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On the instant-controllability of a second order multidimensional differential equation subjected to damping term and impulses

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A R T I C L E I N F O

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A B S T R A C T

In this paper the existence of admissible trajectory control-pairs for an impulsive problem driven by a multidimensional differential equation with a nonlinear Balakrishnan-Taylor type damping term, is investigated. This purpose is achieved rewriting the impulsive problem in an abstract form governed by a semilinear second order differential inclusion in which the nonlinear term also depends on the first derivative. The method used leads to a preliminary study of the existence of mild solutions for a non-impulsive multivalued problem on a closed and bounded interval, stating two new results in non reflexive Banach spaces. Then, the mild solution in $[0, \infty)$ for the impulsive abstract multivalued problem is obtained glueing the solutions defined on the bounded intervals. This approach allows to not require the continuity on the impulsive functions. Applying the abstract impulsive multivalued results we achieve the desired existence of admissible trajectory control-pairs for the impulsive phenomena described by the multidimensional differential equation. The paper concludes with the study of an instant-controllability relatively to a suitable functional for the impulsive problem in exam. The established results improve recent theorems present in the literature and obtained in reflexive Banach spaces and assuming the continuity on the impulsive functions.

1. Introduction

The discussion addressed in this paper deals with the study of the existence of admissible trajectory-control pairs for an impulsive problem governed by the following multidimensional differential equation in which is present a nonlinear Balakrishnan-Taylor type damping term

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$$
\textbf{(E)}\ w''_{tt}(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(\xi)\frac{\partial^2 w}{\partial \xi_i \partial \xi_j}(t,\xi) + \sum_{i=1}^{n} b_i(\xi)\frac{\partial w}{\partial \xi_i}(t,\xi) + c(\xi)w(t,\xi)
$$

$$
+ d(t,\xi) f\left(\int_{\Omega} K(\xi,s) w'_t(t,s)\, ds\right) + v(t,\xi),
$$

$$
t \in [0,\infty) \setminus \{t_k\}_{k\in\mathbb{N}},\ \text{a.e. } \xi \in \Omega
$$

where the control action is given by the condition

$$
v(t,\xi) \in V(t),
$$

being $V : [0,\infty) \to \mathcal{P}(\mathbb{R})$ a suitable multimap and the impulses happen in the fixed instants t_k, $k \in \mathbb{N}$, with $t_k \to \infty$.

It is important to note that the existence of admissible trajectory-control pairs is necessary to obtain the well-poseness for the study of controllability, as it is underlined in the last part of the paper.

A recent result about the existence of mild solutions for an impulsive problem monitored by **(E)**, but without the action-control and in a compact interval, is obtained in [36]. There some results for equations with damping term, usually proven in the setting of one-dimensional spaces (see [2], [23], [31]), are extended to the case of a strongly elliptic linear part in $\mathbb{R}^n$.

As precised in [36], if the linear operator reduces to the Laplacian and the integral term is substituted with the first-order time derivative, equation **(E)** represents the multidimensional telegraph equation. Rutkas and Vlasenko in [38] underline the importance of the impulsive problems driven by telegraph equations to investigate impulsive circuits with transmission lines. Indeed the currents and the voltages of lines satisfy partial differential telegraph equations and the outputs of the lines are exposed to pulse perturbations at discrete times $t_0 < t_1 < \dots \to \infty$.

The damping term, that appears in the model in hand, was initially proposed by Balakrishnan and Taylor in [3] and Bass and Zes in [5]. Starting from these papers, many authors have treated this topic (see, for example, [21], [22], [35], and [45]). In particular, this damping term permits to investigate models that describe flexible structure problems involving infinitely many independent vibrational modes (the “spillover” model).

On the other hand, the theory about impulsive problems started with the pioneering papers [33] and [34] of Milman and Myshkis. Since the impulsive problems driven by differential equations/inclusions describe the dynamics of many evolution processes subjected to sudden changes (inherent to the process described or aimed at modifying it) in fixed istants, they are a natural mathematical model to represent several phenomena in physics, biotechnology, pharmacokinetics, population evolutions, industrial robotics, ecology, reproduction of microorganisms, economics, and so on. This great variety of application fields has led many mathematicians to deep this argument (see, for example, [2], [8], [14], [15], [36]) producing a significant development of the impulsive theory, culminated with the monograph [20].

As it emerges in [38], the study of **(E)** in the unbounded interval $[0,\infty)$ is deemed important. Therefore, we have investigate the existence of impulsive mild solutions for a problem governed by **(E)** in this framework, instead of in the bounded interval considered in [36].

In order to obtain this result, we have rewritten the impulsive problem in an abstract form driven by a semilinear second order differential inclusion

$$
\textbf{(DI)} \qquad x''(t) \in Ax(t) + F(t, x(t), x'(t))
$$

and we initially have studied the existence of mild solutions for a non-impulsive Cauchy problem, guided by **(DI)**, in a bounded interval $[b,c]$, requiring on A and F suitable assumptions which allow to examine the impulsive problem.

The proof of our first result (see Theorem 4.1) relies on the Glicksberg fixed point theorem. Taking inspiration from [36], we are able to require only a hypothesis about the weak compactness instead of the strong one thanks to the use of projections, which allow us to work in a finite dimensional setting (see Theorem 4.1). However, our approach differs from the one presented in [36] in the construction of the approximating operators sequence. Indeed in [36] a sequence of this type is built by associating a sequence of projected problems to the problem studied, while we define a sequence of approximating operators directly linked with the solution operator of the problem in exam. This enables us to establish the existence of a mild solution $q \in \mathcal{C}^1([b,c];X)$, by achieving the existence of a B-measurable selection for the multimap $F(\cdot, q(\cdot), q'(\cdot))$ through a multivalued version of Fatou's Lemma (see Proposition 2.7). This result improves an analogous theorem proven in [36] in reflexive Banach spaces, since we remove the reflexivity condition (see Remark 4.2). Moreover we obtain another result in which the existence of mild solutions for the abstract mentioned problem is obtained without assumptions on the values of the multimap F (see Theorem 4.3)

Then, following the well-established demonstrative approach in [14] and [15], we discuss the existence of mild solutions for an impulsive problem governed by **(DI)** on the real half-line $[0,\infty)$, where the jumps happen in an increasing sequence of times converging to ∞ (see Theorems 5.1 and 5.2). Building upon our Theorems 4.1 and 4.3, where impulsive effects were not considered, we succeed in proving the existence of mild solutions for the impulsive problem. A great advantage of this method is that the new results are proven in Banach spaces not necessary reflexive and without the continuity property on the impulsive functions.

Let us recall that the literature related to the existence of mild solutions to the second order impulsive problems is widely spread in the case in which the right hand side does not depend on the first derivative (see [42], [44], [6] [20], and the references therein) while only a few authors have studied the case in which the first derivative is present (see, for example, [36]).

The abstract impulsive multivalued existence theorems obtained enable us to achieve the desired existence of an admissible trajectory-control pair for the impulsive phenomena described by **(E)**.

Starting from the impulsive problem in exam we proceeded to introduce the notion of instant-controllability monitored by a coercive and lower semicontinuous functional. The results on the existence of admissible pairs led us to claim that the instant-controllability problem under consideration is well posed. In our setting it was then possible to achieve optimality results (see Section 6.2).

In our opinion, the study of this type of controllability can be helpful to monitor the impulses, for example in order to achieve a minimization of the signal distortion or the maximization of the distance reached by the signal, when the model describes the telegraph behaviour.

The paper is structured in the following manner. In Section 2, we gather some well-known contents as definitions, propositions, and theorems useful in the sequel. The problem setting is introduced in Section 3. Section 4 is devoted to the study of existence of mild solutions for a non-impulsive problem governed by **(DI)** in a bounded interval, considering two situations. On one side, we require that the values of the multimap F are convex (see Theorem 4.1). On the opposite side we remove this assumption, working with different hypotheses on F (see Theorem 4.3). Then we compare our results with the recent existence theorem proven in [36] (see Remark 4.2). In Section 5 we establish the existence of mild solutions for the impulsive problem in the real half-line $[0,\infty)$, both assuming convexity on the values of F (see Theorem 5.1) and omitting this property (see Theorem 5.2). In Section 6, as a consequence of the obtained abstract impulsive multivalued existence theorems, we study the instant-controllability relatively to a suitable functional for an impulsive problem driven by **(E)**. Finally we conclude our paper in Section 7.

