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# Exceptional surgeries on hyperbolic knots arising from the 3-chain link

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ABSTRACT

We study some closed connected orientable 3–manifolds obtained by Dehn surgery along the oriented components of the 3-chain link. For such manifolds, we describe exceptional surgeries related to some results from Audoux et al. (2018) and Martelli and Petronio (2006). Then we construct a related family of hyperbolic knots in the 3-sphere, which admit two consecutive Seifert fibered surgeries and two toroidal fillings at distance 3. Such additional examples are not mentioned in the quoted papers.

## 1. Introduction

Dehn surgery provides a nice method to relate the topology of orientable 3-manifolds and knot theory. By Thurston [1], Dehn surgery on a framed hyperbolic knot in the oriented 3-sphere almost always gives rise to a hyperbolic closed orientable 3-manifold. A non-hyperbolic Dehn surgery is called *exceptional*. Thanks to the positive solution to the Geometrization Conjecture [2], any exceptional surgery is either reducible, toroidal, or a small Seifert fibered space. Brittenham and Wu [3] determine all exceptional surgeries on 2-bridge knots. Remarkable results on the classification of exceptional surgeries on hyperbolic pretzel knots, Montesinos knots and alternating knots can be found in Meier [4], Wu [5], and Ichihara and Masai [6], respectively. For Seifert manifolds (notation and classification) see Montesinos [7] and Orlik [8].

In this paper we study some closed connected orientable 3–manifolds obtained by Dehn surgery along the oriented components of the 3-chain link. For such manifolds, we find finite balanced group presentations of the fundamental group which correspond to their spines, and describe exceptional surgeries related to some results from Audoux et al. [9] and Martelli and Petronio [10]. Then we construct a new infinite family of hyperbolic knots in the oriented 3-sphere arising from the 3-chain link, which admit two consecutive Seifert fibered surgeries and two toroidal fillings at distance 3. Such additional examples are not mentioned in the quoted papers. The proofs are based on the combinatorial group theory and the representations of closed manifolds by spines and surgery on framed links. Spines and surgery descriptions of graph manifolds (including Seifert fibered spaces and lens spaces) are discussed in a recent paper of the authors [11], which is most valuable to derive the results presented here. Such results are correctly confirmed by using SnapPea program from Weeks [12].

## 2. Main results

Let $N$ be the complement in the oriented 3-sphere $\mathbb{S}^3$ of the 3-chain link depicted in Fig. 1(a). Let $N(p/q, m/n)$ denote the Dehn filling of $N$ with one boundary component. As usual, the surgery coefficients $(p, q)$ and $(m, n)$ are pairs of coprime integers. For the geometrical classifications of such manifolds (particularly, their hyperbolic structure) we refer to Martelli and Petronio [10]. Here we discuss additional examples which are not mentioned in the cited paper. Furthermore, we reobtain some results from Audoux et al. [9] by using very different techniques based on combinatorial group theory and spine representations of manifolds.

Combining results from [13,14] gives

**Proposition 2.1.** *The fundamental group of the cusped surgery manifold $N(p/q, m/n)$ admits the finite group presentation with generators $a$ and $b$ and one relation $a^q\, b^{m+n}\, a^q\, b^{-n}\, a^{-p-q}\, b^{-n} = 1$. Such a presentation is geometric in the sense that it corresponds to a spine of the manifold. Furthermore, the peripheral meridian and longitude are given by $a^p\, b^n$ and $a^{-q}\, b^{-m-2n}\, a^{-q}\, b^n\, a^p\, b^n$, respectively.*

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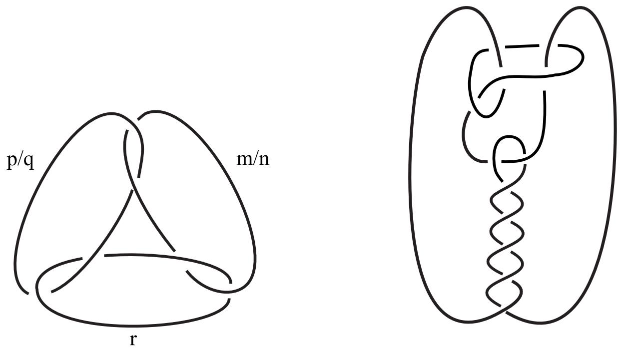


(a) (b)

**Fig. 1.** (a) The 3–chain link. The manifold $N(p/q, m/n)$ with one boundary component. The manifold $M(p/q, m/n, r)$; (b) The hyperbolic knot $K_2$ in $\mathbb{S}^3$ with complement $N(1/2, 3/2)$.

