Even Cycle Decompositions of index 3 by a novel coloring technique*

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Abstract

An even cycle decomposition of an Eulerian graph is a partition of the edge-set into even cycles. We color the even cycles so as two cycles sharing at least one vertex receive distinct colors. If k is the minimum number of required colors in such a coloring, then the even cycle decomposition has index k. We prove that the line graph of every cubic graph of oddness 2 has an even cycle decomposition of index 3. The same property holds for the line graphs of some infinite families of class 2 cubic graphs with arbitrary large oddness. The construction of even cycle decompositions of index 3 in the line graph of a class 2 cubic graph is alternative to the constructions that are known in the literature; that one for the line graph of a cubic graph with arbitrary large oddness is also a new contribution to the more general problem on the existence of even cycle decompositions in the line graph of a bridgeless cubic graph. The constructions are obtained by applying a novel coloring technique on the edges of the line graph.

Keywords: even cycles, decompositions, line graph, class 2 *cubic graphs.*
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1 Introduction.

A classical result by Veblen [19] states that a graph is Eulerian if and only if it has a cycle decomposition, that is, a partition of the edge-set into cycles. Each Euler tour in the graph defines an arbitrary cycle decomposition that does not necessarily satisfy further conditions as, for instance, on the length of the cycles. Cycle decompositions meeting additional conditions might be related to some hard problems in graph theory, see for instance [6, 10, 11].

We consider decompositions into cycles of even length - *even cycle decompositions*, briefly ECD - whose cycles can be colored by k distinct colors

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so as two cycles sharing at least one vertex receive distinct colors. If k is the minimum number of required colors, then we say that the *even cycle decomposition has index k*. The notion of an ECD of index k is connected to the *palette index* of a graph, a chromatic parameter describing a graph by the minimum number of palettes of its vertices [9]. We recall that the *palette* of a vertex v, with respect to a proper edge-coloring, is the set of colors of the edges incident at v. This chromatic parameter has been studied for several families of graphs, see for instance [5, 8, 9].

As for 4-regular graphs, the possible values for the palette index are 1, 3, 4 and 5 [4]; it is 1 if and only if the graph is class 1. It is 3 if and only if the 4-regular graph has an even 2-factor - a 2-factor with no odd cycle - or an ECD of index 3. There exist infinite families of 4-regular graphs with an ECD of index 3 [2]. As far as we know, no example of 4-regular graph having ECDs and palette index larger than 3 is known. Equivalently, no example of 4-regular graph whose edge-set can be partitioned into even cycles and every ECDs has index larger than 3 is known. Motivated by the problem on the existence of such a 4-regular graph, we pursue the study of ECDs in 4-regular line graphs of class 2 cubic graphs that started in [2].

We recall that the *line graph* of a graph G is the graph $L(G)$ whose vertices correspond to the edges of G; and two vertices are adjacent if the corresponding edges are incident in G. 4-regular line graphs of class 2 cubic graphs are an interesting family of class 2 graphs since, among others, they are related to some famous conjectures. For instance, the cycle double cover conjecture [11]; or Thomassen's conjecture on the existence of a Hamiltonian cycle in every 4-connected line graph [18]. In [15] it is also conjectured that the line graph of a bridgeless cubic graph has an ECD; and the conjecture is proved for the line graph of a bridgeless cubic graph with a chordless 2-factor. In [12] it is proved for the line graph of a bridgeless cubic graph of oddness 2 and 4. The *oddness* of a cubic graph G measures the uncolorability of G and is defined as the minimum number of odd cycles in a 2-factor of G taken over the set of all possible 2-factors of G.

The methods used in [12, 15] for the construction of an ECD in the line graph of a cubic graph G do not provide the ECD of $L(G)$ with the smallest index. In fact, for some cubic graphs G we find an ECD of index $k > 3$ even though we know that $L(G)$ has an ECD of index 3. In Section 2, we define a (novel) coloring technique on the edges of $L(G)$ that transforms each odd cycle belonging to a cycle decomposition of $L(G)$ into one path of odd length together with a pattern of even paths (see for instance Figure 1). By this technique we can construct an ECD of index 3 in $L(G)$ provided that the cubic graph G has a perfect matching M satisfying a prescribed property (see Theorem 1). In particular, we can prove the existence of an ECD of index 3 in the line graph of every cubic graph of oddness 2. The same holds for the line graphs of some class 2 cubic graphs with arbitrary large oddness that are known in the literature (see Section 6).

Our proof about the existence of an ECD of index 3 in the line graph of a cubic graph of oddness 2 is alternative to the proof in [12] about the existence of an arbitrary ECD. The new results on the line graphs of class 2 cubic graphs with arbitrary large oddness also give evidence to the conjecture in [15] and a contribution to the more general problem on the existence of ECDs in Eulerian gaphs. This last problem is widely studied in the literature. As far as we know, the existence of ECDs has been proved for 2-connected Eulerian planar graphs of even size [17]; for 2-connected Eulerian graphs of even size containing no subgraph contractible to K_5 [20]. Nevertheless, the result in [20] does not resolve the existence problem for 4-regular graphs since almost all 4-regular graphs have a K_5-minor [14].

Besides the classical graph theory problems, the coloring technique and the colored even cycles presented in this paper can be also used in some applications as, for instance, the self-assembly of DNA molecules [3].

We specify that the graphs considered in this paper are simple, unless otherwise stated. We refer to [1] for graph theory notation and terminology which is not explicitly expressed. In particular, the length $\ell(P)$ of a path P is the number of edges of P; a path is *even* or *odd* according to whether its length is even or odd, respectively. For a path P with endvertices x, y we will also use the notation $x\,P\,y$ and $y\,P\,x$ for the reversed path. We denote by M_C the subset of a perfect matching M of G consisting of the edges that are incident to the vertices in C, where C is a cycle in $G - M$. The subset of the vertex-set of $L(G)$ consisting of the vertices v_e corresponding to the edges e in $C \cup M_C$ will be denoted by $L(C \cup M_C)$.

2 A novel coloring technique

We define a $(1, 2)$-*subgraph* F of an arbitrary graph G as a subgraph whose connected components are either paths or even cycles. If F contains no path, then F is a 2-regular subgraph of G. We say that F is an *even* $(1, 2)$-*subgraph* of G if every component of F has even length; F will be an *even* 2-*regular subgraph* if it contains no path. Accordingly, a partition $\mathcal{F}$ of the edge-set of G into even $(1, 2)$-subgraphs will be called an *even* $(1, 2)$-*decomposition* of G. Because we will interpret a $(1, 2)$-subgraph of G as a colored class of a non-proper edge-coloring of G, an even $(1, 2)$-decomposition $\mathcal{F}$ consisting of $|\mathcal{F}|$ subgraphs will be referred as an *even* $(1, 2)$-*decomposition of index* $|\mathcal{F}|$. For $\mathcal{F}$ consisting of even 2-regular subgraphs, we will speak of an *even cycle decomposition of index* $|\mathcal{F}|$.

In the following, we consider $(1, 2)$-decompositions of the line graph of a class 1 graph G with maximum degree $\Delta(G) = 3$.

Given a class 1 graph G with $\Delta(G) = 3$, any proper 3-edge-coloring θ of G defines a non-proper 3-edge-coloring θ' of the corresponding line graph $L(G)$ whose colored classes form a $(1, 2)$-decomposition of $L(G)$ of index

3. More precisely, let M_1, M_2, M_3 be the colored classes of θ. For every $1 \leq h < k \leq 3$, the union $M_h \cup M_k$ is a $(1,2)$-factor of G, since its connected components are either paths or even cycles. Consequently, $L(M_h \cup M_k)$ is a $(1,2)$-subgraph of $L(G)$.

We set $H_1 = L(M_1 \cup M_2)$, $H_2 = L(M_1 \cup M_3)$, $H_3 = L(M_2 \cup M_3)$ and color by red, green and blue the edges of H_1, H_2 and H_3, respectively. Since $\{H_1, H_2, H_3\}$ is a partition of the edge-set of $L(G)$, a non-proper 3-edge-coloring θ' of $L(G)$, with color-set *red*, *green* and *blue*, is thus defined.

The following definition describes a novel coloring technique on the edges of $L(G)$ that modifies a blue path or cycle in H_3 into a pattern of red and green even paths, so that the number of odd paths in the red and green colored classes H_1, H_2 does not increase.

For a class 2 cubic graph, we will use this technique to modify every blue odd cycle in the corresponding line graph into a pattern of red and green even paths together with exactly one blue odd path. We will connect the blue odd paths and form blue even cycles, so as to obtain an even cycle decomposition of $L(G)$ of index 3. In order to connect the blue odd paths, the coloring technique will be applied to the blue even cycles as well (see Section 4).

Definition 1. Le $P = (u_0, u_1, u_2, \ldots, u_m, u_{m+1})$ be a path in $M_2 \cup M_3$ whose vertices u_i, $1 \leq i \leq m$, have degree 3; and u_0, u_{m+1} have degree 2. For $0 \leq i \leq m+1$, we denote by e_i be the edge of M_1 which is incident to u_i, the edges e_i are not necessarily distinct.

Let $P_{2,i}$ and $P_{4,i}$ be the 2- and 4-path of $L(G)$ defined by $P_{2,i} = (v_{e_i}, v_{u_i u_{i+1}}, v_{e_{i+1}})$, where $0 \leq i \leq m$; $P_{4,i} = (v_{e_i}, v_{u_{i-1} u_i}, v_{u_i u_{i+1}}, v_{u_{i+1} u_{i+2}}, v_{e_{i+1}})$, where $1 \leq i \leq m-1$.

For $u_0 u_1 \in M_2$, we modify the coloring θ' of $L(G)$ on the paths $P_{2,i}$ and $P_{4,i}$ as follows:

- we color by blue the edges $v_{u_0 u_1} v_{e_0}$, $v_{u_{m+1} u_m} v_{e_{m+1}}$;
- for $1 \leq i \leq m-1$, $i \equiv 1 \pmod 4$, we color by green the 4-paths $P_{4,i}$;
- for $1 \leq i \leq m-1$, $i \equiv 3 \pmod 4$, we color by red the 4-paths $P_{4,i}$;
- for $1 \leq i \leq m$, $i \equiv 1 \pmod 4$, we color by red the 2-paths $P_{2,i}$; and leave unchanged the green color for $i \equiv 3 \pmod 4$.

For $u_0 u_1 \in M_3$, we swap the red and green colors.

We apply the coloring technique of Definition 1 to the blue path in Figure 1(a), where the edge $u_0 u_1$ of G belongs to the colored class M_2. We obtain the pattern of red and green even paths of Figure 1(b) or (c) according to whether $m \equiv 2$ or $m \equiv 0 \pmod 4$, respectively.

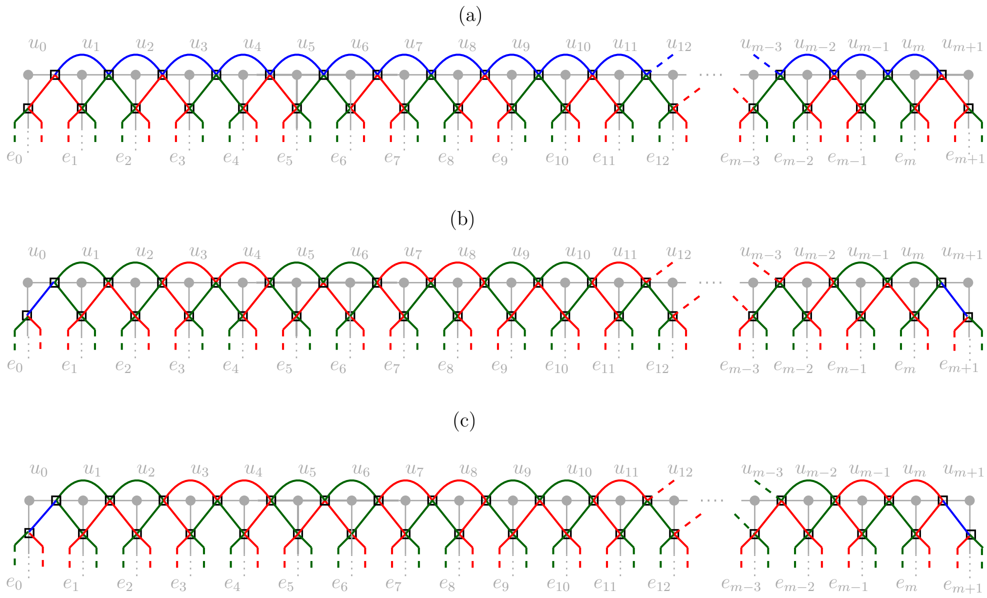


Figure 1: (a) The colored classes H_1, H_2, H_3 of the coloring θ' on the edges of $L(G)$ that are incident to the vertices in $L(P)$. In (b)-(c) we apply the coloring technique of Definition 1; (b) holds for $m \equiv 2 \pmod 4$; (c) holds for $m \equiv 0 \pmod 4$.