2. Preliminaries

In this section we collect notations and properties that we use in all the paper. If $(X, \|\cdot\|_X)$ is a normed space, with X^* its dual space, and τ_w the weak topology on X. It is well-known that the weak topology on

X is completely regular and locally convex (see [16], Proposition 3.4.3) and, on a finite dimensional space, the weak topology and the strong topology coincide.
A normed space X (or the norm $\|\cdot\|_X$ on X) is said to be *locally uniformly convex* (LUR, for short) if for every $\varepsilon > 0$ and $x \in X$, $\|x\|_X = 1$, there exists $\delta(\varepsilon, x) > 0$ such that (see [29], Definition 0.2)

$$\frac{\|x+y\|_X}{2} \leq 1 - \delta(\varepsilon, x), \text{ whenever } \|x+y\|_X \geq \varepsilon \text{ and } \|y\|_X = 1.$$

On the other hand, $\|\cdot\|_X$ is said to be a *weak-Kadec-Klee* norm on X if for every sequence $(x_n)_n$ weakly convergent to x in X such that $\lim_{n\to\infty} \|x_n\|_X = \|x\|_X$, then $\lim_{n\to\infty} \|x_n - x\|_X = 0$. If X is a separable Banach space then X admits an equivalent LUR norm ([25]). Additionally, a Banach space X admits an equivalent LUR norm if and only if it admits an equivalent weak-Kadec-Klee norm ([40]). If X admits a weak-Kadec-Klee norm then $\mathcal{B}(X) = \mathcal{B}(X, \tau_w)$, where $\mathcal{B}(X)$ denotes the Borel σ-field on X and $\mathcal{B}(X, \tau_w)$ stands for the Borel σ-algebra generated by weakly open sets (see [18], Theorem 1.1).
In that follows, $\overline{B}_X(0, r)$ refers to the closed ball in $(X, \|\cdot\|_X)$ centred at the origin with radius $r > 0$, while $\overline{A}^w$ symbolizes the weak closure of a set $A \subset X$. Moreover, a subset A of X is said to be *relatively weakly sequentially compact* if every sequence in A has a subsequence that converges weakly in X (see [32]). According to the Eberlein-Smulian Theorem, this property is equivalent to the notion of relative weak compactness, as established in [16] (see Theorem 3.5.3). In the upcoming sections, we will make use of this version of a result originally attributed to H. Vogt

Proposition 2.1 *([43], Theorem 3). Let A be a relatively weakly compact subset of a Banach space. Then A is weakly closed if and only if A is weakly sequentially closed.*

Now, if J is a closed and bounded interval in $\mathbb{R}$, the notation $\mathcal{M}(J)$ represents the collection of all Lebesgue measurable subsets of J, while μ is usual Lebesgue measure on J and $(J, \mathcal{M}(J), \mu)$ is the relative measure space. Moreover, $\mathcal{C}(J; X)$ stands for the space of all continuous functions provided with the norm $\|\cdot\|_{\mathcal{C}(J;X)}$ of uniform convergence. We recall that a sequence $(f_n)_n$ in $\mathcal{C}(J; X)$ weakly converges to $g \in \mathcal{C}(J; X)$ if and only if $(f_n - g)_n$ is uniformly bounded and $f_n(t) \rightharpoonup g(t)$, $t \in J$ ([27], Theorem 4).
Now, $\mathcal{C}^1(J; X)$ denotes the Banach space of all continuously differentiable functions provided with the norm

$$\|u\|_{\mathcal{C}^1(J;X)} = \max\{\|u\|_{\mathcal{C}(J;X)}, \|u'\|_{\mathcal{C}(J;X)}\}, \; u \in \mathcal{C}^1(J; X). \tag{1}$$

If X, Y are two Banach spaces, a function $u : Y \to X$ is said to be *weakly sequentially continuous* if for every sequence $(y_n)_n$ in Y, $y_n \rightharpoonup y \in Y$, then $u(y_n) \rightharpoonup u(y)$. A function $u : J \to X$ is said to be $(\mathcal{M}(J), \mathcal{B}(X))$-*measurable* if, for all $A \in \mathcal{B}(X)$, $u^{-1}(A) \in \mathcal{M}(J)$ (see [16], Definition 2.1.48), while $u : J \to X$ is said to be *Bochner-measurable* (B-measurable, for short) if there is a sequence of simple functions which converges to u almost everywhere in J (see [16], Definition 3.10.1 (a)).
Taking into account what has been said above, we are able to state the following result

Lemma 2.2. *Let X be a separable Banach space. A map $u : J \to X$ is $(\mathcal{M}(J), \mathcal{B}(X, \tau_w))$-measurable if and only if $u : J \to X$ is B-measurable.*

Proof. The separability implies that the Banach space X admits an equivalent LUR norm and then X has also a Kadec-Klee equivalent norm. Therefore $\mathcal{B}(X) = \mathcal{B}(X, \tau_w)$. So the $(\mathcal{M}(J), \mathcal{B}(X, \tau_w))$-measurability and the $(\mathcal{M}(J), \mathcal{B}(X))$-measurability are equivalent. Hence, by using Corollary 3.10.5 of [16], the thesis holds. □

Now, we denote with $L^1(J; X)$ the space of all X-valued Bochner integrable functions on J with norm $\|u\|_{L^1(J;X)} = \int_J \|u(t)\|_X \, dt$. If $X = \mathbb{R}$, the B-integrability is the $\mathcal{L}$-integrability and we put $\|\cdot\|_1 = \|\cdot\|_{L^1(J;\mathbb{R})}$,

while $L^1_+(J)$ stands for the subset of all non negative functions of $L^1(J)$. In the sequel we denote with $L^{1,\mathrm{loc}}([0,\infty);X)$ the set of all B-integrable functions on the compact intervals of $[0,\infty)$.
As a consequence of the sequential continuity and of the Dominated Convergence Theorem (see [1]), we have the following result.

Proposition 2.3. *Let (Ω,Σ,μ) be a positive measure space, (M,d) be a metric space, X be a Banach space, and $u: M\times\Omega\to X$ a function such that*

- **j)** *$u(t,\cdot)$ is B-measurable, for every $t\in M$;*
- **jj)** *$u(\cdot,s)$ is continuous on M, a.e. $s\in\Omega$;*
- **jjj)** *there exists $\varphi\in L^1_+(\Omega)$ such that $\|u(t,s)\|_X\le\varphi(s)$, $t\in M$, a.e. $s\in\Omega$.*

Then the function $U(t)=\int_\Omega u(t,s)\,ds$, $t\in M$ is continuous on M.

Moreover, by assuming a stronger regularity on the function u, defined on $J\times J$, $J=[a,b]$, we also state

Proposition 2.4. *Let X be a Banach space and $u: J\times J\to X$ be a function such that*

- **i)** *$u(t,\cdot)$ is B-measurable for every $t\in J$;*
- **ii)** *$u(\cdot,s)\in\mathcal{C}^1(J;X)$, a.e. $s\in J$;*
- **iii)** *there exists $\varphi\in L^1_+(J)$ such that $\|u(t,s)\|_X+\|\frac{\partial}{\partial t}u(t,s)\|_X\le\varphi(s)$, $t\in J$, a.e. $s\in J$.*

Then, the function $U: J\to X$, $U(t)=\int_a^b u(t,s)\,ds$, $t\in J$, is continuously differentiable on J and we have

$$U'(t)=\int_a^b\frac{\partial}{\partial t}u(t,s)\,ds,\ t\in J. \tag{2}$$

Proof. First of all, by **ii)** we write

$$\frac{\partial}{\partial t}u(t,s)=\lim_{n\to\infty}\frac{u(t+\frac{1}{n},s)-u(t,s)}{\frac{1}{n}},\ t\in J,\ \text{a.e. } s\in J.$$

Clearly, fixed $t\in J$, from **i)**, we have that $\left(\frac{u(t+\frac{1}{n},\cdot)-u(t,\cdot)}{\frac{1}{n}}\right)_n$ is a sequence of B-measurable functions converging a.e. in J. So the function $\frac{\partial}{\partial t}u(t,\cdot)$ is B-measurable and, from **iii)**, it is B-integrable too.
Now, to prove (2) we fix $t_0\in[a,b]$, $t_n\in[a,b]\setminus\{t_0\}$, $n\in\mathbb{N}$, such that $t_n\to t_0$ for $n\to\infty$, and we consider the sequence

$$\left(\frac{u(t_n,\cdot)-u(t_0,\cdot)}{t_n-t_0}\right)_n. \tag{3}$$

Now, for every $s\in J\setminus N$, where N is the null measure set for which **ii)** and **iii)** hold, and for every $n\in\mathbb{N}$, we can apply the Mean Value Theorem to the function

$$g_n: [\min\{t_0,t_n\},\max\{t_0,t_n\}]\to X$$
$$g_n(t)=u(t,s)$$

and then there exists $\overline{t}^n\in[\min\{t_0,t_n\},\max\{t_0,t_n\}]$ such that (see **iii)**)

$$\frac{\|u(t_n,s)-u(t_0,s)\|_X}{|t_n-t_0|} \le \left\| \frac{\partial}{\partial t} u(\overline{t}^n, s) \right\|_X \le \varphi(s).$$

Moreover, from **ii)** we can say that the sequence $\left(\frac{u(t_n,\cdot)-u(t_0,\cdot)}{t_n-t_0}\right)_n$ is a.e. convergent in J to the map $\frac{\partial u}{\partial t}(t_0,\cdot)$. Since all the hypotheses of the Dominate Convergence Theorem are satisfied by the sequence (3) of B-measurable maps, we can write

$$\lim_{n\to\infty} \frac{U(t_n)-U(t_0)}{t_n-t_0} = \lim_{n\to\infty} \int_a^b \frac{u(t_n,s)-u(t_0,s)}{t_n-t_0}\, ds = \int_a^b \frac{\partial}{\partial t} u(t_0,s)\, ds. \tag{4}$$

Since (4) is true for every sequence $(t_n)_n$ converging to t_0, we have that (2) holds in t_0.
Hence, by the arbitrariness of $t_0 \in J$ we deduce that (2) is true.
Finally, since the map $\frac{\partial u}{\partial t}$ satisfies **j)**, **jj)**, and **jjj)** of Proposition 2.3 we can conclude that $U' \in \mathcal{C}(J;X)$, therefore $U \in \mathcal{C}^1(J;X)$. $\square$

Then, a set $A \subset L^1(J;X)$ (or a sequence $(f_n)_n$, $f_n \in L^1(J;X)$) has the property of *equi-absolute continuity of the integral* if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for every $E \in \mathcal{M}(J)$, $\mu(E) < \delta_\varepsilon$, we have $\int_E \|f(t)\|_X\, dt < \varepsilon$, whenever $f \in A$ (for every $f \in \{f_n : n \in \mathbb{N}\}$), while $A \subset L^1(J;X)$ is *integrably bounded* if there exists $\nu \in L^1_+(J)$ such that $\|f(t)\|_X \le \nu(t)$, a.e. $t \in J$, for every $f \in A$.
Clearly every integrably bounded set has the property of equi-absolute continuity of the integral. The following result establishes that the equi-absolute continuity of the integral is important to characterize the relative weak compactness of bounded sets in $L^1(J;X)$.

