE model is considered in [18]. Recall that the minimization of urban traffic-related air pollution within different uni- and multi-objective optimization frameworks was also studied, among others, in [19–22].

The necessary 1–dimensional results require minor modifications of known theorems and are collected in Section 2. The analytical framework describing a 2–dimensional road together with the corresponding well-posedness and stability results related to (1.1) are in Section 3. The parabolic model (1.2) is treated in Section 4, while the optimal management problems is deferred to Section 5. All proofs are collected in Section 6. Finally, Section 7 provides hints to possible future works.

## 2. IBVP on a 1D network with a flux constraint

Following the classical Lighthill–Whitham [1] and Richards [2] model, the vehicle speed $v = v(\rho)$ at traffic density $\rho$ is prescribed, for all $\rho \in [0, R]$, $R > 0$ being the fixed maximal density. Throughout, we require the following assumption on $v$, stated by means of the corresponding flow $q(\rho) = \rho\, v(\rho)$, whose graph is often referred to as *fundamental diagram*:

**(q)** $q \in \mathbf{C}^2([0, R]; \mathbb{R}_+)$ is such that $q(0) = q(R) = 0$, $q'(\rho) > 0$ for $\rho \in [0, \bar\rho[$ and $q'(\rho) < 0$ for $\rho \in ]\bar\rho, R]$, for a fixed critical density $\bar\rho \in ]0, R[$.

The restrictions $q_{|[0,\bar\rho]}$ of $q$ to $[0, \bar\rho]$ and $q_{|[\bar\rho,R]}$ of $q$ to $[\bar\rho, R]$ are invertible and we denote them by $q^{-1}_{|[0,\bar\rho]}$ and $q^{-1}_{|[\bar\rho,R]}$ respectively.

For completeness, we recall the basic results concerning the following Initial-Boundary Value Problem (IBVP) on the (nontrivial) bounded open interval $I$:

$$\begin{cases} \partial_t \rho + \partial_s q(\rho) = 0 & (t,s) \in [0,T] \times I \\ \rho(0,s) = \rho_o(s) & s \in I \\ q\left(\rho(t,s_{in})\right) = f_{in}(t) & t \in [0,T] \\ q\left(\rho(t,s_{out})\right) = f_{out}(t) & t \in [0,T], \end{cases} \tag{2.1}$$

where we set $s_{in} = \inf I$ and $s_{out} = \sup I$.

Particular care has to be taken to correctly evaluate the effects and the meaning of the boundary conditions. First, as it is usual in the context of hyperbolic conservation laws, a solution to (2.1) needs not to satisfy these conditions at all $t$, not even at a.e. $t$, see [5–7]. Second, due to our considering only fluxes satisfying (q), the boundary conditions in (2.1) can be equivalently understood as *constraints*. Indeed, the trace $(q\circ\rho(t))(s_{in}+)$ at $s_{in}$, respectively $(q\circ\rho(t))(s_{out}-)$ at $s_{out}$, of the flow $q(\rho)$ of the solution $\rho$ is the maximal possible flow of a solution compatible with $(q\circ\rho(t))(s_{in}+) \le f_{in}(t)$, respectively $(q\circ\rho(t))(s_{out}-) \le f_{out}(t)$.

The specific form of the following definition is inspired by [5, Definition 1] and [7, Definition 1].

**Definition 2.1** (*[4, Definition 3.1]*)**.** A *solution* to (2.1) is a map $\rho \in \mathbf{L}^\infty([0,T]\times I;\mathbb{R})$ such that for any test function $\varphi \in \mathbf{C}^1_c(\mathbb{R}^2;\mathbb{R}_+)$ and for any $\kappa \in \mathbb{R}$,

$$\begin{aligned} &\int_0^T \int_I (\rho(t,s)-\kappa)^\pm\, \partial_t\varphi(t,s)\,\mathrm{d}s\,\mathrm{d}t + \int_0^T \int_I \operatorname{sgn}^\pm(\rho(t,s)-\kappa)\,(q(\rho(t,s)) - q(\kappa))\,\partial_s\varphi(t,s)\,\mathrm{d}s\,\mathrm{d}t \\ &+ \int_I \left(\rho_o(s)-\kappa\right)^\pm \varphi(0,s)\,\mathrm{d}s + \int_I (\rho(T,s)-\kappa)^\pm\, \varphi(T,s)\,\mathrm{d}s \\ &+ \|q'\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \int_0^T \left( q^{-1}_{|[0,\bar\rho]}\left(f_{in}(t)\right) - \kappa \right) \varphi(t,s_{in})\,\mathrm{d}t + \|q'\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \int_0^T \left( q^{-1}_{|[\bar\rho,R]}\left(f_{out}(t)\right) - \kappa \right) \varphi(t,s_{out})\,\mathrm{d}t \ge 0\,. \end{aligned}$$

Here and in what follows, we use the standard notation $x^+ = \max\{x,0\}$, $x^- = \min\{x,0\}$, for any $x \in \mathbb{R}$, as well as $\operatorname{sgn}^+ x = \begin{cases} 1 & x>0 \\ 0 & x\le 0 \end{cases}$ and $\operatorname{sgn}^- x = \begin{cases} 0 & x\ge 0 \\ -1 & x<0 \end{cases}$.

The well-posedness of (2.1) and detailed stability estimates are ensured extending [4, Propositions 3.2 and 3.3] to comprise non **BV** initial and boundary data, see also [3, Theorem 2.2]. The following result also provides estimates on the total variation of the nonlinear map $x \mapsto \Psi(\rho(t,x))$, where

$$\Psi(\rho) = \operatorname{sgn}(\rho-\bar\rho)\,(q(\bar\rho)-q(\rho))\,. \tag{2.2}$$

This bound is instrumental in ensuring the existence of traces of $\rho$ and in proving that $q(\rho)$ has bounded total variation, while $x \mapsto \rho(t,x)$ may well have infinite total variation.

**Proposition 2.2.** *Let $q$ satisfy (q), $\rho_o \in \mathbf{L}^1(I;[0,R])$ and $f_{in}, f_{out} \in \mathbf{L}^1([0,T];[0,q(\bar\rho)])$. Then, problem (2.1) admits a unique solution $\rho$ in the sense of Definition 2.1 which satisfies, for all $t \in [0,T]$,*

$$\|\rho(t)\|_{\mathbf{L}^\infty(I;\mathbb{R})} \le \max\left\{ \|\rho_o\|_{\mathbf{L}^\infty(I;\mathbb{R})}, \left\|q^{-1}_{|[0,\bar\rho]}\circ f_{in}\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})}, \left\|q^{-1}_{|[\bar\rho,R]}\circ f_{out}\right\|_{\mathbf{L}^\infty([0,t];\mathbb{R})} \right\}. \tag{2.3}$$

*Moreover, if $\hat\rho_o, \hat f_{in}$ and $\hat f_{out}$ satisfy the same assumptions, the corresponding solution $\hat\rho$ satisfies, for all $t \in [0,T]$,*

$$\left\|\rho(t)-\hat\rho(t)\right\|_{\mathbf{L}^1(I;\mathbb{R})} \le \left\|\rho_o-\hat\rho_o\right\|_{\mathbf{L}^1(I;\mathbb{R})} + \left\|f_{in}-\hat f_{in}\right\|_{\mathbf{L}^1([0,t];\mathbb{R})} + \left\|f_{out}-\hat f_{out}\right\|_{\mathbf{L}^1([0,t];\mathbb{R})}. \tag{2.4}$$

*If in addition, with reference to (2.2), $\Psi(\rho_o) \in \mathbf{BV}(I;\mathbb{R})$ and $f_{in}, f_{out} \in \mathbf{BV}([0,T];[0,q(\bar\rho)])$ then, for all $t \in [0,T]$, the map $\Psi(\rho(t))$ has uniformly bounded total variation in the space variable.*

The proof of Proposition 2.2 is deferred to Section 6.1. We remark that an estimate analogous to (2.4) is also obtained in [3, Theorem 2.2], where, as in the present setting, the incoming flux is assigned as (Dirichlet) boundary datum.

Consider now three 1–dimensional (one-way) roads, where roads 1 and 2 merge into road 3. Denote by $I_i$ the road segment parameterizing the $i$th road, and by $\rho^i$ the corresponding traffic density. All densities vary in the same interval $[0,R]$. Moreover, the speed law $v$ is the same on all the three roads, with $q(\rho) = \rho\, v(\rho)$ satisfying (q). All this leads to consider

$$\begin{cases} \partial_t \rho^i + \partial_s q(\rho^i) = 0 & (t,s) \in [0,T]\times I_i & i = 1,2,3 \\ \rho^i(0,s) = \rho^i_o(s) & s \in I_i & i = 1,2,3 \\ q\left(\rho^i(t,s^i_{in})\right) = f^i_{in}(t) & t \in [0,T] & i = 1,2 \\ q\left(\rho^i(t,s^i_{out})\right) = f^i_{out}(t) & t \in [0,T] & i = 3 \end{cases} \tag{2.5}$$

where $s^i_{in} = \inf I_i$ and $s^i_{out} = \sup I_i$, $i = 1,2,3$.

The regulation of the merge is described by suitable conditions on the outflows $f^1_{out}$, $f^2_{out}$ of the incoming roads and on the inflow $f^3_{in}$ of the outgoing one. For instance, we introduce traffic lights, displaying alternatively red or green lights at the end of

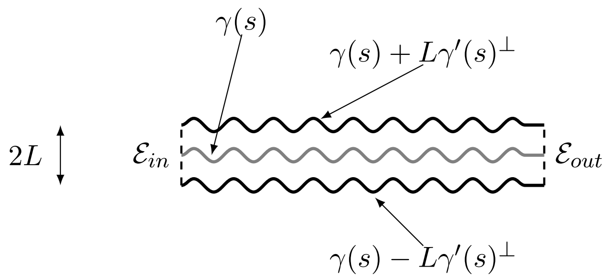


**Fig. 3.1.** Road parameterization introduced in Definition 3.1. The gray line is the support of $s \mapsto \gamma(s)$, while the thick lines support the maps $s \mapsto \gamma(s) \pm L\gamma'(s)^\perp$. The entry $\mathcal{E}_{in}$ and the exit $\mathcal{E}_{out}$ defined in (3.1) are the dashed segments.

each incoming road. This leads to introduce the time intervals $J_j$ where the green light is between roads 1 and 3. Correspondingly, consider the class $\mathcal{W}$ of functions $w$ that can be written as follows, for a suitable $n \in \mathbb{N}$:

$$\begin{array}{rcccl} w & : & [0,T[ & \to & \{0,1\} \\ & & t & \mapsto & \sum_{j=1}^{n} \chi_{J_j}(t) \end{array} \quad \text{where} \quad \begin{array}{l} J_j \subseteq [0,T] \text{ is an interval} \\ J_j \neq \emptyset \\ j \neq k \implies J_j \cap J_k = \emptyset . \end{array} \tag{2.6}$$

**Definition 2.3.** By *solution* to the IBVP (2.5) regulated by $w$ as in (2.6) we mean a triple of maps $\rho^1, \rho^2, \rho^3$ such that, for $i = 1,2,3$, $\rho^i \in \mathbf{L}^\infty([0,T] \times I_i; [0,R])$ solves (2.1) in the sense of Definition 2.1 where, for $t \in [0,T]$,

$$\begin{aligned} f^1_{out}(t) &= w(t)\; q\left(\max\left\{\bar\rho, \rho^3(t, s^3_{in})\right\}\right) \\ f^2_{out}(t) &= (1 - w(t))\; q\left(\max\left\{\bar\rho, \rho^3(t, s^3_{in})\right\}\right) \\ f^3_{in}(t) &= w(t)\; q\big(\min\left\{\rho^1(t, s^1_{out}), \bar\rho\right\}\big) + (1 - w(t))\; q\big(\min\left\{\rho^2(t, s^2_{out}), \bar\rho\right\}\big) . \end{aligned} \tag{2.7}$$

**Theorem 2.4.** *Let $q$ satisfy (q). For any $\rho^i_o \in \mathbf{L}^1(I_i; [0,R])$, $i = 1,2,3$, for any $f^1_{in}, f^2_{in}, f^3_{out} \in \mathbf{L}^1([0,T]; [0, q(\bar\rho)])$ and for any $w$ as in (2.6), the IBVP (2.5) regulated by $w$ admits a unique solution $\rho^1, \rho^2, \rho^3$ in the sense of Definition 2.3. Moreover:*

**(1D.1)** *If $\hat\rho_o{}^i \in \mathbf{L}^1(I_i; [0,R])$, for $i = 1,2,3$, and $\widehat{f^1_{in}}, \widehat{f^2_{in}}, \widehat{f^3_{out}} \in \mathbf{L}^1([0,T]; [0, q(\bar\rho)])$, the corresponding solution $\hat\rho^1, \hat\rho^2, \hat\rho^3$ satisfies for all $t \in [0,T]$,*

$$\sum_{i=1}^{3} \left\|\rho^i(t) - \hat\rho^i(t)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} \le \sum_{i=1}^{3} \left\|\rho^i_o - \hat\rho_o{}^i\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} + \sum_{i=1}^{2} \left\|f^i_{in} - \widehat{f^i_{in}}\right\|_{\mathbf{L}^1([0,t];\mathbb{R})} + \left\|f^3_{out} - \widehat{f^3_{out}}\right\|_{\mathbf{L}^1([0,t];\mathbb{R})} .$$

**(1D.2)** *If, for $i = 1,2,3$, $\Psi(\rho^i_o) \in \mathbf{BV}(I_i; \mathbb{R})$ and $f^1_{in}, f^2_{in}, f^3_{out} \in \mathbf{BV}([0,T]; [0, q(\bar\rho)])$, then, for all $t \in [0,T]$, $\sum_{i=1}^3 \mathrm{TV}\left(\Psi(\rho^i(t))\right)$ is uniformly bounded on $I$.*

**(1D.3)** *If the triple $\hat\rho^1, \hat\rho^2, \hat\rho^3$ solves (2.5) regulated by a map $\hat w$ as in (2.6), then for all $t \in [0,T]$*

$$\sum_{i=1}^{3} \left\|\rho^i(t) - \hat\rho^i(t)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} \le 3\, q(\bar\rho) \left\|w - \hat w\right\|_{\mathbf{L}^1([0,t];\mathbb{R})} . \tag{2.8}$$

The proof of Theorem 2.4 is deferred to Section 6.1. In this connection, recall that the LWR model on a network was first considered in [23] and then developed in [24,25].

## 3. A merge in a 2D traffic flow model

We first introduce the basic definitions and notations to deal with a 2D road, see Fig. 3.1. Any vector $u \in \mathbb{R}^2$ is understood as a *column* vector, i.e., $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. We also set $u^\perp = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}$ and $u^\intercal = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$. If, $u, z \in \mathbb{R}^2$, their scalar product is denoted $u \cdot z = u^\intercal z = u_1 z_1 + u_2 z_2$. Identities useful in the sequel are $u^\perp \cdot z + u \cdot z^\perp = 0$ and $u^\perp \cdot z^\perp = u \cdot z$.

**Definition 3.1.** Let $I \subseteq \mathbb{R}$ be an open nontrivial interval, $L > 0$ and $\gamma \in \mathbf{C}^2(\bar I; \mathbb{R}^2)$ be a simple curve such that $\|\gamma'(s)\| = 1$ and $\|\gamma''(s)\| \le 1/L$ for all $s \in \bar I$. A *Road* is a map

$$\begin{array}{rcccl} \mathcal{R}: & \bar I \times ]-L, L[ & \to & \mathbb{R}^2 & \\ & (s, \ell) & \mapsto & x & \quad \text{where} \quad x = \gamma(s) + \ell\, \left(\gamma'(s)\right)^\perp \end{array}$$

such that $\mathcal{R}(\bar I \times ]-L, L[)$ is simply connected.

On the basis of elementary notions, the above setting means that $s$ is the arc-length, or curvilinear abscissa; $\gamma'(s)$ is the versor tangent to the curve $\gamma$ at $\gamma(s)$ and $\gamma''(s)$ is the curvature. Since the road's width is $2\,L$, the condition $\|\gamma''(s)\| \le 1/L$ avoids *"self superpositions"* of the road.

