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Intrinsic Hölder spaces for fractional kinetic operators

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Abstract. We introduce anisotropic Hölder spaces that are useful for studying the regularity theory for non-local kinetic operators $\mathscr{L}$, whose prototypical example is

$$\mathscr{L}u(t,x,v) = \int_{\mathbb{R}^d} \frac{C_{d,s}}{|v-v'|^{d+2s}}(u(t,x,v') - u(t,x,v))\mathrm{d}v' + \langle v, \nabla_x\rangle + \partial_t,$$

with $(t,x,v) \in \mathbb{R} \times \mathbb{R}^{2d}$. The Hölder spaces are defined in terms of an anisotropic distance relevant to the Galilean geometric structure on $\mathbb{R} \times \mathbb{R}^{2d}$, with respect to which the operator $\mathscr{L}$ is invariant. We prove an intrinsic Taylor-like formula, whose remainder is bounded in terms of the anisotropic distance of the Galilean structure. Our achievements naturally extend analogous known results for purely differential operators on Lie groups.

1. Introduction

We consider Hölder spaces and Taylor-like formulas that are useful for studying the regularity theory of the solutions to $\mathscr{L}u = f$, with $\mathscr{L}$ being a non-local kinetic operator of the form

$$\mathscr{L}u(t,x,v) = \int_{\mathbb{R}^d} K(t,x,v,v')(u(t,x,v') - u(t,x,v))\mathrm{d}v' + Yu, \qquad (1.1)$$

where $(t,x,v) \in \mathbb{R} \times \mathbb{R}^{2d}$,

$$Y = \langle v, \nabla_x\rangle + \partial_t, \qquad v \in \mathbb{R}^d. \qquad (1.2)$$

The integral part is of order $2s$, with $s \in]0,1[$. For example, kernels satisfying the following ellipticity condition

$$\frac{c^-}{|v-v'|^{d+2s}} \leq K(t,x,v,v') \leq \frac{c^+}{|v-v'|^{d+2s}}, \qquad (t,x,v,v') \in \mathbb{R} \times \mathbb{R}^{3d},$$

for some positive constants c^- and c^+, can be considered. Weaker conditions (in average), like in [8], could be considered too. Suitable symmetry conditions are typically

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required as well, such as $K(t,x,v,v+w) = K(t,x,v,v-w)$ or $K(t,x,v,w) = K(t,x,w,v)$, which induce properties typical of differential equations written in non-divergence or in divergence form, respectively (see the discussion in [7]). In [11] a more general cancelation condition is considered in place of the second symmetry assumption above. One notable particular instance of $\mathscr{L}$ is the fractional kinetic Fokker–Planck operator

$$\mathscr{L}_s = (-\Delta_v)^s + Y, \tag{1.3}$$

which is the operator in (1.1) with

$$K(t,x,v,v') = \frac{C_{d,s}}{|v-v'|^{d+2s}}, \tag{1.4}$$

for a suitable positive constant $C_{d,s}$. Note that $\mathscr{L}_s = (-\Delta_v)^s + Y$ is also related to the infinitesimal generator of an α-stable Lévy process, with $\alpha = 2s$.

In the purely diffusive setting, which can be seen as the limiting case $s = 1$, $\mathscr{L}_1 = -\Delta_v + Y$ is a differential hypoelliptic operator. This means that every distributional solution u to the equation $-\Delta_v u + Yu = f$ is a smooth function whenever f is smooth. Indeed, setting the primary vector fields

$$Z_i := \partial_{v_i}, \qquad i = 1, \dots, d, \tag{1.5}$$

we obtain that $\mathscr{L}_1$ writes in the form $\mathscr{L}_1 = -\sum_{i=1}^{d} Z_i^2 + Y$ and the system $Z_1, \dots, Z_d, Y$ satisfies the so-called Hörmander condition, namely

$$\operatorname{rank} \operatorname{Lie}(Z_1, \dots, Z_d, Y) = 2d + 1. \tag{1.6}$$

This is a straightforward consequence of

$$[\partial_{v_j}, Y] = \partial_{x_j}, \qquad j = 1, \dots, d. \tag{1.7}$$

We emphasize that the regularity properties of the Hörmander's operators depend on a non-Euclidean underlying structure (see the survey [2]). In the setting of the kinetic operator $\mathscr{L}$, this structure agrees with the Galilean translation (see (2.1) below).

In this work, we rely on the geometric structure introduced for the differential operator $\mathscr{L}_1$, in order to study the fractional operator $\mathscr{L}_s$ for $0 < s < 1$. We give a definition of *intrinsic Hölder spaces*, which extends the one introduced in [12], and we prove a Taylor polynomial approximation of a function f belonging to this Hölder space.

A clarification is in order regarding the role of our main results with respect to non-local operators. Our study focuses on the Hölder spaces which are the relevant ones for studying the regularity properties of kinetic-type operators with a diffusion of fractional degree. We do not provide results on the operators themselves. Notice that, in analogy with the parabolic geometry, prototype operators with a kernel in the form (1.4) are invariant with respect to the translations and dilations in (2.1)–(2.2), though the latter dictate the geometric structure underpinning the regularity theory of operators with more general kernels. We refer to [4,7,11] and to the references therein, for an overview on state-of-the-art results about kinetic non-local operators and their related equations.

We conclude this section with some remarks about the intrinsic Hölder spaces and the applications of our main results. These spaces are *anisotropic*, as the variables v and x in the Galilean group have a different role. Moreover, the definition of *intrinsic Hölder spaces* is based on the non-Euclidean quasi-distance (2.4) of the underlying Galilean group. Remark 1 contains a brief discussion on our definition of anisotropic Hölder spaces compared with other definitions that appear in the literature. In the purely differential setting, such Hölder spaces were studied by several authors, and a characterization relating the regularity along the vector fields $\partial_{v_1}, \ldots, \partial_{v_d}, Y$ to the existence of appropriate intrinsic Taylor formulas was given in [12]. We refer to the articles [3,5] for similar Taylor formulas on homogeneous groups. The Taylor approximation of a solution to a PDE is a useful tool in the proof of Schauder estimates. We refer to [16], where the regularity of classical solutions to degenerate Kolmogorov equations is obtained by using the method introduced in [17] for uniformly elliptic and parabolic equations. We also recall the article [6] where Taylor approximation results and Schauder estimates for kinetic equations were proved. In the fractional setting, Schauder estimates for the solutions to $\mathscr{L}u = 0$ have been recently proved in [8], and utilized in [7] to study the Boltzmann equation, in suitable Hölder spaces that take into account the intrinsic geometry of the Galilean group. We emphasize that the main results of this note provide an intrinsic characterization of the Hölder spaces considered in [7,8]. Indeed, the latter spaces consist (see [8, Definition 2.3]) of all the functions satisfying the Taylor formulas in our Theorems 1 and 2.

Our main results also apply to a nonlinear non-local version of (1.3), that is

$$\mathscr{L}_{s,p} = (-\Delta_v)^s_p + Y, \tag{1.8}$$

considered for $p \in]1, \infty[$ and $s \in]0, 1[$ in [1]. In this case, the kernel K in (1.1) also depends on the unknown function u and the term $(-\Delta_v)^s_p u$ takes the following form

$$\begin{aligned}&(-\Delta_v)^s_p u(t, x, v)\\ &= \int_{\mathbb{R}^d} \frac{C_{d,p,s}}{|v - v'|^{d+ps}} |u(t, x, v) - u(t, x, v')|^{p-2} \big(u(t, x, v) - u(t, x, v')\big) \mathrm{d}v'.\end{aligned}$$

Also notice that one could consider a Hörmander operator Y in (1.1) with a more general structure than the one in (1.2), like they did in [12] to cover a more general class of ultra-parabolic equations. However useful, this generalization would make the proofs more involved and the main ideas less transparent, so we prefer to state and prove our results only in the kinetic setting.

We finally remark that not only anisotropic spaces of Hölder continuous functions have been considered in literature for the study of kinetic equations of the form $\mathscr{L}u = f$. Indeed, [14] prove intrinsic Taylor expansion for anisotropic Sobolev–Slobodeckij spaces, and prove continuous embeddings into Lorentz and intrinsic Hölder spaces.