Let $M(r) = N(p/q, m/n, r)$ be the closed connected orientable 3-manifold obtained by Dehn filling of $N(p/q, m/n)$ with an integer surgery coefficient $r \in \mathbb{Z}$. See Fig. 1(a). We determine conditions on the parameters for which the manifolds $M(r)$ are Seifert fibered spaces (including lens spaces) or toroidal. An orientable Seifert fiber space is *of type* $(O\,0\,o\,:\,b\,(\alpha_1,\beta_1)\,(\alpha_2,\beta_2)\,(\alpha_3,\beta_3))$ if it has a Seifert fibration over the 2–sphere $\mathbb{S}^2$ with Euler number $b$ and three exceptional fibers of indices $(\alpha_i, \beta_i)$, for $i = 1, 2, 3$. See [7, §4]. This representation also includes lens spaces as a particular subclass. A compact orientable 3–manifold is called a *graph manifold* if it can be obtained by gluing several copies of the manifolds $D \times \mathbb{S}^1$ and $A \times \mathbb{S}^1$ together by homeomorphisms of some components of their boundaries. Here $D$ is the 2–disc and $A$ is an annulus. A closed connected 3–manifold is said to be *toroidal* if it contains an incompressible torus.

By definition of Dehn surgery, Proposition 2.1 directly implies the following result, which also relates with the arguments discussed in [14, §1]:

**Proposition 2.2.** *For any $r \in \mathbb{Z}$, the fundamental group of the surgery manifold $M(r) = N(p/q, m/n, r)$ admits the finite geometric presentation with generators $a$ and $b$ and two relations $a^q\, b^{m+n}\, a^q\, b^{-n}\, a^{-p-q}\, b^{-n} = 1$ and $(a^p\, b^n)^r\, a^{-q}\, b^{-m-2n}\, a^{-q}\, b^n\, a^p\, b^n = 1$.*

Note that the fundamental group of $M(\infty) = N(p/q, m/n, \infty)$ has the geometric presentation with generators $a$ and $b$ and relations $a^p b^n = 1$ and $a^q\, b^{m+n}\, a^q\, b^{-n}\, a^{-p-q}\, b^{-n} = 1$. This is equivalent to the geometric presentation $\langle a, b \,:\, a^p b^n = 1,\ a^q b^m = 1\rangle$. By Theorem 1.2 of [11], it follows that $M(\infty)$ is the lens space $L(\lambda, k)$, where $\lambda = |qn - mp|$ and $k \equiv n' \pmod{\lambda}$ with $0 \le k < \lambda$. The integer $n'$ is completely determined as in [11].

**Proposition 2.3.** *If $qn - mp = \pm 1$, then the surgery manifold $N(p/q, m/n)$ is the complement of a knot in the oriented 3–sphere. This knot is hyperbolic except a finite number of cases listed in Theorem 1.2 from* [10].

The following general theorem implies the results in Tables 1 and 2 of Audoux et al. [9] as particular cases (for which the condition $qn - mp = \pm 1$ is satisfied). For the proof we use the group presentations (written with bold exponents) in the statements of Theorems 1.3 and 2.3, and Corollary 1.10 from [11].