If $I$ is bounded below, respectively above, we set, see Fig. 3.1,

$$\begin{array}{ll} s_{in} = \inf I & \mathcal{E}_{in} = \mathcal{R}(s_{in}, ]-L, L[) ; \\ s_{out} = \sup I & \mathcal{E}_{out} = \mathcal{R}(s_{out}, ]-L, L[) . \end{array} \tag{3.1}$$

Below, we occasionally identify the map $\mathcal{R}$ with its support $\mathcal{R}(\bar I \times ]-L, L[)$.

**Lemma 3.2.** *Let $\mathcal{R}$ be a road in the sense of Definition 3.1. Then $\mathcal{R} : \bar I \times ]-L, L[ \to \mathcal{R}(\bar I \times ]-L, L[)$ is bijective and its inverse is of class $\mathbf{C}^1 \left(\mathcal{R}(I \times ]-L, L[); \mathbb{R}^2\right) \cap \mathbf{C}^0 \left(\mathcal{R}(\bar I \times ]-L, L[); \mathbb{R}^2\right)$.*

The proof is deferred to Section 6.2. The above Lemma 3.2 allows to define on $\mathcal{R}$ the $\mathbf{C}^1$ vector field $r$ by

$$r(x) = \gamma'(s) \iff \exists \ell \in ]-L, L[ \quad \mathcal{R}(s, \ell) = x$$

so that

$$r\left(\mathcal{R}(s, \ell)\right) = \gamma'(s) . \tag{3.2}$$

Consider the IBVP for a conservation law on a bounded road $\mathcal{R}$, understood in the sense of Definition 3.1, where we prescribe the inflow $f_{in}$ and the outflow $f_{out}$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (q(\rho)\, r(x)) = 0 & (t, x) \in [0, T] \times \mathcal{R} \\ \rho(0, x) = \rho_o(x) & x \in \mathcal{R} \\ q\left(\rho\left(t, \xi\right)\right) = f_{in}(t, \xi) & (t, \xi) \in [0, T] \times \mathcal{E}_{in} \\ q\left(\rho\left(t, \xi\right)\right) = f_{out}(t, \xi) & (t, \xi) \in [0, T] \times \mathcal{E}_{out}, \end{cases} \tag{3.3}$$

where $\mathcal{E}_{in}$ and $\mathcal{E}_{out}$ are as in (3.1).

**Definition 3.3.** By *solution* to the IBVP (3.3) on the time interval $[0, T]$ we mean a map $\rho \in \mathbf{L}^\infty([0, T] \times \mathcal{R}; \mathbb{R})$ such that for any test function $\varphi \in \mathbf{C}^1_c(\mathbb{R} \times \mathcal{R}; \mathbb{R}_+)$ and for any $\kappa \in \mathbb{R}$,

$$\begin{aligned} &\int_0^T \int_{\mathcal{R}} (\rho(t, x) - \kappa)^\pm\, \partial_t \varphi(t, x)\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_{\mathcal{R}} \operatorname{sgn}^\pm (\rho(t, x) - \kappa) \left(q\left(\rho(t, x)\right) - q(\kappa)\right) \nabla_x \varphi(t, x) \cdot r(x)\, \mathrm{d}x\, \mathrm{d}t \\ &+ \int_{\mathcal{R}} \left(\rho_o(x) - \kappa\right)^\pm \varphi(0, x)\, \mathrm{d}x + \int_{\mathcal{R}} (\rho(T, x) - \kappa)^\pm \varphi(T, x)\, \mathrm{d}x \\ &+ \|q'\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \int_0^T \int_{\mathcal{E}_{in}} \left(q^{-1}_{|[0,\bar\rho]} \left(f_{in}(t, \xi)\right) - \kappa\right)^\pm \varphi(t, \xi)\, \mathrm{d}\xi\, \mathrm{d}t + \|q'\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \int_0^T \int_{\mathcal{E}_{out}} \left(q^{-1}_{|[\bar\rho,R]} \left(f_{out}(t, \xi)\right) - \kappa\right)^\pm \varphi(t, \xi)\, \mathrm{d}\xi\, \mathrm{d}t \geq 0 . \end{aligned}$$

In the above integral inequality, the usual term with $\nabla_x \cdot r$ is missing since the vector field $r$, defined in (3.2), is divergence free by construction, as proved in Lemma 6.1.

The next Lemma 3.4 relates the 2D IBVP (3.3) to the 1D IBVP (2.1).

**Lemma 3.4.** *Let $\mathcal{R}$ be a road as in Definition 3.1. For any $\rho_o \in \mathbf{L}^1(\mathcal{R}; [0, R])$, $f_{in} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}; [0, q(\bar\rho)])$, $f_{out} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}; [0, q(\bar\rho)])$, the Cauchy problem (3.3), in the sense of Definition 3.3, is equivalent to the family of 1D Cauchy problems parameterized by $\ell \in ]-L, L[$,*

$$\begin{cases} \partial_t \rho^\ell + \partial_\sigma q(\rho^\ell) = 0 & (t, \sigma) \in [0, T] \times I^\ell \\ \rho^\ell(0, \sigma) = \rho^\ell_o(\sigma) & \sigma \in I^\ell \\ q\left(\rho^\ell(t, s^\ell_{in})\right) = f^\ell_{in}(t) & t \in [0, T] \\ q\left(\rho^\ell(t, s^\ell_{out})\right) = f^\ell_{out}(t) & t \in [0, T] \end{cases} \tag{3.4}$$

*understood in the sense of Definition 2.1, where $I^\ell = ]s^\ell_{in}, s^\ell_{out}[$ and for all $\ell \in ]-L, L[$, $s \in I$*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s}(s) = 1 - \ell\, \left(\gamma'(s)\right)^\perp \cdot \gamma''(s) ; \qquad \begin{array}{rcl} s^\ell_{in} & = & s_{in}, \\ s^\ell_{out} & = & s_{out} - \ell \displaystyle\int_{s_{in}}^{s_{out}} \left(\gamma'(s)\right)^\perp \cdot \gamma''(s)\, \mathrm{d}s, \end{array} \tag{3.5}$$

*and moreover for a.e. $t \in [0, T]$, $\ell \in ]-L, L[$, $s \in I$, $\sigma \in I^\ell$,*

$$\begin{array}{ll} \rho^\ell(t, \sigma) = \rho\left(t, \mathcal{R}(s, \ell)\right), & f^\ell_{in}(t) = f_{in}\left(t, \mathcal{R}(s_{in}, \ell)\right), \\ \rho^\ell_o(\sigma) = \rho_o\left(\mathcal{R}(s, \ell)\right), & f^\ell_{out}(t) = f_{out}\left(t, \mathcal{R}(s_{out}, \ell)\right) . \end{array} \tag{3.6}$$

The proof is deferred to Section 6.2.

The next theorem extends the well-posedness results of Proposition 2.2 to the present 2D setting by means of Lemma 3.4.

**Theorem 3.5.** *Let $q$ satisfy (q). For $i = 1, 2, 3$, let $\mathcal{R}^i$ be a road as in Definition 3.1. For any $\rho_o \in \mathbf{L}^1(\mathcal{R}; [0, R])$, $f_{in} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{in}; [0, q(\bar\rho)])$, $f_{out} \in \mathbf{L}^1([0, T] \times \mathcal{E}_{out}; [0, q(\bar\rho)])$, the IBVP (3.3) admits a unique solution $\rho$ in the sense of Definition 3.3 and it satisfies, for $t \in [0, T]$, the estimate*

$$\|\rho(t)\|_{\mathbf{L}^\infty(\mathcal{R};\mathbb{R})} \leq \max \left\{ \|\rho_o\|_{\mathbf{L}^\infty(\mathcal{R};\mathbb{R})}, \left\|q^{-1}_{|[0,\bar\rho]} \circ f_{in}\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{E}_{in};\mathbb{R})}, \left\|q^{-1}_{|[\bar\rho,R]} \circ f_{out}\right\|_{\mathbf{L}^\infty([0,t]\times\mathcal{E}_{out};\mathbb{R})} \right\},$$

*with $\mathcal{E}_{in}$ and $\mathcal{E}_{out}$ as in (3.1). Moreover, if $\hat\rho_o$, $\hat f_{in}$ and $\hat f_{out}$ satisfy the same assumptions, the corresponding solution $\hat\rho$ satisfies, for all $t \in [0,T]$,*

$$\left\|\rho(t) - \hat\rho(t)\right\|_{\mathbf{L}^1(\mathcal{R};\mathbb{R})} \le \left\|\rho_o - \hat\rho_o\right\|_{\mathbf{L}^1(\mathcal{R};\mathbb{R})} + \left\|f_{in} - \hat f_{in}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{E}_{in};\mathbb{R})} + \left\|f_{out} - \hat f_{out}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{E}_{out};\mathbb{R})}.$$

*If in addition*

$$\operatorname*{esssup}_{\ell\in]-L,L[} \mathrm{TV}\left(\Psi\left(\rho_o\left(\mathcal{R}(\cdot,\ell)\right)\right)\right) < +\infty, \quad \operatorname*{esssup}_{\xi\in\mathcal{E}_{in}} \mathrm{TV}\left(f_{in}(\cdot,\xi)\right) < +\infty \quad \textit{and} \quad \operatorname*{esssup}_{\xi\in\mathcal{E}_{out}} \mathrm{TV}\left(f_{out}(\cdot,\xi)\right) < +\infty,$$

*then the following bound holds*

$$\sup_{t\in[0,T]} \operatorname*{esssup}_{\ell\in]-L,L[} \mathrm{TV}\left(\Psi\left(\rho\left(t,\mathcal{R}(\cdot,\ell)\right)\right)\right) < +\infty. \tag{3.7}$$

The proof is deferred to Section 6.2.

Consider 3 roads $\mathcal{R}^1$, $\mathcal{R}^2$ and $\mathcal{R}^3$, understood in the sense of Definition 3.1, with $\mathcal{R}^i : \overline{I_i} \times ]-L,L[ \to \mathbb{R}^2$, $I_i$ being a nontrivial open and bounded real interval. We say we have a *merge* when, with the notation (3.1),

$$\mathcal{E}^1_{out} = \mathcal{E}^2_{out} = \mathcal{E}^3_{in} \qquad \text{and} \qquad \gamma_1''(s^1_{out}) = \gamma_2''(s^2_{out}) = \gamma_3''(s^3_{in}),$$

so that, in particular, also $\gamma_1'(s^1_{out}) = \gamma_2'(s^2_{out}) = \gamma_3'(s^3_{in})$.

Along $\mathcal{R}^i$, for $i = 1,2,3$, the traffic density is $\rho^i = \rho^i(t,x)$, with $\rho^i(t,x) \in [0,R]$, and the traffic flow $q$, as a function of $\rho$, satisfies (q). We are thus led to consider the problem

$$\begin{cases} \partial_t \rho^i + \nabla\cdot\left(q(\rho^i)\, r^i(x)\right) = 0 & (t,x)\in[0,T]\times\mathcal{R}^i \quad i=1,2,3\\ \rho^i(0,x) = \rho^i_o(x) & x\in\mathcal{R}^i \quad i=1,2,3\\ q\left(\rho^i(t,\xi)\right) = f^i_{in}(t,\xi) & (t,\xi)\in[0,T]\times\mathcal{E}^i_{in} \quad i=1,2\\ q\left(\rho^i(t,\xi)\right) = f^i_{out}(t,\xi) & (t,\xi)\in[0,T]\times\mathcal{E}^i_{out} \quad i=3. \end{cases} \tag{3.8}$$

We assume that the merge is regulated by traffic lights displaying alternatively red or green lights at the end of $\mathcal{R}^1$ and $\mathcal{R}^2$. As in the 1D case, the traffic light is represented by a function $w$ as in (2.6). Hence, the traffic flow at the junction is described by conditions on $f^1_{out}$, $f^2_{out}$ and $f^3_{in}$.

**Definition 3.6.** By *solution* to the IBVP (3.8) regulated by $w$ as in (2.6) we mean a triple of maps $\rho^1,\rho^2,\rho^3$ such that for $i=1,2,3$, $\rho^i \in \mathbf{L}^\infty([0,T]\times\mathcal{R}^i;[0,R])$ solves (3.3) in the sense of Definition 3.3, where, for $(t,\ell)\in[0,T]\times]-L,L[$,

$$\begin{aligned} f^1_{out}\left(t,\mathcal{R}^1(s^1_{out},\ell)\right) &= w(t)\, q\left(\max\left\{\bar\rho, \rho^3\left(t,\mathcal{R}^3(s^3_{in},\ell)\right)\right\}\right)\\ f^2_{out}\left(t,\mathcal{R}^2(s^2_{out},\ell)\right) &= (1-w(t))\, q\left(\max\left\{\bar\rho, \rho^3\left(t,\mathcal{R}^3(s^3_{in},\ell)\right)\right\}\right)\\ f^3_{in}\left(t,\mathcal{R}^3(s^3_{in},\ell)\right) &= w(t)\, q\left(\min\left\{\rho^1\left(t,\mathcal{R}^1(s^1_{out},\ell)\right),\bar\rho\right\}\right) + (1-w(t))\, q\left(\min\left\{\rho^2\left(t,\mathcal{R}^2(s^2_{out},\ell)\right),\bar\rho\right\}\right). \end{aligned} \tag{3.9}$$

We now state the main result about the hyperbolic part (3.8) of our model (3.8)–(4.1).

**Theorem 3.7.** *Let $q$ satisfy (q). For $i=1,2,3$, let $\mathcal{R}^i$ be a road as in Definition 3.1. F*

$$\textit{For any} \quad \rho^i_o \in \mathbf{L}^1(\mathcal{R}^i;[0,R]) \quad \textit{and} \quad \begin{aligned} f^1_{in} &\in \mathbf{L}^1([0,T]\times\mathcal{E}^1_{in};[0,q(\bar\rho)]),\\ f^2_{in} &\in \mathbf{L}^1([0,T]\times\mathcal{E}^2_{in};[0,q(\bar\rho)]),\\ f^3_{out} &\in \mathbf{L}^1([0,T]\times\mathcal{E}^3_{out};[0,q(\bar\rho)]), \end{aligned} \tag{3.10}$$

*and for any $w$ as in (2.6), the IBVP (3.8) regulated by $w$ admits a unique solution $\rho^1$, $\rho^2$, $\rho^3$ in the sense of Definition 3.6. Moreover,*

**(2D.1)** *If $\hat\rho_o^{\,i} \in \mathbf{L}^1(\mathcal{R}^i;[0,R])$, for $i=1,2,3$, and $\hat f^1_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^1_{in};[0,q(\bar\rho)])$, $\hat f^2_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^2_{in};[0,q(\bar\rho)])$, $\hat f^3_{out} \in \mathbf{L}^1([0,T]\times\mathcal{E}^3_{out};[0,q(\bar\rho)])$, the corresponding solution $\hat\rho^1,\hat\rho^2,\hat\rho^3$ satisfies for all $t\in[0,T]$,*

$$\sum_{i=1}^3 \left\|\rho^i(t) - \hat\rho^i(t)\right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})} \le \sum_{i=1}^3 \left\|\rho^i_o - \hat\rho_o^{\,i}\right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})} + \sum_{i=1}^2 \left\|f^i_{in} - \hat f^i_{in}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{E}^i_{in};\mathbb{R})} + \left\|f^3_{out} - \hat f^3_{out}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{E}^3_{out};\mathbb{R})}.$$

**(2D.2)** *If moreover for $i=1,2,3$,*

$$\operatorname*{esssup}_{\ell\in]-L,L[} \mathrm{TV}\left(\Psi\left(\rho^i_o\left(\mathcal{R}^i(\cdot,\ell)\right)\right)\right) < +\infty, \quad \begin{aligned} &\operatorname*{esssup}_{\xi\in\mathcal{E}^1_{in}} \mathrm{TV}\left(f^1_{in}(\cdot,\xi)\right) < +\infty,\\ &\operatorname*{esssup}_{\xi\in\mathcal{E}^2_{in}} \mathrm{TV}\left(f^2_{in}(\cdot,\xi)\right) < +\infty,\\ &\operatorname*{esssup}_{\xi\in\mathcal{E}^3_{out}} \mathrm{TV}\left(f^3_{out}(\cdot,\xi)\right) < +\infty, \end{aligned} \tag{3.11}$$

*then the following bound holds*

$$\sum_{i=1}^3 \sup_{t\in[0,T]} \operatorname*{esssup}_{\ell\in]-L,L[} \mathrm{TV}\left(\Psi\left(\rho^i\left(t,\mathcal{R}^i(\cdot,\ell)\right)\right)\right) < +\infty. \tag{3.12}$$

**(2D.3)** *If the triple $\hat\rho^1, \hat\rho^2, \hat\rho^3$ solves (2.5) regulated by a map $\hat w$ as in (2.6), then for all $t \in [0,T]$*

$$\sum_{i=1}^{3} \left\|\rho^i(t) - \hat\rho^i(t)\right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})} \le 6\, L\, q(\bar\rho)\, \left\|w - \hat w\right\|_{\mathbf{L}^1([0,t];\mathbb{R})}. \tag{3.13}$$

The proof of Theorem 3.7 follows from Theorem 3.5, exactly as the proof of Theorem 2.4 follows from that of Proposition 2.2.