This article is organized as follows. Section 2 recalls the non-Euclidean geometry relevant to the kinetic operator $\mathscr{L}_1$. Note that the study of the non-local operator $\mathscr{L}_s$ relies on the same geometric structure. In Sect. 2, we present the definition of intrinsic Hölder spaces and the statement of Theorem 1, which are the main results of this note. Some examples illustrate the meaning of the definitions and of the main results. Section 3 contains some preliminaries necessary for the proof of the main results, which is provided in Sect. 4. Section 5 contains a local version of our main results, which generalizes Theorem 1 in that it applies to the broader geometric framework of *non-homogeneous Lie groups*.

2. Hölder spaces and Taylor polynomials

In this part, we fix some notation for the geometric structure on $\mathbb{R}^{2d+1}$, which will be used in this work. We recall that, remarkably, $\mathscr{L}_s$ in (1.3) has the property of being invariant with respect to left translations in the group $\left(\mathbb{R} \times \mathbb{R}^{2d}, \circ\right)$, where the non-commutative group law "$\circ$" is defined by

$$\begin{aligned} z_1 \circ z_2 &= (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)\,, \\ z_1 &= (t_1, x_1, v_1), z_2 = (t_2, x_2, v_2) \in \mathbb{R} \times \mathbb{R}^{2d}. \end{aligned} \tag{2.1}$$

Precisely, we have

$$\left(\mathscr{L}_s u^{(z_1)}\right)(z_2) = (\mathscr{L}_s u)(z_1 \circ z_2).$$

where

$$u^{(z_1)}(z_2) = u(z_1 \circ z_2).$$

This translation property is often referred to as "Galilean" change of coordinate and is very useful in kinetic theory. A systematic study of the PDEs theory on this group started in [9] in the limiting case $s = 1$. Notice that $(\mathbb{R} \times \mathbb{R}^{2d}, \circ)$ is a group with the identity and the inverse elements

$$\mathrm{Id} = (0, 0, 0), \qquad (t, x, v)^{-1} = (-t, tv - x, -v)\,.$$

Moreover, $\mathscr{L}_s$ *is homogeneous of degree* $\vartheta = 2s$ with respect to the dilations $(D_\lambda)_{\lambda>0}$ on $\mathbb{R} \times \mathbb{R}^{2d}$ given by

$$D_\lambda = \mathrm{diag}\left(\lambda^{\vartheta}, \lambda^{\vartheta+1} I_d, \lambda I_d\right), \tag{2.2}$$

where I_d is the $(d \times d)$ identity matrix, i.e.,

$$\left(Y u^{(\lambda)}\right)(z) = \lambda^{\vartheta} (Y u)(D_\lambda z), \qquad z = (t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d},\ \lambda > 0,$$

where

$$u^{(\lambda)}(z) = u\left(D_\lambda(z)\right).$$

Note that the exponent ϑ in (2.2) equals ps in the case of non-local operator $\mathscr{L}_{s,p}$ in (1.8), hence we do not impose any restriction on the choice of $\vartheta \in]0, +\infty[$.

In the sequel, we will say that Y has formal degree ϑ and that the vector fields $Z_1, \dots, Z_d$ in (1.5) have formal degree 1, in accordance with the terminology adopted in [5, Section 2.4].

Notice that the triplet $\left(\mathbb{R} \times \mathbb{R}^{2d}, \circ, D_\lambda\right)$ forms a homogeneous group. Indeed, it is well defined the so-called homogeneous norm:

$$\|(t, x, v)\| = |t|^{\frac{1}{\vartheta}} + |x|^{\frac{1}{\vartheta+1}} + |v|, \tag{2.3}$$

and we consider the quasi-distance

$$d(z_1, z_2) := \left\| z_2^{-1} \circ z_1 \right\|, \qquad z_1, z_2 \in \mathbb{R} \times \mathbb{R}^{2d}. \tag{2.4}$$

The following properties directly follow from the definition of the quasi-distance

$$d(D_\lambda(z_1), D_\lambda(z_2)) = \lambda\, d(z_1, z_2), \qquad d(z \circ z_1, z \circ z_2) = d(z_1, z_2)$$

for every $z, z_1, z_2 \in \mathbb{R} \times \mathbb{R}^{2d}$ and for every $\lambda > 0$. Note that d is said *quasi-distance* as the following weaker form of triangular inequality holds true: there exists a constant $\kappa \geq 1$ such that

$$d(z_1, z_3) \leq \kappa \left(d(z_1, z_2) + d(z_2, z_3)\right), \qquad z_1, z_2, z_3 \in \mathbb{R} \times \mathbb{R}^{2d}.$$

We recall that in [8] it is considered the *equivalent* distance for $z_1, z_2 \in \mathbb{R} \times \mathbb{R}^{2d}$:

$$\min_{w \in \mathbb{R}^d} \left\{ \max \left(|t_1 - t_2|^{\frac{1}{\vartheta}}, |x_1 - x_2 - w(t_1 - t_2)|^{\frac{1}{1+\vartheta}}, |v_1 - w|, |v_2 - w| \right) \right\}.$$

We next introduce the notions of intrinsic regularity and intrinsic Hölder space. Let X be a Lipschitz vector field on $\mathbb{R} \times \mathbb{R}^{2d}$. For any $z \in \mathbb{R} \times \mathbb{R}^{2d}$, we denote by $\tau \mapsto e^{\tau X}(z)$ the integral curve of X defined as the unique solution to

$$\begin{cases} \frac{d}{d\tau} e^{\tau X}(z) = X\left(e^{\tau X}(z)\right), & \tau \in \mathbb{R}, \\ e^{\tau X}(z)|_{\tau=0} = z. \end{cases}$$

In particular, for a vector $h \in \mathbb{R}^d$ we set $Z_h := h_1 Z_1 + \cdots + h_d Z_d$ and we find

$$e^{\tau Z_h}(t, x, v) = (t, x, v + \tau h), \qquad e^{\tau Y}(t, x, v) = (t + \tau, x + \tau v, v), \quad \tau \in \mathbb{R}, \tag{2.5}$$

for any $(t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$.

Next we recall the general notion of Lie differentiability and Hölder regularity.

Definition 1. Let X be a Lipschitz vector field and u be a real-valued function defined in a neighborhood of $z \in \mathbb{R} \times \mathbb{R}^{2d}$. We say that u is *X-differentiable* in z if the function $\tau \mapsto u\left(e^{\tau X}(z)\right)$ is differentiable in 0. We will refer to the function $z \mapsto Xu(z) := \frac{d}{d\tau} u\left(e^{\tau X}(z)\right)\big|_{\tau=0}$ as *X-Lie derivative of u*, or simply *Lie derivative of u* when the dependence on the field X is clear from the context.

Definition 2. Let $u : \mathbb{R} \times \mathbb{R}^{2d} \to \mathbb{R}$. Then, for $a \in]0, 1]$, we say that $u \in C^{\alpha}_{Z_i}$, $i = 1, \ldots, d$, if

$$\|u\|_{C^{\alpha}_{Z_i}} := \sup_{\substack{z \in \mathbb{R} \times \mathbb{R}^{2d} \\ \tau \in \mathbb{R} \setminus \{0\}}} \frac{\left| u\left(e^{\tau Z_i}(z)\right) - u(z) \right|}{|\tau|^{\alpha}} < \infty.$$

Moreover, for $\alpha \in]0, \boldsymbol{\vartheta}]$, we say that $u \in C^{\alpha}_{Y}$ if

$$\|u\|_{C^{\alpha}_{Y}} := \sup_{\substack{z \in \mathbb{R} \times \mathbb{R}^{2d} \\ \tau \in \mathbb{R} \setminus \{0\}}} \frac{\left| u\left(e^{\tau Y}(z)\right) - u(z) \right|}{|\tau|^{\frac{\alpha}{\boldsymbol{\vartheta}}}} < \infty.$$

Let us introduce the set

$$\mathcal{I} = \{\alpha \in \mathbb{R} \mid \alpha = k + j\boldsymbol{\vartheta} \text{ with } k, j \in \mathbb{N}_0\}.$$

As we shall see in the sequel, α's in $\mathcal{I}$ represent the regularity indices for which there is a *jump* in the regularity of a function u, meaning that new derivatives along Z_i or/and Y appear. In particular, the following statements are true.