Proposition 2.5 *([41] Corollary 9). Let A be a bounded subset of $L^1(J;X)$ having the property of equi-absolute continuity of the integral and, for a.e. $t \in J$, the set $A(t) = \{f(t) : f \in A\}$ is relatively weakly compact. Then A is relatively weakly compact.*

In our discussion we will use multimaps having the following properties.
First of all, if (X,Σ) is a measurable space, Y a topological space, and $\mathcal{P}(Y)$ denotes the family of all nonempty subsets of Y, a multimap $F : X \to \mathcal{P}(Y)$ is said to be *measurable* if for every open set $V \subset Y$ one has $F^-(V) = \{x \in X : F(x) \cap V \neq \varnothing\} \in \Sigma$.

Theorem 2.6 *([16], Theorem 4.3.1). If (X,Σ) is a measurable space, Y is a Polish space and $F : X \to \mathcal{P}(Y)$ is a measurable multimap assuming closed values, then F has a $(\Sigma, \mathcal{B}(Y))$-measurable selection.*

Now, for every sequence $(A_n)_n$, $A_n \subset X$, the *weak-Kuratowski limit superior* of $(A_n)_n$ is defined as (see [24], Definition 7.1.3)

$$w - \limsup_{n\to\infty} A_n = \{x \in X : \; x_{n_k} \rightharpoonup x, \; x_{n_k} \in A_{n_k}, \; n_1 < n_2 < \cdots < n_k < \cdots\}.$$

Proposition 2.7 *([24], Proposition 7.3.9). Let X be a Banach space, $1 \le p < \infty$ and $F : J \to \mathcal{P}(X)$ be a multimap assuming weakly compact values. If $(f_n)_n$, $f_n \in L^p(J;X)$, is a sequence such that*

- **i)** *there exists $f \in L^p(J;X)$ with $f_n \rightharpoonup f$;*
- **ii)** *$f_n(t) \in F(t)$ a.e. $t \in J$, $n \in \mathbb{N}$,*

then

$$f(t) \in \overline{co}\; w - \limsup_{n\to\infty} \{f_n(t)\}, \quad a.e.\ t \in J,$$

where $\overline{co}$ denotes the closure of the convex hull of a set.

If X, Y are normed spaces a multimap $F : X \to \mathcal{P}(Y)$ is *weakly semi-compact* if it maps bounded sets onto relatively weakly compact sets.
A multimap $F : J \to \mathcal{P}(X)$ has a *continuous selection* if there exists a continuous function $f : J \to X$ such that $f(t) \in F(t)$, $t \in J$. A multimap $F : J \to \mathcal{P}(X)$ has a *B-measurable selection* if there exists a B-measurable function $f : J \to X$ such that $f(t) \in F(t)$, a.e. $t \in J$. A multimap $F : J \times X \to \mathcal{P}(X)$ has a *Carathèodory selection* if there exists a function $f : J \times X \to X$ such that

i) for every $t \in J$, $f(t, \cdot)$ is continuous on X;
ii) for every $x \in X$, $f(\cdot, x)$ is $(\mathcal{M}(J), \mathcal{B}(X))$-measurable;
iii) for a.e. $t \in J$ and every $x \in X$, $f(t, x) \in F(t, x)$.

In the sequel we will use the following selection result.

Proposition 2.8 *([9], Theorem 4.4). Let J be a closed and bounded interval and X be a Banach space. Let $G_n, G : J \to \mathcal{P}(X)$ be such that the following holds:*

1. *a.e. $t \in J$, for every $(u_n)_n$, $u_n \in G_n(t)$, there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and $u \in G(t)$ such that $u_{n_k} \rightharpoonup u$;*
2. *there exists a sequence of functions $(y_n)_n$, $y_n : J \to X$, having the property of equi-absolute continuity of the integral, such that $y_n(t) \in G_n(t)$, a.e. $t \in J$, for all $n \in \mathbb{N}$.*

Then there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that $y_{k_n} \rightharpoonup y$ in $L^1(J; X)$, and, moreover, the limit function satisfies $y(t) \in \overline{co}G(t)$, a.e. $t \in J$.

In the previous proposition with *(s-w)sequentially closed graph* we mean that for every $(x_n)_n$, $x_n \in M$, $x_n \to x$ and every $(y_n)_n$, $y_n \in F(x_n)$, $y_n \rightharpoonup y$, then $y \in F(x)$. In particular if M coincides with the Banach space X and the sequence $(x_n)_n$ weakly converges to x, we shortly say that the multimap F has "weakly sequentially closed graph". Moreover, if X and Y are normed spaces, a multimap $F : X \to Y$ having weakly closed graph has also weakly sequentially closed graph. Analogously the "(w-w) sequential continuity" is named "weak sequential continuity".
In this work we will use the following fixed point theorem proved by Glicksberg for multimaps.

Theorem 2.9 *([19], page 171). Let X be a locally convex Hausdorff space and S a convex and compact subset of X. Let $F : S \to \mathcal{P}(S)$ be a multimap such that*

j) *$F(x)$ is convex, for every $x \in S$;*
jj) *F has closed graph.*

Then there exists $\overline{x} \in S$ such that $\overline{x} \in F(\overline{x})$.

In conclusion, we recall the concept of Schauder basis and we collect some important properties that will be useful in the sequel.
A sequence $(e_n)_n$ of vectors in a Banach space X is a Schauder basis for X if for every $x \in X$, there exists a unique sequence $(\alpha_i = \alpha_i(x))_i$ of real numbers, such that (see [36], Definition 1)

$$x = \sum_{i=1}^{\infty} \alpha_i e_i.$$

If X is a n-dimensional vector space then the basis of X is a Schauder basis for the space.
Note that if the Banach space has a Schauder basis, then it is separable (see [36], Remark 1).
If $(e_n)_n$ is a Schauder basis for X, we denote with $X_n = span\{e_1, ..., e_n\}$ the n-dimensional Banach space generated by the first n vectors of the Schauder basis and with $\mathbb{P}_n : X \to X_n$ the natural projection of X onto X_n so defined

$$\mathbb{P}_n\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i e_i.$$

In particular, the projections $\mathbb{P}_n : X \to X_n$ are bounded linear operators (shortly $\mathbb{P}_n \in \mathcal{L}(X, X_n)$) and $\sup_n \|\mathbb{P}_n\|_{\mathcal{L}(X,X_n)} < \infty$, where $\|\mathbb{P}_n\|_{\mathcal{L}(X,X_n)} = \sup\{\|\mathbb{P}_n x\|_{X_n} : \ \|x\|_X = 1\}$ (see [28], Proposition 1.a.2).
In the sequel we assume, w.l.o.g., that $\sup_n \|\mathbb{P}_n\|_{\mathcal{L}(X,X_n)} \leq 1$, i.e. $(\mathbb{P}_n)_n$ is monotone (see [28], p. 2).
Finally, we present a result describing some interesting properties of the operators $\mathbb{P}_n$.

Proposition 2.10 *([7], Lemma 2.1, 2.2, [30], Proposition 2.4). The projection $\mathbb{P}_n : X \to X_n$, $n \in \mathbb{N}$, satisfies the following properties:*

- **a)** $\mathbb{P}_n : (X, \tau_w) \to X_n$ *is continuous;*
- **b)** *if* $x_n \rightharpoonup x$, *then* $\mathbb{P}_n(x_n) \rightharpoonup x$;
- **c)** *if* $f_n \rightharpoonup f$ *in* $L^1(J; X)$, *then* $\mathbb{P}_n f_n \rightharpoonup f$ *in* $L^1(J; X)$;
- **d)** *if* $x_n \to x$, *then* $\mathbb{P}_n(x_n) \to x$;
- **e)** *for every* $x \in X$, $\|\mathbb{P}_n(x) - x\|_X \to 0$.

3. Problem setting

First of all, we recall that a one parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of linear bounded operators mapping the Banach space X into itself is named a *strongly continuous cosine family* if (see [39] or [17])

C1) $C(t + s) + C(s - t) = 2C(t)C(s)$, $t, s \in \mathbb{R}$;
C2) $C(0) = I$;
C3) the map $t \mapsto C(t)x$ is continuous in $\mathbb{R}$, for every $x \in X$.

If $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family, then there exists $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\|_{\mathcal{L}(X)} \leq Me^{\omega|t|}, \ t \in \mathbb{R}, \tag{5}$$

where $\|\cdot\|_{\mathcal{L}(X)}$ denotes the norm of the space $\mathcal{L}(X, X)$ (shortly $\mathcal{L}(X)$). In the sequel we consider a linear closed operator $A : D(A) \subset X \to X$, $D(A)$ dense in X, generating a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, i.e. the operator $A : D(A) \subset X \to X$ is defined by

$$Ax = \frac{d^2}{dt^2}C(\cdot)|_{t=0}x, \ x \in D(A), \tag{6}$$

where $D(A)$ is described by the family $\{C(t)\}_{t\in\mathbb{R}}$ as follows

$$D(A) = \{x \in X : \ C(\cdot)x \text{ is twice continuously differentiable in } \mathbb{R}\}. \tag{7}$$

Moreover, we will use the set so defined

$$E = \{x \in X : \ C(\cdot)x \text{ is continuously differentiable in } \mathbb{R}\} \tag{8}$$

and we will consider the one parameter family $\{S(t)\}_{t\in\mathbb{R}}$ associated with $\{C(t)\}_{t\in\mathbb{R}}$, where $S(t) : X \to X$ is a linear and bounded operator defined by

$$S(t)x = \int_0^t C(s)x\,ds,\ x \in X, \tag{9}$$

for every $t \in \mathbb{R}$. In literature this family is called the *strongly continuous sine family* associated to $\{C(t)\}_{t\in\mathbb{R}}$. Clearly $S(0) = 0$. Additionally, the families $\{S(t)\}_{t\in\mathbb{R}}$ and $\{C(t)\}_{t\in\mathbb{R}}$ have the properties presented in the following proposition.