## 4. The conservative parabolic pollution model

We assume the concentration of air pollutant $u = u(t,x)$ is described by the Cauchy problem

$$\begin{cases} \partial_t u + \nabla\cdot\left(u\,\beta(t,x) - \mu(t,x)\,\nabla u\right) + \kappa(t,x)\,u = g(t,x) & (t,x) \in\, ]0,T]\times\mathbb{R}^2 \\ u(0,x) = u_o(x) & x\in\mathbb{R}^2 . \end{cases} \tag{4.1}$$

The role of the various symbols is as described in the Introduction.

**(HP)** On the functions in (4.1) we assume the following conditions:

**(HP.1)** $\mu \in (\mathbf{C}^1 \cap \mathbf{W}^{1,\infty})([0,T]\times\mathbb{R}^2;\mathbb{R}_+)$ is such that $\nabla\mu$ is Hölder continuous in $x$ uniformly in $t$ and there exist $\hat\mu, \check\mu$ with $\hat\mu > \check\mu > 0$ such that $\mu(t,x) \in [\check\mu, \hat\mu]$ for all $(t,x)\in[0,T]\times\mathbb{R}^2$;

**(HP.2)** $\beta \in (\mathbf{C}^1 \cap \mathbf{W}^{1,\infty})([0,T]\times\mathbb{R}^2;\mathbb{R}^2)$ is such that $\nabla\beta$ is Hölder continuous in $x$ uniformly in $t$;

**(HP.3)** $\kappa \in (\mathbf{C}^0 \cap \mathbf{L}^{\infty})([0,T]\times\mathbb{R}^2;\mathbb{R})$ is Hölder continuous in $x$ uniformly in $t$.

See Definition 6.5 for details on Hölder continuity.

**Definition 4.1.** A function $u \in \mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}))$ is a *weak solution* to (4.1) if for any

$$\begin{array}{ll} \varphi\in\mathbf{C}^0([0,T]\times\mathbb{R}^2;\mathbb{R})\cap\mathbf{C}^1(]0,T[\times\mathbb{R}^2;\mathbb{R}) & \text{such that} \\ \forall\, t\in\, ]0,T[ & x\mapsto\varphi(t,x)\in\mathbf{C}^2(\mathbb{R}^2;\mathbb{R}) \\ \forall\, t\in\, ]0,T[ & \lim_{r\to+\infty}\sup_{\|x\|\ge r}\left|\varphi(t,x)\right| = 0 \\ \forall\, t\in\, ]0,T[ & \lim_{r\to+\infty}\sup_{\|x\|\ge r}\left|\partial_t\varphi(t,x)\right| = 0 \\ \forall\, t\in\, ]0,T[ & \lim_{r\to+\infty}\sup_{\|x\|\ge r}\left\|\nabla\varphi(t,x)\right\| = 0 \\ \forall\, t\in\, ]0,T[ & \lim_{r\to+\infty}\sup_{\|x\|\ge r}\left\|\nabla^2\varphi(t,x)\right\| = 0 \end{array} \tag{4.2}$$

the following equality holds

$$\int_0^T\int_{\mathbb{R}^2} u\left(\partial_t\varphi + \nabla\cdot(\mu\,\nabla\varphi) + \beta\cdot\nabla\varphi - \kappa\,\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^2} u_o(x)\,\varphi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^2} u(T,x)\,\varphi(T,x)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^2} g\,\varphi\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.3}$$

Below, we extensively use the Green function $\Gamma$ associated to the homogeneous version of (4.1), refer to Lemma 6.6 for further details.

**Theorem 4.2.** *Assume that* **(HP)** *holds. Let $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ and $g\in\mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$. Then, Problem (4.1) admits a unique weak solution in the sense of Definition 4.1. Moreover:*

**(P.1)** *The following representation formula holds:*

$$u(t,x) = \int_{\mathbb{R}^2}\Gamma(t,x,0,\xi)\,u_o(\xi)\,\mathrm{d}\xi + \int_0^t\int_{\mathbb{R}^2}\Gamma(t,x,\tau,\xi)\,g(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{4.4}$$

*where $\Gamma$ is defined in Lemma 6.6.*

**(P.2)** *If also $\hat u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ and $\hat g \in \mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$, calling $\hat u$ the corresponding solution, the following estimate holds:*

$$\left\|u(t)-\hat u(t)\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} \le 4\,\pi\,C\,\hat\mu\left(\left\|u_o-\hat u_o\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} + \left\|g-\hat g\right\|_{\mathbf{L}^1([0,t]\times\mathbb{R}^2;\mathbb{R})}\right) \tag{4.5}$$

*where $C$ is defined in* **(Γ.3)** *of Lemma 6.6.*

**(P.3)** *If $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R}_+)$ and $g, \hat g \in \mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$ satisfy $g \ge \hat g$, then the corresponding solutions $u, \hat u$ satisfy $u \ge \hat u$.*

The proof is deferred to Section 6.3.

Consider now the case of a measure source. Call $\mathcal{M}$ the set of Radon measures on $\mathbb{R}^2$ and denote by $\|\mu\|_{\mathcal{M}}$ the total variation norm, see [26, Chapter 7, § 3, p.216].

**Definition 4.3.** A function $u \in \mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})$ and $u\in\mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R}^2;\mathbb{R}))$ is a *weak solution* to (4.1) with $g:[0,T]\mapsto\mathcal{M}$ being measurable, if for any test function $\varphi$ as in (4.2) the following equality holds

$$\int_0^T\int_{\mathbb{R}^2} u\left(\partial_t\varphi + \nabla\cdot(\mu\,\nabla\varphi) + \beta\cdot\nabla\varphi - \kappa\,\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^2} u_o(x)\,\varphi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^2} u(T,x)\,\varphi(T,x)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^2}\varphi(t,x)\,\mathrm{d}g_t(x)\,\mathrm{d}t = 0$$

where $g_t(x) = g(t,x)$.

**Theorem 4.4.** *Assume that* **(HP)** *holds. Let* $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ *and* $g : [0,T] \to \mathcal{M}$ *be measurable with*

$$\sup_{\tau\in[0,T]} \|g_\tau\|_{\mathcal{M}} < +\infty . \tag{4.6}$$

*Then, Problem* (4.1) *admits a unique weak solution in the sense of Definition* 4.3. *Moreover:*

**(M.1)** *The following representation formula holds:*

$$u(t,x) = \int_{\mathbb{R}^2} \Gamma(t,x,0,\xi)\, u_o(\xi)\, \mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t,x,\tau,\xi)\, \mathrm{d}g_\tau(\xi)\, \mathrm{d}\tau \tag{4.7}$$

*where* $\Gamma$ *is defined in Lemma* 6.6.

**(M.2)** *If also* $\hat{u}_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ *and* $\hat{g} : [0,T] \to \mathcal{M}$ *is measurable and satisfies* (4.6), *calling* $\hat{u}$ *the corresponding solution, the following estimate holds:*

$$\|u(t) - \hat{u}(t)\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} \le 4\,\pi\, C\, \hat{\mu} \left( \|u_o - \hat{u}_o\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} + t \sup_{[0,t]} \|g_\tau - \hat{g}_\tau\|_{\mathcal{M}} \right) \tag{4.8}$$

*where* $C$ *is defined in* **(Γ.3)** *of Lemma* 6.6.

**(M.3)** *If* $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R}_+)$ *and* $g, \hat{g} : [0,T] \to \mathcal{M}$ *satisfy* (4.6) *and* $g_\tau \ge \hat{g}_\tau$ *for all* $\tau \in [0,T]$, *then the corresponding solutions* $u, \hat{u}$ *satisfy* $u \ge \hat{u}$.

## 5. The coupled problem

### 5.1. Well-posedness

We now couple the traffic model (3.8) to the pollution diffusion model (4.1), choosing a source term $g$ in (4.1) that depends on the traffic evolution described by (3.8). To this aim, for $i = 1,2,3$, extend each $\rho^i$ to $\mathbb{R}^2$ assigning the value 0 for all $x \in \mathbb{R}^2 \setminus \mathcal{R}_i$.

A first reasonable choice is

$$g(t,x) = G\left(t, \rho(t,x), q(t,x)\right) \quad \text{where} \quad \begin{array}{rcl} \rho &=& (\rho^1,\rho^2,\rho^3) \\ q &=& \left(q(\rho^1), q(\rho^2), q(\rho^3)\right) \end{array} \tag{5.1}$$

meaning that the production of pollutant at $(t,x)$ depends on the traffic density $\rho(t,x)$ and on the traffic flow $q(t,x)$ at $(t,x)$.

**Theorem 5.1.** *Let* $q$ *satisfy* (q). *For* $i = 1,2,3$, *let* $\mathcal{R}^i$ *be a road as in Definition* 3.1. *Assume that, for* $i = 1,2,3$, $\rho^i_o \in \mathbf{L}^1(\mathcal{R}^i;[0,R])$, $f^1_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^1_{in};[0,q(\bar\rho)])$, $f^2_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^2_{in};[0,q(\bar\rho)])$, $f^3_{out} \in \mathbf{L}^1([0,T]\times\mathcal{E}^3_{out};[0,q(\bar\rho)])$, *and* $w \in \mathcal{W}$ *is as in* (2.6). *Assume that* **(HP)** *holds,* $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R}_+)$ *and moreover*

**(G)** $G : [0,T] \times [0,R]^3 \times [0,q(\bar\rho)]^3 \to \mathbb{R}$ *is such that*

$$\begin{array}{ll} \forall\, t \in [0,T] & (\rho,q) \mapsto G(t,\rho,q) \in \mathbf{C}^{0,1}([0,R]^3 \times [0,q(\bar\rho)]^3;\mathbb{R}) \\ \forall (\rho,q) \in [0,R]^3 \times [0,q(\bar\rho)]^3 & t \mapsto G(t,\rho,q) \in \mathbf{L}^1([0,T];\mathbb{R}) \end{array}$$

*Then, there exists a unique solution* $(\rho^1,\rho^2,\rho^3,u)$ *to the coupled problem* (3.8)–(4.1)–(5.1).

The proof follows from Theorems 3.7 and 4.2, which can be applied since the source term $g$ provided by (G) is in $\mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$.

Remark that physical reasons induce to consider only positive functions $G$ in (G). However, this assumption is not necessary in Theorem 5.1. The proof is deferred to Section 6.4.

We now take into consideration the pollution produced by variations in the traffic flow, such as those due to accelerations or brakings. To this aim, we consider source terms in (4.1) that depend on the total variation of the traffic flow. The analytical basis for this relies on the information provided by Theorem 3.7.

**Proposition 5.2.** *Under the same assumptions of Theorem* 3.7, *including also the requirement in* **(2D.2)**, *each component* $\rho^i$ *of the solution to* (3.8)–(2.6) *defines for* $i = 1,2,3$, *the map* $g^i : [0,T] \to \mathcal{M}$ *setting, for any test function* $\varphi \in \mathbf{C}^\infty_c([0,T]\times\mathbb{R}^2;\mathbb{R})$,

$$g^i_t(\varphi) = \int_{-L}^{L} \int_I \varphi\left(t, \mathcal{R}^i(s,\ell)\right) \left(1 - \ell\left(\gamma'(s)\right)^\perp \cdot \gamma''(s)\right) \mathrm{d}\mu^{i,\ell}_t(s)\, \mathrm{d}\ell \tag{5.2}$$

*where* $\mu^{i,\ell}_t$ *is the total variation measure of the distributional derivative of the map* $s \mapsto q\left(\rho^i\left(t,\mathcal{R}^i(s,\ell)\right)\right)$. *Moreover, for* $i = 1,2,3$, *the measure* $g^i_t$ *is supported in* $\mathcal{R}^i$ *and the map* $g = g^1 + g^2 + g^3$ *satisfies* (4.6).

The measure $g^i$ essentially replaces the total variation of the distributional derivative of the map $x \mapsto q\left(\rho^i(t,x)\right)$.

**Theorem 5.3.** *Let the assumptions of Proposition* 5.2 *hold. Assume that* **(HP)** *holds and let* $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R}_+)$. *Let* $g$ *be defined as in Proposition* 5.2. *Then, there exists a unique solution* $(\rho^1,\rho^2,\rho^3,u)$ *to the coupled problem* (3.8)–(4.1)–(5.1).

The proof follows applying subsequently Theorem 3.7, Proposition 5.2 and Theorem 4.4.

### 5.2. Optimal management

To limit the effects of pollutant, the controller acts on the evolution of traffic through the function $w$ in (2.6) that regulates the merge, or also on the inflows $f^1_{in}, f^2_{in}$ and on the outflow $f^3_{out}$. The target is the minimization of a functional of the type

$$\mathcal{F}(w, f^1_{in}, f^2_{in}, f^3_{out}) = \int_0^T \int_{\mathbb{R}^2} p(t,x)\, u(t,x)\, \mathrm{d}x\, \mathrm{d}t \tag{5.3}$$

where $u$ is the pollutant's density and $p$ is a suitable weight assigning to different regions different relevance according, for instance, to population density.

**Theorem 5.4.** *Let $q$ satisfy (q). For $i = 1,2,3$, let $\mathcal{R}^i$ be a road as in Definition 3.1. Assume that, for $i = 1,2,3$, $\rho^i_o \in \mathbf{L}^1(\mathcal{R}^i;[0,R])$. Assume that* **(HP)** *and (G) hold, $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R}_+)$ and moreover $\mathcal{F}$ is as in (5.3) with $p \in \mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})$.*

*Let $\rho^1, \rho^2, \rho^3, u$ be the unique solution to the coupled problem (3.8)–(4.1)–(5.1) regulated by $w$ in $\mathcal{W}$ as in (2.6) with inflows $f^1_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^1_{in};[0,q(\bar\rho)])$, $f^2_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^2_{in};[0,q(\bar\rho)])$, and outflow $f^3_{out} \in \mathbf{L}^1([0,T]\times\mathcal{E}^3_{out};[0,q(\bar\rho)])$. Then, the map*

$$\mathcal{F} : \mathcal{W} \times \prod_{i=1}^{2} \mathbf{L}^1([0,T]\times\mathcal{E}^i_{in};[0,q(\bar\rho)]) \times \mathbf{L}^1([0,T]\times\mathcal{E}^3_{out};[0,q(\bar\rho)]) \to \mathbb{R} \tag{5.4}$$

*in (5.3) is $\mathbf{L}^1$–Lipschitz continuous.*

The proof is deferred to Section 6.4.

As for $G$ in (G), physical reasons induce to consider only positive functions $p$ in the functional $\mathcal{F}$ (5.3). We remark that these positivity assumptions on both $G$ and $p$ are not necessary in the proof of Theorem 5.4.

Theorem 5.4 allows to prove the existence of optimal management strategies as soon as $w$, $f^1_{in}$, $f^2_{in}$ and $f^3_{out}$ are suitably selected in a compact set, thanks to Weierstrass Theorem.

An example tailored to the present physical setting is as follows. For a $N \in \mathbb{N}$, introduce the sets $\mathbf{PC}_N([0,T];\{0,1\})$, respectively $\mathbf{PC}_N([0,T];\{0,q(\bar\rho)\})$, of piecewise constant functions defined on $[0,T]$ with values in $\{0,1\}$, respectively $\{0,q(\bar\rho)\}$, that change value at most $N$ times, i.e., with at most $N$ jumps. This choice corresponds to traffic lights at the merge and at $\mathcal{E}^1_{in}, \mathcal{E}^2_{in}, \mathcal{E}^3_{out}$ that alternate between green and red lights at most $N$ times in $[0,T]$.