- If $\alpha \in]0, 1 \wedge \boldsymbol{\vartheta}]$ there are no derivatives with respect to any of the vector fields $Z_1, \ldots, Z_d$ or Y.
- If $\alpha \in]1 \wedge \boldsymbol{\vartheta}, 1 \vee \boldsymbol{\vartheta}]$, there are two cases: if $\boldsymbol{\vartheta} < 1$, then only derivatives along Y appear, up to order j with $\boldsymbol{\vartheta} j < \alpha$; if $\boldsymbol{\vartheta} > 1$, then only derivatives along the vector fields $Z_1, \ldots, Z_d$ appear, up to order k with $k < \alpha$. Of course, if $\boldsymbol{\vartheta} = 1$, the interval is empty and all the fields have the same formal degree (the gradation is the same as in the Heisenberg group).
- If $\alpha > 1 \vee \boldsymbol{\vartheta}$, then there exist derivatives along $Z_1, \ldots, Z_d$ and Y.
- In Theorem 1 we show that if $\alpha > 1 + \boldsymbol{\vartheta}$, then the derivatives ∂_{x_j} also appear for $i = 1, \ldots, d$.

Now we define the intrinsic Hölder spaces on the homogeneous group $\left(\mathbb{R} \times \mathbb{R}^{2d}, \circ, D_{\lambda}\right)$, by extending the definitions of Hölder spaces given in [12]. Namely, our procedure is recursive with respect to the (ordered) indices in $\mathcal{I}$. In particular, in the second step, which is for $\alpha \in]1 \wedge \boldsymbol{\vartheta}, 1 \vee \boldsymbol{\vartheta}]$, the definition splits in two different cases depending on whether $\boldsymbol{\vartheta} < 1$ or $\boldsymbol{\vartheta} > 1$. If $\boldsymbol{\vartheta} = 1$, $\alpha \in]1 \wedge \boldsymbol{\vartheta}, 1 \vee \boldsymbol{\vartheta}]$ is empty and one moves on to the third step.

Definition 3. Let $u : \mathbb{R} \times \mathbb{R}^{2d} \to \mathbb{R}$ and $\alpha > 0$. Then,

(i) if $\alpha \in]0, 1 \wedge \boldsymbol{\vartheta}]$, $u \in C^{\alpha}$ if the semi-norm

$$\|u\|_{C^{\alpha}} := \|u\|_{C^{\alpha}_{Y}} + \sum_{i=1}^{d} \|u\|_{C^{\alpha}_{Z_i}}$$

is finite;

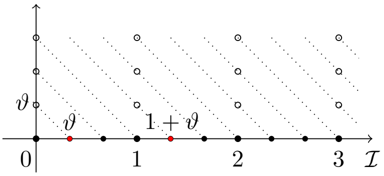

Figure 1. The set $\mathcal{I}$ for $\vartheta = 1/3$

(ii) if $\alpha \in]1 \wedge \vartheta, 1 \vee \vartheta]$, $u \in C^\alpha$ if the semi-norm

$$\|u\|_{C^\alpha} := \begin{cases} \|Yu\|_{C^{\alpha-\vartheta}} + \sum_{i=1}^d \|u\|_{C^\alpha_{Z_i}}, & \text{if } \vartheta < 1 \\ \|u\|_{C^\alpha_Y} + \sum_{i=1}^d \|Z_i u\|_{C^{\alpha-1}}, & \text{if } \vartheta > 1 \end{cases}$$

is finite. If $\vartheta = 1$, $\alpha \in]1 \wedge \vartheta, 1 \vee \vartheta]$ is empty and this case can be skipped;

(iii) if $\alpha > 1 \vee \vartheta$, $u \in C^\alpha$ if the semi-norm

$$\|u\|_{C^\alpha} := \|Yu\|_{C^{\alpha-\vartheta}} + \sum_{i=1}^d \|Z_i u\|_{C^{\alpha-1}}$$

is finite.

Example 1. Figure 1 describes the pairs $(k, j\vartheta)$ with $k, j \in \mathbb{N}_0$ and $\vartheta = 1/3$. The filled dots form the set $\mathcal{I}$, the points ϑ and $1 + \vartheta$ are highlighted.

Let us consider $u \in C^\alpha$. Then, we have:

- If $\alpha \in]0, 1/3]$, there are no derivatives with respect to either Z_i or Y.
- If $\alpha \in]1/3, 2/3]$, the Lie derivative Yu exists and belongs to $C^{\alpha-1/3}$. Furthermore, u belongs to $C^\alpha_{Z_i}$. Note that the definition is well-posed. Indeed, the index $\alpha - 1/3 \in]0, 1/3]$ and thus the space $C^{\alpha-1/3}$ has already been defined; also $\alpha \leq 1$ and thus $C^\alpha_{Z_i}$ is defined too.
- If $\alpha \in]2/3, 1]$, Yu belongs to $C^{\alpha-1/3}$ where $\alpha - 1/3 \in]1/3, 2/3]$. In particular there exists $Y^2u \in C^{\alpha-2/3}$.
- If $\alpha \in]1, 4/3]$, there exist the derivatives along the fields Z_i, which are the Euclidean derivatives $\partial_{v_1}u, \ldots, \partial_{v_d}u$. Furthermore, such derivatives belong to $C^{\alpha-1}$ where $\alpha - 1 \in]0, 1/3]$. Also, the Lie derivatives Yu, Y^2u and Y^3u exist and belong to $C^{\alpha-1/3}$, $C^{\alpha-2/3}$ and $C^{\alpha-1}$, respectively.
- If $\alpha \in]4/3, 5/3]$, then $Y^4u \in C^{\alpha-4/3}$, and $\partial_{v_i}u \in C^{\alpha-1}$. Moreover, the mixed derivatives $\partial_{v_i}Yu$ and $Y\partial_{v_i}u$ exist and belong to $C^{\alpha-4/3}$, and so do the commutators $[\partial_{v_i}, Y]u$. Indeed we have $Yu \in C^{\alpha-1/3}$ where $\alpha - 1/3 \in]1, 4/3]$, and $\partial_{v_i}u \in C^{\alpha-1}$ where $\alpha - 1 \in]1/3, 2/3]$.

Once more, the last step is crucial as it can be proved that $[\partial_{v_i}, Y]u = \partial_{x_i}u$ (see Theorem 1). By induction, there exist the n-th order Euclidean derivatives with respect to x so long as $\alpha > n(\vartheta + 1)$.

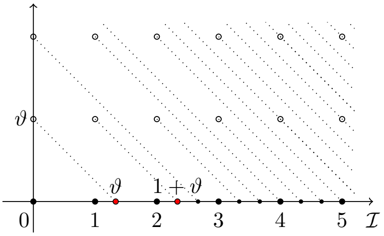

Figure 2. The set $\mathcal{I}$ for $\boldsymbol{\vartheta} = 4/3$

Example 2. Figure 2 describes the pairs $(k, j\boldsymbol{\vartheta})$ with $k, j \in \mathbb{N}_0$ and $\boldsymbol{\vartheta} = 4/3$. The filled dots form the set $\mathcal{I}$.

Let us consider $u \in C^\alpha$. Then, we have:

- If $\alpha \in]0, 1]$, there are no derivatives with respect to either Z_i or Y.
- If $\alpha \in]1, 4/3]$, the 1st order derivatives $Z_i u = \partial_{v_i} u$ exist and belong to $C^{\alpha-1}$. Plus, $u \in C^\alpha_Y$.
- If $\alpha \in]4/3, 2]$, the Lie derivative Yu exists and belongs to $C^{\alpha-4/3}$. Moreover, $\partial_{v_1} u, \dots, \partial_{v_d} u \in C^{\alpha-1}$.
- If $\alpha \in]2, 7/3]$, the 2nd order derivatives $\partial_{v_i v_j} u$, for $i, j = 1, \dots, d$, exist and belong to $C^{\alpha-2}$. Furthermore, $\partial_{v_i} u \in C^{\alpha-1}_Y$ and $Yu \in C^{\alpha-4/3}$.
- If $\alpha \in]7/3, 8/3]$, then the mixed derivatives $\partial_{v_i} Yu$ and $Y\partial_{v_i} u$ exist and belong to $C^{\alpha-7/3}$, and so do the commutators $[\partial_{v_i}, Y]u$. Also, $Yu \in C^{\alpha-4/3}$ and $\partial_{v_i v_j} u \in C^{\alpha-2}$.