We denote by φ the new coloring of $L(G)$ obtained from Definition 1. Let H'_1, H'_2, H'_3 the red, green and blue classes of φ that are obtained from the red, green and blue classes H_1, H_2, H_3 of θ, respectively. In the following lemma, we prove that the coloring technique of Definition 1 leaves invariant or decreases the number of odd paths in the red and green classes H'_1, H'_2.

Lemma 1. *If the endvertices of every path in H_1 and H_2 correspond to edges of G belonging to the same colored class M_1, then the red and green colored classes H'_1, H'_2 of φ are even $(1,2)$-factors of $L(G)$.*

Moreover, v_{e_0}, $v_{e_{m+1}}$ are the endvertices of even paths in H'_1 and H'_2, possibly the same path, or do not belong to $V(H'_1)$ and $V(H'_2)$ at all.

Proof. We uncolor the paths $P_{2,i}$, $1 \leq i \leq m$, in H_1 and H_2 so that the red and green components of H_1 and H_2 visiting the vertices v_{e_j}, $0 \leq j \leq m+1$, result in paths P' with at least one endvertex in $\{v_{e_j} : 0 \leq j \leq m+1\}$. Each red path P' has even length, possibly length 0. In fact, vertices corresponding to edges in M_1 alternate to vertices corresponding to edges in M_2, since $H_1 = L(M_1 \cup M_2)$; by the assumptions, the endvertices of P' correspond to edges in M_1; hence P' has even length. Analogously for the green paths P'.

The coloring φ connects the red paths P' by the even paths $P_{4,i}$, $P_{2,j}$, where $1 \leq i \leq m-1$, $i \equiv 3 \pmod 4$, and $1 \leq j \leq m$, $j \equiv 1 \pmod 4$, assuming that $u_0u_1 \in M_2$. Hence the resulting paths or cycles have even length, see for instance Figure 1(b)-(c). Analogously for the green paths P'. For $u_0u_1 \in M_3$, the red and green are swapped.

It is easy to see that v_{e_0}, $v_{e_{m+1}}$ are the endvertices of even paths in H'_1 and H'_2, possibly the same path or of length 0. In this last case, v_{e_0}, $v_{e_{m+1}}$ do not belong to $V(H'_1)$ and $V(H'_2)$. □

3 Auxiliary graphs and colorings.

In this section, G is a bridgeless cubic graph, M and $G-M$ will denote a perfect matching of G and its complementary 2-factor, respectively. We give the auxiliary graphs of a method that, combined with the coloring technique of Definiton 1, provides an even cycle decomposition of index 3 in $L(G)$. The auxiliary graphs are ordered sets of cycles contained in the cycle-multigraph of G relative to M togheter with the graphs G', $L(G')$. The cycles and the graphs G', $L(G')$ are endowed with a prescribed coloring as we are going to explain.

We define the *cycle-multigraph of G relative to M*, denoted by $\mathcal{C}(G,M)$, as the multigraph obtained from $G-M$ by contracting each cycle of $G-M$ into a vertex and removing the resulting loops. A vertex of $\mathcal{C}(G,M)$ corresponding to the cycle C of $G-M$ will be denoted by v_C. Vertices of $\mathcal{C}(G, M)$ corresponding to cycles of odd length will be called *odd cycle-vertices*; vertices corresponding to cycles of even length will be called *even cycle-vertices*. Since G is connected, the graph $\mathcal{C}(G, M)$ is also connected and contains an even number of odd cycle-vertices. The following notations are in order.

Notation 2. For a cycle-vertex v_C in $\mathcal{C}(G, M)$ having degree $d \geq 2$, we label two incident edges by f, g. Then M contains at least two edges, say uu', vv', providing the edges f, g while contracting the cycles of $G - M$, where $u, v \in V(C)$, $u', v' \notin V(C)$.

The cycle C can be represented as the vertex-disjoint union of the paths $u\,P^C\,v = (u, u_1, \ldots, u_m, v)$, $w_1\,P^C\,w_n = (w_1, \ldots, w_n)$ togheter with the edges uw_1, vw_n. For an odd cycle C, the path $u\,P^C\,v$ will denote the subpath of odd length.

The edges M that are incident to u, v will be labeled by f, g as in $\mathcal{C}(G, M)$, respectively.

Notation 3. For a cycle-vertex v_C in $\mathcal{C}(G, M)$ having degree $d \geq 4$, we label four incident edges by f_1, f_2, f_3, f_4. Then M contains at least four edges, say $x_i x_i'$, $i = 1, \ldots, 4$, providing the edges f_i while contracting the cycles of $G - M$, and such that $x_i \in V(C)$, $x_i' \notin V(C)$, for $i = 1, \ldots, 4$.

The cycle C can be represented as the union of the paths $x_1\,P^C\,x_2 = (x_1, u_1, \ldots, u_m, x_2)$, $x_2\,P^C\,x_3 = (x_2, z_1, \ldots, z_n, x_3)$, $x_3\,P^C\,x_4 = (x_3, u'_{m'}, \ldots, u'_1, x_4)$, $x_4\,P^C\,x_1 = (x_4, z'_{n'}, \ldots, z'_1, x_1)$ that are pairwise internally disjoint.

The paths $x_i\,P^C\,x_{i+1}$, $x_{i+1}\,P^C\,x_{i+2}$, where $1 \leq i \leq 4$, will be called consecutive paths, as they share the endvertex x_{i+1} (subscripts are read modulo 4).

The edges of M that are incident to x_i will be labeled by f_i, $i = 1, \ldots, 4$, as in $\mathcal{C}(G, M)$.

In our construction, we assume that the odd cycle-vertices of $\mathcal{C}(G, M)$ can be partitioned into ordered sets of cycles; each ordered set, say $\{\Gamma_0, \ldots, \Gamma_r\}$, $r \geq 0$, spans an even number of odd cycle-vertices and satisfies the following property.

Property 4. (i) for every $i, j \in \{0, \ldots, r\}$, $i \neq j$, the vertex-sets $V(\Gamma_i)$, $V(\Gamma_j)$ have a non-empty intersection if and only if $|j - i| = 1$ and in this last case they share exactly one even cycle-vertex; moreover Γ_0, Γ_r contain at least one odd cycle-vertex, all other cycles Γ_i, $1 \leq i \leq r-1$, contain an even number of odd cycle-vertices, possibly none;

(ii) for a cycle-vertex $v_C \in V(\Gamma_i) \cap V(\Gamma_{i+1})$, whose incident edges are denoted according to Notation 3, the cycles Γ_i, Γ_{i+1} contain the edges f_j, f_{j+1} and f_{j+2}, f_{j+3}, respectively, where $j = 1$ or $j = 2$ (subscripts are read modulo 4).

(iii) there is a 2-edge coloring of $\mathcal{B} = \Gamma_0 \cup \ldots \cup \Gamma_r$, say by *red* and *green*, such that the odd cycle-vertices in $\mathcal{B}$ are incident to edges of distinct colors; even cycle-vertices of degree 2 in $\mathcal{B}$ are incident to edges of the same color; the edges that are incident to a cycle-vertex of degree 4 in $\mathcal{B}$ receive the same color if Γ_0, Γ_r contain an even number of odd cycle-vertices, otherwise f_j, f_{j+3} are red and f_{j+1}, f_{j+2} are green (or vice versa).

The following lemma shows that a cycle-multigraph having a prescribed Eulerian subgraph contains an ordered set of cycles satisfying the Property 4. In Section 5, we will prove that this is always the case for the cycle-graph of a cubic graph of oddness 2; the same holds for a wide range of cubic graphs of oddness larger than 2 that are known in the literature (see Section 6).

Lemma 2. *Assume that $\mathcal{C}(G, M)$ has an Eulerian subgraph $\mathcal{S}$ of maximum degree 4 containing exactly two odd cycle-vertices, say v_D, $v_{D'}$, having degree 2 in $\mathcal{S}$. Then $\mathcal{C}(G, M)$ contains an ordered set of cycles $\{\Gamma_0, \ldots, \Gamma_r\}$ with $v_D \in V(\Gamma_0)$, $v_{D'} \in V(\Gamma_r)$ that satisfies the Property 4.*

Proof. If the Eulerian subgraph is a cycle, then we can color its edges by red and green so that the Property 4(iii) is satisfied, since it contains an even number of odd cycle-vertices; the assertion follows.

Assume that the Eulerian subgraph $\mathcal{S}$ in $\mathcal{C}(G, M)$ contains at least one vertex of degree 4. For an even cycle-vertex v_C of degree 4 in $\mathcal{S}$, we label its incident edges according to Notation 3 and split it into two copies, say v_C^1 and v_C^2, that are incident to f_1, f_3 and f_2, f_4, respectively. The resulting graph is 2-regular and we denote its connected components by $\hat{\Gamma}_0, \ldots, \hat{\Gamma}_s$, $s \geq 0$. Actually, each cycle $\hat{\Gamma}_i$, $0 \leq i \leq s$, is also a cycle of $\mathcal{S}$ visiting a vertex v_C of degree 4 in $\mathcal{S}$ by the edges f_1, f_3 or f_2, f_4. We use $\hat{\Gamma}_0, \ldots, \hat{\Gamma}_s$ to find a red path and a green path whose union provides the cycles satisfying the Property 4. Without loss of generality, we can assume that $v_D \in V(\hat{\Gamma}_0)$ and $v_{D'} \in V(\hat{\Gamma}_t)$, where $t \leq s$.

Let $\mathbf{\Gamma}$ be the multigraph whose vertices are the cycles $\hat{\Gamma}_0, \ldots, \hat{\Gamma}_s$; and where two vertices, corresponding to the cycles $\hat{\Gamma}_a$, $\hat{\Gamma}_b$, $0 \leq a, b \leq s$, are connected by exactly γ edges if and only if there exist γ even cycle-vertices v_C of degree 4 in $\mathcal{S}$ such that $\hat{\Gamma}_a$ contains exactly one copy of v_C, say v_C^1, and $\hat{\Gamma}_b$ contains the other copy v_C^2. Since $\mathcal{S}$ is connected, $\mathbf{\Gamma}$ is also connected. Hence, it contains at least one path connecting the vertices corresponding to $\hat{\Gamma}_0$ and $\hat{\Gamma}_t$. Let $\hat{\Gamma}_0, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_t$ be the cycles corresponding to the vertices of the shortest path in $\mathbf{\Gamma}$ connecting the vertices related to $\hat{\Gamma}_0$ and $\hat{\Gamma}_t$. Since we are considering the shortest path in $\mathbf{\Gamma}$, the cycles $\hat{\Gamma}_i$, $\hat{\Gamma}_j$, with $1 \leq i + 1 < j \leq t$, share no cycle-vertex of $\mathcal{S}$ (otherwise we find a shorter path in $\mathbf{\Gamma}$, a contradiction). By construction of $\mathbf{\Gamma}$, the cycles $\hat{\Gamma}_i$, $\hat{\Gamma}_{i+1}$ share at least one cycle-vertex of $\mathcal{S}$.

For a cycle-vertex v_C of $\hat{\Gamma}_i$, we define the closest cycle-vertices of v_C in $\hat{\Gamma}_{i+1}$ as the vertices v_A, v_B in $V(\hat{\Gamma}_i) \cap V(\hat{\Gamma}_{i+1})$ such that the subpath $\hat{P}$ of $\hat{\Gamma}_i$, with endvertices v_A, v_B and visiting v_C, share no internal vertex with $\hat{\Gamma}_{i+1}$; $v_A = v_B$ if $\hat{\Gamma}_i$, $\hat{\Gamma}_{i+1}$ intersect in exactly one vertex. We will split $\hat{P}$ into the two subpaths with endvertices v_A, v_C and v_C, v_B.

Let v_{C^0}, $v_{\overline{C}^0}$ be the closest cycle-vertices of v_D in $\hat{\Gamma}_1$, possibly $v_{C^0} = v_{\overline{C}^0}$. We denote by $\hat{P}_{0,1}$ and $\hat{P}_{0,2}$ the subpaths of $\hat{\Gamma}_0$ with endvertices v_D, v_{C^0} and v_D, $v_{\overline{C}^0}$, respectively, sharing no internal vertex with $\hat{\Gamma}_1$.