**Corollary 5.5.** *Under the same assumptions of Theorem 5.4, the map $\mathcal{F}$ in (5.4) restricted to $\mathbf{PC}_N([0,T];\{0,1\})\times\mathbf{PC}_N([0,T];\{0,q(\bar\rho)\})^3$ admits a point of global minimum.*

Alternatively, Theorem 5.4 also comprises the case of assigned – possibly non maximal – inflows and outflows.

**Corollary 5.6.** *Under the same assumptions of Theorem 5.4, fix $f^1_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^1_{in};[0,q(\bar\rho)])$, $f^2_{in} \in \mathbf{L}^1([0,T]\times\mathcal{E}^2_{in};[0,q(\bar\rho)])$, and $f^3_{out} \in \mathbf{L}^1([0,T]\times\mathcal{E}^3_{out};[0,q(\bar\rho)])$. Then, the map $w \mapsto \mathcal{F}(w, f^1_{in}, f^2_{in}, f^3_{out})$ with $\mathcal{F}$ as in (5.4) admits a point of global minimum in $\mathbf{PC}_N([0,T];\{0,1\})$.*

## 6. Technical details

### 6.1. Proofs related to the 1D hyperbolic traffic model

**Proof of Proposition 2.2.** We distinguish different steps.

**Claim 1** (*The* **BV** *Case*)**.** *We assume in addition that $\rho_o \in \mathbf{BV}(I;[0,R])$ and $(q_{|[0,\bar\rho]})^{-1}(f_{in})$, $(q_{|[\bar\rho,R]})^{-1}(f_{out}) \in \mathbf{BV}([0,T];\mathbb{R})$. Then, the existence of solutions and the estimate (2.3) follow from [4, Proposition 3.3].*

Assume that $\rho$ and $\hat\rho$ are solutions to (2.1) in the sense of Definition 2.1. To prove the bound (2.4), we refer to the detailed proof in [27, Theorem 4.3], devoted to the multi-dimensional case. Exploit the fact that in problem (2.1) the fluxes at the boundaries are assigned. The key point is the estimate of the term

$$-\operatorname{sgn}(\operatorname{tr}\rho - \operatorname{tr}\hat\rho)\left[q(\operatorname{tr}\rho) - q(\operatorname{tr}\hat\rho)\right]\cdot\nu, \tag{6.1}$$

$\nu$ being the exterior normal vector, here $\nu = \pm 1$, and in the present setting the traces correspond to the two boundary terms, one at $s_{in}$ and one at $s_{out}$. Let us focus on the left boundary at $s_{in}$. Recall that for a.e. $t \in [0,T]$

$$\begin{aligned}(\operatorname{tr}\rho)(t,s_{in}) &\in \left\{(q_{|[0,\bar\rho]})^{-1}\left(f_{in}(t)\right)\right\} \cup \left[(q_{|[\bar\rho,R]})^{-1}\left(f_{in}(t)\right), R\right],\\ (\operatorname{tr}\hat\rho)(t,s_{in}) &\in \left\{(q_{|[0,\bar\rho]})^{-1}\left(\hat f_{in}(t)\right)\right\} \cup \left[(q_{|[\bar\rho,R]})^{-1}\left(\hat f_{in}(t)\right), R\right].\end{aligned}$$

A case by case analysis shows that

$$-\operatorname{sgn}\left((\operatorname{tr}\rho)(t,s_{in}) - (\operatorname{tr}\hat\rho)(t,s_{in})\right)\left[q\left((\operatorname{tr}\rho)(t,s_{in})\right) - q\left((\operatorname{tr}\hat\rho)(t,s_{in})\right)\right]\cdot\nu \le \left|f_{in}(t) - \hat f_{in}(t)\right|.$$

This term is then integrated over the time interval $[0,t]$ leading to the second term in the right hand side of (2.4). The term related to the right boundary $s_{out}$ is treated in an entirely similar way.

**Claim 2** (*The General Case*). *Introduce a sequence of mollifiers $\eta_n(\xi) = n\,\hat\eta(n\,\xi)$ with $\hat\eta \in \mathbf{C}_c^\infty(\mathbb{R};\mathbb{R})$, $\hat\eta \geq 0$, $\operatorname{spt}\hat\eta = [-1,1]$ and $\int_{\mathbb{R}} \hat\eta = 1$. Define*

$$\hat\rho_o^n(x) = \begin{cases} \rho_o(x) & x \in I \\ 0 & x \in \mathbb{R}\setminus I \end{cases} \qquad \rho_o^n(x) = \left(\hat\rho_o^n * \eta_n\right)(x) \qquad x \in I,$$
$$\hat f_{in}^n(t) = \begin{cases} \min\left\{f_{in}(t), q\left(\bar\rho - \frac{1}{n}\right)\right\} & t \in [0,T] \\ 0 & t \in \mathbb{R}\setminus[0,T] \end{cases} \qquad f_{in}^n(t) = \left(\hat f_{in}^n * \eta_n\right)(t) \qquad t \in [0,T],$$
$$\hat f_{out}^n(t) = \begin{cases} \min\left\{f_{out}(t), q\left(\bar\rho + \frac{1}{n}\right)\right\} & t \in [0,T] \\ 0 & t \in \mathbb{R}\setminus[0,T] \end{cases} \qquad f_{out}^n(t) = \left(\hat f_{out}^n * \eta_n\right)(t) \qquad t \in [0,T].$$

*Clearly, $\rho_o^n \in \mathbf{W}^{1,\infty}(I;[0,R])$ and $\lim_{n\to+\infty}\rho_o^n = \rho_o$ in $\mathbf{L}^1(I;[0,R])$. At the same time, also $f_{in}^n \in \mathbf{W}^{1,\infty}\left([0,T];[0,q(\bar\rho - \frac{1}{n})]\right)$, so that $(q_{|[0,\bar\rho]})^{-1}\circ f_{in}^n \in \mathbf{W}^{1,\infty}([0,T];[0,\bar q])$ since $(q_{|[0,\bar\rho]})^{-1} \in \mathbf{W}^{1,\infty}\left([0,q(\bar\rho-\frac{1}{n})];[0,\bar\rho-\frac{1}{n}]\right)$. Moreover, the map $(q_{|[0,\bar\rho]})^{-1}(f_{in}^n)$ converges to $(q_{|[0,\bar\rho]})^{-1}(f_{in})$ in $\mathbf{L}^1([0,T];\mathbb{R})$. The outflow $\hat f_{out}^n$ is treated similarly.*

Note also, for later use, that by construction

$$\left\|\rho_o^n\right\|_{\mathbf{L}^\infty(I;\mathbb{R})} \leq \left\|\rho_o\right\|_{\mathbf{L}^\infty(I;\mathbb{R})} \quad\text{and}\quad \begin{array}{rcl} \left\|f_{in}^n\right\|_{\mathbf{L}^\infty([0,T];\mathbb{R})} & \leq & \left\|f_{in}\right\|_{\mathbf{L}^\infty([0,T];\mathbb{R})}, \\ \left\|f_{out}^n\right\|_{\mathbf{L}^\infty([0,T];\mathbb{R})} & \leq & \left\|f_{out}\right\|_{\mathbf{L}^\infty([0,T];\mathbb{R})}. \end{array} \tag{6.2}$$

Moreover, $\rho_o^n \in \mathbf{BV}(I;[0,R])$, $(q_{|[0,\bar\rho]})^{-1}\left(f_{in}^n(t)\right) \in \mathbf{BV}([0,T];\mathbb{R})$ and $(q_{|[\bar\rho,R]})^{-1}\left(f_{out}^n(t)\right) \in \mathbf{BV}([0,T];\mathbb{R})$ so that Claim 1 applies.

Call $\rho_n$ the corresponding solution to (2.1). By (2.4), the sequence $\rho_n$ is a Cauchy sequence in $\mathbf{L}^1([0,T]\times I;\mathbb{R})$. Call $\rho_\infty$ the corresponding $\mathbf{L}^1$ limit. Since Definition 2.1 is stable with respect to $\mathbf{L}^1$ convergence and thanks to (q), we obtain that $\rho$ solves (2.1) in the sense of Definition 2.1. Also (2.4) immediately follows. Since $\rho_n$ satisfies (2.3), (6.2) ensures that $\rho_\infty$ satisfies (2.3).

Finally, the **BV** estimate follows from [3, Theorem 2.2]. □

**Proof of Theorem 2.4.** We assume that, with a suitable reparameterization, $\sup I_1 = \sup I_2 = \inf I_3$.

**Existence and uniqueness of solution to (2.5)–(2.6).** A solution to (2.5) regulated by (2.6) can be iteratively constructed as follows. Assume for simplicity that $J_1 = [0,\tau_1[$. Then, for all $t \in J_1$ consider separately the problems

$$\overset{s\in\,]s_{in}^1,\,s_{out}^3[}{\begin{cases} \partial_t\rho + \partial_s q(\rho) = 0 \\ \rho(0,s) = \begin{cases} \rho_o^1(s) & s \in I_1 \\ \rho_o^3(s) & s \in I_3 \end{cases} \\ q\left(\rho(t,s_{in}^1)\right) = f_{in}^1(t) \\ q\left(\rho(t,s_{out}^3)\right) = f_{out}^3(t) \end{cases}} \qquad \overset{s\in I_2}{\begin{cases} \partial_t\rho^2 + \partial_s q(\rho^2) = 0 \\ \rho(0,s) = \rho_o^2(s) \\ q\left(\rho^2(t,s_{in})\right) = f_{in}^2(t) \\ q\left(\rho^2(t,s_{out})\right) = 0\,. \end{cases}} \tag{6.3}$$

Both of them fit into Proposition 2.2 and thus admit unique solutions $\rho$ and $\rho^2$ defined for $t \in J_1$. Thanks to (2.7), we immediately have that, setting $\rho^1 = \rho_{|I_1}$ and $\rho^3 = \rho_{|I_3}$, the triple $\rho^1,\rho^2,\rho^3$ is the unique solution to (2.5) regulated by $w$ as in (2.6) for $t \in J_1$.

This procedure can be iterated on the time interval $[\sup J_1, \inf J_2[$, gluing road 2 with road 3, taking as initial datum at time $\tau_1$, $\rho^1(\tau_1)$ and $\rho(\tau_1,s) = \begin{cases} \rho^2(\tau_1,s) & s \in I_2 \\ \rho^3(\tau_1,s) & s\in I_3 \end{cases}$. Hence, (2.5)–(2.6) admits a unique solution in the sense of Definition 2.3.

**Proof of (1D.1).** Use the construction above to define the solutions $\rho^1,\rho^2,\rho^3$ and $\hat\rho^1,\hat\rho^2,\hat\rho^3$. On each time interval, the estimate (2.4) applies. Note that this estimate is additive both in time and in space over non overlapping time or space intervals.

**Proof of (1D.2).** The construction above of $\rho^1,\rho^2,\rho^3$ allows to iteratively use the bound on the total variation of $\Psi(\rho^1),\Psi(\rho^2),\Psi(\rho^3)$ in Proposition 2.2.

**Proof of (1D.3).** Introduce a partition of $[0,T]$ in intervals $[\tau_j,\tau_{j+1}[$ for $i = 1,\dots,N$ and values $\vartheta_1,\dots,\vartheta_N$, $\hat\vartheta_1,\dots,\hat\vartheta_N$ in $\{0,1\}$ such that

$$\begin{array}{rcl} w & = & \sum_{j=1}^N \vartheta_j\,\chi_{[\tau_j,\tau_{j+1}[}\,, \\ \hat w & = & \sum_{j=1}^N \hat\vartheta_j\,\chi_{[\tau_j,\tau_{j+1}[} \end{array} \quad\text{and}\quad \begin{array}{l} \forall\, j \in \{1,\dots,N-1\} \quad \vartheta_j \neq \vartheta_{j+1} \text{ or } \hat\vartheta_j \neq \hat\vartheta_{j+1} \\ 0 = \tau_1 < \tau_2 < \dots < \tau_N < \tau_{N+1} = T\,. \end{array}$$

Assume $t \in [\tau_1,\tau_2[$. Then, these cases are possible:

**1.** $\vartheta_1 = 0$, $\hat\vartheta_1 = 0$**:** then, problem (2.5) can be solved considering roads 2 and 3 as a unique road in the time interval $[\tau_1,\tau_2[$, the outflow of road 1 being $f_{out}^1 = 0$. Hence, we have $\sum_{i=1}^3 \left\|\rho^i(t) - \hat\rho^i(t)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} = 0$.

**2.** $\vartheta_1 \neq \hat\vartheta_1$**:** by the stability estimate (2.4) we have for $i = 1,2$

$$\left\|\rho^i(t) - \hat\rho^i(t)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} \leq \left\|\rho^i(\tau_1) - \hat\rho^i(\tau_1)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} + q(\bar\rho)\left|\vartheta_1 - \hat\vartheta_1\right|(t-\tau_1)$$

$$
\begin{aligned}
&= q(\bar\rho)\,\|w-\widehat w\|_{\mathbf{L}^1([0,t];\mathbb{R})}\\
\left\|\rho^3(t)-\widehat\rho^3(t)\right\|_{\mathbf{L}^1(I_3;\mathbb{R})} &\le \left\|\rho^3(\tau_1)-\widehat\rho^3(\tau_1)\right\|_{\mathbf{L}^1(I_3;\mathbb{R})} + q(\bar\rho)\left|\vartheta_1-\widehat\vartheta_1\right|(t-\tau_1)\\
&= q(\bar\rho)\,\|w-\widehat w\|_{\mathbf{L}^1([0,t];\mathbb{R})}\,.
\end{aligned}
$$

**3.** $\vartheta_1 = 1$, $\widehat\vartheta_1 = 1$**:** problem (2.5) can be solved considering roads 1 and 3 as a unique road in the time interval $[\tau_1,\tau_2[$, the outflow of road 2 being $f^2_{out} = 0$. Hence, we have $\sum_{i=1}^3 \|\rho^i(t)-\widehat\rho^i(t)\|_{\mathbf{L}^1(I_i;\mathbb{R})} = 0$.

Proceed now iteratively and assume that $t\in[\tau_j,\tau_{j+1}[$. Then,

**1.** $\vartheta_j = 0$, $\widehat\vartheta_j = 0$**:** problem (2.5) can be solved considering roads 2 and 3 as a unique road in the time interval $[\tau_j,\tau_{j+1}[$, the outflow of road 1 being $f^1_{out} = 0$. Hence, we have $\sum_{i=1}^3 \|\rho^i(t)-\widehat\rho^i(t)\|_{\mathbf{L}^1(I_i;\mathbb{R})} \le \sum_{i=1}^3 \left\|\rho^i(\tau_j)-\widehat\rho^i(\tau_j)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})}$ and observe that, clearly, $\|w-\widehat w\|_{\mathbf{L}^1([\tau_j,t];\mathbb{R})} = 0$.

**2.** $\vartheta_j \ne \widehat\vartheta_j$**:** by the stability estimate (2.4) we have for $i = 1,2$

$$
\begin{aligned}
\|\rho^i(t)-\widehat\rho^i(t)\|_{\mathbf{L}^1(I_i;\mathbb{R})} &\le \left\|\rho^i(\tau_j)-\widehat\rho^i(\tau_j)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})} + q(\bar\rho)\left|\vartheta_j-\widehat\vartheta_j\right|(t-\tau_j)\\
\left\|\rho^3(t)-\widehat\rho^3(t)\right\|_{\mathbf{L}^1(I_3;\mathbb{R})} &\le \left\|\rho^3(\tau_j)-\widehat\rho^3(\tau_j)\right\|_{\mathbf{L}^1(I_3;\mathbb{R})} + q(\bar\rho)\left|\vartheta_j-\widehat\vartheta_j\right|(t-\tau_j)
\end{aligned}
$$

and observe that $\left|\vartheta_j-\widehat\vartheta_j\right|(t-\tau_j) = \|w-\widehat w\|_{\mathbf{L}^1([\tau_j,t];\mathbb{R})}$.