In the sequel, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$ will denote a multi-index. As usual

$$|\beta| := \sum_{j=1}^d \beta_j \quad \text{and} \quad \beta! := \prod_{j=1}^d (\beta_j!)$$

are called the length and the factorial of β, respectively. Moreover, for any $x \in \mathbb{R}^d$, we set

$$x^\beta = x_1^{\beta_1} \dots x_d^{\beta_d} \quad \text{and} \quad \partial^\beta = \partial_x^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_d}^{\beta_d}.$$

Finally, for $u \in C^\alpha$ we let $T_\alpha u(z_0, \cdot)$ its intrinsic Taylor polynomial around $z_0 = (t_0, x_0, v_0)$ defined as

$$T_\alpha u(z_0; z) := \sum_{0 \le \boldsymbol{\vartheta} k + (1+\boldsymbol{\vartheta})|\gamma| + |\beta| < \alpha} \frac{Y^k \partial_v^\beta \partial_x^\gamma u(z_0)}{k!\, \gamma!\, \beta!} (t - t_0)^k \big(x - x_0 - (t - t_0) v_0\big)^\gamma (v - v_0)^\beta.$$

Example 3. For $d = 1$ and $\boldsymbol{\vartheta} = 4/3$ as in Example 2, we have

– If $\alpha \in]0, 1]$, then we set $\alpha_1 := 1$ and we have $T_\alpha u(z_0; z) = T_{\alpha_1} u(z_0; z) = u(z_0)$;

- If $\alpha \in]1, 4/3]$ ($\alpha_2 := 4/3$), then $T_\alpha u(z_0; z) = T_{\alpha_2} u(z_0; z) = T_{\alpha_1} u(z_0; z) + \big(\partial_v u(z_0)\big)(v - v_0)$;
- If $\alpha \in]4/3, 2]$ ($\alpha_3 := 2$), then $T_\alpha u(z_0; z) = T_{\alpha_3} u(z_0; z) = T_{\alpha_2} u(z_0; z) + \big(Y u(z_0)\big)(t - t_0)$;
- If $\alpha \in]2, 7/3]$ ($\alpha_4 := 7/3$), then $T_\alpha u(z_0; z) = T_{\alpha_4} u(z_0; z) = T_{\alpha_3} u(z_0; z) + \frac{1}{2}\big(\partial_v^2 u(z_0)\big)(v - v_0)^2$;
- If $\alpha \in]7/3, 8/3]$, then $T_\alpha u(z_0; z) = T_{\alpha_4} u(z_0; z) + \big(\partial_x u(z_0)\big)(x - x_0 - (t - t_0)v_0) + \big(Y\partial_v u(z_0)\big)(t - t_0)(v - v_0)$.

Theorem 1. *For any $\alpha > 0$ and for any $u \in C^\alpha$ the following statements are true:*

HÖLDER SPACES CHARACTERIZATION. *For any $k \in \mathbb{N}_0$ and $\gamma, \beta \in \mathbb{N}_0^d$ with $0 \leq \vartheta k + (1 + \vartheta)|\gamma| + |\beta| < \alpha$, the derivatives $Y^k \partial_v^\beta \partial_x^\gamma u$ exist and*

$$\|Y^k \partial_v^\beta \partial_x^\gamma u\|_{C^{\alpha - \vartheta k - (1+\vartheta)|\gamma| - |\beta|}} \leq \|u\|_{C^\alpha}. \tag{2.6}$$

TAYLOR FORMULA. *There exists a positive constant $c > 0$, only dependent on α, such that*

$$|u(z) - T_\alpha u(z_0; z)| \leq c\|u\|_{C^\alpha} \|z_0^{-1} \circ z\|^\alpha, \qquad z, z_0 \in \mathbb{R} \times \mathbb{R}^{2d}. \tag{2.7}$$

Remark 1. If $\alpha \in]0, 1 \wedge \vartheta]$, estimate (2.7) restores the definition of Hölder continuous function given in Definition 1.2 in [15]

$$|u(z) - u(z_0)| \leq c\|u\|_{C^\alpha} \left\|z_0^{-1} \circ z\right\|^\alpha, \qquad z, z_0 \in \mathbb{R} \times \mathbb{R}^{2d}.$$

For a comparison between intrinsic and Euclidean Hölder continuity we refer to Proposition 2.1 in [15].

Remark 2. By (2.6)–(2.7), it is straightforward to see that $\partial_v^\beta \partial_x^\gamma u$ and $\partial_x^\gamma \partial_v^\beta u$ are both continuous on $\mathbb{R} \times \mathbb{R}^{2d}$ for any $\gamma, \beta \in \mathbb{N}_0^d$ with $0 \leq (1 + \vartheta)|\gamma| + |\beta| < \alpha$, hence they agree. By contrast, Y^k does not commute with $\partial_v^\beta u$.

3. Preliminaries

The method of the proof relies on the construction of a finite sequence of integral curves of the vector fields $Y, Z_1, \ldots, Z_d$ which steer a point $z_0 = (t_0, x_0, v_0)$ to any other point $z = (t, x, v)$. We then rely on the usual Taylor expansion of the functions $\tau \mapsto u\left(e^{\tau Y}(z_0)\right)$ and $\tau \mapsto u\left(e^{\tau Z_h}(z_0)\right)$ introduced in (2.5), to obtain a good approximation of u near z_0. The approximation of u along the integral curves of the commutators $[Y, Z_i]$ is obtained by using a rather classical technique from control theory. We first consider the approximation along the integral curve $e^{\tau Y}(z)$.

Remark 3. Let $n \in \mathbb{N}_0$, $\gamma \in]0, \vartheta]$ and $u \in C^{\vartheta n + \gamma}$. Then, by Definition 3, we have $Y^j u \in C_Y^{\vartheta(n-j)+\gamma}$ with $j = 1, \ldots, n$. Therefore, by mean-value theorem along the

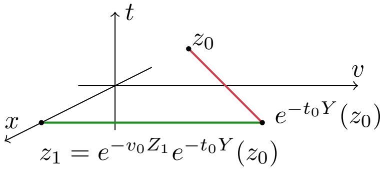

Figure 3. The path $e^{-v_0Z_1}e^{-t_0Y}(z_0)$

vector field Y, for any $z = (t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$ and $\tau \in \mathbb{R}$, there exists $\delta \in]0, 1[$ such that

$$u\big(e^{\tau Y}(z)\big) - u(z) - \sum_{j=1}^{n} \frac{\tau^j}{j!} Y^j u(z) = \frac{\tau^n}{n!}\Big(Y^n u\big(e^{\delta\tau Y}(z)\big) - Y^n u(z)\Big),$$

and thus, Definition 2 yields

$$\begin{aligned}\Big|u\big(e^{\tau Y}(z)\big) - T_{\vartheta n+\gamma} u\big(z; e^{\tau Y}(z)\big)\Big| &= \Big|u\big(e^{\tau Y}(z)\big) - u(z) - \sum_{j=1}^{n} \frac{\tau^j}{j!} Y^i u(z)\Big| \\ &\leq \frac{1}{n!}\|u\|_{C^{\vartheta n+\gamma}} |\tau|^{n+\frac{\gamma}{\vartheta}},\end{aligned}$$

for $\tau \in \mathbb{R}$ and $z \in \mathbb{R} \times \mathbb{R}^{2d}$.

Since the vector fields $Z_1, \ldots, Z_d$ have constant coefficients, the usual Euclidean Taylor theorem with Lagrange reminder plainly gives the following result.

Remark 4. Let $n \in \mathbb{N}_0$, $\gamma \in]0, 1]$ and $u \in C^{n+\gamma}$. Then, by Definition 3, we have $\partial_v^\beta u \in C_{Z_i}^{\gamma+n-|\beta|}$ for any $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq n$ and $i = 1, \ldots, d$. Therefore, recalling Definition 2, mean-value theorem yields

$$\Big|u(t, x, v + h) - T_{n+\gamma} u\big(z; t, x, v + h\big)\Big| \leq \frac{1}{n!}\|u\|_{C^{n+\gamma}} |h|^{n+\gamma},$$

for $z = (t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$ and $h \in \mathbb{R}^d$.