For $i = 1, \ldots, t-1$, let v_{C^i}, $v_{\overline{C}^i}$ be the closest cycle-vertices of $v_{C^{i-1}}$, $v_{\overline{C}^{i-1}}$ in $\hat{\Gamma}_{i+1}$, respectively. We denote by $\hat{P}_{i,1}$ and $\hat{P}_{i,2}$ the subpaths of $\hat{\Gamma}_i$ with endvertices $v_{C^{i-1}}$, v_{C^i} and $v_{\overline{C}^{i-1}}$, $v_{\overline{C}^i}$, respectively, sharing no internal vertex with $\hat{\Gamma}_{i+1}$. The paths $\hat{P}_{i,1}$, $\hat{P}_{i,2}$ are internally disjoint but might share the vertices v_{C^i}, $v_{\overline{C}^i}$. It is understood that v_{C^i}, $v_{\overline{C}^i}$ are the two closest cycle-vertices of $v_{C^{i-1}}$ or $v_{\overline{C}^{i-1}}$, if $v_{C^{i-1}} = v_{\overline{C}^{i-1}}$. For $v_{C^{t-1}}$ and $v_{\overline{C}^{t-1}}$, we can find two internally disjoint subpaths of $\hat{\Gamma}_t$ with endvertices $v_{C^{t-1}}$, $v_{D'}$ and $v_{\overline{C}^{t-1}}$, $v_{D'}$, say $\hat{P}_{t,1}$ $\hat{P}_{t,2}$.

Because the paths $\hat{P}_{i,1}$, $0 \leq i \leq t$, are pairwise internally disjoint and $\hat{P}_{i,1}$, $\hat{P}_{i+1,1}$ share the endvertex v_{C^i}, their union is a path with endvertices v_D, $v_{D'}$ that we color by red. Analogously, the paths $\hat{P}_{i,2}$, $\hat{P}_{i+1,2}$ form a path with endvertices v_D, $v_{D'}$ that we color by green.

The union of the red and green path provides the ordered set of cycles $\{\Gamma_0, \ldots \Gamma_r\}$ that satisfies the Property 4. Two cycles Γ_j, Γ_{j+1} intersect in the even cycle-vertex $v_{C^i} = v_{\overline{C}^i}$ for some $i \in \{0, \ldots, t-1\}$, hence the Property 4(i) is satisfied. Because the edges of $\hat{\Gamma}_i$ that are incident to $v_{C^i} = v_{\overline{C}^i}$ are f_1, f_3 or f_2, f_4, the red edges a, b that are incident to $v_{C^i} = v_{\overline{C}^i}$ are f_j, f_{j+1}, where $1 \leq j \leq 4$, since $a \in E(\hat{\Gamma}_i)$ and $b \in E(\hat{\Gamma}_{i+1})$ (the subscripts are read modulo 4). Consequently, the green edges that are incident to v_C are f_{j+2}, f_{j+3}. Hence the Property 4(ii)-(iii) are satisfied and the assertion follows.

□

By Notation 2 and 3, we define the auxiliary graph G' as follows.

Definition 5. Let C be an odd cycle of $G - M$ whose corresponding odd cycle-vertex v_C has degree $d \geq 2$ in $\mathcal{C}(G, M)$. According to Notation 2, the cycle C is the the vertex-disjoint union of the paths $u\,P^C\,v = (u, u_1, \ldots, u_m, v)$, $w_1\,P^C\,w_n = (w_1, \ldots, w_n)$ togheter with the edges uw_1, vw_n.

We define G' as the graph obtained from G by deleting the edges uw_1, vw_n in every odd cycle C of $G - M$.

It is easy to see that G' is class 1, since in every odd cycle C of $G - M$ at least one edge has been removed. Moreover, its maximum degree is $\Delta(G') \leq 3$. The graphs G' and $L(G')$ inherit the edge-coloring of G and $L(G)$ defined by M. More specifically, the following lemma holds.

Lemma 3. *The perfect matching M induces a proper 3-edge-coloring θ of G' with color-set $\{a_1, a_2, a_3\}$, where the edges of M are colored by a_1, that ones of $G' - M$ are colored alternately by a_2, a_3 in such a way that $\theta(uu_1) = \theta(vu_m) = \theta(w_1w_2) = a_3$ and $\theta(w_{n-1}w_n) = a_2$, for every odd cycle C of $G - M$ whose vertices are labeled according to Notation 2. Consequently, the palettes of the vertices are contained in $\{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\}$.*

For $i = 1, 2, 3$, let M_i be the colored class of θ consisting of the edges of color a_i. Then θ defines a non-proper 3-edge-coloring θ' of $L(G')$ whose colored classes $H_1 = L(M_1 \cup M_2)$, $H_2 = L(M_1 \cup M_3)$ are even $(1, 2)$-factors of $L(G')$. The endvertices of the paths in H_1 correspond to the edges of $M = M_1$ that are incident to the vertices u, v, w_1 of every odd cycle C of $G - M$; the endevertices of the paths in H_2 correspond to the edges of $M = M_1$ that are incident to the vertex w_n of every odd cycle C of $G - M$.

Proof. Since G' is obtained from G by deleting the edges uw_1, vw_n in every odd cycle C of $G - M$ which is colored alternately by a_2, a_3, no vertex of G' has palette $\{a_2, a_3\}$. Notice that we can color alternately the edges of G' in such a way that $\theta(uu_1) = \theta(vu_m) = \theta(w_1w_2) = a_3$ and $\theta(w_{n-1}w_n) = a_2$, since $u\,P^C\,v$ has odd length and $w_1\,P^C\,w_n$ has even length, see Notation 2. Hence, the palettes of the vertices are contained in $\{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\}$.

By the remarks in Section 1, the colored classes H_1, H_2, H_3 are $(1, 2)$-factors of $L(G')$. Since the palettes of the vertices of G' are contained in $\{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\}$, the paths in $M_1 \cup M_2$ and in $M_1 \cup M_3$ start and end with edges of M_1, that is, they have odd length. Consequently, the paths in H_1, H_2 have even length and the endvertices correspond to edges of $M = M_1$. More specifically, the endvertices of the paths in H_1 correspond to the edges of $M = M_1$ that are incident to the vertices u, v, w_1 of every odd cycle C of $G - M$, since $\theta(uu_1) = \theta(vu_m) = \theta(w_1w_2) = a_3$. The endevertices of the paths in H_2 correspond to the edges of $M = M_1$ that are incident to the vertex w_n of every odd cycle C of $G - M$, since $\theta(w_{n-1}w_n) = a_2$.

Notice that the vertices of G' with palette $\{a_1\}$, if any, provide isolated vertices, or equivalently paths of null length, that will be not considered in in H_1, H_2. □

Remark 6. The whole set of edges of $M = M_1$ that are incident to the vertices u, v, w_1, w_n of the odd cycles C of $G - M$ are not necessarily distinct. Hence, an endvertex of a path in H_1 might be also an endvertex of a path in H_2.

We give an example showing the role of the auxiliary graphs G', $L(G')$ and the process that led to obtain the even cycle decomposition of index 3 in $L(G)$ by the sets of cycles satisfying the Property 4.

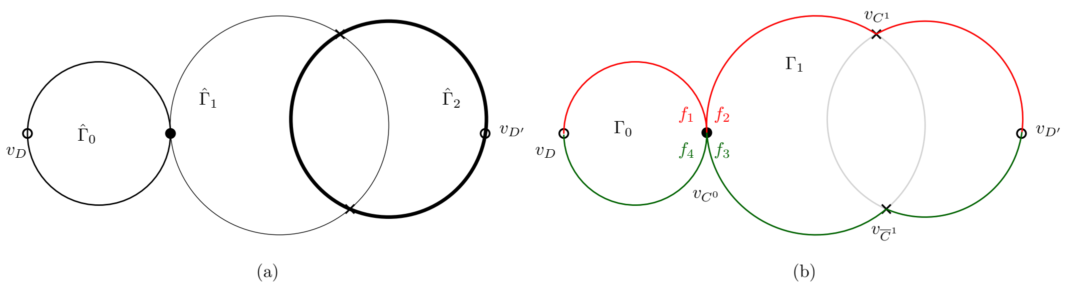


Figure 2: (a) The graph $\mathcal{S} = \mathcal{C}(G, M)$ of Example 7 with odd cycle-vertices v_D, $v_{D'}$; even cycle-vertices of degree 4 are marked by a cross; the edges are partitioned into the cycles $\hat{\Gamma}_i$, $0 \le i \le 2$. (b) The ordered set of cycles $\{\Gamma_0, \Gamma_1\}$ endowed with the $\{red,\ green\}$-coloring of the Property 4; v_{C^0} is the closest cycle-vertex of v_D in $\hat{\Gamma}_1$; v_{C^1}, $v_{\overline{C}^1}$ are the closest cycle-vertices of v_{C^0} in $\hat{\Gamma}_2$.

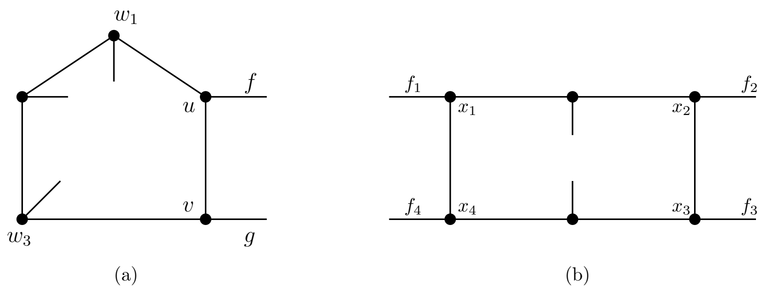


Figure 3: The cycles corresponding to the cycle-vertices of Example 7: (a) holds for the odd cycle-vertices v_D, $v_{D'}$; (b) holds for the even cycle-vertices of degree 4 in $\mathcal{S}$.

Example 7. Assume that $\mathcal{C}(G, M)$ is the Eulerian graph $\mathcal{S}$ in Figure 2(a) having exactly two odd cycle-vertices v_D, $v_{D'}$ of degree 2. The edge-set of $\mathcal{S}$ is partitioned into the cycles $\hat{\Gamma}_i$, $0 \le i \le 2$, providing the ordered set of cycles $\{\Gamma_0, \Gamma_1\}$ of $\mathcal{C}(G, M)$ satisfying the Property 4; their union $\mathcal{B} = \Gamma_0 \cup \Gamma_1$ is endowed with the 2-coloring of the Property 4(iii), see Figure 2(b). For the sake of brevity, we assume that the odd cycles D, D' and the even cycles C are the 5- and 6-cycles in Figure 3.

We replace each cycle-vertex v_C by the corresponding cycle C in such a way that the edges of $\mathcal{B}$ that are incident to v_C correspond to the edges of G with the same label. For $C = D, D'$, we also remove the edges uw_1, $vw_n = vw_3$ and obtain the graph G' defined in Definition 5. We construct the line graph $L(G')$ endowed with the coloring θ' of Lemma 3 by replacing the cycle-vertices v_C with the graphs in Figure 4. Notice that a vertex v_e of

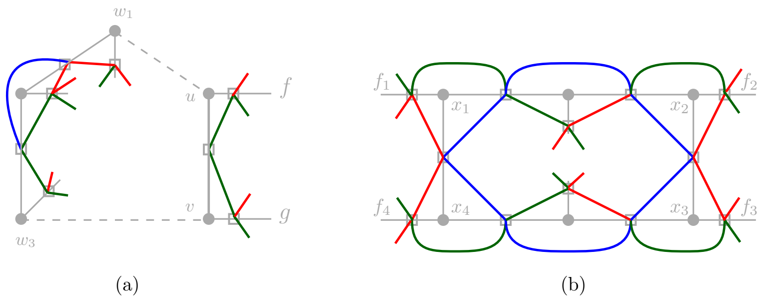


Figure 4: The line graph $L(G')$ of Example 7 endowed with the coloring θ' defined in Lemma 3 is obtained by replacing the cycle-vertices v_C in $\mathcal{S}$ by the graphs in the figure: (a) holds for $C = D$, D' where uw_1, vw_3 have been removed; (b) holds for the even cycles C with v_C of degree 4 in $\mathcal{S}$.

$L(G')$, corresponding to the edge $e \in M = M_1$, is incident to two red edges and two green edges, provided that the endvertices of e are different from the vertices u, v, w_1, w_3 of the odd cycles D, D' of $G - M$. By Lemma 3 and Remark 6, if the edge $e \in M = M_1$ is incident to u, v, w_1 (respectively, w_3) then v_e is incident to λ green edges and μ red edges, where $1 \leq \lambda \leq 2$ and $0 \leq \mu \leq 1$ (respectively, $0 \leq \lambda \leq 1$ and $1 \leq \mu \leq 2$).

We apply Definition 1 and obtain the 3-edge-coloring φ of $L(G)$ by modifying the colored classes H_1, H_2, H_3 of θ' as follows. We consider the cycles C of $G - M$ with corresponding cycle-vertex v_C in $\mathcal{B} = \Gamma_0 \cup \Gamma_1$; the edges of $L(G')$ that are incident to the vertices in $L(C)$ are colored according to the edge-coloring θ' of Lemma 3.