**3.** $\vartheta_j = 1$, $\widehat\vartheta_j = 1$**:** problem (2.5) can be solved considering roads 1 and 3 as a unique road in the time interval $[\tau_j,\tau_{j+1}[$, the outflow of road 2 being $f^2_{out} = 0$. Hence, we have $\sum_{i=1}^3 \|\rho^i(t)-\widehat\rho^i(t)\|_{\mathbf{L}^1(I_i;\mathbb{R})} \le \sum_{i=1}^3 \left\|\rho^i(\tau_j)-\widehat\rho^i(\tau_j)\right\|_{\mathbf{L}^1(I_i;\mathbb{R})}$ and observe that, clearly, $\|w-\widehat w\|_{\mathbf{L}^1([\tau_j,t];\mathbb{R})} = 0$.

The bound (2.8) follows. □

### 6.2. Proofs related to the 2D hyperbolic traffic model

**Lemma 6.1.** *Let $\mathcal{R}$ be a road as in Definition 3.1. Then, the following identities hold:*

$$
\begin{aligned}
D\mathcal{R}(s,\ell) &= \begin{bmatrix} \gamma'(s)+\ell\left(\gamma''(s)\right)^\perp & \left(\gamma'(s)\right)^\perp \end{bmatrix} &(6.4)\\
\det D\mathcal{R}(s,\ell) &= 1-\ell\,\left(\gamma'(s)\right)^\perp\cdot\gamma''(s) &(6.5)\\
[D\mathcal{R}(s,\ell)]^{-1} &= \frac{1}{1-\ell\,(\gamma'(s))^\perp\cdot\gamma''(s)}\begin{bmatrix} \left(\gamma'(s)\right)^\intercal\\ \left(\left(\gamma'(s)\right)^\perp-\ell\,\gamma''(s)\right)^\intercal \end{bmatrix} &(6.6)\\
\nabla_x s &= \frac{1}{1-\ell\,(\gamma'(s))^\perp\cdot\gamma''(s)}\left(\gamma'(s)\right)^\intercal. &(6.7)
\end{aligned}
$$

*Moreover, if $r$ is as defined in (3.2), then*

$$
\nabla_x\cdot r = 0\,. \tag{6.8}
$$

In connection with (6.5), observe that, since $\gamma'(s)\cdot\gamma''(s) = 0$, we have

$$
\left|\left(\gamma'(s)\right)^\perp\cdot\gamma''(s)\right| = \left|\gamma'(s)\cdot\left(\|\gamma''(s)\|\gamma'(s)\right)\right| = \|\gamma''(s)\|\,,
$$

which is the *curvature* of $\gamma$ at $\gamma(s)$.

**Proof of Lemma 6.1.** The proofs of (6.4), (6.5) and (6.6) are immediate. Relation (6.7) is the first line in (6.6). To verify (6.8), use (6.7) and $\partial_\ell r = 0$, for $x = \mathcal{R}(s,\ell)$, so that

$$
\begin{aligned}
\nabla_x\cdot r &= \partial_{x_1} r_1 + \partial_{x_2} r_2\\
&= \partial_s\gamma_1'\,\partial_{x_1}s + \partial_s\gamma_2'\,\partial_{x_2}s\\
&= \begin{bmatrix}\partial_{x_1}s & \partial_{x_2}s\end{bmatrix}\begin{bmatrix}\partial_s\gamma_1'\\ \partial_s\gamma_2'\end{bmatrix}\\
&= \nabla_x s\,\gamma''\\
&= \frac{1}{1-\ell\,(\gamma')^\perp\cdot\gamma''}\,(\gamma')^\intercal\,\gamma''\\
&= 0\,,
\end{aligned}
$$

completing the proof. □

For later use, we provide the following equalities, where $\varphi : \mathbb{R} \times \mathcal{R} \to \mathbb{R}_+$ is smooth.

$$\nabla_{(s,\ell)}\varphi\left(\mathcal{R}(s,\ell)\right) = \nabla_x \varphi(x)_{|x=\mathcal{R}(s,\ell)} \, [D\mathcal{R}(s,\ell)] \,. \tag{6.9}$$

$$\nabla_x \varphi\left(\mathcal{R}^{-1}(x)\right) = \nabla_{(s,\ell)}\varphi(s,\ell)_{|(s,\ell)=\mathcal{R}^{-1}(x)} \, [D\mathcal{R}(s,\ell)] \,. \tag{6.10}$$

**Proof of Lemma 3.2.** By construction, $\mathcal{R} \in \mathbf{C}^1(\bar I \times ]-L,L[, \mathbb{R}^2)$. $\mathcal{R}$ is locally invertible in each point of $I \times ]-L,L[$, by (6.5)

$$\det D\mathcal{R}(s,\ell) > 1 - L \left\|\gamma'(s)\right\| \left\|\gamma''(s)\right\| = 1 - L \left\|\gamma''(s)\right\| > 0$$

which shows that $\det D\mathcal{R}(s,\ell) \neq 0$. The Implicit Function Theorem ensures local invertibility at any $(s,\ell) \in I \times ]-L,L[$.

Since $\mathcal{R}(I \times ]-L,L[)$ is simply connected, the map $\mathcal{R} : I \times ]-L,L[ \to \mathcal{R}(I \times ]-L,L[)$ is globally invertible and its inverse is of class $\mathbf{C}^1$.

The conditions $\|\gamma'\| = 1$, $\|\gamma''\| \leq 1/L$, $|\ell| < L$ and (6.4) ensure that $\mathcal{R}$ is uniformly continuous and can be uniquely extended to $\bar I \times ]-L,L[$. $\mathcal{R}$ is injective also on $\bar I \times ]-L,L[$, otherwise $\mathcal{R}(\bar I \times ]-L,L[)$ cannot be simply connected. □

**Lemma 6.2.** *Let $f \in \mathbf{BV}(]-L,L[;\mathbb{R})$. Then,*

$$\forall \eta \in \mathbf{C}^1([-L,L];\mathbb{R}_+) \quad \int_{-L}^{L} f(\ell)\,\eta(\ell)\,\mathrm{d}\ell \geq 0 \implies f(\ell) \geq 0 \text{ for a.e. } \ell \in [-L,L]\,.$$

**Proof of Lemma 6.2.** If $f$ is continuous, then the conclusion follows by a standard elementary procedure. Assume $f \in \mathbf{BV}(]-L,L[;\mathbb{R})$ and, by contradiction, that $f(\ell) < 0$ for $\ell \in E$, $E$ being a subset of $]-L,L[$ of positive measure. Call $\eta_n$ a sequence in $\mathbf{C}^1([-L,L];\mathbb{R}_+)$ such that $\left\|\eta_n - \chi_E\right\|_{\mathbf{L}^1([-L,L];\mathbb{R})} \to 0$ as $n \to +\infty$, $\chi_E$ being the characteristic function of $E$. Then,

$$0 > \int_E f(\ell)\,\mathrm{d}\ell = \lim_{n\to+\infty} \int_{-L}^{L} f(\ell)\,\eta_n(\ell)\,\mathrm{d}\ell \geq 0\,,$$

where the latter inequality follows from the assumption. □

**Lemma 6.3.** *Let $K \subset \mathbb{R}$ be a compact interval and $\varphi \in \mathbf{C}^1(K\times[0,1]^2;\mathbb{R}_+)$. Then, for every $\epsilon > 0$, there exist functions $f_1, \ldots, f_n, g_1, \ldots, g_n \in \mathbf{C}^1([0,1];[0,1])$ and $c_{ij} \in \mathbf{C}^1(K;\mathbb{R}_+)$ for $i,j = 1,\ldots,n$ such that*

$$\left|\varphi(t,x,y) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(t)\,f_i(x)\,g_j(y)\right| < \epsilon \quad \text{for all} \quad (t,x,y) \in K \times [0,1]^2.$$

**Proof.** Define

$$\begin{array}{cccc} Y & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & (2|x|+1)\,(|x|-1)^2\,\chi_{[-1,1]}(x)\,. \end{array} \tag{6.11}$$

Clearly, $Y$ is even, non negative and in $\mathbf{C}^1(\mathbb{R};\mathbb{R})$. Moreover, for all $x \in [-1,1]$, $Y$ is such that $Y(x-1)+Y(x)+Y(x+1) = 1$, since at any $x \in [-1,1]$ only two summands in the latter sum are non 0.

Choose $n$ so that $1/n$ is a modulus of uniform continuity of $\varphi$ on $[0,1]^2$ and set for $t \in K$

$$c_{ij}(t) = \varphi\left(t, \frac{i}{n}, \frac{j}{n}\right), \quad f_i(x) = Y(n\,x - i)\,, \quad g_j(y) = Y(n\,y - j)\,. \tag{6.12}$$

The proof follows by direct computations, since $\sum_{i=1}^n f_i(x) = \sum_{j=1}^n g_j(y) = 1$ for $(x,y) \in [0,1]^2$. □

**Proof of Lemma 3.4.** In this proof, we extensively use the change of variable $x = \mathcal{R}(s,\ell)$, see Lemma 3.2 for its properties, and, with the notation (3.5), the change of variables $(\sigma,\ell) = \mathcal{S}(s,\ell)$ where

$$\begin{array}{rccl} \mathcal{S} : & \bar I \times ]-L,L[ & \to & \left\{(\sigma,\ell) \in \mathbb{R}^2 : \sigma \in \overline{I^\ell} \text{ and } \ell \in ]-L,L[\right\} \\ & (s,\ell) & \mapsto & (\sigma,\ell) \quad \text{where } \frac{\mathrm{d}\sigma}{\mathrm{d}s}(s) = 1 - \ell\,\left(\gamma'(s)\right)^\perp \cdot \gamma''(s)\,. \end{array} \tag{6.13}$$

Note that $\mathcal{S}$ is a diffeomorphism of class $\mathbf{C}^1(I\times]-L,L[;\mathbb{R}^2) \cap \mathbf{C}^0(\bar I\times]-L,L[;\mathbb{R}^2)$ and the same computations as in the proof of Lemma 3.2 ensure that $s^\ell_{out} > s^\ell_{in}$.

Let $\rho$ solve (3.3) in the sense of Definition 3.3. Let $\varphi \in \mathbf{C}^1_c(\mathbb{R}\times\mathcal{R};\mathbb{R}_+)$ be a test function and call

$$\tilde\varphi(t,s,\ell) = \varphi\left(t,\mathcal{R}(s,\ell)\right) \quad \text{and} \quad \varphi^\ell(t,\sigma) = \tilde\varphi\left(t,\mathcal{S}^{-1}(\sigma,\ell)\right)\,. \tag{6.14}$$

Clearly, $\varphi^\ell \in \mathbf{C}^1_c(\mathbb{R}\times I^\ell;\mathbb{R}_+)$. Moreover, setting for instance $\Phi^\ell(t,\sigma) = \eta(\ell)\,\hat\varphi(t,\sigma)$ with $\eta \in \mathbf{C}^1_c(]-L,L[;\mathbb{R}_+)$ and $\hat\varphi \in \mathbf{C}^1_c(\mathbb{R}\times I^\ell;\mathbb{R}_+)$, reversing the changes of variables $\mathcal{R}$ and (6.13), through (6.14) one gets that $\Phi^\ell = \varphi^\ell$ for a $\varphi \in \mathbf{C}^1_c(\mathbb{R}\times\mathcal{R};\mathbb{R}_+)$.

Similarly, also recalling (3.6), denote for a.e. $t \in [0,T]$, $\ell \in ]-L,L[$, $s \in I$, $\sigma \in I^\ell$,

$$\tilde\rho(t,s,\ell) = \rho\left(t,\mathcal{R}(s,\ell)\right) \quad \text{and} \quad \rho^\ell(t,\sigma) = \tilde\rho\left(t,\mathcal{S}^{-1}(\sigma,\ell)\right)\,. \tag{6.15}$$

Compute each line in the integral inequality in Definition 3.3, using first the change of variables $x = \mathcal{R}(s,\ell)$ and then the change of variables (6.13).

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}} (\rho(t,x)-\kappa)^{\pm}\, \partial_t\varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I (\rho(t,\mathcal{R}(s,\ell))-\kappa)^{\pm}\, \partial_t\varphi(t,\mathcal{R}(s,\ell))\, |\det D\mathcal{R}(s,\ell)|\, \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I (\tilde\rho(t,s,\ell)-\kappa)^{\pm}\, \partial_t\tilde\varphi(t,s,\ell)\, \left(1-\ell\, \left(\gamma'(s)\right)^{\perp}\cdot\gamma''(s)\right) \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_{I^\ell} \left(\rho^\ell(t,\sigma)-\kappa\right)^{\pm}\, \partial_t\varphi^\ell(t,\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_{-L}^{L} \left(\int_0^T \int_{I^\ell} \left(\rho^\ell(t,\sigma)-\kappa\right)^{\pm}\, \partial_t\varphi^\ell(t,\sigma)\, \mathrm{d}\sigma\, \mathrm{d}t\right) \mathrm{d}\ell\,,
\end{aligned}$$

where we used the notations (6.14) and (6.15).

Set temporarily $F(t,x) = \mathrm{sgn}^{\pm}(\rho(t,x)-\kappa)(q(\rho(t,x))-q(\kappa))$ and $\tilde F(t,s,\ell) = F(t,\mathcal{R}(s,\ell))$. Then, the second line in the integral inequality in Definition 3.3, thanks to (6.9), (6.10) and the notations (6.14)–(6.15), becomes

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}} \mathrm{sgn}^{\pm}(\rho(t,x)-\kappa)\,(q(\rho(t,x))-q(\kappa))\, \nabla_x\varphi(t,x)\cdot r(x)\, \mathrm{d}x\, \mathrm{d}t\\
&= \int_0^T \int_{\mathcal{R}} F(t,x)\, \nabla_x\varphi(t,x)\cdot r(x)\, \mathrm{d}x\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I F(t,\mathcal{R}(s,\ell))\, \nabla_{(s,\ell)}\varphi(t,\mathcal{R}(s,\ell))\, [D\mathcal{R}(s,\ell)]^{-1}\cdot r(\mathcal{R}(s,\ell)) \det D\mathcal{R}(s,\ell)\, \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I \tilde F(t,s,\ell)\, \nabla_{(s,\ell)}\tilde\varphi(t,s,\ell) \begin{bmatrix} (\gamma'(s))^{\intercal}\\ \left((\gamma'(s))^{\perp}-\ell\,\gamma''(s)\right)^{\intercal}\end{bmatrix} \gamma'(s)\, \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I \tilde F(t,s,\ell)\, \nabla_{(s,\ell)}\tilde\varphi(t,s,\ell) \begin{bmatrix} \gamma'(s)\cdot\gamma'(s)\\ -\ell\,\gamma'(s)\cdot\gamma''(s)\end{bmatrix} \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I \tilde F(t,s,\ell)\, \nabla_{(s,\ell)}\tilde\varphi(t,s,\ell) \begin{bmatrix} 1\\ 0\end{bmatrix} \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I \tilde F(t,s,\ell)\, \partial_s\tilde\varphi(t,s,\ell)\, \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I \mathrm{sgn}^{\pm}(\rho(t,\mathcal{R}(s,\ell))-\kappa)\,(q(\rho(t,\mathcal{R}(s,\ell)))-q(\kappa))\, \partial_s\tilde\varphi(t,s,\ell)\, \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_I \mathrm{sgn}^{\pm}(\tilde\rho(t,s,\ell)-\kappa)\,(q(\tilde\rho(t,s,\ell))-q(\kappa))\, \partial_s\tilde\varphi(t,s,\ell)\, \mathrm{d}s\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \int_{I^\ell} \mathrm{sgn}^{\pm}\left(\rho^\ell(t,\sigma)-\kappa\right) \left(q\left(\rho^\ell(t,\sigma)\right)-q(\kappa)\right)\, \partial_\sigma\varphi^\ell(t,\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\ell\, \mathrm{d}t\\
&= \int_{-L}^{L} \left(\int_0^T \int_{I^\ell} \mathrm{sgn}^{\pm}\left(\rho^\ell(t,\sigma)-\kappa\right) \left(q\left(\rho^\ell(t,\sigma)\right)-q(\kappa)\right)\, \partial_\sigma\varphi^\ell(t,\sigma)\, \mathrm{d}\sigma\, \mathrm{d}t\right) \mathrm{d}\ell\,.
\end{aligned}$$