In view of Theorem 1, we consider two points $z_0, z \in \mathbb{R}^{1+2d}$ and we note that Remarks 3 and 4 provide us with a bound of $|u(z_0) - u(z_1)|$ in terms of $\|z_0^{-1} \circ z\|$, where z_1 is a specific point of $\mathbb{R}^{1+2d}$ whose components t and v agree with the components of z. The following picture illustrates the situation in the case $d = 1$ and $z = (0, 0, 0)$.

We finally recover the regularity of the function u with respect to the variable x by the usual trajectory defined as concatenations of integral curves of $Z_1, \ldots, Z_m$ and Y, as Fig. 4 shows.

The following proposition provides us with a quantitative estimate of the increment of a function $u \in C^\alpha$ with respect to the variable x.

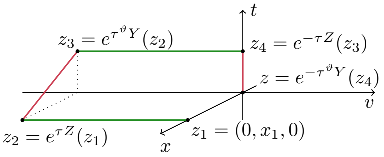

Figure 4. The path $e^{-\tau^\vartheta Y} e^{-\tau Z} e^{\tau^\vartheta Y} e^{\tau Z}(z_1)$

Proposition 1. *Let* $u \in C^\alpha$ *with* $\alpha \in]0, 1+\boldsymbol{\vartheta}]$, *then*

$$|u(t,x,v) - u(t,x+h,v)| \le c\, \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\boldsymbol{\vartheta}+1}}, \tag{3.1}$$

for $z = (t,x,v) \in \mathbb{R} \times \mathbb{R}^{2d}$ *and* $h \in \mathbb{R}^d$.

Proof. Assume $h \neq 0$ and set $w := h/|h|$, $\tau := |h|^{\frac{1}{\boldsymbol{\vartheta}+1}}$. Recall the notation in (2.5), put

$$z_2 = e^{\tau Z_w}(t,x,v), \qquad z_3 = e^{\tau^{\boldsymbol{\vartheta}} Y}(z_2) \qquad z_4 = e^{-\tau Z_w}(z_3), \tag{3.2}$$

and note that

$$e^{-\tau^{\boldsymbol{\vartheta}} Y}(z_4) = \big(t, x + \tau^{\boldsymbol{\vartheta}+1} w, v\big) = \big(t, x+h, v\big). \tag{3.3}$$

With this notation, we express the left hand side of (3.1) as follows

$$\begin{aligned}
u(t,x+h,v) - u(t,x,v) = {} & \boxed{u(t,x+h,v) - u(z_4) - \sum_{\boldsymbol{\vartheta} \le \boldsymbol{\vartheta} j < \alpha} \frac{(-\tau^{\boldsymbol{\vartheta}})^j}{j!} Y^j u(z_4)}_{(1)} \\
& + \boxed{u(z_4) - u(z_3) - \sum_{1 \le k < \alpha} \frac{(-\tau)^k Z_w^k u(z_3)}{k!}}_{(2)} \\
& + \boxed{u(z_3) - u(z_2) + \sum_{\boldsymbol{\vartheta} \le \boldsymbol{\vartheta} j < \alpha} \frac{(-\tau^{\boldsymbol{\vartheta}})^j}{j!} Y^j u(z_3)}_{(3)} \\
& + \boxed{u(z_2) - u(t,x,v) + \sum_{1 \le k < \alpha} \frac{(-\tau)^k}{k!} Z_w^k u(z_2)}_{(4)} \\
& + \boxed{\sum_{\boldsymbol{\vartheta} \le \boldsymbol{\vartheta} j < \alpha} \frac{(-\tau^{\boldsymbol{\vartheta}})^j}{j!} \Big(Y^j u(z_4) - Y^j u(z_3)\Big)}_{(5)} \\
& + \boxed{\sum_{1 \le k < \alpha} \frac{(-\tau)^k}{k!} \Big(Z_w^k u(z_3) - Z_w^k u(z_2)\Big)}_{(6)} \\
& =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}$$

where the indices j, k in the above summations are non-negative integers. Note that if $\alpha \le \vartheta$, then no derivatives $Y^j u$ appear in I_1, I_3 and the term I_5 does not appear. Thus, in the following we consider separately this case and $\vartheta < \alpha \le 1 + \vartheta$. Analogously, if $\alpha \le 1$, no derivatives of the form $Z_w^k u$ appear in I_2, I_4. We claim that, in every case, the terms $I_1, \ldots, I_6$ are bounded by $\|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}$, up to multiplying by a positive constant.

Consider first I_1, I_3 and I_5 with $\vartheta < \alpha \le 1 + \vartheta$. We apply Remark 3 to I_1 and I_3, and we find

$$|I_1|, |I_3| \le \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}. \tag{3.4}$$

Moreover, $\alpha \le 1 + \vartheta$ also yields $Y^j u \in C_{Z_i}^{\alpha - j\vartheta}$, with $0 < \alpha - j\vartheta \le 1$ for every j such that $\vartheta \le \vartheta j < \alpha$. Then

$$|\tau|^{j\vartheta} \left| Y^j u(z_4) - Y^j u(z_3) \right| \le \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}, \qquad \vartheta \le \vartheta j < \alpha,$$

because of the very definition of C^α space. As a consequence,

$$|I_5| \le e\, \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}. \tag{3.5}$$

If $0 < \alpha \le \vartheta$ instead, we set $I_5 = 0$, so that (3.5) trivially holds. Moreover, (3.4) follows again from the definition of C^α. In both cases (3.4) and (3.5) hold.

The argument for I_2, I_4 and I_6 is analogous. If $1 < \alpha \le 1 + \vartheta$, then Remark 4 yields

$$|I_2|, |I_4| \le \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}, \tag{3.6}$$

and $\alpha \le \vartheta + 1$ implies $Z_w^k u \in C_Y^{\alpha - k}$, with $0 < \alpha - k \le \vartheta$ for every $k \in \mathbb{N}$ such that $k < \alpha$. Then

$$|\tau|^{k} \left| Z_w^k u(z_2) - Z_w^k u(z_3) \right| \le \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}, \qquad 1 \le k < \alpha,$$

and this inequality yields

$$|I_6| \le e\, \|u\|_{C^\alpha} |h|^{\frac{\alpha}{\vartheta+1}}. \tag{3.7}$$

If $0 < \alpha \le 1$, we set $I_6 = 0$ and the bound (3.6) holds by the definition of C^α.

The conclusion of the proof of (3.1) is then a direct consequence of (3.4)–(3.5) and (3.6)–(3.7). □

4. Proofs

The proof of Theorem 1 is divided into two steps:

1. (preliminary result) assuming true the first statement of Theorem 1 for some $\alpha > 0$, we prove that also the second statement is true for the same α;
2. we prove the first statement of Theorem 1 for a general $\alpha > 0$, by means of a suitable induction procedure.

4.1. The preliminary result

We assume the first statement of Theorem 1 to be true for fixed $\alpha > 0$ and prove the second one.
Step I: We fix

$$z_1 = (t_1, x_1, v_1) = e^{(t-t_0)Y}(z_0) = (t, x_0 + (t-t_0)v_0, v_0)$$

and we prove that

$$|u(z) - T_\alpha u(z_1, z)| \le c\|u\|_{C^\alpha}\big(|x - x_1|^{\frac{1}{\vartheta+1}} + |v - v_1|\big)^\alpha = c\|u\|_{C^\alpha}\,\|z_1^{-1} \circ z\|^\alpha. \tag{4.1}$$

Set $z_2 := (t_1, x_1, v)$. We rearrange the first term as

$$\begin{aligned} u(z) - T_\alpha u(z_1, z) = {} & \boxed{u(z) - T_\alpha u(z_2, z)}_{(1)} \\ & + \boxed{T_\alpha u(z_2, z) - T_\alpha u(z_1, z)}_{(2)} =: I_1 + I_2. \end{aligned}$$

We have

$$I_1 = u(z) - \sum_{(1+\vartheta)|\gamma|<\alpha} \frac{1}{\gamma!}\partial_x^\gamma u(z_2)(x - x_1)^\gamma.$$

By assumption, for any multi-index γ in the summation above with highest order, namely $|\gamma| = \lceil \alpha/(1+\vartheta) \rceil - 1$, we have $\partial_x^\gamma u \in C^{\alpha-(1+\vartheta)|\gamma|}$. Therefore, Proposition 1 together with Euclidean Taylor formula yield

$$|I_1| \le c\,\|u\|_{C^\alpha}\,|x - x_1|^{\frac{\alpha}{1+\vartheta}} \le c\,\|u\|_{C^\alpha}\,\|z_1^{-1} \circ z\|^\alpha.$$