Proposition 3.1 *([39], Propositions 2.1, 2.2). The families $\{S(t)\}_{t\in\mathbb{R}}$ and $\{C(t)\}_{t\in\mathbb{R}}$ satisfy the following properties:*

- **a)** $C(t) = C(-t)$, $t \in \mathbb{R}$;
- **b)** $S(t) = -S(-t)$, $t \in \mathbb{R}$;
- **c)** *for every* $x \in X$, *the map* $t \mapsto S(t)x$ *is continuous;*
- **d)** $S(s+t) + S(s-t) = 2S(s)C(t)$, $t, s \in \mathbb{R}$;
- **e)** $S(s+t) = S(s)C(t) + S(t)C(s)$, $t, s \in \mathbb{R}$;
- **f)** $\|S(t) - S(\hat{t})\|_{\mathcal{L}(X)} \le M|\int_{\hat{t}}^t e^{\omega|s|}\,ds|$, $t, \hat{t} \in \mathbb{R}$, *where* w *and* M *are the constants presented in* (5);
- **g)** $C(s), S(s), C(t), S(t)$ *commute, for every* $t, s \in \mathbb{R}$;
- **h)** $S(t)x \in E$, $t \in \mathbb{R}, x \in X$;
- **i)** $S(t)x \in D(A)$, $\lim_{t\to 0} AS(t)x = 0$, $\frac{d}{dt}C(t)x = AS(t)x$, *and* $\frac{d^2}{dt^2}S(t)x = AS(t)x$, $t \in \mathbb{R}$, $x \in E$;
- **j)** $C(t)x \in D(A)$, $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$, *and* $AS(t)x = S(t)Ax$, $x \in D(A)$, $t \in \mathbb{R}$;
- **k)** $C(t+s) - C(t-s) = 2AS(t)S(s)$, $t, s \in \mathbb{R}$.

We note that, from **f)**, for every $t \in \mathbb{R}$, we have

$$\|S(t)\|_{\mathcal{L}(X)} \le \begin{cases} M\frac{|e^{\omega|t|}-1|}{\omega}, & \omega \neq 0 \\ M|t|, & \omega = 0. \end{cases} \tag{10}$$

Then, taking into account (10) and (5), for every compact interval $J \subset \mathbb{R}$ we can write

$$\|S(t)\|_{\mathcal{L}(X)} \le L_1^J,\ \|C(t)\|_{\mathcal{L}(X)} \le L_2^J,\ t \in J, \tag{11}$$

where L_1^J, L_2^J are two positive constants.

Additionally, for every $x \in X$, by using the Fundamental Theorem of Calculus for vector valued functions we have (see (9) and **C3)**)

$$S(\cdot)x \in \mathcal{C}^1(\mathbb{R}; X) \tag{12}$$

and

$$\frac{d}{dt}S(t)x = C(t)x,\ t \in \mathbb{R}. \tag{13}$$

Remark 3.2. In Lemma 4 of [36] is proved that if A is the generator of a cosine family $\{C(t)\}_{t\in\mathbb{R}}$, then A generates a C_0-semigroup $\{T(t)\}_{t\ge 0}$ and $\{C(t)\}_{t\in\mathbb{R}}$ satisfies (5) if and only if $\|T(t)\|_{\mathcal{L}(X)} \le 2Me^{w^2 t}$, $t \ge 0$.

Now, we give the concept of mild solution for the Cauchy problem

$$(\mathbf{PC})_b \begin{cases} x''(t) \in Ax(t) + F(t, x(t), x'(t)), \ t \in I = [b, c] \\ x(b) = y_1 \\ x'(b) = y_2 \end{cases}$$

where A is the operator defined in (6)-(7), $I = [b, c]$, $0 \leq b < c$, $F : I \times X \times X \to \mathcal{P}(X)$ is a multimap, $y_1 \in E$ (see (8)), and $y_2 \in X$.

Definition 3.1 *([36], Proposition 2).* A function $x : I \to X$ is said to be a *mild solution* for $(\mathbf{PC})_\mathbf{b}$ if x is a $\mathcal{C}^1$-function such that

$$x(t) = C(t-b)y_1 + S(t-b)y_2 + \int_b^t S(t-s)f(s)\,ds, \ t \in I \tag{14}$$

where $f \in S^1_{F(\cdot,x(\cdot),x'(\cdot))} = \{f \in L^1(I; X) : \ f(t) \in F(t, x(t), x'(t)), \text{ a.e. } t \in I\}$.

Let us note that, for every $f \in L^1(I; X)$ and every $y_1 \in E$, a map defined as in (14) is actually a $\mathcal{C}^1$-function in I, as it is possible to deduce from the following version, in the case $b \geq 0$, of Lemma II.4.2 proved in [17], where $\{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family and $\{S(t)\}_{t\in\mathbb{R}}$ is defined as in (9) (see **i)** of Proposition 3.1, (13) and proof of Lemma II.4.1 of [17]).

Proposition 3.3. *If X is a Banach space, $f \in L^1(I; X)$, $y_1 \in E$, and $y_2 \in X$, then the function $x : I \to X$ so defined*

$$x(t) = C(t-b)y_1 + S(t-b)y_2 + \int_b^t S(t-s)f(s)\,ds, \ t \in I$$

is continuously differentiable in I, with

$$x'(t) = AS(t-b)y_1 + C(t-b)y_2 + \int_b^t C(t-s)f(s)\,ds, \ t \in I.$$

Next, we present the following results that describes some properties of the operators $\mathfrak{C}_C, \mathfrak{S}_S : L^1(I; X) \to \mathcal{C}(I; X)$ respectively so defined

$$\mathfrak{C}_C f(t) = \int_b^t C(t-s)f(s)\,ds, \ t \in I, \ f \in L^1(I; X), \tag{15}$$

$$\mathfrak{S}_S f(t) = \int_b^t S(t-s)f(s)\,ds, \ t \in I, \ f \in L^1(I; X), \tag{16}$$

where $\{C(t)\}_{t\in\mathbb{R}}$ is the cosine family and $\{S(t)\}_{t\in\mathbb{R}}$ is the sine family.

Proposition 3.4. *Let $\{C(t)\}_{t\in\mathbb{R}}$ be a cosine family and $\{S(t)\}_{t\in\mathbb{R}}$ be a sine family, $I = [b, c]$, $0 \leq b < c$, then the operators $\mathfrak{C}_C, \mathfrak{S}_S : L^1(I; X) \to \mathcal{C}(I; X)$ are linear, bounded, weakly continuous, and so weakly sequentially continuous.*

Proof. We prove the result for the operator $\mathfrak{C}_C$.
Clearly $\mathfrak{C}_C$ is linear. Moreover, taking into account (11) we have that

$$\|\mathfrak{C}_C f\|_{\mathcal{C}(J;X)} = \max_{t\in I} \| \int_b^t C(t-s) f(s)\, ds\|_X \leq L_2^I \|f\|_{L^1(I;X)}, \; f \in L^1(I;X),$$

i.e. $\mathfrak{C}_C$ is bounded. Hence we can conclude that $\mathfrak{C}_C$ is weakly continuous (see [10], Theorem 3.10) and so it is also weakly sequentially continuous (see [4], Definition 1.1).
Analogously we obtain the properties for the operator $\mathfrak{S}_S$. □

Finally, we enunciate the following existence theorem for the Cauchy problem $(\mathbf{PC})_\mathbf{b}$ proved by M. Pavlackova and V. Taddei.

Theorem 3.5 *([36], Theorem 1). Let X be a reflexive Banach space having a Schauder basis, $A : D(A) \subset X \to X$ be a infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, $y_1 \in E$, where the set E is defined in (8), and $y_2 \in X$. Let $F : I \times X \times X \to \mathcal{P}(X)$ be a multimap satisfying*

F1)$_S$ *for every $(t, x, y) \in I \times X \times X$ $F(t, x, y)$ is nonempty, convex, closed, and bounded;*
F2) *for every $(x, y) \in X \times X$, $F(\cdot, x, y)$ has a B-measurable selection;*
F3)$_S$ *for a.e. $t \in I$, $F(t, \cdot, \cdot) : (X, \tau_w) \times (X, \tau_w) \to (X, \tau_w)$ is upper semicontinuous (shortly $F(t, \cdot, \cdot)$ weakly u.s.c.);*
F4) *for every $n \in \mathbb{N}$, there exists $\varphi_n \in L^1_+(I)$ such that*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_1}{n} = 0$$

and

$$\|F(t, \overline{B}_X(0, n), \overline{B}_X(0, n))\| \leq \varphi_n(t), \; a.e.\ t \in I.$$

Then the problem $(\mathbf{PC})_\mathbf{b}$ has at least one mild solution.