**Theorem 2.1.** *For any $r \in \mathbb{Z}$, let us consider the closed orientable surgery manifold $M(r) = N(p/q, m/n, r)$. Then we have*

(**i**) If $r = 0$, then $M(r)$ is homeomorphic to the graph manifold

$$(D \quad (2,1) \quad (3,-1)) \cup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D \quad (n, 3n+m) \quad (q, p+q)).$$

(**ii**) If $r = -1$, then $M(r)$ is the Seifert fibered space of type

$$(O\,0\,o\,:\,-1 \quad (m+3n, -n) \quad (p+3q, -q) \quad (2,-1)).$$

In particular, if $p = -5$, $q = 2$, $m = 1 - 2k$ and $n = 5k - 2$, $k \neq 0$, then $M(-1)$ is the lens space $L(49k - 19, 49) \cong L(49k - 19, 31k - 12)$.

(**iii**) If $r = -2$, then $M(r)$ is the Seifert fibered space of type

$$(O\,0\,o\,:\,-1 \quad (m+2n, -m-n) \quad (p+2q, q) \quad (3,-1)).$$

In particular, if $p = -5$, $q = 2$, $m = 1 - 2k$ and $n = 5k - 2$, $k \neq 0$, then $M(-2)$ is the lens space $L(49k - 18, 49) \cong L(49k - 18, 19k - 7)$.

(**iv**) If $r = -3$, then $M(r)$ is homeomorphic to the graph manifold

$$(D \quad (2,1) \quad (p+q, q)) \cup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D \quad (2,1) \quad (m+n, n)).$$

In particular, if $p = -n - 1$, $q = n$ and $m = -n + 1$, $n \in \mathbb{Z}\backslash\{\pm 1, 0, \pm 2\}$, then $M(-3)$ is the lens space $L(4n^2 + 3, 2n^2 + n + 2)$. If $p = -n + 1$, $q = n - 2$ and $m = -n + 1$, $n \in \mathbb{Z}\backslash\{\pm 1, 0, 2, 3\}$, then $M(-3)$ is the lens space $L(4n^2 - 8n - 1, 2n^2 - 3n)$.

**Proof.** (**i**) By Proposition 2.2, the fundamental group of $M(0)$ has a group presentation with generators $a$, $b$ and relations $a^q b^{m+n} a^q b^{-n} a^{-p-q} b^{-n} = 1$ and $a^{-q} b^{-m-2n} a^{-q} b^n a^p b^n = 1$. The inverse of the first relation coincides with the second relation (with bold exponents) in the statement of Theorem 2.3 from [11] with $\mathbf{p} = n$, $\mathbf{n} = p + q$, $\mathbf{q} = q$, and $\mathbf{2\,p\text{ - }m} = -m - n$. By cyclic permutation the second relation becomes $b^n a^p b^n a^{-q} b^{-m-2n} a^{-q} = 1$, which coincides with the first relation of Theorem 2.3 from [11] with $\mathbf{p} = n$, $\mathbf{n\text{ - }q} = p$, $\mathbf{q} = q$, and $\mathbf{p\text{- }m} = -m - 2n$, hence $\mathbf{m} = 3n + m$. Now the result follows from Theorem 2.3 of [11].

(**ii**) By Proposition 2.2, the fundamental group of $M(-1)$ has a group presentation with generators $a$, $b$ and relations $a^q b^{m+n} a^q b^{-n} a^{-p-q} b^{-n} = 1$ and $b^{-n} a^{-p} a^{-q} b^{-m-2n} a^{-q} b^n a^p b^n = 1$. The latter becomes $a^{-q} b^n a^{-q} b^{-m-2n} = 1$, which coincides with the second relation in Theorem 1.3 of [11] with $y = a$, $x = b$, $\mathbf{n} = -q$, $\mathbf{m} = n$, $\mathbf{r} = 1$, and $\mathbf{m} + \mathbf{p\,s} = -m - 2n$. Substituting $a^q b^{-n} = b^{-m-2n} a^{-q}$ into the first relation of the initial group presentation gives $a^q b^{m+n} b^{-m-2n} a^{-q} a^{-p-q} b^{-n} = 1$, or equivalently $a^q b^{-n} a^q a^{-p-3q} b^{-n} = 1$, hence $b^{-m-2n} a^{-p-3q} b^{-n} = 1$, i.e., $b^{m+3n} a^{p+3q} = 1$. This relation coincides with the first relation in Theorem 1.3 of [11] with $x = b$, $y = a$, $\mathbf{p} = m + 3n$ and $\mathbf{q} = p + 3q$, hence $\mathbf{s} = -1$. The result follows from Theorem 1.3 of [11].