For the second term, we have

$$I_2 = \sum_{(1+\vartheta)|\gamma|<\alpha} \frac{1}{\gamma!}(x - x_1)^\gamma \big(\partial_x^\gamma u(z_2) - T_{\alpha-(1+\vartheta)|\gamma|}(\partial_x^\gamma u)(z_1, z_2)\big).$$

By assumption $\partial_x^\gamma u \in C^{\alpha-(1+\vartheta)|\gamma|}$, and thus, by applying Remark 4 with $h = v - v_0$ to each term in the summation we obtain

$$|I_2| \le c\,\|u\|_{C^\alpha} \sum_{(1+\vartheta)|\gamma|<\alpha} |x - x_1|^\gamma |v - v_0|^{\alpha-(1+\vartheta)|\gamma|} \le c\,\|u\|_{C^\alpha}\,\|z_1^{-1} \circ z\|^\alpha.$$

Step II: we conclude the proof of (2.7). Note that

$$\begin{aligned} z_1^{-1} \circ z &= (0, x - x_0 - (t-t_0)v_0, v - v_0), \\ z_0^{-1} \circ z &= (t - t_0, x - x_0 - (t-t_0)v_0, v - v_0), \end{aligned}$$

so that $\|z_1^{-1} \circ z\| \le \|z_0^{-1} \circ z\|$. Therefore, by (4.1), we have

$$\begin{aligned} |u(z) - T_\alpha(z_0, z)| &= |u(z) - T_\alpha(z_1, z)| + |T_\alpha(z_1, z) - T_\alpha(z_0, z)| \\ &\le c\, \|u\|_{C^\alpha} \|z_0^{-1} \circ z\|^\alpha + |T_\alpha(z_1, z) - T_\alpha(z_0, z)|. \end{aligned}$$

Rearranging the Taylor polynomials, we can write

$$\begin{aligned} &T_\alpha(z_1, z) - T_\alpha(z_0, z) \\ &\quad = \sum_{(1+\vartheta)|\gamma|+|\beta|<\alpha} \frac{1}{\gamma!\,\beta!} \Big(\partial_x^\gamma \partial_v^\beta u(z_1) - T_{\alpha-(1+\vartheta)|\gamma|-|\beta|}(\partial_x^\gamma \partial_v^\beta u)(z_0, z_1) \Big) \\ &\qquad \times \big(x - x_0 - (t - t_0) v_0 \big)^\gamma (v - v_0)^\beta. \end{aligned}$$

By Remarks 2 and 3, we obtain

$$|T_\alpha(z_1, z) - T_\alpha(z_0, z)| \le c\, \|u\|_{C^\alpha} \|z_0^{-1} \circ z\|^\alpha.$$

This concludes the proof of the second statement of Theorem 1, assuming that the first one holds true.

4.2. Proof of Theorem 1

If $\alpha \le 1 + \vartheta$, then (2.6) does not contain derivatives with respect to x, thus it stems from Definition 3.

We prove (2.6) for $\alpha > 1 + \vartheta$. Notice that it is enough to show that, for any $i = 1, \dots, d$, we have

$$\partial_{x_i} u = [Z_i, Y] u. \tag{4.2}$$

Indeed, with (4.2) at hand,

$$\partial_{x_i} u \in C^{\alpha - (\vartheta + 1)}$$

and the whole (2.6) follows, once more, by Definition 3, and Remark 2, combined with a plain induction argument.

To show (4.2), we consider any $\alpha > 1 + \vartheta$, we let $n := \lceil \frac{\alpha}{\vartheta+1} \rceil - 1$, so that $\alpha \in]n(1 + \vartheta), (n + 1)(1 + \vartheta)]$, and we assume (2.6) true for $\bar\alpha := \alpha - (\vartheta + 1)$. We claim that

$$\frac{u(t, x + \delta\, e_i, v) - u(t, x, v)}{\delta} \to [Z_i, Y] u(t, x, v) \quad \text{as} \quad \delta \to 0, \tag{4.3}$$

for every $(t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$. Here e_i denotes the i-th element of the canonical basis of $\mathbb{R}^d$. To prove (4.3), we rely on an argument similar to that used in the proof of Proposition 1. We fix $(t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$, $i \in \{1, \dots, d\}$, $\delta \neq 0$, and we recall the notation in (2.5). If $\delta > 0$ we set $w := e_i$, otherwise we set $w := -e_i$. For simplicity we only consider the case $\delta > 0$, the case $\delta < 0$ being completely analogous. Set $\tau := \delta^{\frac{1}{1+\vartheta}}$,

$$\begin{aligned} z_2 &:= e^{\tau Z_w}(t,x,v) = e^{\tau Z_i}(t,x,v), \\ z_3 &:= e^{\tau^{\vartheta} Y}(z_2) \qquad z_4 := e^{-\tau Z_w}(z_3) = e^{-\tau Z_i}(z_3), \end{aligned}$$

and note that we have

$$e^{-\tau^{\vartheta} Y}(z_4) = \big(t, x + \tau^{\vartheta+1} e_i, v\big) = \big(t, x + \delta\, e_i, v\big).$$

We proceed as in the proof of Proposition 1, the unique difference being that $\alpha > \vartheta + 1$ here. We have

$$\begin{aligned} u(t,x+\delta\, e_i,v) - u(t,x,v) = {} & \boxed{u(t,x+\delta\, e_i,v) - u(z_4) - \sum_{\vartheta \le j\vartheta < \alpha} \frac{(-\tau^{\vartheta})^j}{j!} Y^j u(z_4)}_{(1)} \\ & + \boxed{u(z_4) - u(z_3) - \sum_{1 \le k < \alpha} \frac{(-\tau)^k Z_i^k u(z_3)}{k!}}_{(2)} \\ & + \boxed{u(z_3) - u(z_2) + \sum_{\vartheta \le j\vartheta < \alpha} \frac{(-\tau^{\vartheta})^j}{j!} Y^j u(z_3)}_{(3)} \\ & + \boxed{u(z_2) - u(t,x,v) + \sum_{1 \le k < \alpha} \frac{(-\tau)^k}{k!} Z_i^k u(z_2)}_{(4)} \\ & + \boxed{\sum_{\vartheta \le j\vartheta < \alpha} \frac{(-\tau^{\vartheta})^j}{j!} \Big(Y^j u(z_4) - Y^j u(z_3) \Big)}_{(5)} \\ & + \boxed{\sum_{1 \le k < \alpha} \frac{(-\tau)^k}{k!} \Big(Z_i^k u(z_3) - Z_i^k u(z_2) \Big)}_{(6)} \\ =: {} & I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Note that all the sums in the above boxes are not void, since $\alpha > \vartheta + 1$. By Remark 3 and 4, we find that

$$|I_1| + |I_2| + |I_3| + |I_4| \le c_\alpha \|u\|_{C^\alpha} |\delta|^{\frac{\alpha}{\vartheta+1}}, \tag{4.4}$$

for some positive constant c_α. Concerning I_5 and I_6, we have

$$\begin{aligned} I_5 + I_6 = & \boxed{\sum_{\vartheta \le j\vartheta < \alpha} \frac{(-\tau^\vartheta)^j}{j!}\left(Y^j u(z_4) - Y^j u(z_3) - \sum_{0<k<\alpha - j\vartheta} \frac{(-\tau)^k}{k!} Z_i^k Y^j u(z_3)\right)}_{(5.1)} \\ & + \boxed{\sum_{1\le k<\alpha} \frac{(-\tau)^k}{k!}\left(Z_i^k u(z_3) - Z_i^k u(z_2) + \sum_{0<j\vartheta<\alpha-k} \frac{(-\tau^\vartheta)^j}{j!} Y^j Z_i^k u(z_3)\right)}_{(6.1)} \\ & + \boxed{\sum_{\substack{k>0, j>0 \\ j\vartheta + k < \alpha}} \frac{(-\tau^\vartheta)^j}{j!} \frac{(-\tau)^k}{k!} \left(v_{jk}(z_3) - T_{\alpha_{jk}} v_{jk}(z; z_3)\right)}_{(7)} \\ & + \boxed{\sum_{\substack{k>0, j>0 \\ j\vartheta + k < \alpha}} \frac{(-\tau^\vartheta)^j}{j!} \frac{(-\tau)^k}{k!} T_{\alpha_{jk}} v_{jk}(z; z_3)}_{(8)} \quad =: I_{5.1} + I_{6.1} + I_7 + I_8. \end{aligned} \tag{4.5}$$

where

$$\alpha_{jk} := \alpha - (j\vartheta + k), \qquad \text{and} \qquad v_{jk} := [Z_i^k, Y^j]u.$$