For an odd cycle $C = D, D'$, we delete uv and apply Definition 1 to the path $uw_1 \cup w_1\, P^C\, w_3$, we obtain a colored subgraph of $L(G' \cup uw_1)$ that we complete to the colored subgraph of $L(G)$ in Figure 5(a). Analogously, for the even cycles C with v_C in $\mathcal{S}$ we delete a suitable set S of edges and apply Definition 1 to the resulting paths; we obtain a colored subgraph of $L(G - S)$ that we complete to the colored subgraph of $L(G)$ in Figure 5(b)-(d). In particular, for the cycle $C = C^0$ we use the graph in Figure 5(c) or (d) according to the length of the path $x_1\, P^C\, x_4$: we use (c) if $x_1\, P^C\, x_4$ has length 1; we use (d) if $x_1\, P^C\, x_4$ has length 2. In Figure 6 and 7 we represent $L(G)$ endowed with φ. In Figure 6, the graph $L(C^0 \cup M_{C^0})$ has been replaced by the graph in Figure 5(c); in Figure 7 by the graph in Figure 5(d). The graphs in Figure 5(c)-(d) behave differently. The former has two blue odd paths: one connecting to the left blue odd path and forming with it an even blue cycle; the other is connected to the right blue paths and forms with them another even blue cycle. The graph in Figure 5(d) has two blue even paths both connecting to the left and right blue paths and contributing in forming an even blue cycle. In Section 4, the underlying cycles of the colored graphs

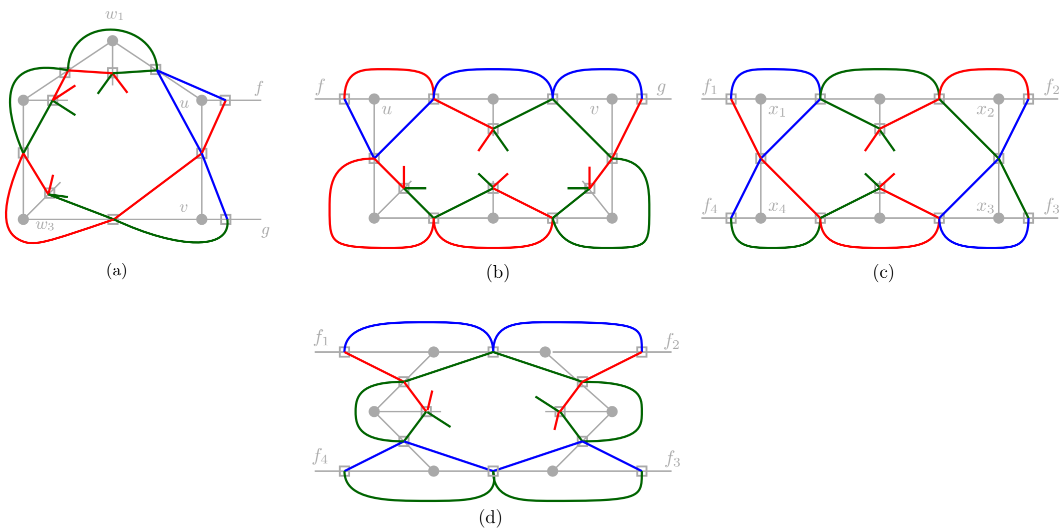


Figure 5: The colored subgraphs of $L(G)$ in Example 7 obtained by applying Definition 1 to the cycles C of $G - M$ with v_C in $\mathcal{B}$: (a) holds for the odd cycles D, D' if the edges f, g incident to v_D, $v_{D'}$ in $\Gamma_0 \cup \Gamma_1$ are red and green, respectively; we swap the colors red and green if f is green and g is red in $\mathcal{B}$. (b) holds for the even cycles C with v_C of degree 2 in $\mathcal{B}$ whose incident edges f, g are colored by red, we swap the colors red and green if f, g are green. (c)-(d) hold for the even cycle C^0 with v_{C^0} of degree 4 in $\mathcal{B}$.

in Figure 5(c)-(d) will be called 'separating cycles' and 'junction cycles', respectively. The underlying cycles of the colored graphs in Figure 5(a)-(b) will be called 'fitting cycles', as they have just one blue path - odd in case (a) and even in case (b) - connecting to the left and right blue pahts.

As remarked, the blue colored class is an even 2-regular subgraph of $L(G)$, since it contains no odd cycle. The same holds for the red and green colored classes: it follows from the properties of the graphs in Figure 5 and from Lemma 1, as the colored classes of φ have been obtained from the colored classes of θ' by applying Defintion 1.

4 Fitting, junction and separating cycles.

As in Section 3, we assume that G is a bridgeless cubic graph having a perfect matching M and a cycle-graph $\mathcal{C}(G, M)$ whose odd cycle-vertices can be partitioned into sets of cycles satisfying the Property 4. G' will denote the graph defined in Definition 5. We give the properties characterizing the fitting, junction and separating cycles that we met in Example 7.

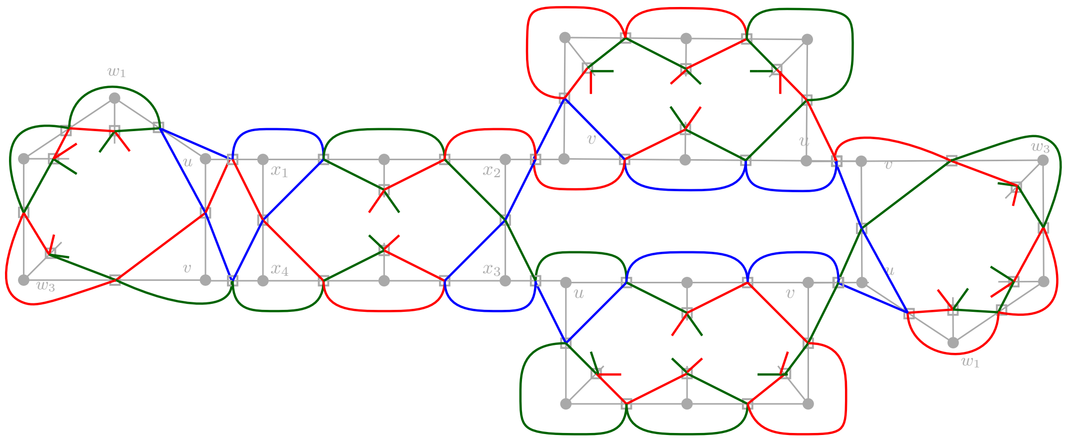


Figure 6: The graph $L(G)$ in Example 7 endowed with the coloring φ where $L(C^0 \cup M_{C^0})$ has been replaced by the graph in Figure 5(c).

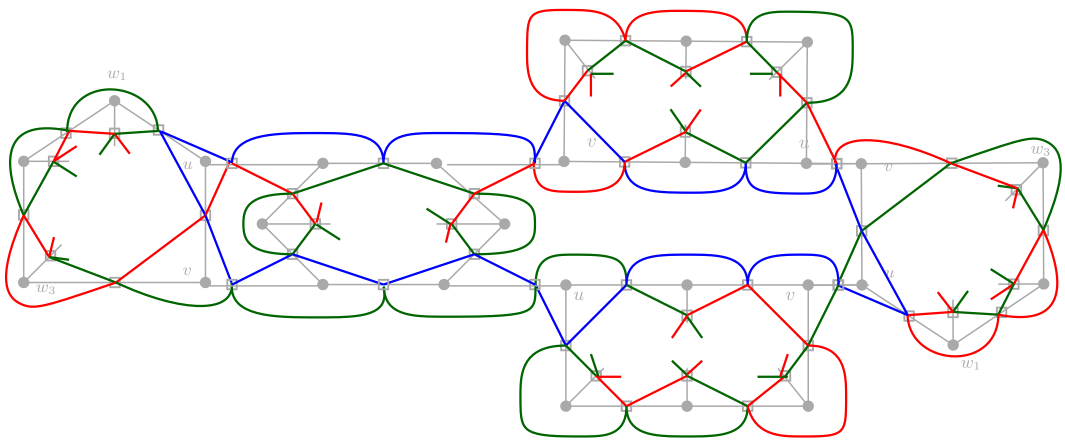


Figure 7: The graph $L(G)$ in Example 7 endowed with the coloring φ where $L(C^0 \cup M_{C^0})$ has been replaced by the graph in Figure 5(d).

Lemma 4. Fitting cycles. *Let C be a cycle of $G - M$ whose corresponding cycle-vertex v_C belongs to $V(\mathcal{B})$, where $\mathcal{B} = \Gamma_0 \cup \ldots \cup \Gamma_r$ and $\{\Gamma_0 \cup \ldots \cup \Gamma_r\}$ is the ordered set of cycles of $\mathcal{C}(G, M)$ satisfying the Property 4; v_C has degree 2 in $\mathcal{B}$; vertices and edges of C are labeled according to Notation 2.*

We modify the edge-coloring θ' on the edges of $L(G')$ that are incident to the vertices in $L(C)$ by applying Definition 1. We obtain a coloring of $L(G)$ satisfying following properties:

(1) *$L(C \cup \{f, g\})$ contains a blue path with endvertices v_f, v_g having even or odd length according to whether C has even or odd length, respectively;*

(2) *if the edge $e \in \{f, g\}$ is colored by red (respectively, green) in $\mathcal{B}$, then there is a red (respectively, green) even path with vertices in $L(C \cup M_C)$ connecting v_e to a vertex $v_{e'}$ where $e' \in M_C$, $e \neq e'$.*

The cycle C will be called a fitting cycle.

Proof. We modify the coloring θ' on the edges of $L(G')$ that are incident to the vertices in $L(C)$ by applying Definition 1 to a suitable odd path of C, which is obtained by removing a prescribed set of edges of C. For an odd path of length larger than 1, Definition 1 provides a pattern of red and green even paths together with two blue edges, see for instance Figure 1; for an odd path of length 1, it provides just two blue incident edges.

As for the edges of $M = M_1$ that are incident to the vertices u_1, u_m, w_1, w_n of C, we will use the notation e_1, e_m, h_1, h_n, respectively.

We distinguish the following cases: v_C is an odd cycle-vertex; v_C is an even cycle-vertex.

Case 1: v_C is an odd cycle-vertex.

According to Notation 2, that path $u\, P^C\, v$ is the subpath of C having odd length; we recall that in G' the edges uw_1, $vw_n \in E(C)$ have been removed. By the Property 4(iii), the edges f, g have distinct colors in $\mathcal{B}$, say red and green, respectively. We modify the coloring θ' as follows.

If $\ell(u\, P^C\, v) > 1$, then we apply Definition 1 to the path $u\, P^C\, v$ so that the following edges of $L(G)$ become blue: $v_a v_b$ with $(a, b) = (f, uu_1)$, (g, vu_m). We add the blue 2-paths (v_a, v_b, v_c), where $(a, b, c) = (w_1w_2, uw_1, uu_1)$, $(w_{n-1}w_n, vw_n, vu_m)$, and obtain an odd blue path with endvertices v_f, v_g. We also add the red 2-path (v_a, v_b, v_c) where $(a, b, c) = (h_1, uw_1, f)$ and the green 2-path (v_a, v_b, v_c) where $(a, b, c) = (h_n, vw_n, g)$, that connect to the red (respectively, green) even paths in $L(G')$ with endvertices corresponding to the edges of $M = M_1$ that are incident to u, v, w_1 (respectively, to w_n), see Lemma 3. The resulting coloring of $L(G)$ satisfies the properties (1)-(2), that is, the cycle C is fitting. See for instance the 7-cycle in Figure 8(b).

If $u\, P^C\, v = (u, v)$, then in G' we delete the edge uv and add the edge uw_1; and apply Definition 1 to the path $uw_1 \cup w_1\, P^C\, w_n$. The following edges of $L(G)$ become blue: $v_a v_b$ with $(a, b) = (f, uw_1)$, $(w_{n-1}w_n, h_n)$. We recolor the edge $v_{w_{n-1}w_n}v_{h_n}$ by adding the red 4-path $(v_a, v_b, v_c, v_d, v_e)$ where $(a, b, c, d, e) = (h_n, w_{n-1}w_n, vw_n, uv, f)$; and the green 2-path (v_a, v_b, v_c) where $(a, b, c) = (h_n, vw_n, g)$ We also add the blue 2-path (v_{uw_1}, v_{uv}, v_g) that combined with the blue edge $v_f v_{uw_1}$ form an odd blue path. The resulting coloring of $L(G)$ satisfies the properties (1)-(2), that is, C is a fitting cycle. See for instance the 5-cycle in Figure 5(a) of Example 7 or the 3-cycle in Figure 8(a).

Case 2: v_C is an even cycle-vertex.

By the Property 4(iii), the edges f, g have the same color in B, say red.