4. Mild solutions for non-impulsive problems on a bounded interval

In this section we study the existence of mild solutions for the Cauchy problem $(\mathbf{PC})_\mathbf{b}$ driven by a semilinear second order differential inclusion perturbed by a weakly semi-compact multimap.
Drawing inspiration from [36], we use the projections $\mathbb{P}_n : X \to X_n$, $n \in \mathbb{N}$, to obtain the existence of mild solutions for $(\mathbf{PC})_\mathbf{b}$, but with a different approach. Instead of considering, as in [36], a family of projected problems linked to $(\mathbf{PC})_\mathbf{b}$, we associate a sequence of approximating operators to the solution operator of $(\mathbf{PC})_\mathbf{b}$. This allows us to establish the existence of a mild solution $q \in \mathcal{C}^1(I; X)$, by achieving directly the existence of a B-measurable selection for the multimap $F(\cdot, q(\cdot), q'(\cdot))$ via a multivalued version of Fatou's Lemma (see Proposition 2.7).

Theorem 4.1. *Let X be a Banach space having a Schauder basis, $I = [b, c] \subset \mathbb{R}^+_0$, $A : D(A) \subset X \to X$ be a infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, $y_1 \in E$, where the set E is defined in (8), and $y_2 \in X$. Let $F : I \times X \times X \to \mathcal{P}(X)$ be a multimap satisfying*

F1) *for every $(t, x, y) \in I \times X \times X$, $F(t, x, y)$ is nonempty and convex;*

F2) *for every* $(x,y) \in X \times X$, $F(\cdot,x,y)$ *has a B-measurable selection;*
F3) *for a.e.* $t \in I$, $F(t,\cdot,\cdot) : X \times X \to \mathcal{P}(X)$ *has weakly sequentially closed graph;*
F4) *for every* $n \in \mathbb{N}$, *there exists* $\varphi_n \in L^1_+(I)$ *such that*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_1}{n} = 0$$

and

$$\|F(t, \overline{B}_X(0,n), \overline{B}_X(0,n))\| \le \varphi_n(t), \text{ a.e. } t \in I.$$

F5) *for a.e.* $t \in I$, *the multimap* $F(t,\cdot,\cdot)$ *is weakly semi-compact.*

Then the Cauchy problem $(\mathbf{PC})_\mathbf{b}$ *has at least one mild solution.*

Proof. First of all, we have that

(I) $F(t,x,y)$ is closed, for a.e. $t \in I$ and every $x, y \in X$.

Indeed, denoted by N the null measure set such that **F3)** and **F5)** hold in $I \setminus N$, we fix $t \in I \setminus N$ and $x, y \in X$. Clearly, from **F5)** $F(t,x,y)$ is relatively weakly compact. By virtue of **F3)** the set $F(t,x,y)$ is also weakly sequentially closed. So Proposition 2.1 and **F1)** imply that $F(t,x,y)$ is closed.

From now on we assume w.l.o.g. that the Schauder basis on X is monotone and we split the proof into steps.

Step 1. Characterization of a sequence of approximating operators for the solution operator of $(\mathbf{PC})_\mathbf{b}$.

For every $n \in \mathbb{N}$, we consider the operator $\Gamma_n : \mathcal{C}^1(I, X_n) \to \mathcal{P}(\mathcal{C}^1(I, X_n))$ so defined

$$\begin{aligned}\Gamma_n q =& \{y \in \mathcal{C}^1(I;X_n) : \ y(t) = \mathbb{P}_n[C(t-b)y_1] + \mathbb{P}_n[S(t-b)y_2] \\ &+ \int_b^t \mathbb{P}_n[S(t-s)f(s)]\,ds, \ t \in I, \ f \in S^1_{F(\cdot,q(\cdot),q'(\cdot))}\},\end{aligned} \tag{17}$$

where $S^1_{F(\cdot,q(\cdot),q'(\cdot))} = \{f \in L^1(I;X) : \ f(t) \in F(t,q(t),q'(t)), \text{ a.e. } t \in I\}$, $X_n = span\{e_1, ...e_n\}$, and $\mathbb{P}_n : X \to X_n$ is the natural projection.
Let us prove that the operator Γ_n is well posed. Fixed $\overline{q} \in \mathcal{C}^1(I;X_n)$, we show that $\Gamma_n \overline{q} \ne \varnothing$.
First of all, since $\overline{q} \in \mathcal{C}^1(I;X_n)$, there exists $r_{\overline{q}} \in \mathbb{N}$ such that

$$\|\overline{q}\|_{\mathcal{C}^1(I;X_n)} = \max_{t\in I}\{\|\overline{q}(t)\|_{X_n}, \|\overline{q}'(t)\|_{X_n}\} \le r_{\overline{q}}. \tag{18}$$

Put

$$M_{\overline{q}} = \overline{B}_X(0, r_{\overline{q}}), \tag{19}$$

we show that $F_{|I\times M_{\overline{q}}\times M_{\overline{q}}} : I \times M_{\overline{q}} \times M_{\overline{q}} \to \mathcal{P}(X)$ satisfies all the assumptions of Theorem 4.2 of [12], by considering on $M_{\overline{q}} \times M_{\overline{q}}$ the metric induced by the norm.
Clearly, **a)** and **b)** of Theorem 4.2 of [12] hold (see **F1)** and **F2)**).

In particular, from **F3)** we have that for a.e. $t \in I$, $F_{|I\times M_{\overline{q}}\times M_{\overline{q}}}(t,\cdot,\cdot)$ has $(s\text{-}w)$sequentially closed graph, i.e. **c)** of Theorem 4.2 of [12] is true.
Next, also **d)** of Theorem 4.2 of [12] holds. Indeed, fixed $t \in I\setminus N$, where N is the null measure set relative to **F5)**, if $((x_k, y_k))_k$ is a convergent sequence in $M_{\overline{q}} \times M_{\overline{q}}$, the set $\bigcup_k F(t, x_k, y_k)$ is relatively weakly compact, being a subset of $F(t, M_{\overline{q}}, M_{\overline{q}})$ (see (19) and **F5)**).
To prove **e)** of Theorem 4.2 of [12], we note that from **F4)** there exists $\varphi_{r_{\overline{q}}} \in L^1_+(I)$ such that

$$\|F(t, M_{\overline{q}}, M_{\overline{q}})\| \leq \varphi_{r_{\overline{q}}}(t), \text{ a.e. } t \in I.$$

Thanks to Theorem 4.2 of [12], fixed the B-measurable map $u : I \to M_{\overline{q}} \times M_{\overline{q}}$, $u(t) = (\overline{q}(t), \overline{q}'(t))$, $t \in I$, there exists a B-measurable function $f_{\overline{q}} : I \to X$ such that

$$f_{\overline{q}}(t) \in F(t, \overline{q}(t), \overline{q}'(t)), \text{ a.e. } t \in I.$$

Now, from (18) and **F4)** we can say that $f_{\overline{q}}$ is B-integrable. Therefore,

$$f_{\overline{q}} \in S^1_{F(\cdot,\overline{q}(\cdot),\overline{q}'(\cdot))}. \tag{20}$$

Next, we consider the map $g_{\overline{q}} : I \to X$ so defined

$$g_{\overline{q}}(t) = C(t-b)y_1 + S(t-b)y_2 + \int_b^t S(t-s)f_{\overline{q}}(s)\,ds, \ t \in I.$$

Since $y_1 \in E$, we can say that $g_{\overline{q}}$ is continuously differentiable on I and we have (see Proposition 3.3)

$$g'_{\overline{q}}(t) = AS(t-b)y_1 + C(t-b)y_2 + \int_b^t C(t-s)f_{\overline{q}}(s)\,ds, \ t \in I.$$

Now, if $y_{\overline{q}} : I \to X_n$ is so defined $y_{\overline{q}}(t) = \mathbb{P}_n[g_{\overline{q}}(t)]$, being $\mathbb{P}_n$ linear and bounded, we can deduce

$$y'_{\overline{q}}(t) = \mathbb{P}_n[g'_{\overline{q}}(t)] = \mathbb{P}_n[AS(t-b)y_1] + \mathbb{P}_n[C(t-b)y_2] + \int_b^t \mathbb{P}_n[C(t-s)f_{\overline{q}}(s)]\,ds, \ t \in I. \tag{21}$$

Hence $y_{\overline{q}} \in \mathcal{C}^1(I; X_n)$ (see **C3)** and **i)** of Proposition 3.1) and so, recalling (20), $\Gamma_n\overline{q} \neq \varnothing$. Therefore Γ_n is well posed.

Now, by using Theorem 2.9, we prove the existence of a fixed point for Γ_n.

Step 1a. There exists $\overline{p} \in \mathbb{N}$ such that

$$\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})) \subset \overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}). \tag{22}$$

Assume, by contradiction that

$$\exists\ q_p \in \mathcal{C}^1(I; X_n),\ \|q_p\|_{\mathcal{C}^1(I;X_n)} \leq p :\ \exists\ y_{q_p} \in \Gamma_n q_p \text{ with } \|y_{q_p}\|_{\mathcal{C}^1(I;X_n)} > p,\ p \in \mathbb{N}. \tag{23}$$

Then, for every $p \in \mathbb{N}$, there exists $t_p \in I$ such that (see (1))

(A) $\|y_{q_p}(t_p)\|_{X_n} > p$

or

(B) $\|y'_{q_p}(t_p)\|_{X_n} > p$.