Consider $I_{5.1}$ first. By Remark 4, we obtain

$$\left| Y^j u(z_4) - Y^j u(z_3) - \sum_{0<k<\alpha - j\vartheta} \frac{(-\tau)^k}{k!} Z_i^k Y^j u(z_3) \right| \le \|u\|_{C^\alpha} |\tau|^{\alpha - j\vartheta}.$$

The same argument applies to $I_{6.1}$, in this case we use Remark 3. By collecting the above inequalities we find that there exists a positive constant c'_α such that

$$|I_{5.1}| + |I_{6.1}| \le c'_\alpha \|u\|_{C^\alpha} |\delta|^{\frac{\alpha}{\vartheta+1}}.$$

Concerning I_7, we first note that, by Definition 3, $v_{jk} \in C^{\alpha_{jk}}$. Moreover, only terms with $0 < \alpha_{jk} \le \bar{\alpha} = \alpha - (\vartheta + 1)$ appear there. Then, because of the induction hypothesis, in particular by (2.7), we have

$$\left| v_{jk}(z_3) - T_{\alpha_{jk}} v_{jk}(z; z_3) \right| \le c \|v_{jk}\|_{C^{\alpha_{jk}}} \|z^{-1} \circ z_3\|^{\alpha_{jk}}.$$

We then conclude that there exists a positive constant c''_α such that

$$|I_7| \le c''_\alpha \|u\|_{C^\alpha} |\delta|^{\frac{\alpha}{\vartheta+1}}. \tag{4.6}$$

We are left with the term I_8. We note that

$$\begin{aligned} I_8 = \delta\, [Z_i, Y] u(z) + & \sum_{\substack{k>0, j>0 \\ \vartheta+1 < m'(\vartheta+1) + (j'+j)\vartheta + k' + k < \alpha}} \delta^{\frac{m'(\vartheta+1)+(j'+j)\vartheta+k'+k}{\vartheta+1}} \\ & \times \left(\bar{c}_{m'j'k'jk} \partial_{x_i}^{m'} Y^{j'} Z_i^{k'+k} Y^j u + \tilde{c}_{m'j'k'jk} \partial_{x_i}^{m'} Y^{j'} Z_i^{k'} Y^j Z_i^k u \right)(z), \end{aligned}$$

where every constant $\bar{c}_{m'j'k'jk}$ and $\tilde{c}_{m'j'k'jk}$ is obtained by collecting the coefficients of the Taylor polynomials of the v_{jk} functions. As a consequence of the fact that $\frac{m'(\vartheta+1)+(j'+j)\vartheta+k'+k}{\vartheta+1} > 1$ for every term of the above sum, we finally obtain

$$\frac{I_8}{\delta} \to [Z_i, Y]u(z) \qquad \text{as} \quad \delta \to 0. \tag{4.7}$$

Moreover, since $1+\vartheta < \alpha \le 2(1+\vartheta)$, (4.4), (4.5) and (4.6) yield

$$\frac{I_1+I_2+I_3+I_4}{\delta} \to 0, \qquad \frac{I_{5.1}+I_{6.1}}{\delta} \to 0, \qquad \frac{I_7}{\delta} \to 0 \qquad \text{as} \quad \delta \to 0. \tag{4.8}$$

The proof of (4.3) follows from (4.7) and (4.8).

5. Extensions: non-homogeneous and local case

In some applications, the operator Y in (1.1) appears in a more general form than (1.2), namely

$$Y = \left\langle B(x,v)^\top, \nabla_{(x,v)} \right\rangle + \partial_t, \qquad (x,v) \in \mathbb{R}^{2d},$$

with B being a $(2d \times 2d)$-matrix with real entries that admits the following block decomposition:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where each block is a $(d \times d)$-matrix and B_{12} has rank d. For instance, in the kinetic model originally introduced in [10], the term B_{12} is the $d \times d$ identity matrix, while B_{22} is non-null and depends on the viscosity of the liquid. In mathematical finance, B_{22} may represent an interest rate in the pricing of path-dependent derivatives.

Note that the Hörmander condition (1.6) is still satisfied: in particular, (1.7) becomes

$$B_{12}\nabla_x = [\nabla_v, Y] - B_{22}\nabla_v. \tag{5.1}$$

The integral curve of the vector field Y now reads as

$$e^{\tau Y}(t,x,v) = \left(t+\tau, e^{\tau B}(x,v)^\top\right), \qquad (t,x,v) \in \mathbb{R} \times \mathbb{R}^{2d}, \quad \tau \in \mathbb{R}.$$

Accordingly, the relevant non-commutative group law "$\circ$" in (5.2) is replaced by

$$\begin{gathered} z_1 \circ z_2 = \left(t_1+t_2, (x_2,v_2)^\top + e^{t_2 B}(x_1,v_1)^\top\right), \\ z_1 = (t_1,x_1,v_1), z_2 = (t_2,x_2,v_2) \in \mathbb{R} \times \mathbb{R}^{2d}. \end{gathered} \tag{5.2}$$

Notice that $(\mathbb{R} \times \mathbb{R}^{2d}, \circ)$ remains a group, with the identity and the inverse elements that now read as

$$\mathrm{Id} = (0,0,0), \qquad (t,x,v)^{-1} = \left(-t, -e^{-tB}(x,v)^\top\right).$$

In particular, we have

$$(t_0, x_0, v_0)^{-1} \circ (t, x, v) = \big(t - t_0, (x, v)^\top - e^{(t-t_0)B}(x_0, v_0)^\top\big). \tag{5.3}$$

Despite the fact that $\mathscr{L}_s$ is no longer homogeneous with respect to the dilations $(D_\lambda)_{\lambda>0}$ in (2.2), the homogeneous norm (2.3) remains well behaved with respect to the stratification induced by the Hörmander condition, in particular by (5.1).

A local version of Theorem 1 can be proved in this more general setting, with the Hölder spaces C^α being exactly as in Definition 3. However, it is more natural to localize the definition of Hölder spaces as well. We follow the approach in [12,13] for the case $\vartheta = 2$. Let Ω be a domain in $\mathbb{R} \times \mathbb{R}^{2d}$. For any $z \in \Omega$ we set

$$\delta_z := \sup\left\{\bar\delta \in]0,1] \mid e^{\delta Z_1}(z), \ldots, e^{\delta Z_d}(z), e^{\delta Y}(z) \in \Omega \text{ for any } \delta \in [-\bar\delta, \bar\delta]\right\}.$$

If Ω_0 is a bounded domain with $\overline{\Omega}_0 \subseteq \Omega$, we set $\delta_{\Omega_0} := \min_{z \in \overline{\Omega}_0} \delta_z$. Note that $\delta_{\Omega_0} \in]0,1]$. Now we first localize Definition 2 of Hölder regularity along the Hörmander fields.

Definition 4. Let $u : \mathbb{R} \times \mathbb{R}^{2d} \to \mathbb{R}$. Then, for $a \in]0,1]$, we say that $u \in C^\alpha_{Z_i}(\Omega)$, $i = 1, \ldots, d$ if, for any bounded domain Ω_0 with $\overline{\Omega}_0 \subseteq \Omega$, we have

$$\|u\|_{C^\alpha_{Z_i}(\Omega_0)} := \sup_{\substack{z \in \Omega_0 \\ 0<|\tau|<\delta_{\Omega_0}}} \frac{\left|u\left(e^{\tau Z_i}(z)\right) - u(z)\right|}{|\tau|^\alpha} < \infty.$$

Moreover, for $\alpha \in]0, \vartheta]$, we say that $u \in C^\alpha_Y(\Omega)$ if, for any bounded domain Ω_0 with $\overline{\Omega}_0 \subseteq \Omega$, we have

$$\|u\|_{C^\alpha_Y(\Omega_0)} := \sup_{\substack{z \in \Omega_0 \\ 0<|\tau|<\delta_{\Omega_0}}} \frac{\left|u\left(e^{\tau Y}(z)\right) - u(z)\right|}{|\tau|^{\frac{\alpha}{\vartheta}}} < \infty.$$

With Definition 4 at hand, we can now localize the notion of intrinsic Hölder spaces C^α with the following definition, which is completely analogous to Definition 3 and thus is written in a more compact form for sake of brevity.