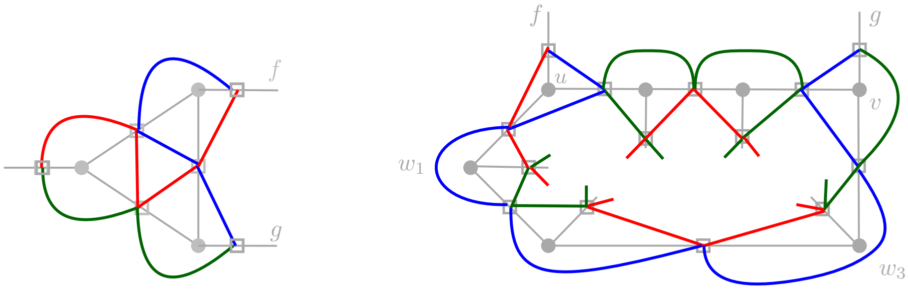
If $u\,P^C\,v = (u, v)$, then we delete the edge uv and apply Definition 1 to the path $uw_1 \cup w_1\,P^C\,w_n \cup vw_n$ so that the edges $v_f v_{uw_1}$, $v_g v_{vw_n}$ of $L(G)$ become blue. We add the red 2-path (v_f, v_{uv}, v_g); and the blue 2-path $(v_{uw_1}, v_{uv}, v_{vw_n})$. The resulting coloring of $L(G)$ satisfies the properties (1)-(2), that is, the cycle C is fitting. See for instance the 4-cycle in Figure 9(a).

If $u\,P^C\,v$ has odd length $\ell > 1$, then we delete the edges uu_1, vu_m and apply Definition 1 to the path $uw_1 \cup w_1\,P^C\,w_n \cup vw_n$ so that the edges $v_f v_{uw_1}$, $v_g v_{vw_n}$ of $L(G)$ become blue. We add the blue 2-paths (v_a, v_b, v_c) where $(a, b, c) = (uw_1, uu_1, u_1u_2)$, $(vw_n, vu_m, u_{m-1}u_m)$ that, combined with the blue edges $v_f v_{uw_1}$, $v_g v_{vw_n}$ and the blue even path $L((u_1, \ldots, u_m))$ form an even blue path with endvertices v_f, v_g. We also add the red 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f, uu_1, e_1)$, (g, vu_m, e_m). The resulting coloring of $L(G)$ satisfies the properties (1)-(2), that is, the cycle C is fitting. See for instance the 8-cycle in Figure 9(c).

As for $u\,P^C\,v$ of even length $\ell \geq 2$, we delete the edges uu_1, vw_n and apply Definition 1 to the path $uw_1 \cup w_1\,P^C\,w_n$. The following edges of $L(G)$ become blue: $v_a v_b$ with $(a, b) = (f, uw_1)$, $(h_n, w_{n-1}w_n)$. We recolor the edge $v_{h_n} v_{w_{n-1}w_n}$ by adding the green 4-path $(v_{h_n}, v_{w_{n-1}w_n}, v_{vw_n}, v_{vu_m}, v_{e_m})$; and the red 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f, uu_1, e_1)$, (g, vw_n, h_n). We also add the blue edge $v_g v_{vu_m}$ and the blue 2-path $(v_{uw_1}, v_{uu_1}, v_{u_1u_2})$ that combined with the odd blue path $L((u_1, \ldots, u_m, v))$ form an even blue path with endvertices v_f, v_g. The resulting coloring of $L(G)$ satisfies the properties (1)-(2), that is, the cycle C is fitting. See for instance the 8-cycle in Figure 5(b) or the 4- and 6-cycles in Figure 9(b) and (d). □

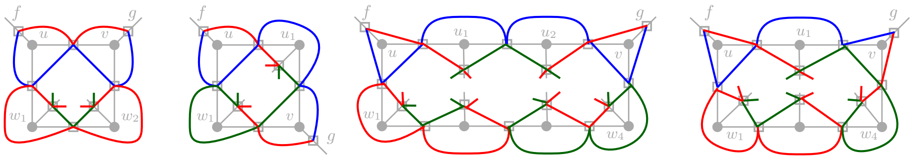
A remark is in order before we introduce the notion of 'junction and separating cycles'. More specifically, since in Notation 3 the edges f_i, $i = 1, \ldots, 4$, are labeled clockwise, without loss of generality we can assume that $j = 2$ in the Property 4(ii), that is, for a cycle-vertex v_C belonging to the cycles Γ_i, Γ_{i+1}, the cycles Γ_i and Γ_{i+1} contain the edges f_1, f_4 and f_2, f_3, respectively. The definition of 'junction and splitting cycles' can thus be stated as follows.

Lemma 5. Junction and separating cycles. *Let C be a cycle of $G - M$ whose corresponding cycle-vertex v_C belongs to $V(\mathcal{B})$, where $\mathcal{B} = \Gamma_0 \cup \ldots \cup \Gamma_r$*



(a) (b)

Figure 8: An odd cycle C in Lemma 4.



(a) (b) (c) (d)

Figure 9: An even cycle C in Lemma 4

and $\{\Gamma_0 \cup \ldots \cup \Gamma_r\}$ is the order set of cycles of $\mathcal{C}(G, M)$ satisfying the Property 4; v_C has degree 4 in $\mathcal{B}$; vertices and edges of C are labeled according to Notation 3.

We modify the edge-coloring θ' on the edges of $L(G')$ that are incident to the vertices in $L(C)$ by applying Definition 1. We obtain a coloring of $L(G)$ such that for a red (respectively, green) edge e in $\mathcal{B}$, where $e \in \{f_i : 1 \leq i \leq 4\}$, there is a red (respectively, green) even path with vertices in $L(C \cup M_C)$ connecting v_e to a vertex $v_{e'}$ where $e' \in M_C$, $e \neq e'$. Moreover, one of the following properties holds:

(1) *there are two blue paths in $L(C \cup \{f_i : 1 \leq i \leq 4\})$ with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, respectively, both having even or odd length;*

(2) *there are two blue paths in $L(C \cup \{f_i : 1 \leq i \leq 4\})$ with endvertices v_a, v_b, where $\{a, b\} = \{f_1, f_4\}$, $\{f_2, f_3\}$, having even or odd length according to whether a, b have the same color or not in $\mathcal{B}$, respectively.*

In the former case, the cycle C will be called a junction cycle; in the latter case, a separating cycle.

Proof. According to the Property 4(iii), the edges f_i, $1 \leq i \leq 4$, might have the same color in $\mathcal{B}$; or f_1, f_2 are red and f_3, f_4 are green in $\mathcal{B}$, since we can assume $j = 2$ in the Property 4(ii). We will distinguish the two cases and in each case we will consider the four possibilities corresponding to the possible lengths of the paths partitioning the edge-set of an even cycle C, see Notation 3. More specifically, we will consider the following cases: each path has odd length; each path has even length; just two vertex-disjoint paths $x_i\, P^C\, x_{i+1}$, $x_{i+2}\, P^C\, x_{i+3}$ have odd length; just two consecutive paths $x_i\, P^C\, x_{i+1}$, $x_{i+1}\, P^C\, x_{i+2}$ have odd length, where $1 \le i \le 4$ and the subscripts are read modulo 4.

We will use the following notation for the edges of $M = M_1$ that are incident to the vertices of the paths $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$, $x_3\, P^C\, x_4$, $x_4\, P^C\, x_1$: e_i, h_j, $e'_{i'}$, $h'_{j'}$ for the edges of $M = M_1$ with endvertex u_i, z_j, $u'_{i'}$, $z'_{j'}$, respectively, where $1 \le i \le m$, $1 \le j \le n$, $1 \le i' \le m'$ and $1 \le j' \le n'$.

In each case, we apply Definition 1 to suitable odd paths of C for obtaining the desired coloring of $L(G)$. The odd paths are obtained by deleting suitable sets of edges in C and are endowed the the coloring θ' defined in Lemma 3. We recall that for an odd path of length larger than 1, Definition 1 provides a pattern of red and green even paths together with two blue edges, see for instance Figure 1; for an odd path of length 1, it provides just two blue incident edges.

Case 1: the edges f_1, f_2, f_3, f_4 have the same color in $\mathcal{B}$.

We assume that the edges f_i, $1 \le i \le 4$, are red and apply Definition 1 as follows

Case 1(a): each path $x_i\, P^C\, x_{i+1}$ has odd length.

If there exists at least one pair of vertex-disjoint paths partitioning $E(C)$, say $x_4\, P^C\, x_1$, $x_2\, P^C\, x_3$, whose paths have length larger than 1, then we delete the edges x_1u_1, x_2u_m, $x_3u'_{m'}$, $x_4u'_1$ and apply Definition 1 to the odd paths $x_4\, P^C\, x_1$, $x_2\, P^C\, x_3$, so that the following edges of $L(G)$ become blue: v_av_b where $(a,b) = (f_1, x_1z'_1)$, $(f_4, x_4z'_{n'})$, (f_2, x_2z_1), (f_3, x_3z_n). We add the blue 2-paths (v_a, v_b, v_c) where $(a,b,c) = (x_1z'_1, x_1u_1, u_1u_2)$, $(x_2z_1, x_2u_m, u_{m-1}u_m)$, $(x_3z_n, x_3u'_{m'}, u'_{m'-1}u'_{m'})$, $(x_4z'_{n'}, x_4u'_1, u'_1u'_2)$. We also add the red 2-paths (v_a, v_b, v_c) where $(a,b,c) = (f_1, x_1u_1, e_1)$, (f_2, x_2u_m, e_m), $(f_3, x_3u'_{m'}, e'_{m'})$, $(f_4, x_4u'_1, e'_1)$. We obtain the desired coloring of $L(G)$ having two even blue paths with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, that is, the cycle C is a junction cycle. See for instance the graph in Figure 10(d).

If every pair of vertex-disjoint paths partitioning $E(C)$ contains a path having length 1, then we can set $\ell(x_1\, P^C\, x_2) = \ell(x_2\, P^C\, x_3) = 1$. We remove the edges $x_4z'_{n'}$, x_2x_3 and apply Defintion 1 to the path $x_3\, P^C\, x_4$ so that the following edges of $L(G)$ become blue: v_av_b where $(a,b) = (f_3, x_3u'_{m'})$, $(f_4, x_4u'_1)$. We recolor v_av_b where $(a,b) = (f_3, x_3u'_{m'})$ by adding the red 4-path $(v_a, v_b, v_c, v_d, v_e)$ where $(a,b,c,d,e) = (f_3, x_3u'_{m'}, x_2x_3, x_2x_1, f_2)$; and the red 2-path (v_a, v_b, v_c) where $(a,b,c) =(f_4, x_4z'_{n'}, h'_{n'})$. We color by blue the edge v_av_b where $(a,b) = (f_1, x_1x_2)$; we also add the blue 2-path (v_a, v_b, v_c) where $(a,b,c) =(f_2, x_2x_3, f_3)$, $(z'_{n'-1}z'_{n'}, z'_{n'}x_4, x_4u'_1)$. We obtain the desired coloring of $L(G)$ having two even blue paths with endvertices v_{f_1}, v_{f_4} and

v_{f_2}, v_{f_3}, that is, the cycle C is a separating cycle. See for instance the graphs in Figure 10(a)-(c).

Case 1(b): each path $x_i\, P^C\, x_{i+1}$ has even length.

We remove the edges x_1u_1, $x_4z'_{n'}$, x_2z_1, $x_3u'_{m'}$ and apply Defintion 1 to the odd paths $(x_1, z'_1, \ldots, z'_{n'})$, $(z_1, \ldots, z_n, x_3)$. The following edges of $L(G)$ become blue: v_av_b where $(a, b) = (f_1, x_1z_1)$, $(h'_{n'}, z'_{n'-1}z'_{n'})$, (h_1, z_1z_2), (f_3, x_3z_n). We recolor the edges v_av_b where $(a, b) = (h'_{n'}, z'_{n'-1}z'_{n'})$, (h_1, z_1z_2) by adding the green 4-path (v_a, v_b, v_c, v_d) where $(a, b, c, d) = (h'_{n'}, z'_{n'-1}z'_{n'}, x_4z'_{n'}, x_4u'_1, e'_1)$, $(h_1, z_1z_2, x_2z_1, x_2u_m, e_m)$. We also add the red 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f_1, x_1u_1, e_1)$, (f_2, x_2z_1, h_1), $(f_3, x_3u'_{m'}, e'_{m'})$, $(f_4, x_4z'_{n'}, h'_{n'})$; the blue 2-paths (v_a, v_b, v_c) where $(a, b, c) = (x_1z'_1, x_1u_1, u_1u_2)$, $(x_3z_n, x_3u'_{m'}, u'_{m'-1}u'_{m'})$; and the blue edges v_av_b where $(a, b) = (f_2, x_2u_m)$, $(f_4, x_4u'_1)$. We obtain the desired coloring of $L(G)$ having two even blue paths with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, that is, the cycle C is a junction cycle. See for instance the graphs in Figure 11(a)-(b).