Now, the first summand in the third line in the integral inequality in Definition 3.3 is

$$\begin{aligned}
&\int_{\mathcal{R}} \left(\rho_o(x)-\kappa\right)^{\pm} \varphi(0,x)\, \mathrm{d}x\\
&= \int_{-L}^{L} \int_I \left(\rho_o(\mathcal{R}(s,\ell))-\kappa\right)^{\pm} \varphi(0,\mathcal{R}(s,\ell))\, |\det D\mathcal{R}(s,\ell)|\, \mathrm{d}s\, \mathrm{d}\ell\\
&= \int_{-L}^{L} \int_{I^\ell} \left(\rho_o^\ell(\sigma)-\kappa\right)^{\pm} \varphi^\ell(0,\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\ell\\
&= \int_{-L}^{L} \left(\int_{I^\ell} \left(\rho_o^\ell(\sigma)-\kappa\right)^{\pm} \varphi^\ell(0,\sigma)\, \mathrm{d}\sigma\right) \mathrm{d}\ell\,,
\end{aligned}$$

where we used (3.6). The second summand in the same line is entirely analogous. The fourth line in the integral inequality in Definition 3.3, by (3.6) and (6.14), is

$$\begin{aligned}
&\int_0^T \int_{\mathcal{E}_{in}} \left(q^{-1}_{|[0,\bar\rho]}\left(f_{in}(t,\xi)\right)-\kappa\right)^{\pm} \varphi(t,\xi)\, \mathrm{d}\xi\, \mathrm{d}t\\
&= \int_0^T \int_{-L}^{L} \left(q^{-1}_{|[0,\bar\rho]}\left(f_{in}\left(t,\mathcal{R}(s_{in},\ell)\right)\right)-\kappa\right)^{\pm} \varphi\left(t,\mathcal{R}(s_{in},\ell)\right)\, \mathrm{d}\ell\, \mathrm{d}t
\end{aligned}$$

$$= \int_0^T \int_{-L}^{L} \left( q^{-1}_{|[0,\bar\rho]} \left( f^\ell_{in}(t) \right) - \kappa \right)^{\pm} \varphi^\ell(t, s^\ell_{in}) \,\mathrm{d}\ell \,\mathrm{d}t$$
$$= \int_{-L}^{L} \left( \int_0^T \left( q^{-1}_{|[0,\bar\rho]} \left( f^\ell_{in}(t) \right) - \kappa \right)^{\pm} \varphi^\ell(t, s^\ell_{in}) \,\mathrm{d}t \right) \mathrm{d}\ell \,.$$

Finally, the fifth line is entirely analogous.

By the above computations, for any $\eta \in \mathbf{C}^1_c(]-L, L[; \mathbb{R}_+)$, for any $\ell \in ]-L, L[$ and any $\hat\varphi \in \mathbf{C}^1_c(\mathbb{R} \times I^\ell; \mathbb{R}_+)$, choosing a function $\varphi$ so that $\varphi^\ell(t,\sigma) = \eta(\ell)\,\hat\varphi(t,\sigma)$, we have

$$\begin{aligned}
&\int_{-L}^{L} \Bigg[ \int_0^T \int_{I^\ell} \left( \rho^\ell(t,\sigma) - \kappa \right)^{\pm} \partial_t \hat\varphi(t,\sigma) \,\mathrm{d}\sigma \,\mathrm{d}t \\
&+ \int_0^T \int_{I^\ell} \operatorname{sgn}^{\pm} \left( \rho^\ell(t,\sigma) - \kappa \right) \left( q\left(\rho^\ell(t,\sigma)\right) - q(\kappa) \right) \partial_\sigma \hat\varphi(t,\sigma) \,\mathrm{d}\sigma \,\mathrm{d}t \\
&+ \int_{I^\ell} \left( \rho^\ell_o(\sigma) - \kappa \right)^{\pm} \hat\varphi(0,\sigma) \,\mathrm{d}\sigma + \int_{I^\ell} \left( \rho^\ell(T,\sigma) - \kappa \right)^{\pm} \hat\varphi(T,\sigma) \,\mathrm{d}\sigma \\
&+ \int_0^T \left( q^{-1}_{|[0,\bar\rho]} \left( f^\ell_{in}(t) \right) - \kappa \right)^{\pm} \hat\varphi(t, s_{in}) \,\mathrm{d}t \\
&+ \int_0^T \left( q^{-1}_{|[\bar\rho,R]} \left( f^\ell_{out}(t) \right) - \kappa \right)^{\pm} \hat\varphi(t, s^\ell_{out}) \,\mathrm{d}t \Bigg] \eta(\ell) \,\mathrm{d}\ell \geq 0 \,.
\end{aligned}$$

An application of Lemma 6.2, thanks to the arbitrariness of $\eta$, shows that $\rho^\ell$ solves (3.4), for a.e. $\ell \in ]-L, L[$.

Conversely, assume $\rho^\ell$ solves (3.4), for any $\ell \in ]-L, L[$. Then, any $\varphi \in \mathbf{C}^1_c(\mathbb{R} \times \mathcal{R}; \mathbb{R}_+)$ yields a function $\varphi^\ell$ as defined in (6.14) which is of class $\mathbf{C}^1$ in all three variables $(t, \ell, \sigma)$ and compactly supported. Call $\varphi^\ell$ the map defined in (6.14) with $s^\ell_{out}$ as in (3.5). Then, a direct consequence of Lemma 6.3 ensures that for any $\epsilon > 0$ there exist functions $F_1, \ldots, F_n \in \mathbf{C}^1_c(]-L, L[; \mathbb{R}_+)$, $G^\ell_1, \ldots, G^\ell_n \in \mathbf{C}^1(\overline{I^\ell}; \mathbb{R}_+)$ and $c_{ij} \in \mathbf{C}^1_c(]-\infty, T[; \mathbb{R}_+)$ such that for all $t \in ]-\infty, T[$, $\ell \in ]-L, L[$ and $\sigma \in \overline{I^\ell}$

$$\left| \varphi^\ell(t,\sigma) - \sum_{i=1}^n \sum_{j=1}^n c_{ij}(t)\, F_i(\ell)\, G^\ell_j(\sigma) \right| < \epsilon \,.$$

By the above computations, the left hand side of the inequality in Definition 3.3 satisfies

$$\begin{aligned}
&\int_0^T \int_{\mathcal{R}} (\rho(t,x) - \kappa)^{\pm} \partial_t \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \int_{\mathcal{R}} \operatorname{sgn}^{\pm} (\rho(t,x) - \kappa) (q(\rho(t,x)) - q(\kappa)) \nabla_x \varphi(t,x) \cdot r(x) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_{\mathcal{R}} \left( \rho_o(x) - \kappa \right)^{\pm} \varphi(0,x) \,\mathrm{d}x + \int_{\mathcal{R}} (\rho(T,x) - \kappa)^{\pm} \varphi(T,x) \,\mathrm{d}x \\
&+ \|q'\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \int_0^T \int_{\mathcal{E}_{in}} \left( q^{-1}_{|[0,\bar\rho]} \left( f_{in}(t,\xi) \right) - \kappa \right)^{\pm} \varphi(t,\xi) \,\mathrm{d}\xi \,\mathrm{d}t \\
&+ \|q'\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \int_0^T \int_{\mathcal{E}_{out}} \left( q^{-1}_{|[\bar\rho,R])} \left( f_{out}(t,\xi) \right) - \kappa \right)^{\pm} \varphi(t,\xi) \,\mathrm{d}\xi \,\mathrm{d}t \geq \\
&- C\,\epsilon + \sum_{i,j=1}^n \int_{-L}^{L} \Bigg[ \int_0^T \int_{I^\ell} \left( \rho^\ell(t,\sigma) - \kappa \right)^{\pm} \partial_t c_{ij}(t)\, G^\ell_j(\sigma) \,\mathrm{d}\sigma \,\mathrm{d}t \\
&+ \int_0^T \int_{I^\ell} \operatorname{sgn}^{\pm} \left( \rho^\ell(t,\sigma) - \kappa \right) \left( q\left(\rho^\ell(t,\sigma)\right) - q(\kappa) \right) c_{ij}(t)\, \partial_\sigma G^\ell_j(\sigma) \,\mathrm{d}\sigma \,\mathrm{d}t \\
&+ \int_{I^\ell} \left( \rho^\ell_o(\sigma) - \kappa \right)^{\pm} c_{ij}(0)\, G^\ell_j(\sigma) \,\mathrm{d}\sigma + \int_{I^\ell} \left( \rho^\ell(T,\sigma) - \kappa \right)^{\pm} c_{ij}(T)\, G^\ell_j(\sigma) \,\mathrm{d}\sigma \\
&+ \int_0^T \left( q^{-1}_{|[0,\bar\rho]} \left( f^\ell_{in}(t) \right) - \kappa \right)^{\pm} c_{ij}(t)\, G^\ell_j(s_{in}) \,\mathrm{d}t \\
&+ \int_0^T \left( q^{-1}_{|[\bar\rho,R]} \left( f^\ell_{out}(t) \right) - \kappa \right)^{\pm} c_{ij}(t)\, G^\ell_j(s^\ell_{out}) \,\mathrm{d}t \Bigg] F_i(\ell) \,\mathrm{d}\ell \geq -C\,\epsilon \,,
\end{aligned}$$

for a constant $C$ dependent only on $R$, $\bar\rho$, $s_{in}$, $s_{out}$, $L$ and norms of $q$. To get to the latter inequality above, use the fact that, for any $\ell$, $\rho^\ell$ solves (3.4) in the sense of Definition 2.1. □

**Proof of Theorem 3.5.** The existence and uniqueness of a solution to (3.3) is a consequence of Lemma 3.4 and of Proposition 2.2. The $\mathbf{L}^\infty$ estimate directly follows from (2.3).

To prove the $\mathbf{L}^1$ Lipschitz estimate, compute:

$$
\begin{aligned}
& \left\|\rho(t)-\widehat{\rho}(t)\right\|_{\mathbf{L}^1(\mathcal{R};\mathbb{R})} \\
= & \int_{\mathcal{R}} \left|\rho(t,x)-\widehat{\rho}(t,x)\right| \mathrm{d}x \\
= & \int_{-L}^{L}\int_I \left|\rho\left(t,\mathcal{R}(s,\ell)\right)-\widehat{\rho}\left(t,\mathcal{R}(s,\ell)\right)\right| \left(1-\ell\left(\gamma'(s)\right)^{\perp}\cdot\gamma''(s)\right) \mathrm{d}s\,\mathrm{d}\ell && \text{[By (6.5)]} \\
= & \int_{-L}^{L}\int_{I^\ell} \left|\rho^\ell(t,\sigma)-\widehat{\rho}^\ell(t,\sigma)\right| \mathrm{d}\sigma\,\mathrm{d}\ell && \text{[By (3.6)]} \\
= & \int_{-L}^{L} \left\|\rho^\ell(t)-\widehat{\rho}^\ell(t)\right\|_{\mathbf{L}^1(I^\ell;\mathbb{R})} \mathrm{d}\ell \\
\le & \int_{-L}^{L} \left(\left\|\rho_o^\ell-\widehat{\rho}_o^{\,\ell}\right\|_{\mathbf{L}^1(I^\ell;\mathbb{R})}+\left\|f_{in}^\ell-\widehat{f}_{in}^\ell\right\|_{\mathbf{L}^1([0,t];\mathbb{R})}+\left\|f_{out}^\ell-\widehat{f}_{out}^\ell\right\|_{\mathbf{L}^1([0,t];\mathbb{R})}\right) \mathrm{d}\ell && \text{[By (2.4)]} \\
= & \left\|\rho_o-\widehat{\rho}_o\right\|_{\mathbf{L}^1(\mathcal{R};\mathbb{R})}+\left\|f_{in}-\widehat{f}_{in}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{E}_{in};\mathbb{R})}+\left\|f_{out}-\widehat{f}_{out}\right\|_{\mathbf{L}^1([0,t]\times\mathcal{E}_{out};\mathbb{R})} && \text{[By (3.6) and (6.5)]}
\end{aligned}
$$

where the latter equality follows from the changes of coordinates (3.5) and $x=\mathcal{R}(s,\ell)$.

The proof of the bound (3.7) follows immediately from Proposition 2.2. □

### *6.3. Proofs related to the parabolic pollution model*

**Definition 6.4.** A *strong solution* to (4.1) is a map $u\in\mathbf{C}^0([0,T]\times\mathbb{R}^2;\mathbb{R})$ such that

$$\forall\, t\in\,]0,T[\quad x\mapsto u(t,x)\in\mathbf{C}^2(\mathbb{R}^2;\mathbb{R}) \qquad\text{and}\qquad \forall\, x\in\mathbb{R}^2\quad t\mapsto u(t,x)\in\mathbf{C}^1(]0,T[;\mathbb{R})$$

which satisfies the equation in (4.1) at every $(t,x)\in\,]0,T[\,\times\mathbb{R}^2$ and the initial condition in (4.1) at every $x\in\mathbb{R}^2$.

Recall the following definition.

**Definition 6.5.** Let $k\in\mathbb{N}\setminus\{0\}$. A function $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}^k$ is *Hölder continuous* if there exist $M>0$ and $\alpha\in\,]0,1]$ such that $\left\|f(t_1,x_1)-f(t_2,x_2)\right\|\le M\left(\left|t_1-t_2\right|^{\alpha/2}+\left\|x_1-x_2\right\|^{\alpha}\right)$ for all $(t_1,x_1),(t_2,x_2)\in[0,T]\times\mathbb{R}^2$.

A function $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}^k$ is *Hölder continuous in $x$ uniformly in $t$* if there exist $M>0$ and $\alpha\in\,]0,1]$ such that $\left\|f(t,x_1)-f(t,x_2)\right\|\le M\left\|x_1-x_2\right\|^{\alpha}$ for all $t\in[0,T]$ and $x_1,x_2\in\mathbb{R}^2$.

**Lemma 6.6.** *[9, Chapter 1, Section 6, Theorem 10 and Section 7, Theorem 12] Let* **(HP)** *hold. Then, there exists a function*

$$\Gamma\in\mathbf{C}^0\left(\left\{(t,x,\tau,\xi)\in[0,T]\times\mathbb{R}^2\times[0,T]\times\mathbb{R}^2 : t>\tau\right\};\mathbb{R}_+\right) \tag{6.16}$$

*such that*

- **(Γ.1)** *For all $(\tau,\xi)\in[0,T]\times\mathbb{R}^2$, $(t,x)\mapsto\Gamma(t,x,\tau,\xi)$ is a strong solution, in the sense of Definition 6.4, to the parabolic equation in (4.1) with $g=0$.*
- **(Γ.2)** *For all $x\in\mathbb{R}^2$ and for all $u_o\in\mathbf{C}^0(\mathbb{R}^2;\mathbb{R})$, $\lim_{t\to\tau+}\int_{\mathbb{R}^2}\Gamma(t,x,\tau,\xi)\,u_o(\xi)\,\mathrm{d}\xi=u_o(x)$.*
- **(Γ.3)** *There exists a constant $C>0$ such that for all choices of the variables $(t,x,\tau,\xi)$ in $\left\{(t,x,\tau,\xi)\in[0,T]\times\mathbb{R}^2\times[0,T]\times\mathbb{R}^2 : t>\tau\right\}$*

$$\Gamma(t,x,\tau,\xi)<\frac{C}{(t-\tau)}\exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(t-\tau)}\right), \tag{6.17}$$

$$\left|\partial_{x_i}\Gamma(t,x,\tau,\xi)\right|<\frac{C}{(t-\tau)^{3/2}}\exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(t-\tau)}\right) \qquad i=1,2, \tag{6.18}$$

$$\left|\partial^2_{x_i,x_j}\Gamma(t,x,\tau,\xi)\right|<\frac{C}{(t-\tau)^{2}}\exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(t-\tau)}\right) \qquad i,j=1,2. \tag{6.19}$$

*Moreover, if*

- **(R.1)** *$g\in\mathbf{C}^0([0,T]\times\mathbb{R}^2;\mathbb{R})$; for every bounded set $B\subset\mathbb{R}^2$ there exists $M_B>0$ and $\alpha_B\in\,]0,1]$ such that $\left|g(t,x_1)-g(t,x_2)\right|\le M_B\left\|x_1-x_2\right\|^{\alpha_B}$ for all $t\in[0,T]$ and $x_1,x_2\in B$ and, for suitable $C>0$ and $h\in\,]0,1/(4\,T\,\hat\mu)]$, satisfies $|g(t,x)|\le C\,e^{h\|x\|^2}$ for all $(t,x)\in[0,T]\times\mathbb{R}^2$;*
- **(R.2)** *$u_o\in\mathbf{C}^0(\mathbb{R}^2;\mathbb{R})$ and, for suitable $C>0$ and $h\in\,]0,1/(4\,T\,\hat\mu)]$, satisfies $\left|u_o(x)\right|\le C\,e^{h\|x\|^2}$ for all $x\in\mathbb{R}^2$;*

*then, the Cauchy problem (4.1) admits the strong solution*

$$u(t,x)=\int_{\mathbb{R}^2}\Gamma(t,x,0,\xi)\,u_o(\xi)\,\mathrm{d}\xi+\int_0^t\int_{\mathbb{R}^2}\Gamma(t,x,\tau,\xi)\,g(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau\,. \tag{6.20}$$

**Proof of Lemma 6.6.** Observe that (4.1) takes the form $\partial_t u - \sum_{ij} a_{ij}\, \partial^2_{ij} u - b \cdot \nabla u - c\, u = -f$ as soon as $a_{ii} = \mu$, $a_{ij} = 0$, $b = \nabla\mu - \beta$, $c = -\kappa + \nabla \cdot \beta$, $f = -g$.