For the particular case in (**ii**), $M(-1)$ is the Seifert manifold of type $(O\,0\,o\,:\,-1\,(13k-5, -5k+2)\,(1,-2)\,(2,-1))$, which is homeomorphic to $(O\,0\,o\,:\,b\,(\alpha_1,\beta_1)\,(\alpha_2,\beta_2))$, where $b = -1$, $(\alpha_1, \beta_1) = (13k - 5, -5k + 2)$ and $(\alpha_2, \beta_2) = (2, -1)$. This is homeomorphic to the lens space $L(\xi, \eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1| = |-49k + 19|$, $\eta = \bar{m}\alpha_2 + \bar{n}\beta_2$ and $\bar{m}\alpha_1 - \bar{n}(b\alpha_1 + \beta_1) = 1$, hence $\bar{m} = 18$, $\bar{n} = -13$ and $\eta = 49$, as requested.

(**iii**) By Proposition 2.2, the fundamental group of $M(-2)$ has a group presentation with generators $a$, $b$ and relations $a^q b^{m+n} a^q b^{-n} a^{-p-q} b^{-n} = 1$ and $(a^p b^n)^{-2} a^{-q} b^{-m-2n} a^{-q} b^n a^p b^n = 1$. The last relation is equivalent to $b^{m+2n} a^{p+2q} = 1$, which coincides with the relation $x^{\mathbf{p}} y^{\mathbf{q}} = 1$ in Theorem 1.3 of [11] with $x = b$, $y = a$, $\mathbf{p} = m + 2n$ and $\mathbf{q} = p + 2q$. Substituting $a^{-p-q} = a^q b^{m+2n}$ into the first relation of the initial group presentation yields $(a^q b^{m+n})^2 a^q b^{-n} = 1$, which coincides with the second relation $(y^{\mathbf{n}} x^{\mathbf{m}})^{\mathbf{r}} y^{\mathbf{n}} x^{\mathbf{m\,+\,ps}} = 1$ in Theorem 1.3 of [11] with $x = b$, $y = a$, $\mathbf{n} = q$, $\mathbf{m} = m + n$, $\mathbf{r} = 2$, and $\mathbf{m} + \mathbf{ps} = -n$, hence $\mathbf{s} = -1$. Then the result follows from Theorem 1.3 of [11]. The particular case in (**iii**) can be derived similarly as in (**ii**).

(**iv**) By Proposition 2.2, the fundamental group of $M(-3)$ has a group presentation with generators $a$, $b$ and relations $a^q b^{m+n} a^q b^{-n} a^{-p-q} b^{-n} = 1$ and $(a^p b^n)^{-3} a^{-q} b^{-m-2n} a^{-q} b^n a^p b^n = 1$. The last relation is equivalent to $a^{p+q} b^n a^{p+q} b^{m+2n} = 1$, which coincides with first relation in the statement of Corollary 1.10 from [11] with $x = a$, $y = b$, $\mathbf{p} = p + q$, $\mathbf{q} = n$, and $\mathbf{n} + \mathbf{q} = m + 2n$. Substituting $b^n a^{p+q} b^n = a^{-p-q} b^{-m-n}$ into the inverse of the first relation in the initial presentation yields $b^{m+n} a^q b^{m+n} a^{p+2q} = 1$, which coincides with the second relation in Corollary 1.10 of [11] with $x = a$, $y = b$, $\mathbf{n} = m + n$, $\mathbf{m} = q$, and $\mathbf{m} + \mathbf{p} = p + 2q$. Then the result follows from Corollary 1.10 of [11].