First of all, taking into account that $y_1 \in E$ and of **i)** of Proposition 3.1, we deduce that

$$\|AS(t)y_1\|_X = \left\|\frac{d}{dt}C(t)y_1\right\|_X \leq \max_{t\in I}\left\|\frac{d}{dt}C(t)y_1\right\|_X := L,\ t \in I. \tag{24}$$

Now, in case **(A)**, since $\|\mathbb{P}_n\|_{\mathcal{L}(X,X_n)} \leq 1$ and $y_{q_p} \in \Gamma_n q_p$, we can write (see (11))

$$\begin{aligned} p < \|y_{q_p}(t_p)\|_{X_n} &\leq \|\mathbb{P}_n C(t_p - b)y_1\|_{X_n} + \|\mathbb{P}_n S(t_p - b)y_2\|_{X_n} + \int_b^{t_p} \|\mathbb{P}_n S(t_p - s)f_{q_p}(s)\|_{X_n}\, ds \\ &\leq L_2^I\|y_1\|_X + L_1^I\|y_2\|_X + L_1^I \int_b^{t_p} \|f_{q_p}(s)\|_X\, ds, \end{aligned} \tag{25}$$

where $f_{q_p} \in S^1_{F(\cdot,q_p(\cdot),q'_p(\cdot))}$. Next, since $\|q_p\|_{\mathcal{C}^1(I;X_n)} \leq p$ (see (23)), by **F4)** there exists $\varphi_p \in L^1_+(I)$ such that

$$\|f_{q_p}(s)\|_X \leq \varphi_p(s),\ \text{a.e. } s \in I.$$

Then, from (25), we have

$$p < \|y_{q_p}(t_p)\|_{X_n} \leq L_2^I\|y_1\|_X + L_1^I\|y_2\|_X + L_1^I\|\varphi_p\|_1,$$

and so

$$1 - \frac{L_2^I\|y_1\|_X + L_1^I\|y_2\|_X}{p} < \frac{L_1^I\|\varphi_p\|_1}{p}. \tag{26}$$

Analogously, in the case **(B)**, by using (24), (11) and **F4)**, from (21) we obtain

$$1 - \frac{L + L_2^I\|y_2\|_X}{p} < \frac{L_2^I\|\varphi_p\|_1}{p}. \tag{27}$$

Since at least one between (26) and (27) is true for infinitely many $p \in \mathbb{N}$, by using the first part of **F4)**, we obtain one of the following contradictions

$$1 \leq \liminf_{p\to\infty} \frac{L_1^I\|\varphi_p\|_1}{p} = 0$$

or

$$1 \leq \liminf_{p\to\infty} \frac{L_2^I\|\varphi_p\|_1}{p} = 0.$$

Therefore, (22) is true.

Step 1b. The set $\overline{co}\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$ is a convex and compact subset of $\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})$, where $\overline{p}\in\mathbb{N}$ is fixed as in **Step 1a.**

To this aim we first establish that $\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$ is relatively compact.
Fixed $(y_k)_k$ a sequence in $\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$, we will prove that exists a subsequence that uniformly converges to a function y in $\mathcal{C}^1(I;X_n)$.
For every $k\in\mathbb{N}$, since $y_k\in\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$ there exist $q_k\in\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})$ and $f_k\in S^1_{F(\cdot,q_k(\cdot),q'_k(\cdot))}$ such that

$$y_k(t)=\mathbb{P}_n[C(t-b)y_1]+\mathbb{P}_n[S(t-b)y_2]+\int_b^t\mathbb{P}_n[S(t-s)f_k(s)]\,ds,\ t\in I$$

and

$$y'_k(t)=\mathbb{P}_n[AS(t-b)y_1]+\mathbb{P}_n[C(t-b)y_2]+\int_b^t\mathbb{P}_n[C(t-s)f_k(s)]\,ds,\ t\in I. \tag{28}$$

Now, taking into account the monotonicity of $\mathbb{P}_n$, of (11), (24), and of **F4)**, the following inequalities hold

$$\|y_k(\overline{t})-y_k(t)\|_{X_n}\le\int_{\min\{\overline{t},t\}}^{\max\{\overline{t},t\}}\|y'_k(s)\|_{X_n}\,ds\le(L+L_2^I\|y_2\|_X+L_2^I\|\varphi_{\overline{p}}\|_1)|\overline{t}-t|,\ t,\overline{t}\in I,\ k\in\mathbb{N}.$$

Thus, $(y_k)_k$ is equicontinuous and equibounded in I. Hence, by the classical Ascoli-Arzelà Theorem there exist a subsequence $(y_{k_m})_m$ of $(y_k)_k$ and $y\in\mathcal{C}(I;X_n)$ such that

$$y_{k_m}\to y\text{ in }\mathcal{C}(I;X_n) \tag{29}$$

On the other hand, since every y_{k_m} is continuously differentiable on I, we can show that there exists a subsequence of $(y'_{k_m})_m$ that uniformly converges to $z\in\mathcal{C}(I;X_n)$.
To this aim, we first prove that there exists a subsequence of $(f_{k_m})_m$, weakly converging to a function in $L^1(I;X)$.
Let us consider the set

$$A=\{f_{k_m}:\ m\in\mathbb{N}\}\subset L^1(I;X). \tag{30}$$

Since $q_{k_m}\in\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})$, $m\in\mathbb{N}$, we write

$$f_{k_m}(t)\in F(t,q_{k_m}(t),q'_{k_m}(t))\subset F(t,\overline{B}_{X_n}(0,\overline{p}),\overline{B}_{X_n}(0,\overline{p})),\ \text{a.e. } t\in I, \tag{31}$$

then, by **F4)** there exists $\varphi_{\overline{p}}\in L^1_+(I)$ such that

$$\|f_{k_m}(t)\|_X\le\varphi_{\overline{p}}(t),\ \text{a.e. } t\in I,\ m\in\mathbb{N}.$$

Therefore A is integrably bounded, that implies the boundedness in $L^1(I;X)$ and the property of equi-absolute continuity of the integral of A.
Next, fixed $t\in I\setminus\overline{N}$, where $\overline{N}$ is the null measure set for which (31) and **F5)** hold in $I\setminus\overline{N}$, we have the inclusion

$$A(t) = \{f_{k_m}(t) : \ m \in \mathbb{N}\} \subset F(t, \overline{B}_{X_n}(0,\overline{p}), \overline{B}_{X_n}(0,\overline{p}))$$

and, from **F5)**, the relative weak compactness of $A(t)$. Therefore, by using Proposition 2.5 we conclude that the set A is relatively weakly compact. So there exist a subsequence of $(f_{k_m})_m$, named again $(f_{k_m})_m$, and $f \in L^1(I;X)$ such that

$$f_{k_m} \rightharpoonup f. \tag{32}$$

Now, thanks to the weak sequential continuity of $\mathfrak{C}_C$ (see Proposition 3.4) we have (see (15))

$$\int_b^t C(t-s)f_{k_m}(s)\,ds \rightharpoonup \int_b^t C(t-s)f(s)\,ds, \ t \in I.$$

Recalling that the projection $\mathbb{P}_n$ is linear and bounded, being $\dim X_n < +\infty$, we can write

$$\int_b^t \mathbb{P}_n[C(t-s)f_{k_m}(s)]\,ds \to \int_b^t \mathbb{P}_n[C(t-s)f(s)]\,ds, \ m \to \infty, \ t \in I,$$

therefore, from (28), we deduce

$$y'_{k_m}(t) \to \mathbb{P}_n[AS(t-b)y_1] + \mathbb{P}_n[C(t-b)y_2] + \int_b^t \mathbb{P}_n[C(t-s)f(s)]\,ds := z(t), \ t \in I. \tag{33}$$

Then, put $t^* \in I$ such that $\max_{t\in I} \|y'_{k_m}(t) - z(t)\|_{X_n} = \|y'_{k_m}(t^*) - z(t^*)\|_{X_n}$, by using (33) at the point t^*, we can write

$$y'_{k_m} \to z \text{ in } \mathcal{C}(I;X_n). \tag{34}$$

Taking into account (29) and (34), we conclude that $z = y'$. So

$$y_{k_m} \to y \text{ in } \mathcal{C}^1(I;X_n).$$

Therefore the set $\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$ is relatively compact in $\mathcal{C}^1(I;X_n)$. Hence we can deduce that the set $\overline{co}\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$ is convex, compact, and a subset of $\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})$.

Step 1c. The restriction of the multioperator $\Gamma_{n|B} : B \to \mathcal{P}(B)$ has closed graph, where $B = \overline{co}\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p}))$ (see **Step 1b.**).

To this aim it is sufficient to show that $\Gamma_{n|B}$ has sequentially closed graph.
Now, fixed two sequences $(q_k)_k$ and $(y_k)_k$ in B such that $q_k \to q$, $y_k \to y$ in $\mathcal{C}^1(I;X_n)$ with

$$y_k \in \Gamma_{n|B}q_k, \ k \in \mathbb{N}, \tag{35}$$

we want to prove that $y \in \Gamma_{n|B}q$.
At first, the convergence of $(q_k)_k$ to q in $\mathcal{C}^1(I;X_n)$ implies

$$q_k(t) \to q(t), \ q'_k(t) \to q'(t), \ t \in I. \tag{36}$$

Now, we establish that the multimaps $G_k : I \to \mathcal{P}(X)$, $k \in \mathbb{N}$ and $G : I \to \mathcal{P}(X)$, respectively so defined

$$G_k(t) = F(t, q_k(t), q_k'(t)),\ t \in I, \tag{37}$$

$$G(t) = F(t, q(t), q'(t)),\ t \in I, \tag{38}$$

satisfy all the assumptions of Proposition 2.8.
If N is the null measure set for which **F3)** and **F5)** hold, let us fix $t \in I \setminus N$ and a sequence $(u_k)_k$, such that

$$u_k \in G_k(t),\ k \in \mathbb{N}. \tag{39}$$

Now, we have that $q_k(t), q_k'(t) \in \overline{B}_{X_n}(0, \overline{p})$, $k \in \mathbb{N}$. So we can say that the set $\{u_k : k \in \mathbb{N}\}$ is relatively weakly compact, being a subset of $F(t, \overline{B}_{X_n}(0, \overline{p}), \overline{B}_{X_n}(0, \overline{p}))$ (see (39), (37) and **F5)**). Hence there exists a subsequence $(u_{k_m})_m$ weakly convergent to $u \in X$. Moreover, since, for every $m \in \mathbb{N}$, $u_{k_m} \in F(t, q_{k_m}(t), q_{k_m}'(t))$, from **F3)** and (36) we can conclude that $u \in F(t, q(t), q'(t)) = G(t)$ (see (38)). So **1.** of Proposition 2.8 holds.
Then, from (35), we note that for every $k \in \mathbb{N}$, there exists $f_k \in S^1_{F(\cdot, q_k(\cdot), q_k'(\cdot))}$ such that

$$y_k(t) = \mathbb{P}_n[C(t-b)y_1] + \mathbb{P}_n[S(t-b)y_2] + \int_b^t \mathbb{P}_n[S(t-s)f_k(s)]\, ds,\ t \in I. \tag{40}$$