Definition 5. Let $u : \Omega \to \mathbb{R}$ and $\alpha > 0$. Then, $u \in C^\alpha(\Omega)$ if, for any bounded domain Ω_0 with $\overline{\Omega}_0 \subseteq \Omega$, the semi-norm defined recursively as

$$\|u\|_{C^\alpha(\Omega_0)} := \begin{cases} \|u\|_{C^\alpha_Y(\Omega_0)} + \sum_{i=1}^d \|u\|_{C^\alpha_{Z_i}(\Omega_0)} & \text{if } \alpha \in]0, 1 \wedge \vartheta], \\ \|Yu\|_{C^{\alpha-\vartheta}(\Omega_0)} + \sum_{i=1}^d \|u\|_{C^\alpha_{Z_i}(\Omega_0)} & \text{if } \alpha \in]1 \wedge \vartheta, 1 \vee \vartheta] \\ & \quad \text{and } \vartheta < 1, \\ \|u\|_{C^\alpha_Y(\Omega_0)} + \sum_{i=1}^d \|Z_i u\|_{C^{\alpha-1}(\Omega_0)} & \text{if } \alpha \in]1 \wedge \vartheta, 1 \vee \vartheta] \\ & \quad \text{and } \vartheta > 1, \\ \|Yu\|_{C^{\alpha-\vartheta}(\Omega_0)} + \sum_{i=1}^d \|Z_i u\|_{C^{\alpha-1}(\Omega_0)} & \text{if } \alpha > 1 \vee \vartheta \end{cases}$$

is finite.

Remark 5. It is easy to see that $C^{\alpha}(\Omega)$ is a decreasing family in α, meaning that $C^{\alpha}(\Omega) \subset C^{\alpha'}(\Omega)$ for any $\alpha > \alpha' > 0$.

We now have the following local version of Theorem 1. Before, for $u \in C^{\alpha}(\Omega)$, we define its intrinsic Taylor polynomial centered at $z_0 = (t_0, x_0, v_0)$ as

$$T_{\alpha} u(z_0; z) := \sum_{0 \le \vartheta k + (1+\vartheta)|\gamma| + |\beta| < \alpha} \frac{Y^k \partial_v^{\beta} \partial_x^{\gamma}}{k!\, \gamma!\, \beta!} \big((t_0, x_0, v_0)^{-1} \circ (t, x, v) \big)^{(k,\gamma,\beta)},$$

with $z = (t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$, $\circ$ as in (5.3), and where we adopted the multi-product notation

$$(s, h, y)^{(k,\gamma,\beta)} = s^k h_1^{\gamma_1} \dots h_d^{\gamma_d} y_1^{\beta_1} \dots y_d^{\beta_d}, \qquad (s, h, y) \in \mathbb{R} \times \mathbb{R}^{2d}.$$

Note that the term *polynomial* here is slightly abused, as $T_{\alpha} u(z_0; z)$ is not necessarily a polynomial in the time-increment.

Theorem 2. *For any $\alpha > 0$ and for any $u \in C^{\alpha}(\Omega)$ the following statements are true:*

HÖLDER SPACES CHARACTERIZATION. *For any $k \in \mathbb{N}_0$ and $\gamma, \beta \in \mathbb{N}_0^d$ with $0 \le \vartheta k + (1+\vartheta)|\gamma| + |\beta| < \alpha$, the derivatives $Y^k \partial_v^{\beta} \partial_x^{\gamma} u$ exist on Ω and*

$$\| Y^k \partial_v^{\beta} \partial_x^{\gamma} u \|_{C^{\alpha - \vartheta k - (1+\vartheta)|\gamma| - |\beta|}(\Omega_0)} \le \| u \|_{C^{\alpha}(\Omega_0)}$$

for any bounded domain Ω_0 with $\overline{\Omega}_0 \subseteq \Omega$.

TAYLOR FORMULA. *For any $\zeta_0 \in \Omega$, there exist two bounded domains U, Ω_0 such that $\zeta_0 \in \overline{U} \subset \overline{\Omega_0} \subset \Omega$, and a positive constant c, only dependent on B, α and U, such that*

$$|u(z) - T_{\alpha} u(z_0; z)| \le c \| u \|_{C^{\alpha}(\Omega_0)} \| z_0^{-1} \circ z \|^{\alpha}, \qquad z, z_0 \in U.$$

In the homogeneous case, namely when the blocks B_{11}, B_{21}, B_{22} are null, the proof of Theorem 2 is substantially identical to that of Theorem 1 with the only additional complexity being that U and Ω_0 have to be chosen in a way that all the integral curves employed to connect z_0 and z are contained in Ω. On the other hand, in the general non-homogeneous case, there is a substantial additional difficulty stemming from the fact that the discretization of (5.1) with the integral curves of $Z_1, \dots, Z_d$ and Y is more involved than in the homogeneous case. In the rest of the section we give an account of this additional complexity and recall a result from [13] that allows to overcome it.

Let $z = (t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d}$, $w \in \mathbb{R}^d$ and recall the points z_2, z_3, z_4 as defined in (3.2), namely

$$z_2 = e^{\tau Z_w}(t, x, v), \qquad z_3 = e^{\tau^{\vartheta} Y}(z_2), \qquad z_4 = e^{-\tau Z_w}(z_3),$$

and set

$$z_5 = e^{-\tau^{\vartheta} Y}(z_4), \qquad z_6 = e^{-\tau^{\vartheta+1} Z_{B_{22} w}}(z_5).$$

A direct computation shows that, for $\tau \in \mathbb{R}$, we have

$$z_6 = \big(t, x + \tau^{\vartheta+1} B_{12} w, v\big) - \tau^{2\vartheta+1}\left(0, \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{\vartheta n}}{(n+2)!} B^{n+2}(0, w)^{\top}\right). \quad (5.4)$$

Assume that our aim is to move along the x variable only, by an increment $h \in \mathbb{R}^d$. If we adjust the leading order increment in (5.4) by choosing $w = B_{1,2}^{-1} h/|h|$ and $\tau = |h|^{\frac{1}{\vartheta+1}}$ (recall that $B_{1,2}$ has maximum rank), then we are off by an error term of order $\tau^{2\vartheta+1}$. Note that this error involves both the velocity and position variables as the blocks of B are all non-null. Therefore, if we make a further correction in the velocity variables by considering

$$g_{w,\tau}(z) := e^{\tau^{2\vartheta+1} Z_{w'}}(z_6), \text{ with } w' = w'(\tau, w) = \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{\vartheta n}}{(n+2)!} B_{22}^{n+2} w, \quad (5.5)$$

we fix the velocity variables but the increment in the position variables still differs from h by an error term of order $\delta^{2\vartheta+1}$. The next lemma allows to connect $z = (t, x, v)$ to $(t, x + h, v)$ moving along the curves above, while keeping w bounded and $|\tau|$ controlled in terms of $|h|^{\frac{1}{\vartheta+1}}$. It coincides with [13, Lemma 3.2], which has been proved for $\vartheta = 2$. We recall this result without proof, as dealing with a general $\vartheta > 0$ requires no modification.

Lemma 1. *There exists $\varepsilon > 0$, only dependent on B, such that: for any $h \in \mathbb{R}^d$ with $|h| \le \varepsilon$, there exist $w \in \mathbb{R}^d$, $\tau \ge 0$ with*

$$|w| = 1, \qquad |\tau| \le \frac{2}{\|B_{12}\|} |h|^{\frac{1}{\vartheta+1}},$$

such that

$$g_{w,\tau}(z) = (t, x + h, v), \qquad z = (t, x, v) \in \mathbb{R} \times \mathbb{R}^{2d},$$

where $g_{w,\tau}(z)$ is as in (5.5).

With Lemma 1 at hand, which plays the role of identity (3.3) that we had in the case $Y = \langle v, \nabla_x \rangle + \partial_t$, the proof of Theorem 2 is essentially the same as in the homogeneous case.

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