Case 1(c): just two vertex-disjoint paths $x_i\, P^C\, x_{i+1}$, $x_{i+2}\, P^C\, x_{i+3}$ have odd length.

Without loss of generality, we can assume that $x_1\, P^C\, x_2$, $x_3\, P^C\, x_4$ have odd length. Consequently, $x_2\, P^C\, x_3$, $x_4\, P^C\, x_1$ have even length. We remove the edges x_1u_1, $x_4z'_{n'}$, x_2u_m, x_3z_n and apply Definition 1 to the odd paths $(x_1, z'_1, \ldots, z'_{n'})$, $(x_2, z_1, \ldots, z_n)$. The following edges of $L(G)$ become blue: v_av_b where $(a, b) = (f_1, x_1z'_1)$, $(z'_{n'-1}z'_{n'}, h'_{n'})$, (f_2, x_2z_1), $(z_{n-1}z_n, h_n)$. We recolor the edges v_av_b where $(a, b) = (z'_{n'-1}z'_{n'}, h'_{n'})$, $(z_{n-1}z_n, h_n)$ by adding the green 4-paths $(v_a, v_b, v_c, v_d, v_e)$ where $(a, b, c, d, e) = (h_n, z_{n-1}z_n, x_3z_n, x_3u'_{m'}, e'_{m'})$, $(h'_{n'}, z'_{n'-1}z'_{n'}, x_4z'_{n'}, x_4u'_1, e'_1)$ if $\ell(x_3\, P^C\, x_4) > 2$. Notice that the two green 4-paths form a 6-path if $\ell(x_3\, P^C\, x_4) = 1$. We also add the red 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f_1, x_1u_1, e_1)$, (f_2, x_2u_m, e_m), (f_3, x_3z_n, h_n), $(f_4, x_4z'_{n'}, h'_{n'})$; the blue 2-paths (v_a, v_b, v_c) where $(a, b, c) = (x_1z'_1, x_1u_1, u_1u_2)$, $(x_2z_1, x_2u_m, u_{m-1}u_m)$; and the blue edges v_av_b where $(a, b) = (f_3, x_3u'_{m'})$, $(f_4, x_4u'_1)$. We thus obtain the desired coloring of $L(G)$ having two even blue paths with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, that is, the cycle C is a junction cycle. See for instance the graphs in Figure 11(c)-(d).

Case 1(d): just two consecutive paths $x_i\, P^C\, x_{i+1}$, $x_{i+1}\, P^C\, x_{i+2}$ have odd length.

Without loss of generality, we can assume that $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$ have odd length. Consequently, $x_3\, P^C\, x_4$, $x_4\, P^C\, x_1$ have even length.

If at least one of the paths $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$ has length larger than 1, say $x_2\, P^C\, x_3$, then we remove the edges x_1u_1, x_2u_m, $x_3u'_{m'}$, $x_4z'_{n'}$ and apply Definition 1 to the odd paths $(x_1, z'_1, \ldots, z'_{n'})$, $x_2\, P^C\, x_3$. The following edges of $L(G)$ become blue: v_av_b where $(a, b) = (f_1, x_1z'_1)$, $(h'_{n'}, z'_{n'-1}z'_{n'})$, (f_2, x_2z_1), (f_3, x_3z_n). We recolor the edge v_av_b where $(a, b) = (h'_{n'}, z'_{n'-1}z'_{n'})$ by adding the green 4-path $(v_a, v_b, v_c, v_d, v_e)$ where $(a, b, c, d, e) = (h'_{n'}, z'_{n'-1}z'_{n'}, x_4z'_{n'}, x_4u'_1, e'_1)$. We also add the red 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f_1, x_1u_1, e_1)$, (f_2, x_2u_m, e_m),

$(f_3, x_3u'_{m'}, e'_{m'})$, $(f_4, x_4z'_{n'}, h'_{n'})$; the blue 2-paths (v_a, v_b, v_c) where $(a, b, c) = (x_1z'_1, x_1u_1, u_1u_2)$, $(x_2z_1, x_2u_m, u_{m-1}u_m)$, $(x_3z_n, x_3u'_{m'}, u'_{m'-1}u'_{m'})$; and the blue edge v_av_b where $(a, b) = (f_4, x_4u'_1)$. We obtain the desired coloring of $L(G)$ having two even blue paths with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, that is, the cycle C is a junction cycle. See for instance the graphs in Figure 12(a)-(b).

If both paths $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$ have length 1, then we remove the edges $x_1z'_1$, $x_4u'_1$ and apply Definition 1 to the odd path $(z'_1, \dots, z'_{n'}, x_4)$ so that the following edges of $L(G)$ become blue: v_av_b where $(a, b) = (h'_1, z'_1z'_2)$, $(f_4, x_4z'_{n'})$. We recolor the edge v_av_b where $(a, b) = (h'_1, z'_1z'_2)$ by adding the green 6-path $(v_{h'_1}, v_{z'_1z'_2}, v_{x_1z'_1}, v_{x_1x_2}, v_{x_2x_3}, v_{x_3u'_{m'}}, v_{e'_{m'}})$. We also add the red 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f_1, x_1z'_1, h'_1)$, $(f_4, x_4u'_1, e'_1)$, (f_2, x_2x_3, f_3); the blue 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f_1, x_1x_2, f_2)$, $(x_4z'_{n'}, x_4u'_1, u'_1u'_2)$; and the blu edge $v_{f_3}v_{x_3u'_{m'}}$. We obtain the desired coloring of $L(G)$ having two even blue paths with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, that is, the cycle C is a junction cycle. See for instance the graph in Figure 12(c).

Case 2: f_1, f_2 are red and f_3, f_4 are green in $\mathcal{B}$.

The coloring of $L(G)$ is obtained from the colored graphs of Case 1(a)-(d) by swapping the red and green color as we are going to explain.

Case 2(a): each path $x_i\, P^C\, x_{i+1}$ has odd length.

For the vertex-disjoint paths $x_4\, P^C\, x_1$, $x_2\, P^C\, x_3$ having length larger than 1, we obtain the desired coloring of $L(G)$ from the corresponding coloring of Case 1(a) as follows: we color by green the 2-paths (v_a, v_b, v_c) where $(a, b, c) = (f_3, u'_{m'}, e'_{m'})$, $(f_4, x_4u'_1, e'_1)$; and swap the red and green color on the edges of $L(G)$ that are incident to the vertices in $L(x_3\, P^C\, x_4)$. The cycle C is a junction cycle.

For the case where every pair of vertex-disjoint paths contains a path having length 1, we obtain the desired coloring of $L(G)$ from the corresponding coloring of Case 1(a) as follows: we color by green the 4-path $(v_a, v_b, v_c, v_d, v_e)$ where $(a, b, c, d, e) = (f_2, x_1x_2, x_2x_3, x_3u'_{m'}, f_3)$; and add 1 modulo 4 to the subscritpts of the labels x_1, x_2, x_3, x_4 and f_1, f_2, f_3, f_4. The cycle C is a junction cycle.

See for instance the graphs in Figure 13(a)-(b) that are obtained from the graphs in Figure 10(a)-(d) as described above. Notice that the new green paths are red in Case 1(a).

Case 2(b): each path $x_i\, P^C\, x_{i+1}$ has even length.

We obtain the desired coloring of $L(G)$ from the corresponding coloring of Case 1(b) as follows. We leave invariant the coloring on the edges of $L(G)$ that are incident to the vertices in $L(x_1\, P^C\, x_2 \cup x_2\, P^C\, x_3)$. We swap the red and green color on the edges of $L(G)$ that are incident to the vertices in $L(x_3\, P^C\, x_4 \cup x_4\, P^C\, x_1)$. The cycle C is a junction cycle. For instance, the graph in Figure 13(c) is obtained from the graph in Figure 11(b) as described above.

Case 2(c): just two vertex-disjoint paths $x_i\,P^C\,x_{i+1}$, $x_{i+2}\,P^C\,x_{i+3}$ have odd length.

The desired coloring of $L(G)$ is obtained from the corresponding coloring of Case 1(c) as follows: we swap the red and green color on the edges of $L(G)$ that are incident to the vertices in $L(x_2\,P^C\,x_3 \cup x_3\,P^C\,x_4 \cup x_4\,P^C\,x_1)$; we leave invariant the color on the edges of $L(G)$ that are incident to the vertices in $L(x_1\,P^C\,x_2)$. The cycle C is a junction cycle. For instance, the graph in Figure 13(d) is obtained from the graph in Figure 11(d) as described above.

Case 2(d): just two consecutive paths $x_i\,P^C\,x_{i+1}$, $x_{i+1}\,P^C\,x_{i+2}$ have odd length.

As for the case where at least one of the paths $x_1\,P^C\,x_2$, $x_2\,P^C\,x_3$ has length larger than 1, we obtain the desired coloring of $L(G)$ from the corresponding coloring of Case 1(d) as follows. We swap the red and green color on the edges of $L(G)$ that are incident to the vertices in $L(x_2\,P^C\,x_3 \cup x_3\,P^C\,x_4 \cup x_4\,P^C\,x_1)$; and leave invariant the color on the edges of $L(G)$ that are incident to the vertices in $L(x_1\,P^C\,x_2)$. For instance, the graph in Figure 14(a) is obtained from the graph in Figure 12(b) as described above.

As for the case where both paths $x_1\,P^C\,x_2$, $x_2\,P^C\,x_3$ have length 1, we remove the edges x_1x_2, $x_4z'_{n'}$ of G and apply Definition 1 to the odd path $(x_1, z'_1, \ldots, z'_{n'})$. The following edges of $L(G)$ become blue: (v_a, v_b) where $(a, b) = (f_1, x_1z'_1)$, $(h'_{n'}, z'_{n'-1}z'_{n'})$. We recolor the edge (v_a, v_b) where $(a, b) = (h'_{n'}, z'_{n'-1}z'_{n'})$ by adding the green 4-path $(v_a, v_b, v_c, v_d, v_e)$ where $(a, b, c, d, e) = (h'_{n'}, z'_{n'-1}z'_{n'}, x_4z'_{n'}, x_4u'_1, e'_1)$. We also add the green 2-path $(v_{f_2}, v_{x_2x_3}, v_{f_3})$; the red 4-path $(v_a, v_b, v_c, v_d, v_e)$ where $(a, b, c, d, e) = (f_1, x_1x_2, x_2x_3, x_3u'_{m'}, e'_{m'})$; the red 2-path $(v_{f_4}, v_{x_4z'_{n'}}, v_{h'_{n'}})$; the blue 2-paths (v_a, v_b, v_c) where $(a, b, c) = (x_1z'_1, x_1x_2, f_2)$, $(f_4, x_4u'_1, u'_1u'_2)$; and the blue edge v_av_b where $(a, b) = (f_3, x_3u'_{m'})$. We obtain the desired coloring of $L(G)$ having two odd blue paths with endvertices v_{f_1}, v_{f_2} and v_{f_3}, v_{f_4}, that is, the cycle C is a junction cycle. See for instance the graphs in Figure 14(b). □

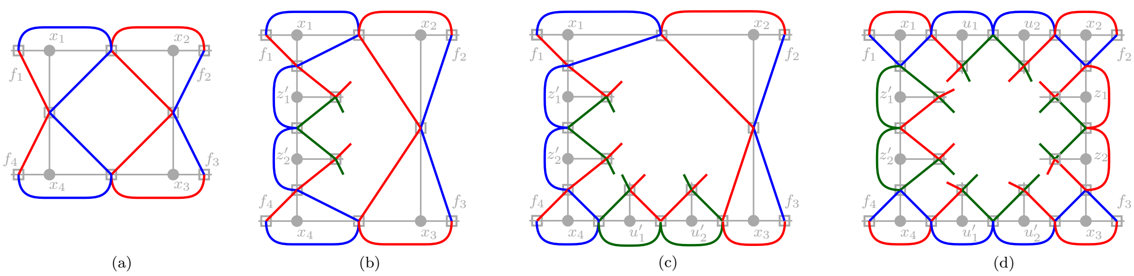


Figure 10: A cycle C in Lemma 5 where the edges f_i, $1 \le i \le 4$, have the same color and each path $x_i\,P^C\,x_{i+1}$ has odd length.