The sequence $(f_k)_k$ satisfies **2.** of Proposition 2.8. Indeed (see (37))

$$f_k(t) \in G_k(t) = F(t, q_k(t), q_k'(t)),\ \text{a.e. } t \in I$$

and, from **F4)**, there exists $\varphi_{\overline{p}} \in L^1_+(I)$ such that

$$\|f_k(t)\|_X \le \varphi_{\overline{p}}(t),\ \text{a.e. } t \in I,\ k \in \mathbb{N}$$

i.e. $(f_k)_k$ is integrably bounded. Then it has the property of equi-absolute continuity of the integral.
Since all the hypotheses of Proposition 2.8 are satisfied by the multimaps $G_k, G : I \to \mathcal{P}(X)$, $k \in \mathbb{N}$, there exist a subsequence $(f_{k_m})_m$ of $(f_k)_k$ and $f \in L^1(I; X)$ such that

$$f_{k_m} \rightharpoonup f, \tag{41}$$

and

$$f(t) \in \overline{co}G(t) = \overline{co}F(t, q(t), q'(t)),\ a.e.\ t \in I. \tag{42}$$

Next, if N^* is a null measure set such that the property **(I)** and (42) hold in $I \setminus N^*$, we have $f(t) \in F(t, q(t), q'(t))$, for every $t \in I \setminus N^*$ (see **F1)** and **(I)**). Hence, since $f \in L^1(I; X)$, we can conclude that

$$f \in S^1_{F(\cdot, q(\cdot), q'(\cdot))}. \tag{43}$$

Now, recalling the weak continuity of the operator $\mathfrak{S}_S : L^1(I; X) \to \mathcal{C}(I; X)$ (see (16) and Proposition 3.4), from (41) we write

$$\mathfrak{S}_S(f_{k_m}) \rightharpoonup \mathfrak{S}_S(f),$$

then, since $\mathbb{P}_n$ is linear and bounded and $\dim X_n < +\infty$, we obtain for $m \to \infty$

$$\int_b^t \mathbb{P}_n[S(t-s)f_{k_m}(s)]\,ds \to \int_b^t \mathbb{P}_n[S(t-s)f(s)]\,ds,\ t \in I.$$

Therefore, we can say that there exists $\tilde{y} \in \mathcal{C}^1(I;X_n)$ such that (see (40))

$$y_{k_m}(t) \to \mathbb{P}_n[C(t-s)y_1] + \mathbb{P}_n[S(t-s)y_2] + \int_b^t \mathbb{P}_n[S(t-s)f(s)]\,ds := \tilde{y}(t),\ t \in I. \tag{44}$$

On the other hand, since $y_k \to y$ in $\mathcal{C}^1(I;X_n)$, we have $y_{k_m}(t) \to y(t)$, for all $t \in I$. From the uniqueness of the limit implies that the functions $y, \tilde{y} : I \to X_n$ are the same.
Finally, recalling the definition of Γ_n, from (44) and (43) we deduce $y \in \Gamma_{n|B}q$. Therefore $\Gamma_{n|B}$ has (sequentially) closed graph.

Clearly, from **F1)**, the multimap $\Gamma_{n|B}$ assumes convex values.

The arguments above presented, allow us to say that all the hypotheses of Theorem 2.9 hold. Hence, for every $n \in \mathbb{N}$, the map $\Gamma_{n|B}$ has a fixed point $q_n \in B = \overline{co}\Gamma_n(\overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})) \subset \overline{B}_{\mathcal{C}^1(I;X_n)}(0,\overline{p})$.
As a consequence of the technique used, the sequence of fixed points $(q_n)_n$ is such that

$$q_n \in \overline{B}_{\mathcal{C}^1(I;X)}(0,\overline{p}),\ n \in \mathbb{N}. \tag{45}$$

Step2. Limiting procedure.
We want to prove that the sequence of fixed points $(q_n)_n$, satisfying (45), admits a subsequence that weakly converges to a fixed point q for the solution-operator relative to the Cauchy problem $(\mathbf{PC})_\mathbf{b}$, i.e. $q \in \mathcal{C}^1(I;X)$ is a mild solution for $(\mathbf{PC})_\mathbf{b}$.
Since, for every $n \in \mathbb{N}$, q_n is a fixed point of Γ_n, we have that $q_n \in \mathcal{C}^1(I;X_n)$ (see (17))

$$q_n(t) = \mathbb{P}_n[C(t-b)y_1] + \mathbb{P}_n[S(t-b)y_2] + \int_b^t \mathbb{P}_n[S(t-s)f_n(s)]\,ds,\ t \in I,$$

where $f_n \in S^1_{F(\cdot,q_n(\cdot),q_n'(\cdot))}$.
At first, we consider the set $A = \{f_n :\ n \in \mathbb{N}\} \subset L^1(I;X)$. Reasoning as in **Step 1b.** (see from (30) to (32)), from Proposition 2.5 we have that A is relatively weakly compact. So there exist $(f_{n_k})_k \subset (f_n)_n$ and $f \in L^1(I;X)$ such that

$$f_{n_k} \rightharpoonup f. \tag{46}$$

Now, we prove that

$$\int_b^t \mathbb{P}_{n_k}[S(t-s)f_{n_k}(s)]\,ds \rightharpoonup \int_b^t S(t-s)f(s)\,ds,\ t \in I. \tag{47}$$

First of all, thanks to Proposition 3.4, we claim that (see (46))

$$\mathfrak{S}_S f_{n_k} \rightharpoonup \mathfrak{S}_S f,$$

hence, fixed $t \in I$, we have (see [27], Theorem 4)

$$\int_b^t S(t-s)f_{n_k}(s)\,ds \rightharpoonup \int_b^t S(t-s)f(s)\,ds$$

and by using **b)** of Proposition 2.10 we can write

$$\mathbb{P}_{n_k}\left[\int_b^t S(t-s)f_{n_k}(s)\,ds\right] \rightharpoonup \int_b^t S(t-s)f(s)\,ds.$$

So, from the linearity of the integral we conclude that (47) holds.
Therefore, again from **b)** of Proposition 2.10, there exists $q : I \to X$ such that

$$q_{n_k}(t) \rightharpoonup C(t-b)y_1 + S(t-b)y_2 + \int_b^t S(t-s)f(s)\,ds := q(t),\ t \in I \tag{48}$$

and, by Proposition 3.3 we have $q \in \mathcal{C}^1(I;X)$ with

$$q'(t) = AS(t-b)y_1 + C(t-b)y_2 + \int_b^t C(t-s)f(s)\,ds,\ t \in I.$$

Moreover, reasoning as before, it is possible to prove that

$$q'_{n_k}(t) \rightharpoonup q'(t),\ t \in I. \tag{49}$$

Finally, in order to prove that $q : I \to X$ is a mild solution for $(\mathbf{PC})_\mathbf{b}$, we have only to show that $f(t) \in F(t, q(t), q'(t))$, a.e. $t \in I$. To this aim, let N^* be the null measure set such that

$$f_{n_k}(t) \in F(t, q_{n_k}(t), q'_{n_k}(t)),\ t \in I \setminus N^*, k \in \mathbb{N} \tag{50}$$

and **F5)** holds in $I \setminus N^*$.
We want to apply Proposition 2.7 to the sequence $(f_{n_k})_k$ and to the multimap $\Phi : I \to \mathcal{P}(X)$ defined as

$$\Phi(t) = \begin{cases} \overline{\bigcup_k F(t, q_{n_k}(t), q'_{n_k}(t))}^w, & t \in I \setminus N^* \\ \{0\}, & t \in N^* \end{cases} \tag{51}$$

Taking into account (45) we have that, for every $t \in I \setminus N^*$, the set $\Phi(t)$ satisfies

$$\Phi(t) = \overline{\bigcup_k F(t, q_{n_k}(t), q'_{n_k}(t))}^w \subset \overline{F(t, \overline{B}_X(0,\overline{p}), \overline{B}_X(0,\overline{p}))}^w,$$

hence it is a weakly closed subset of a weakly compact set (see **F5)**). Therefore Φ assumes weakly compact values. Moreover, we note that hypotheses **i)** and **ii)** of Proposition 2.7 are obviously satisfied by $(f_{n_k})_k$ and Φ (see (46) and (50), (51) respectively).