For the first particular case in (**iv**), $M(-3)$ is the graph manifold

$$(D\ (2,1)\ (-1,n)) \cup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D\ (2,1)\ (1,n)),$$

which is homeomorphic to $(O\,0\,o\,:\,-1\,(2,1)\,(1,-n)\,(2n+1,2))$ or, equivalently, $(O\,0\,o\,:\,-n\,(2,1)\,(2n+1,-2))$. This is the lens space $L(\xi,\eta)$, where $\xi = |b\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1|$, $\eta = \bar{m}\alpha_2 + \bar{n}\beta_2$ and $\bar{m}\alpha_1 - \bar{n}(b\alpha_1 + \beta_1) = 1$, being $b = -n$, $(\alpha_1,\beta_1) = (2,1)$ and $(\alpha_2,\beta_2) = (2n+1,-2)$. Then it follows that $\xi = 4n^2 + 3$, $\bar{m} = n$, $\bar{n} = -1$ and $\eta = n(2n+1) - (-2) = 2n^2 + n + 2$, as claimed. The second particular case can be derived similarly. □

Now we discuss an infinite family of hyperbolic knots in $\mathbb{S}^3$, which is not considered in [9,10]. Then we have

**Theorem 2.2.** *For every integer $n \geq 2$, there are infinitely many hyperbolic knots $K_n$ in the oriented 3–sphere whose $0$- and $(-3)$-surgeries give toroidal graph manifolds, and $(-1)$- and $(-2)$-surgeries give Seifert fiber spaces over $\mathbb{S}^2$ with exactly three exceptional fibers*

**Proof.** For every $n \in \mathbb{Z}$, $n \geq 2$, let $K_n$ be the hyperbolic knot in $\mathbb{S}^3$ with complement $N((n-1)/n,(n+1)/n)$. For example, the hyperbolic knot $K_2$ is depicted in Fig. 1(b). Now the result follows by using Theorem 2.1. □

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Data availability

No data was used for the research described in the article.

## References

[1] W. Thurston, The geometry and topology of 3-manifolds, in: Lect. Notes, Princeton Univ. Press, Princeton, NJ, 1980.
[2] G. Perelman, Ricci flow with surgery on three-manifolds, 2003, arXiv:math.DG/0303109.
[3] M. Brittenham, Y.Q. Wu, The classification of exceptional Dehn surgeries on 2-bridge knots, Comm. Anal. Geom. 9 (1) (2001) 97–113.
[4] J. Meier, Small Seifert fibered surgery on hyperbolic pretzel knots, Algebr. Geom. Topol. 14 (1) (2014) 439–487.
[5] Y.-Q. Wu, The classification of toroidal Dehn surgeries on Montesinos knots, Comm. Anal. Geom. 19 (2011) 305–345.
[6] K. Ichihara, H. Masai, Exceptional surgeries on alternating knots, Comm. Anal. Geom. 24 (2) (2016) 337–377.
[7] J.M. Montesinos, Classical tessellations and three-manifolds, in: Universitext, Springer-Verlag, Berlin-Heidelberg, 1987.
[8] P. Orlik, Seifert manifolds, in: Lect. Notes in Math. 291, Springer-Verlag, Berlin-Heidelberg-New York, 1972.
[9] B. Audoux, A.G. Lecuona, F. Roukema, On hyperbolic knot in $\mathbb{S}^3$ with exceptional surgeries at maximal distance, Alg. Geom. Topol. 18 (4) (2018) 2371–2417.
[10] B. Martelli, C. Petronio, Dehn filling of the “magic” 3-manifold, Comm. Anal. Geom. 14 (5) (2006) 969–1026.
[11] A. Cavicchioli, F. Spaggiari, Spines and surgery descriptions of graph manifolds, Topology Appl. 339 (2023) 108579, Part A.
[12] J.R. Weeks, SnapPea: a computer program for creating and studying hyperbolic 3-manifolds, available for free download from www.geom.umn.edu.
[13] A. Cavicchioli, F. Spaggiari, Surgery presentations of cube manifolds reducible to two generators, Publ. Math. Debrecen 106/1–2 (2025) 29–52.
[14] M. Takahashi, On the presentations of the fundamental groups of 3-manifolds, Tsukuba J. Math. 13 (1989) 175–189.