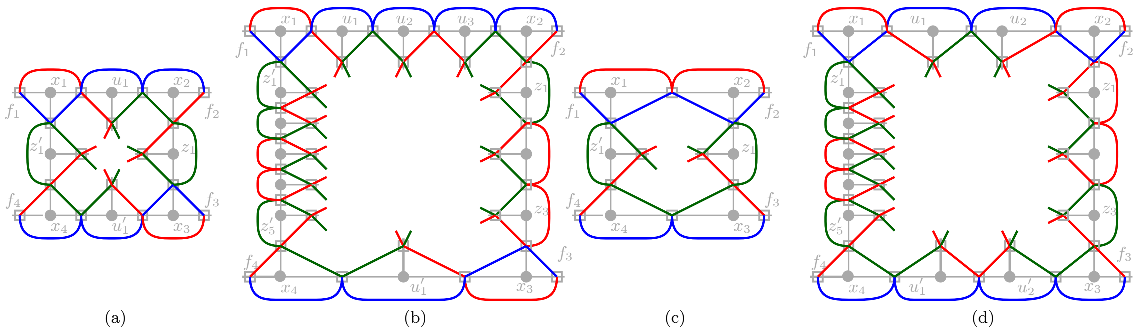


Figure 11: A cycle C in Lemma 5 where the edges f_i, $1 \leq i \leq 4$, have the same color: (a)-(b) hold for the case where each path $x_i\, P^C\, x_{i+1}$ has even length; (c)-(d) hold for the case where each path $x_i\, P^C\, x_{i+1}$, $i = 1, 3$ has odd length.

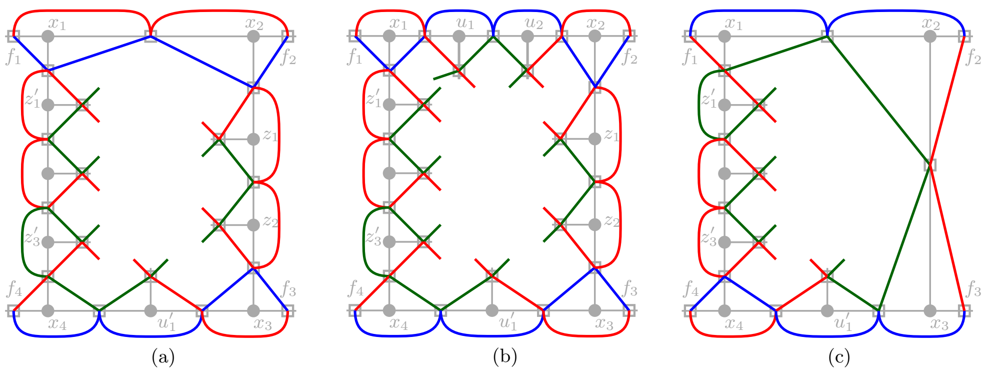


Figure 12: A cycle C in Lemma 5 where the edges f_i, $1 \leq i \leq 4$, have the same color: (a)-(b) hold for the case where at least one of the paths $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$ has length larger than 1; (c) holds for the case where both paths $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$ have length 1.

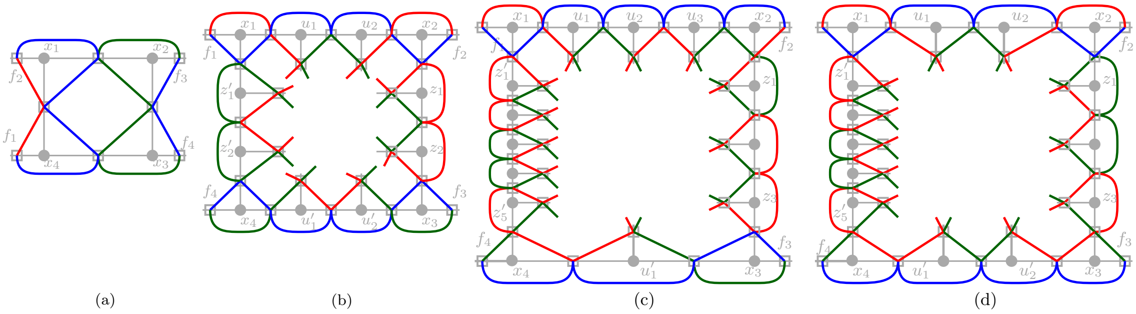


Figure 13: A cycle C in Lemma 5 where the edges f_1, f_2 are red, f_3, f_4 are green: (a)-(b) hold for the case where every path $x_i\, P^C\, x_{i+1}$ has odd length; (c) holds for the case where every path $x_i\, P^C\, x_{i+1}$ has even length; (d) holds for $x_1\, P^C\, x_2$, $x_3\, P^C\, x_4$ of odd length.

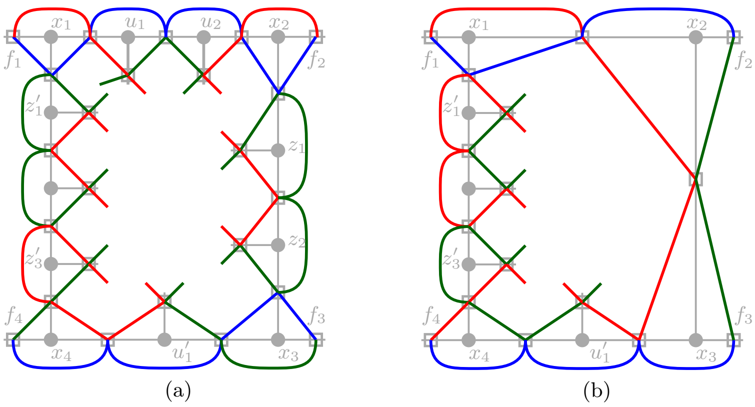


Figure 14: A cycle C in Lemma 5 where the edges f_1, f_2 are red, f_3, f_4 are green and $x_1\, P^C\, x_2$, $x_2\, P^C\, x_3$ have odd length: (a) holds for the case where at least one of the odd two paths has length larger than 1; (b) holds for the case where both paths have length 1.

5 The main theorem.

For a bridgeless cubic graph G, we give a sufficient condition for the existence of an even cycle decomposition of index 3 in the corresponding line graph $L(G)$. As a corollary of this result, the line graph of a bridgeless cubic graph of oddness 2 has an even cycle decomposition of index 3. We also show that the same holds for the smallest snarks of oddness 4, see Figure 15. We recall that, in the literature, a snark is a bridgeless cubic graph of class 2 with girth at least 5 and satisfying additional conditions on the cyclic edge-connectivity.

Theorem 1. *Let G be a bridgeless cubic graph having a perfect matching M whose cycle-multigraph $\mathcal{C}(G, M)$ admits a partition of the odd cycle-vertices into sets of cycles spanning an even number of odd cycle-vertices and satisfying the Property 4. Then $L(G)$ has an even cycle decomposition of index 3.*

Proof. The proof is a generalization of Example 7. Let G' be the class 1 graph of Definition 5. For a set $\{\Gamma_0, \ldots, \Gamma_r\}$ of cycles in $\mathcal{C}(G, M)$ spanning an even number of odd cycle-vertices and satisfying the Property 4, we set $\mathcal{B} = \Gamma_0 \cup \ldots \cup \Gamma_r$. We consider the line graph $L(G')$ endowed with the coloring θ' defined in Lemma 3. As in Example 7, we firstly modify the coloring θ' on the edges of $L(G')$ that are incident to the vertices in $L(C)$, where C is a cycle of $G - M$ with v_C in $\mathcal{B}$. Secondly, we complete the new coloring of $L(G')$ to a coloring φ of $L(G)$. More specifically, in $\mathcal{B}$ we replace every cycle-vertex v_C of degree 2 by the corresponding fitting cycle; and every cycle-vertex of degree 4 by the corresponding junction or separating cycle. The coloring φ colors the edges of $L(G)$ that are incident to the vertices in $L(C)$ according to Lemma 4 if C is a fitting cycle; or Lemma 5 if C is a junction or a separating cycle.

By Lemma 4 and 5, the red, green and blue colored classes H'_1, H'_2, H'_3 of φ are obtained from the red, green and blue colored classes H_1, H_2, H_3 of θ', respectively. By construction, every vertex of $L(G)$ is incident to exactly two edges of the same color. Hence, every colored class of φ is a 2-regular subgraph of $L(G)$. See for instance Figure 6, 7. We show that they are even 2-regular subgraphs.

The colored classes H'_1, H'_2 are obtained from H_1, H_2 by firstly applying Definition 1 and secondly by adding red and green even paths with endvertices v_e, $v_{e'}$, where $e \in \{f, g\}$ or $e \in \{f_i : 1 \leq i \leq 4\}$, $e' \in M_C$, $e \neq e'$. By applying Definition 1 to H_1 and H_2, the red and green edges still form two even $(1, 2)$-subgraphs of $L(G)$, since Lemma 1 and 3 hold. By adding the red and green even paths with endvertices v_e, $v_{e'}$, the red and green edges form two even 2-regular subgraphs H'_1, H'_2 of $L(G)$.

As for the blue colored class H'_3, a cycle C of $G - M$ with v_C not belonging to $\mathcal{B}$ provides exactly one blue even cycle of H'_3 as its edges are colored alternately by a_2, a_3 (see Lemma 3).

We consider those cycles C with corresponding cycle-vertex v_C in $\mathcal{B}$. By Lemma 4, a fitting cycle C contributes with exactly one blue path in forming a blue cycle of H'_3; the blue path has even or odd length according to whether C has even or odd length, respectively. By Lemma 5, a junction cycle C contributes with exactly two blue paths in forming the same blue cycle of H'_3; both blue paths have even or odd length. By Lemma 5, a separating cycle C contributes with exactly two blue paths P_1, P_2 in forming two distinct blue cycles of H'_3. The length of P_1, P_2 depends on the coloring of the edges of $\mathcal{B}$ that are incident to v_C: P_1, P_2 have even length if the edges of $\mathcal{B}$ that are incident to v_C have the same color, or equivalently, if Γ_0, Γ_r contain an even number of odd cycle-vertices, see Property 4(iii); P_1, P_2 have odd length if v_C is incident to two red edges, say f_1, f_2, and two green edges, say f_3, f_4, or equivalently, if Γ_0, Γ_r contain an odd number of odd cycle-vertices.

Consequently, if no cycle C with v_C in $\mathcal{B}$ is a separating cycle, then $\mathcal{B}$ provides exactly one blue even cycle, since it spans an even number of odd cycle-vertices. See for instance Figure 7.

Assume that $\mathcal{B}$ contains the cycle-vertices v_{C^i} corresponding to the separating cycles C^i, $0 \leq i \leq m$. Then $\mathcal{B}$ provides $(m+1)$ blue even cycles of H'_3, see for instance Figure 6. More specifically, let Γ_{j_i}, Γ_{j_i+1} be the cycles in $\{\Gamma_0, \ldots, \Gamma_r\}$ intersecting in v_{C^i}, where $0 \leq i \leq m$. The cycle C^i contributes with the two blue paths P_1, P_2: P_1 connects to the blue paths provided by the cycles C with $v_C \in \cup_{s=j_{i-1}+1}^{j_i} V(\Gamma_s)$ and together form a blue cycle; P_2 connects to the blue paths provided by the cycles C with $v_C \in \cup_{s=j_i+1}^{j_{i+1}} V(\Gamma_s)$ and together form another blue cycle.

If every cycle Γ_i, $0 \leq i \leq r$, contains an even number of odd cycle-vertices, then every blue cycle is even since every blue path P_1, P_2 has even length. The same holds if Γ_0, Γ_r contain an odd number of odd cycle-vertices and all other cycles Γ_i, $1 \leq i \leq r-1$, contain an even number of odd cycle-vertices - see Property 4 - since every blue path P_1, P_2 has odd length. It is thus proved that the blue colored class H'_3 is an even 2-regular subgraph of $L(G)$. The assertion follows. □

As a corollary of Theorem 1, we have the following result.

Proposition 8. *The line graph of a bridgeless cubic graph of oddness* 2 *has an even cycle decomposition of index* 3*.*

Proof. Let G be a bridgeless cubic graph having a perfect matching M whose cycle-multigraph $\mathcal{C}(G, M)$ has exactly two odd cycle-vertices, say v_D, $v_{D'}$. Since Menger's Theorem holds, there exists two edge-disjoint paths in $\mathcal{C}(G, M)$ with endvertices v_D, $v_{D'}$ whose union satisfies Lemma 2. The assertion follows from Theorem 1. □

We also tested Theorem 1 on the smallest snarks having oddnes 4 [13].