The existence of $\Gamma$, together with ($\Gamma$.1) and ($\Gamma$.2), follow from [9, Chapter 1, Section 6, Theorem 10]. The continuity of $\Gamma$ follows from [9, Chapter 1, Section 6], where the results for bounded domains in [9, Section 4] are extended to unbounded domains. The positivity of $\Gamma$ follows from [9, Chapter 2, Section 4, Theorem 11]. The estimates (6.17) and (6.18) on $\Gamma$ are explained after [9, Chapter 1, Section 6, Theorem 11]. The estimate (6.19) is obtained in a similar manner, exploiting [9, Chapter 1, (4.11)], with, in his notation, $\mu = (n+2)/2$.

The existence of the solution to (4.1) and its expression follow from [9, Chapter 1, Section 7, Theorem 12]. □

**Lemma 6.7** (*[9, Chapter 1, Section 9, Theorem 16]*). *Let* **(HP)**, *(R.1) and (R.2) hold. Assume moreover that $D^2_x\mu$ is in $(\mathbf{C}^0 \cap \mathbf{L}^\infty)([0,T]\times\mathbb{R}^2)$ and Hölder continuous in $x$ uniformly in $t$. Then, (6.20) is the unique strong solution to (4.1).*

**Proof of Lemma 6.7.** To apply [9, Chapter 1, Section 9, Theorem 16] we need to verify the boundedness condition [9, Chapter 1, Section 9, Formula (9.1)], namely that there exists a $k > 0$ such that $\int_0^T \int_{\mathbb{R}^2} |u(t,x)| e^{-k\|x\|^2}\, dx\, dt < +\infty$. Indeed,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} |u(t,x)| e^{-k\|x\|^2}\, dx\, dt && \text{[Use (6.20)]}\\
\le{}& \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Gamma(t,x,0,\xi)\, |u_o(\xi)| e^{-k\|x\|^2}\, d\xi\, dx\, dt && \text{[Use (6.17) and (R.2)]}\\
&+ \int_0^T \int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}^2} \Gamma(t,x,\tau,\xi)\, |g(\tau,\xi)| e^{-k\|x\|^2}\, d\xi\, d\tau\, dx\, dt && \text{[Use (6.17) and (R.1)]}\\
\le{}& \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{C}{t} e^{-\frac{\|x-\xi\|^2}{4\tilde\mu t}} e^{h\|\xi\|^2} e^{-k\|x\|^2}\, d\xi\, dx\, dt && (6.21)\\
&+ \int_0^T \int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}^2} \frac{C}{(t-\tau)} e^{-\frac{\|x-\xi\|^2}{4\tilde\mu (t-\tau)}} e^{h\|\xi\|^2} e^{-k\|x\|^2}\, d\xi\, d\tau\, dx\, dt\,. && (6.22)
\end{aligned}
$$

The latter terms (6.21) and (6.22) can be treated similarly. By (R.2) for an $\epsilon > 0$ we have $h \le (1-\epsilon)/(4\tilde\mu T)$ and set $\xi - x = y$, so that

$$
\begin{aligned}
[(6.21)] \le{}& \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{t} e^{-\frac{\|y\|^2}{4\tilde\mu t}} e^{h\|x+y\|^2} e^{-k\|x\|^2}\, dy\, dx\, dt\\
\le{}& \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{t} e^{-\frac{\|y\|^2}{4\tilde\mu t}+h\|y\|^2} e^{-k\|x\|^2+h\|x\|^2}\, dy\, dx\, dt\\
\le{}& \int_0^T \int_{\mathbb{R}^2} \frac{1}{t} e^{-\frac{\epsilon\|y\|^2}{4\tilde\mu t}}\, dy\, dt \int_{\mathbb{R}^2} e^{-(k-h)\|x\|^2}\, dx\,,\\
[(6.22)] \le{}& \int_0^T \int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}^2} \frac{C}{(t-\tau)} e^{-\frac{\|y\|^2}{4\tilde\mu (t-\tau)}} e^{h\|x+y\|^2} e^{-k\|x\|^2}\, dy\, d\tau\, dx\, dt\\
\le{}& \int_0^T \int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}^2} \frac{C}{(t-\tau)} e^{-\frac{\|y\|^2}{4\tilde\mu (t-\tau)}+h\|y\|^2} e^{-k\|x\|^2+h\|x\|^2}\, dy\, d\tau\, dx\, dt\\
\le{}& \int_0^T \int_{\mathbb{R}^2}\int_0^t \frac{C}{(t-\tau)} e^{-\frac{\epsilon\|y\|^2}{4\tilde\mu (t-\tau)}}\, d\tau\, dy\, dt \int_{\mathbb{R}^2} e^{-(k-h)\|x\|^2}\, dx\,,
\end{aligned}
$$

and both are finite for $k > h$. □

**Lemma 6.8.** *Under the same assumptions of Lemma 6.7, suppose in addition that $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ and $g \in \mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$. Then, every strong solution to (4.1) in the sense of Definition 6.4 is also a weak solution in the sense of Definition 4.1.*

**Proof.** By Lemmas 6.6 and 6.7, the unique strong solution $u$ to (4.1) can be written as in (6.20). The $\mathbf{L}^1$ in space regularity and the $\mathbf{L}^1$ continuity required in Definition 4.1 follow from (6.17), since $u_o \in \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ and $g \in \mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$.

We now have to show that $u$ written as in (6.20) satisfies the integral equality (4.3). First, observe that the requirements in (4.2) imply that $\varphi$, $\partial_t\varphi$, $\nabla\varphi$, $\nabla^2\varphi$, $\varphi(0,\cdot)$ and $\varphi(T,\cdot)$ are all in $\mathbf{L}^\infty$ on $\mathbb{R}^2$. Hence, the integrals appearing in (4.3) are all finite.

A standard procedure allows to obtain (4.3) by subsequent integrations by parts, exploiting the hypotheses (4.2) on the test function $\varphi$ and ($\Gamma$.1) in Lemma 6.6. □

**Proof of Theorem 4.2.** We deal first with existence and, separately, with uniqueness. The other claims are proved subsequently.

**Existence:** For $n \in \mathbb{N}\setminus\{0\}$, call $\eta_n$ a mollifier in $\mathbf{C}^\infty_c(\mathbb{R}^3;\mathbb{R}_+)$ with the properties $\operatorname{spt}\eta_n \subseteq B_{\mathbb{R}^3}(0,1/n)$, $\int_{\mathbb{R}^3}\eta_n(t,x)\, dt\, dx = 1$ so that (extending $g$ to $\mathbb{R}^3$ with value 0 outside $[0,T]\times\mathbb{R}^2$)

$$
(g * \eta_n) \in \mathbf{C}^1([0,T]\times\mathbb{R}^2;\mathbb{R}) \quad \text{and} \quad \lim_{n\to+\infty} g * \eta_n = g \text{ in } \mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})\,. \tag{6.23}
$$

Similarly, for $n \in \mathbb{N}\setminus\{0\}$, call $\vartheta_n$ a mollifier in $\mathbf{C}^\infty_c(\mathbb{R}^2;\mathbb{R}_+)$ with the properties $\operatorname{spt}\vartheta_n \subseteq B_{\mathbb{R}^2}(0,1/n)$, $\int_{\mathbb{R}^2}\vartheta_n(t,x)\, dt\, dx = 1$ so that

$$
(u_o * \vartheta_n) \in \mathbf{C}^1(\mathbb{R}^2;\mathbb{R}) \quad \text{and} \quad \lim_{n\to+\infty} u_o * \vartheta_n = u_o \text{ in } \mathbf{L}^1(\mathbb{R}^2;\mathbb{R})\,. \tag{6.24}
$$

Note that $g * \eta_n$ satisfies (R.1) and $u_o * \vartheta_n$ satisfies (R.2). Hence, the problem

$$\begin{cases} \partial_t u + \nabla \cdot (u\,\beta(t,x) - \mu(t,x)\,\nabla u) + \kappa(t,x)\,u = (g * \eta_n)(t,x) & (t,x) \in \,]0,T] \times \mathbb{R}^2 \\ u(0,x) = (u_o * \vartheta_n)(x) & x \in \mathbb{R}^2 . \end{cases} \tag{6.25}$$

admits a strong solution $u_n$ which belongs to $\mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$, due to the construction of $g * \eta_n$ and $u_o * \eta_n$. The function $u_n$ can be written as in (6.20):

$$u_n(t,x) = \int_{\mathbb{R}^2} \Gamma(t,x,0,\xi)\,(u_o * \vartheta_n)(\xi)\,\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t,x,\tau,\xi)\,(g * \eta_n)(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau . \tag{6.26}$$

The sequence $t \mapsto u_n(t)$ is a Cauchy sequence for the uniform convergence in $\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$, indeed using (6.17)

$$\begin{aligned} & \left\|u_n(t) - u_m(t)\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} && (6.27)\\ \le\; & \frac{C}{t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,t}\right) \left((u_o * \vartheta_n)(\xi) - (u_o * \vartheta_m)(\xi)\right)\,\mathrm{d}\xi\,\mathrm{d}x \\ & + \int_0^t \frac{C}{t-\tau}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(t-\tau)}\right) \left((g * \eta_n)(\tau,\xi) - (g * \eta_m)(\tau,\xi)\right)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}\tau \\ \le\; & \frac{C}{t}\int_{\mathbb{R}^2} \exp\left(-\frac{\|\xi\|^2}{4\,\hat\mu\,t}\right)\mathrm{d}\xi \int_{\mathbb{R}^2} \left((u_o * \vartheta_n)(\xi) - (u_o * \vartheta_m)(\xi)\right)\,\mathrm{d}\xi \\ & + \int_0^t \int_{\mathbb{R}^2} \frac{C}{t-\tau} \exp\left(-\frac{\|\xi\|^2}{4\,\hat\mu\,(t-\tau)}\right)\mathrm{d}\xi \int_{\mathbb{R}^2} \left((g * \eta_n)(\tau,\xi) - (g * \eta_m)(\tau,\xi)\right)\,\mathrm{d}\xi\,\mathrm{d}\tau \\ \le\; & 4\pi C\hat\mu\left(\left\|(u_o * \vartheta_n) - (u_o * \vartheta_m)\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} + \left\|(g * \eta_n) - (g * \eta_m)\right\|_{\mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})}\right) . && (6.28) \end{aligned}$$

Recall that $(u_o * \vartheta_n)$ is a Cauchy sequence in $\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})$ by (6.24) and $(g * \eta_n)$ is a Cauchy sequence in $\mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$ by (6.23).

Call $u \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})\right)$ the $\mathbf{L}^1$ uniform limit of the $u_n$. Clearly, $u \in \mathbf{L}^1([0,T]\times\mathbb{R}^2;\mathbb{R})$. To prove that $u$ is a weak solution to (4.1) in the sense of Definition 4.1, since $u_n$ is a strong – and hence also weak – solution to (4.1), compute

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} u\left(\partial_t\varphi + \nabla\cdot(\mu\,\nabla\varphi) + \beta\cdot\nabla\varphi - \kappa\,\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\mathbb{R}^2} u_o(x)\,\varphi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^2} u(T,x)\,\varphi(T,x)\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2} g\,\varphi\,\mathrm{d}x\,\mathrm{d}t \\ = & \int_0^T \int_{\mathbb{R}^2} (u-u_n)\left(\partial_t\varphi + \nabla\cdot(\mu\,\nabla\varphi) + \beta\cdot\nabla\varphi - \kappa\,\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\mathbb{R}^2} (u_o - u_o * \vartheta_n)(x)\,\varphi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^2} (u-u_n)(T,x)\,\varphi(T,x)\,\mathrm{d}x \\ & + \int_0^T \int_{\mathbb{R}^2} (g - g * \eta_n)\,\varphi\,\mathrm{d}x\,\mathrm{d}t \\ \to & \; 0 \quad \text{as } n \to +\infty \end{aligned}$$

by the Dominated Convergence Theorem.

**Uniqueness:** This proof follows the lines of [28, Lemma 5.1]. Let $u_1, u_2$ be two weak solutions to (4.1) in the sense of Definition 4.1. Then, by a standard approximation argument, the difference $u = u_2 - u_1$ satisfies for all $\varphi$ as in (4.2) and for all $\tau \in \,]0,T[$

$$\int_0^\tau \int_{\mathbb{R}^2} u\left(\partial_t\varphi + \nabla\cdot(\mu\,\nabla\varphi) + \beta\cdot\nabla\varphi - \kappa\,\varphi\right)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^2} u(\tau,x)\,\varphi(\tau,x)\,\mathrm{d}x = 0 . \tag{6.29}$$

For an arbitrary $\omega \in \mathbf{C}_c^0(\mathbb{R};\mathbb{R})$, consider the (backward) solution to the parabolic problem

$$\begin{cases} \partial_t\varphi + \nabla\cdot(\mu\,\nabla\varphi + \beta\,\varphi) - (\kappa + \nabla\cdot\beta)\varphi = 0 & (t,x) \in \,]0,\tau[\, \times \mathbb{R}^2 \\ \varphi(\tau,x) = \omega(x) & x \in \mathbb{R}^2 . \end{cases} \tag{6.30}$$

This equation fits into (4.1) (with time reversed) and Lemma 6.6 applies. Therefore, (6.30) admits a strong solution which can be written as in (6.20).