Proposition 9. *The line graphs of the smallest snarks of oddness* 4 *- see Figure 15 - have an even cycle decomposition of index* 3*.*

Proof. For each graph G in Figure 15, the bold edges define a perfect matching M whose complementary 2-factor has exactly four odd cycles. The cycle-multigraph $\mathcal{C}(G, M)$ of the graphs in Figure 15(a)-(b) has a cycle consisting of the four odd cycle-vertices. The cycle-multigraph of the graph G in Figure 15(c) admits a partition of the odd cycle-vertices into two dipoles, each consisting of two edges and two odd cycle-vertices. The assertion follows from Lemma 2 and Theorem 1. □

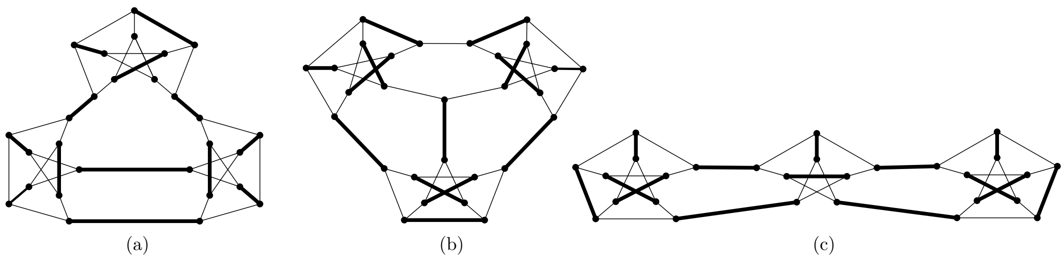


Figure 15: The smallest snarks of oddness 4.

6 Snarks of arbitrary large oddness.

We apply Theorem 1 to the infinite families of snarks that are constructed in [7], [13] and are characterized by an arbitrary large oddness (larger than 2). The perfect matching M, that will be used to prove the existence of an even cycle decomposition of index 3 in the corresponding line graph, does not necessarily provide the oddness of the snark.

The snarks in [7] are obtained by the *semi blowup* and *blowup* of a pair (G, H), where G is a bridgeless cubic graph and H is a 2-regular subgraph of G. Both operations use the base block B, which is obtained from the Petersen graph by removing a pair of adjacent vertices.

The semi blowup and blowup of (G, H) are described by Figure 16: for every k-cycle $C = (v_1, \ldots, v_k)$ of H, they consider k copies of B, remove each edge $v_i v_{i+1}$ of C and connect v_i to the i-th and $(i+1)$-th copy of B.

Proposition 10. *Let G be a bridgeless cubic graph and let M be a perfect matching of G. Let H be a 2-regular subgraph of the complementary* 2*-factor $G - M$ containing no odd cycle and such that in the cycle-multigraph $\mathcal{C}(G, M)$ the odd cycle-vertices corresponding to the odd cycles of $G - M$, if any, can be partitoned into sets of cycles satisfying the Property 4.*

Then the line graph of the semiblowup and blowup of (G, H) *has an even cycle decomposition of index* 3.

Proof. The bold edges in Figure 16(a) and (b) together with the edges of M form a perfect matching M^* of the semi blowup and blowup of (G, H), respectively. The complementary 2-factor of M^* contains the odd cycles of $G - M$, if any.

For every k-cycle C of H, it also contains k cycles of length 9 for the semi blowup of (G, H); and k cycles of length 11 for the blow up of (G, H). In the cycle-multigraph relative to M^* the odd cycle-vertices corresponding to the odd cycles coming from the same k-cycle C of H form a k-cycle. As k is even, the Property 4 is satisfied. Since the same property is satisfied for the odd cycle-vertices corresponding to the odd cycles not belonging to H, the assertion follows from Theorem 1. □

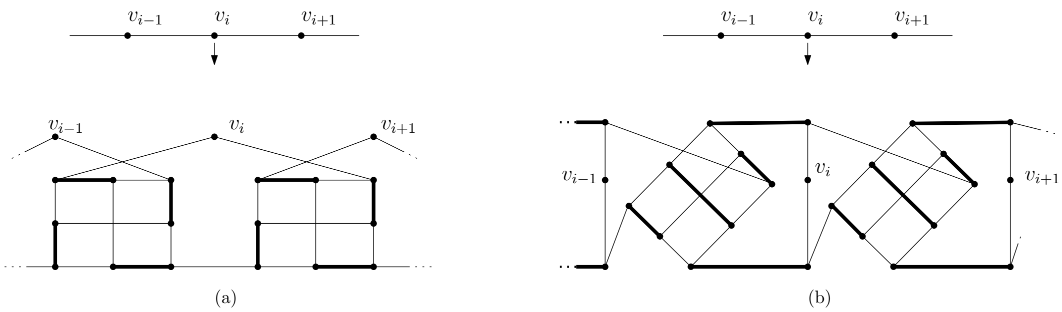


Figure 16: (a)-(b) The semi blowup and blowup of (G, H), respectively.

We consider the snarks with cyclic connectivity 4 and 5 that are constructed in [13]. These graphs are obtained from the base blocks P_4^v, P_4^e, P_5^{vvv}, P_5^{ev} in Figure 17. The edges that are incident to 1-valent vertices will be called *terminal edges* and are partitioned into pairs as described in [13]. The basic operation is the join of terminal edges: two terminal edges are joined if the corresponding 1-valent vertices are identified so as to form a 2-valent vertex that will be suppressed.

The join of the terminal edges in P_4^v and P_4^e defines the graphs N_1, N_2 that are used to construct the snarks with cyclic connectivity 4; the join of the terminal edges in P_5^{vvv} and P_5^{ev} defines the graph Z that will be used to construct the snarks with cyclic connectivity 5.

More specifically, the graph N_1 is obtained from the join of a pair of terminal edges from P_4^v with a pair of terminal edges from P_4^e. The graph N_2 is obtained from two copies of P_4^v and one copy of P_4^e; each pair of terminal edges from P_4^e is joined to a pair of terminal edges in a different copy of P_4^v. For the construction of a snark with cyclic connectivity 4, in [13] the authors arrange into a circuit an arbitrary number of copies of N_1 and

N_2; and join one pair of terminal edges from each copy to a pair of terminal edges of succeeding copy, so as the pairs of the first and last copy are joined. As remarked in [13], several non-isomorphic graphs can be obtained from the same number of copies of N_1, N_2.

The graph Z is obtained from two copies of P_5^{ev} and one copy of P_5^{vvv}; each pair of terminal edges from P_5^{vvv} is joined to a pair of terminal edges in a different copy of P_5^{ev}, so as the terminal edges of Z can be partitioned into two triples and one singleton. A snark with cyclic connectivity 5 is obtained by joining the terminal edges from r, r even, disjoint copies of Z so as to form a graph with cyclic connectivity 5.

For the line graph of a snark constructed as described above, the following result holds.

Proposition 11. *Let G be a snark with cyclic connectivity 4 or 5 constructed in [13]. Then $L(G)$ has an even cycle decomposition of index 3.*

Proof. By construction of G, the bold edges of the base blocks in Figure 17 define a perfect matching M of G.

As for G with cyclic connectivity 4, the complementary 2-factor $G - M$ contains exactly one 8-cycle from each copy of P_4^v; and exactly two 5-cycles from each copy of P_4^e. In the cycle-multigraph $\mathcal{C}(G, M)$, each pair of odd cycle-vertices corresponding to the odd cycles of $G - M$ coming from the same copy of P_4^v forms a dipole with two vertices and two edges. Hence, $\mathcal{C}(G, M)$ admits a partition of the odd cycle-vertices into dipoles. Since each dipole is a cycle satisfying the Property 4 - or, equivalently, Lemma 2 - the existence of an even cycle decomposition of index 3 in $L(G)$ follows from Theorem 1.

As for G with cyclic connectivity 5, the complementary 2-factor $G - M$ contains exactly one 25-cycle from each copy of Z. In the cycle-multigraph $\mathcal{C}(G, M)$, the odd cycle-vertices form a cycle. Hence, the Property 4 is satisfied and the assertion follows from Theorem 1. □

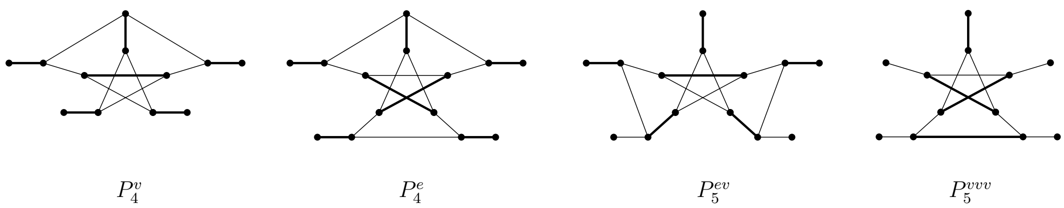


Figure 17: The base blocks of the construction in [13].

7 Final Remarks

Our method of constructing an ECD of index 3 in the line graph of a cubic graph G is based on the existence of a cycle-multigraph $\mathcal{C}(G, M)$ whose odd cycle-vertices are partitoned into Eulerian subgraphs of maximum degree 4; each subgraph spans an even number of odd cycle-vertices and every odd cycle-vertex has degree 2 (see Property 4).

This approach can be extended to cycle-multigraphs having Eulerian subgraphs of maximum degree larger than 4 and where the degree of an odd cycle-vertex might be larger than 2. This entails new definitions for the cycles in addition to the ones in Section 4 of a fitting, junction and separating cycle. A generalization of our approach would prove the existence of an ECD of index 3 in the line graph of every cubic graph of oddness 4 that does not satisfy Theorem 1, if any; or in other snark families that are known in the literature as, for instance, the snarks with cyclic connectivity 6 that are constructed in [13] by a suitable superposition. Graphs with cyclic connectivity 6 - and some others - are considered in the PhD thesis of the second author, we decided to show just some constructions in this paper in order to describe the main idea of our approach.

A final remark about the existence of a 4-regular graph with every ECDs of index larger than 3 is in order. The results of this paper and the possible generalizations of our method suggest that it seems to be hard to find such a 4-regular graph in the family of 4-regular line graphs, actually in any family of 4-regular graphs. The non-existence would imply that the 4-regular graphs with palette index 4, 5 have no ECD, which would contribute to the problem on the existence of Eulerian graphs with no ECD, see for instance [10, 16].

References

[1] J.A. Bondy, U.S.R. Murty. Graph Theory, Springer, 2008.

[2] A. Bonisoli, S. Bonvicini, Even cycles and even 2-factors in the line graph of a simple graph, Electron. J. Combin. 24 (2017) P4.15.

[3] S. Bonvicini, M.M. Ferrari, On the minimum number of bond-edge types and tile types: An approach by edge-colorings of graphs, Discrete Appl. Math. 227 (2020), 1–13.

[4] S. Bonvicini, G. Mazzuoccolo, Edge-colorings of 4-regular graphs with the minimum number of palettes, Graphs Combin. 32 (2016), 1293–1311.

[5] C.J. Casselgren, P.A. Petrosyan, Some results on the palette index of graphs, Discrete Math. Theor. Comput. Sci. 21(3) (2019), dmtcs: 4509.

[6] H. Fleischner, (Some of) the many uses of Eulerian graphs in graph theory (plus some applications), Discrete Math. 230 (2001), 23–43.

[7] J. Ḧagglund, On snarks that are far from being 3-edge-colorable, Electron. J. Combin. 23(2) (2016), P2.6.

[8] M. Horňák, J. Hudák, On the palette index of complete bipartite graphs, Discuss. Math. Graph Theory 38(2) (2018), 463–476.

[9] M. Horňák, R. Kalinowski, M. Meszka, M. Woźniak, Edge colorings and the number of palettes, Graphs Combin. 30 (2014), 619–626.

[10] B. Jackson, On circuit covers, circuit decompositions and Euler tours of graphs, in: Surveys in Combinatorics, volume 187 of London Math. Soc. Lecture Note Ser, Cambridge University Press, 1993, pp. 191-210.

[11] F. Jaeger, A survey of the cycle double cover conjecture, Ann. Discrete Math. 27 (1985), 1-12.

[12] W. Liu, H. You, Q. Cui, On even cycle decompositions of line graphs of cubic graphs, Discrete Math. 343 (2020), 111904.

[13] R. Lukot'ka, E. Máčajová, J. Mazák, M. Škoviera, Small snarks with large oddness, Electron. J. Combin. 22(1) (2015), P1.51.

[14] K. Markström, Complete minors in cubic graphs with few short cycles and random cubic graphs, Ars Combin. 70 (2004), 289–295.

[15] K. Markström, Even cycle decompositions of 4-regular graphs and line graphs, Discrete Math. 312 (2012), 2676-2681.

[16] R. Rizzi, On 4-connected graphs without even cycle decompositions, Discrete Math. 234 (2001), 181-186.

[17] P. D. Seymour, Even circuits in planar graphs, J. Combin. Theory Ser. B, 31, 1982, 327–338.

[18] C. Thomassen, Reflections on graph theory, J. Graph Theory 10 (1986), 309–324.

[19] O. Veblen, An Application of Modular Equations in Analysis Situs, Ann. of Math. (2) 14(1) (1912), 86–94.

[20] C. Q. Zhang, On even circuit decompositions of Eulerian graphs, J. Graph Theory 18(1) (1994), 51–57 .