We now show that $\varphi$ satisfies (4.2). Indeed, the $\mathbf{C}^1 - \mathbf{C}^2$ regularity follows from the definition of strong solution. To verify the conditions at infinity, use (6.20) with $g = 0$, bearing in mind that time is reversed, starting form $\tau$ and going backward to 0. Assume that $\operatorname{spt}\omega \subseteq B(0,\bar r)$ for a suitable $\bar r > 0$. Compute for $r \gg \bar r$, using (6.17):

$$\begin{aligned} \sup_{\|x\|\ge r} |\varphi(t,x)| = & \sup_{\|x\|\ge r} \left|\int_{\mathbb{R}^2} \Gamma(\tau-t,x,0,\xi)\,\omega(\xi)\,\mathrm{d}\xi\right| \\ \le & \sup_{\|x\|\ge r} \int_{\operatorname{spt}\omega} \frac{C}{(\tau-t)} \exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(\tau-t)}\right) |\omega(\xi)|\,\mathrm{d}\xi \\ \le & \sup_{\|x\|\ge r} \int_{B(0,\bar r)} \frac{C}{(\tau-t)} \exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(\tau-t)}\right) \mathrm{d}\xi\; \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})} \end{aligned}$$

$$\begin{aligned} &\leq \frac{C}{(\tau-t)} \int_{B(0,\bar r)} \exp\left(-\frac{(r-\bar r)^2}{4\,\hat\mu\,(\tau-t)}\right) \mathrm{d}\xi\, \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})} \\ &\leq \frac{C}{(\tau-t)}\, \mathcal{L}\left(B(0,\bar r)\right)\, \exp\left(-\frac{(r-\bar r)^2}{4\,\hat\mu\,(\tau-t)}\right) \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}, \end{aligned}$$

$\mathcal{L}$ being the Lebesgue measure on $\mathbb{R}^2$. The latter quantity vanishes as $r \to +\infty$. We need to repeat similar computations to bound $\|\nabla\varphi\|$, using as above (6.20) with $g = 0$ and (6.18):

$$\begin{aligned} \sup_{\|x\|\geq r} \|\nabla\varphi(t,x)\| &= \sup_{\|x\|\geq r} \left\| \int_{\mathbb{R}^2} \nabla\Gamma(\tau-t,x,0,\xi)\,\omega(\xi)\,\mathrm{d}\xi \right\| \\ &\leq \sup_{\|x\|\geq r} \int_{\mathrm{spt}\,\omega} \frac{C\sqrt{n}}{(\tau-t)^{3/2}} \exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(\tau-t)}\right) |\omega(\xi)|\,\mathrm{d}\xi \\ &\leq \frac{C\sqrt{n}}{(\tau-t)^{3/2}}\, \mathcal{L}\left(B(0,\bar r)\right)\, \exp\left(-\frac{(r-\bar r)^2}{4\,\hat\mu\,(\tau-t)}\right) \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}. \end{aligned}$$

Similarly, to bound $\left\|\nabla^2\varphi\right\|$, we use as above (6.20) with $g = 0$ and (6.19):

$$\begin{aligned} \sup_{\|x\|\geq r} \left\|\nabla^2\varphi(t,x)\right\| &= \sup_{\|x\|\geq r} \left\| \int_{\mathbb{R}^2} \nabla^2\Gamma(\tau-t,x,0,\xi)\,\omega(\xi)\,\mathrm{d}\xi \right\| \\ &\leq \sup_{\|x\|\geq r} \int_{\mathrm{spt}\,\omega} \frac{C\sqrt{n}}{(\tau-t)^{2}} \exp\left(-\frac{\|x-\xi\|^2}{4\,\hat\mu\,(\tau-t)}\right) |\omega(\xi)|\,\mathrm{d}\xi \\ &\leq \frac{C\sqrt{n}}{(\tau-t)^{2}}\, \mathcal{L}\left(B(0,\bar r)\right)\, \exp\left(-\frac{(r-\bar r)^2}{4\,\hat\mu\,(\tau-t)}\right) \|\omega\|_{\mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R})}. \end{aligned}$$

The bound on $\|\partial_t\varphi\|$ follows from the previous ones, since, by (6.30), $\partial_t\varphi = -\nabla\cdot(\mu\,\nabla\varphi + \beta\,\varphi) + (\kappa + \nabla\cdot\beta)\varphi$ and **(HP)** holds. We thus obtain from (6.29) that $\int_{\mathbb{R}^2} u(\tau,x)\,\omega(x)\,\mathrm{d}x = 0$ for any $\omega \in \mathbf{C}^0_c(\mathbb{R}^2;\mathbb{R})$, which implies that $u \equiv 0$, completing the proof of uniqueness.

**Representation:** Formula (4.4) follows from (6.26) and the $\mathbf{L}^1$ convergence of the $u_n$ to $u$.

**Stability:** Thanks to (4.4), the estimate (4.5) is obtained repeating the same computations as in (6.27)–(6.28), substituting $u_n$ with $u$.

**Monotonicity:** It is an immediate consequence of (4.4) and (6.16). □

**Proof of Theorem 4.4.** This result extends Theorem 4.2, we detail the key differences.

**Existence:** Introduce the mollifiers $\eta_n$ and the solutions

$$u_n(t,x) = \int_{\mathbb{R}^2} \Gamma(t,x,0,\xi)\,u_o(\xi)\,\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}^2} \Gamma(t,x,\tau,\xi)\,(g_\tau * \eta_n)(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{6.31}$$

to (4.1) with the map $(t,x) \mapsto (g_t * \eta_n)(x)$ as source term. Since $g_t * \eta_n$ converges to $g_t$ in $\mathcal{D}'$, we can pass to the limit $n \to +\infty$ in (6.31) and define the function $u$ as in (4.7), so that for all $(t,x) \in [0,T]\times\mathbb{R}^2$, $\lim_{n\to+\infty} u_n(t,x) = u(t,x)$. Observe that

$$\begin{aligned} \left\|u_n(t)\right\|_{\mathbf{L}^1} &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Gamma(t,x,0,\xi)\left|u_o(\xi)\right|\mathrm{d}\xi\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}^2} \Gamma(t,x,\tau,\xi)\left|(g_\tau * \eta_n)(\xi)\right|\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}x \\ &\leq \frac{C}{t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\|x-\xi\|^2/(4\hat\mu t)}\left|u_o(\xi)\right|\mathrm{d}\xi\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_0^t\int_{\mathbb{R}^2} \frac{C}{t-\tau}\, e^{-\|x-\xi\|^2/(4\hat\mu(t-\tau))}\left|(g_\tau * \eta_n)(\xi)\right|\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}x \\ &\leq \frac{C}{t}\int_{\mathbb{R}^2} e^{-\|z\|^2/(4\hat\mu t)}\int_{\mathbb{R}^2}\left|u_o(z+x)\right|\mathrm{d}x\,\mathrm{d}z + \int_0^t\int_{\mathbb{R}^2}\frac{C}{t-\tau}\, e^{-\|z\|^2/(4\hat\mu(t-\tau))}\int_{\mathbb{R}^2}\left|(g_\tau * \eta_n)(z+x)\right|\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}\tau \\ &\leq C\left\|u_o\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} + C\int_0^t\int_{\mathbb{R}^2}\left|(g_\tau * \eta_n)(\xi)\right|\mathrm{d}\xi\,\mathrm{d}\tau \\ &\leq C\left\|u_o\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} + C\int_0^t \left\|g_\tau\right\|_{\mathcal{M}}\,\mathrm{d}\tau\,\left\|\eta_n\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})}, \end{aligned}$$

where we used [26, Proposition 8.68, p.282]. Thanks to (4.6), apply Fatou Lemma and obtain that $u(t) \in \mathbf{L}^1(\mathbb{R}^2,\mathbb{R})$ for all $t \in [0,T]$.

The continuity in time of $t \mapsto u(t)$ is proved exactly as in Theorem 4.2.

Proving that $u$ satisfies (4.8) and solves (4.1) is done exactly as in Theorem 4.2, as also the uniqueness of $u$ in the class of weak solutions.

**Stability and Monotonicity.** Using the representation (4.7), we have

$$\left\|u(t)-\hat u(t)\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})} \leq \int_{\mathbb{R}^2} \Gamma(t,x,0,\xi)\left|u_o(\xi)-\hat u_o(\xi)\right|\mathrm{d}\xi + \int_0^t\int_{\mathbb{R}^2}\Gamma(t,x,\tau,\xi)\left|\,\mathrm{d}g_\tau(\xi) - \,\mathrm{d}\hat g_\tau(\xi)\right|\mathrm{d}\tau$$

and (4.8) follows by [26, Proposition 8.68, p.282]. Monotonicity is immediate. □

### 6.4. Proofs related to the coupled problem

**Proof of Proposition 5.2.** Each $g_t^i$ in (5.2) is a finite measure by **(2D.2)** in Theorem 3.7, by Lemma 3.2 and by Definition 3.1. Clearly, the distribution defined by (5.2) has order 0, hence it is a Radon measure, see [29, § 1.3, Remark on p.10].

The definition (5.2) also implies that $g_t^i$ is supported in $\mathcal{R}^i$. Moreover, also by [26, Ex. 1.4, Chap. 3],

$$\begin{aligned}
\sup_{t\in[0,T]} \|g_t^i\|_{\mathcal{M}} &= \sup\left\{\left|g_t^i(\varphi)\right| : \varphi \in \mathbf{C}_c^\infty([0,T]\times\mathbb{R}^2;\mathbb{R}),\ |\varphi|\le 1\right\} \\
&\le \int_{-L}^{L}\int_I \left|1-\ell\left(\gamma'(s)\right)^\perp\cdot\gamma''(s)\right| \mathrm{d}\left|\mu_t^{i,\ell}(s)\right| \mathrm{d}\ell && \text{[By (5.2)]} \\
&\le 4\,L\,\mathrm{TV}\left(q\left(\rho^i\left(t,\mathcal{R}^i(\cdot,\ell)\right)\right)\right) && \text{[By Definition 3.1]} \\
&< +\infty && \text{[By (2.2) and (3.12)]}
\end{aligned}$$

completing the proof. □

**Proof of Theorem 5.4.** We first fix the inflows $f_{in}^1 \in \mathbf{L}^1([0,T]\times\mathcal{E}_{in}^1;[0,q(\bar\rho)])$, $f_{in}^2 \in \mathbf{L}^1([0,T]\times\mathcal{E}_{in}^2;[0,q(\bar\rho)])$, and outflow $f_{out}^3 \in \mathbf{L}^1([0,T]\times\mathcal{E}_{out}^3;[0,q(\bar\rho)])$ and choose $w,\hat w$ in $\mathcal{W}$. With the notation in (5.1), call $(\rho,u)$, respectively $(\hat\rho,\hat u)$, the solution to the coupled problem (3.8)–(4.1)–(5.1) regulated by $w$, respectively $\hat w$, and $f_{in}^1, f_{in}^2, f_{out}^3$. Then,

$$\begin{aligned}
&\left|\mathcal{F}\left(\hat w, f_{in}^1, f_{in}^2, f_{out}^3\right) - \mathcal{F}\left(w, f_{in}^1, f_{in}^2, f_{out}^3\right)\right| \\
\le&\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\int_0^T \left\|\hat u(t)-u(t)\right\|_{\mathbf{L}^1(\mathbb{R}^2;\mathbb{R})}\mathrm{d}t \\
\le&\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\int_0^T 4\,\pi\,C\,\hat\mu\left\|\hat g-g\right\|_{\mathbf{L}^1(0,t]\times\mathbb{R}^2;\mathbb{R})}\mathrm{d}t && \text{[By (4.5)]} \\
=&4\,\pi\,C\,\hat\mu\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \\
&\times\int_0^T\int_0^t\int_{\mathbb{R}^2}\left|G\left(\tau,\hat\rho(\tau,x),q\left(\hat\rho(\tau,x)\right)\right)-G\left(\tau,\rho(\tau,x),q\left(\rho(\tau,x)\right)\right)\right|\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t && \text{[By (5.1)]} \\
\le&4\,\pi\,C\,\hat\mu\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\mathbf{Lip}(G)\left(1+\mathbf{Lip}(q)\right) \\
&\times\int_0^T\int_0^t\sum_{i=1}^3\left\|\hat\rho^i(\tau)-\rho^i(\tau)\right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})}\mathrm{d}\tau\,\mathrm{d}t && \text{[By (G), (q)]} \\
\le&24\,\pi\,C\,\hat\mu\,L\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\mathbf{Lip}(G)\left(1+\mathbf{Lip}(q)\right) \\
&\times q(\bar\rho)\int_0^T\int_0^t\left\|\hat w-w\right\|_{\mathbf{L}^1([0,\tau];\mathbb{R})}\mathrm{d}\tau\,\mathrm{d}t && \text{[By (3.13)]} \\
\le&\pi\,C\,\hat\mu\,L\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\mathbf{Lip}(G)\left(1+\mathbf{Lip}(q)\right)q(\bar\rho)\,T^2\left\|\hat w-w\right\|_{\mathbf{L}^1([0,T];\mathbb{R})},
\end{aligned}$$

which proves that the map $\mathcal{F}$, as in (5.4), is $\mathbf{L}^1$–Lipschitz continuous with respect to $w\in\mathcal{W}$.

Let now $w \in \mathcal{W}$ be fixed. Choose the inflows and outflows $f_{in}^1, f_{in}^2, f_{out}^3, \hat f_{in}^1, \hat f_{in}^2, \hat f_{out}^3$. With the notation in (5.1), call $(\rho,u)$, respectively $(\hat\rho,\hat u)$, the solution to the coupled problem (3.8)–(4.1)–(5.1) regulated by $w$ and $f_{in}^1, f_{in}^2, f_{out}^3$, respectively $\hat f_{in}^1, \hat f_{in}^2, \hat f_{out}^3$. Then, proceeding exactly as above,

$$\begin{aligned}
&\left|\mathcal{F}\left(w,\hat f_{in}^1,\hat f_{in}^2,\hat f_{out}^3\right)-\mathcal{F}\left(w,f_{in}^1,f_{in}^2,f_{out}^3\right)\right| \\
\le&4\,\pi\,C\,\hat\mu\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\mathbf{Lip}(G)\left(1+\mathbf{Lip}(q)\right) \\
&\times\int_0^T\int_0^t\sum_{i=1}^3\left\|\hat\rho^i(\tau)-\rho^i(\tau)\right\|_{\mathbf{L}^1(\mathcal{R}^i;\mathbb{R})}\mathrm{d}\tau\,\mathrm{d}t \\
\le&4\,\pi\,C\,\hat\mu\,\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\mathbf{Lip}(G)\left(1+\mathbf{Lip}(q)\right) \\
&\times\int_0^T\int_0^t\left(\sum_{i=1}^2\left\|\hat f_{in}^i-f_{in}^i\right\|_{\mathbf{L}^1([0,\tau]\times\mathcal{E}_{in}^i;\mathbb{R})}+\left\|\hat f_{out}^3-f_{out}^3\right\|_{\mathbf{L}^1([0,\tau]\times\mathcal{E}_{out}^3;\mathbb{R})}\right)\mathrm{d}\tau\,\mathrm{d}t && \text{[By (2D.1)]} \\
\le&\pi\,C\,\hat\mu\,\|p\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\mathbf{Lip}(G)\left(1+\mathbf{Lip}(q)\right)\,T^2 \\
&\times\left(\sum_{i=1}^2\left\|\hat f_{in}^i-f_{in}^i\right\|_{\mathbf{L}^1([0,T]\times\mathcal{E}_{in}^i;\mathbb{R})}+\left\|\hat f_{out}^3-f_{out}^3\right\|_{\mathbf{L}^1([0,T]\times\mathcal{E}_{out}^3;\mathbb{R})}\right),
\end{aligned}$$

which proves that the map $\mathcal{F}$, as in (5.4), is $\mathbf{L}^1$–Lipschitz continuous with respect to the inflows and outflow.

Combining together the above results concludes the proof. □

## 7. Conclusions

A first possible extension of the present results consists in tackling more general networks. As long as further merges are introduced, the present techniques can be easily extended, essentially thanks to the finite speed of propagation inherent to (3.3).

As soon as diverges are considered, the analytical approach depends on the specific modeling choice concerning drivers' route preferences, see [25, § 3.2] for more details.

We stress that the techniques here adopted to deal with 2D roads can be carried to the 3D case of interest, for instance, in blood flow model. Indeed, in a first approximation, one may assume that the pulsating of veins or arteries is assigned and then the 3D extension of Definition 3.1 comes straightforward, also in a time dependent setting.

The extension of (q) to comprise the so called *triangular* fundamental diagram is merely technical.

In the literature, other models deal with the 2D description of vehicular traffic. In [30], for instance, a diffusion term is added in the direction orthogonal to that of the main traffic. In [31], the second dimension is achieved through a limit on the number of lanes. Here, the modeling considers roads consisting of a single lane which, analytically, directly yields a purely hyperbolic equation and no limits are required. The extension to 2, or more, lanes involves mainly technical difficulties and requires a modeling of the lane change mechanism.

The treatment of the pollutant equation (1.2) is also amenable to further developments. The techniques here adopted are extensions of the classical *strong* ones in [9] to the *weak* case based on Definition 4.1, here introduced. Moreover, the recent results in [32] may allow to pass to Neumann boundary conditions, while remaining in the functional setting of $\mathbf{L}^1$.


## Acknowledgments

The authors thank the referees for their careful reading of the manuscript.

RMC and ER were partly supported by the GNAMPA 2023 project *Analytical Techniques in Biological, Fluid Dynamics and Vehicular Traffic Models*. RMC and ER acknowledge the PRIN 2022 project *Modeling, Control and Games through Partial Differential Equations* (D53D23005620006), funded by the European Union - Next Generation EU. ER was partly supported by the DISMI–UniMoRe FAR–DIP2022_CDR project.


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