Therefore, we can conclude that

$$f(t) \in \overline{co}\ w - \limsup_{k\to\infty}\{f_{n_k}(t)\},\ \text{a.e. } t \in I. \tag{52}$$

Denoted by $\overline{N}$ the null measure set for which (50), (52), **(I)**, and **F3)** hold in $I \setminus \overline{N}$, from (50) we can write

$$\overline{co}\ w - \limsup_{k\to\infty}\{f_{n_k}(t)\} \subset \overline{co}\ w - \limsup_{k\to\infty} F(t, q_{n_k}(t), q'_{n_k}(t)),\ t \in I \setminus \overline{N}. \tag{53}$$

Now, we want to prove that

$$w - \limsup_{k\to\infty} F(t, q_{n_k}(t), q'_{n_k}(t)) \subset F(t, q(t), q'(t)),\ t \in I \setminus \overline{N}. \tag{54}$$

Fixed $t \in I \setminus \overline{N}$ and $z \in w - \limsup_{k\to\infty} F(t, q_{n_k}(t), q'_{n_k}(t))$, there exists $z_{n_{k_p}} \in F(t, q_{n_{k_p}}(t), q'_{n_{k_p}}(t))$ such that

$$z_{n_{k_p}} \rightharpoonup z.$$

So, taking into account (48) and (49), **F3)** implies that $z \in F(t, q(t), q'(t))$. From the arbitrariness of z we conclude that (54) is true.
Moreover, in virtue of **F1)** and **(I)** the set $F(t, q(t), q'(t))$ is closed and convex in X. So, since (54) holds for every $t \in I \setminus \overline{N}$, we can claim

$$\overline{co}\ w - \limsup_{k\to\infty} F(t, q_{m_k}(t), q'_{m_k}(t)) \subset F(t, q(t), q'(t)),\ t \in I \setminus \overline{N}. \tag{55}$$

Finally, thanks to (52), (53), (55), we say that the map $f \in L^1(I; X)$ satisfies $f(t) \in F(t, q(t), q'(t))$ a.e. $t \in I$, i.e.

$$f \in S^1_{F(\cdot, q(\cdot), q'(\cdot))}.$$

Therefore the map $q : I \to X$ is a fixed point for the solution-operator $\Gamma : \mathcal{C}^1(I; X) \to \mathcal{P}(\mathcal{C}^1(I; X))$

$$\begin{aligned}\Gamma x =&\{y \in \mathcal{C}^1(I; X) :\ y(t) = C(t-b)y_1 + S(t-b)y_2 \\ &+ \int_b^t S(t-s)f(s)\, ds,\ t \in I,\ f \in S^1_{F(\cdot, x(\cdot), x'(\cdot))}\},\ x \in \mathcal{C}^1(I; X),\end{aligned}$$

i.e. q is a mild solution for the Cauchy problem $(\mathbf{PC})_\mathbf{b}$. □

Remark 4.2. Let us note that Theorem 4.1 improves Theorem 3.5 of [36].
First of all, the reflexivity on the Banach space X is omitted. Moreover, from **F5)**, for us the values of F are bounded only a.e. on I, instead of everywhere on I (see $\mathbf{F1)}_\mathbf{S}$ of Theorem 3.5). Then hypotheses $\mathbf{F1)}_\mathbf{S}$ and $\mathbf{F3)}_\mathbf{S}$ of Theorem 3.5 imply **F3)** of Theorem 4.1. Indeed, since the weak topology of a normed space is regular, taking into account [24], Proposition 2.17, we deduce that for a.e. $t \in I$, $F(t, \cdot, \cdot)$ has weakly closed graph and so it has weakly sequentially closed graph.
In the setting of reflexive Banach spaces, considered in Theorem 3.5, we highlight that **F4)** implies **F5)** of Theorem 4.1. Indeed, fixed $t \in I \setminus N$ (where N is the null measure set for which **F4)** holds) and a bounded set $B \subset X \times X$, there exists $n \in \mathbb{N}$ such that $B \subset \overline{B}_X(0, n) \times \overline{B}_X(0, n)$. Then, from **F4)**, there exists $\varphi_n \in L^1_+(I)$ such that

$$F(t, B) \subset F(t, \overline{B}_X(0, n) \times \overline{B}_X(0, n)) \subset \overline{B}_X(0, \varphi_n(t)).$$

Therefore, from the weak compactness of $\overline{B}_X(0, \varphi_n(t))$ in the reflexive Banach space X, we deduce the relative weak compactness of $F(t, B)$.

Moreover, the following example shows that, even in reflexive Banach spaces, Theorem 4.1 may work while Theorem 3.5 does not.

Example. Let $X = \mathbb{R}^2$, $I = [0,1]$ and $F : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$ be a multimap so defined

$$F(t,(x,y),(z,w)) = \begin{cases} \{(s, s\sqrt{\|x\|_{\mathbb{R}^2}}) : \ s \in [0,1]\}, \ t \in]0,1] \\ [0,\infty) \times [0,\infty) \qquad\qquad , \ t = 0 \end{cases}$$

Let us note that F satisfies all the assumptions of Theorem 4.1.
Indeed, clearly **F1)** holds. Moreover, for every $((x,y),(z,w)) \in \mathbb{R}^2 \times \mathbb{R}^2$, the map $f : t \mapsto f(t) = 0_{\mathbb{R}^2}$, $t \in [0,1]$, is a B-measurable selection of $F(\cdot,(x,y),(z,w))$, i.e. **F2)** is true.
Now, to say that **F3)** holds let us fix $t \in]0,1]$ and $(((x_n,y_n),(z_n,w_n)))_n$ a sequence in $\mathbb{R}^2 \times \mathbb{R}^2$ such that

$$((x_n,y_n),(z_n,w_n)) \to ((x,y),(z,w)) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{56}$$

Let $((k_n,h_n))_n$ be a sequence such that $(k_n,h_n) \in F(t,(x_n,y_n),(z_n,w_n))$, $n \in \mathbb{N}$, and

$$(k_n,h_n) \to (k,h). \tag{57}$$

Now, for every $n \in \mathbb{N}$ we have $h_n = k_n\sqrt{\|x\|_{\mathbb{R}^2}}$. Then, from the convergence of $(x_n)_n$ and $(k_n)_n$ respectively to x and k (see (56) and (57)), we deduce that $(k_n,h_n) \to (k, k\sqrt{\|x\|_{\mathbb{R}^2}}) \in [0,1] \times \mathbb{R}$. By the uniqueness of the limit we conclude that $(k,h) = (k, k\sqrt{\|x\|_{\mathbb{R}^2}}) \in F(t,(x,y),(z,w))$. Therefore **F3)** is proved.
Next, for every $n \in \mathbb{N}$, we consider

$$\begin{aligned} F(t, \overline{B}_{\mathbb{R}^2}(0,n), \overline{B}_{\mathbb{R}^2}(0,n)) &= \{(s, s\sqrt{\|x\|_{\mathbb{R}^2}}), \ (x,y),(z,w) \in \overline{B}_{\mathbb{R}^2}(0,n), \ s \in [0,1]\} \\ &= \{(s, s\sqrt{\|x\|_{\mathbb{R}^2}}), \ \|x\|_{\mathbb{R}^2} \le n, \ s \in [0,1]\}, \ t \in]0,1]. \end{aligned}$$

So we have that there exists $\varphi_n : [0,1] \to \mathbb{R}$ such that

$$\|F(t, \overline{B}_{\mathbb{R}^2}(0,n), \overline{B}_{\mathbb{R}^2}(0,n))\| = \sup_{\|x\|_{\mathbb{R}^2} \le n, \ s \in [0,1]} \|(s, s\sqrt{\|x\|_{\mathbb{R}^2}})\|_\infty \le \sqrt{n} =: \varphi_n(t), \ t \in]0,1].$$

Moreover, the sequence $(\varphi_n)_n$ satisfies

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_1}{n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0.$$

Therefore, **F4)** is also true.
Finally, **F5)** holds too. Indeed, fixed $t \in]0,1]$ and $B \times C$ a bounded subset of $\mathbb{R}^2 \times \mathbb{R}^2$, there exists $R > 0$ such that

$$F(t, B \times C) = \{(s, s\sqrt{\|x\|_{\mathbb{R}^2}}) : \ ((x,y),(z,w)) \in B \times C, \ s \in [0,1]\} \subset [0,1] \times \overline{B}_{\mathbb{R}}(0,R).$$

Hence $F(t, B \times C)$ is relatively compact.
In conclusion, the multimap F fulfils all the assumptions of Theorem 4.1, but not the ones of Theorem 3.5 (clearly F has not the property $\mathbf{F1)}_{\mathbf{S}}$).

Finally we would also like to point out that, again in the setting of reflexive spaces, if the multimaps have closed and convex values and satisfy **F4)**, as required in Theorem 3.5, it is possible to state that $\mathbf{F3)}_{\mathbf{S}}$ of Theorem 3.5 and **F3)** of Theorem 4.1 are equivalent. Taking into account what has been stated above, it is sufficient to show that **F3)** of Theorem 4.1 implies **F3)$_\mathbf{S}$** of Theorem 3.5. Indeed fixed $t \in I \setminus N$, where N is the null measure set for which **F3)** and **F4)** hold, and $(\overline{x}, \overline{y}) \in X \times X$, there exists $n \in \mathbb{N}$ such that $(\overline{x}, \overline{y}) \in B = \overline{B}_{X\times X}((0,0), n)$. Clearly, being $\overline{F}(t, B)$ a subset of the weakly compact set $\overline{B}_X(0, \varphi_n(t))$ (see **F4)**), the multimap $F_{|B}(t, \cdot, \cdot) : B \to \mathcal{P}(X)$ is locally weakly compact and assumes weakly compact values (see **F1)$_\mathbf{S}$**). Furthermore, by using **F3)** of Theorem 4.1, thanks to the reflexivity of X we can say that $F_{|B}(t, \cdot, \cdot)$ has weakly closed graph (see Proposition 2.1). Hence, Theorem 1.1.5 of [26] implies that $F_{|B}(t, \cdot, \cdot)$ is weakly upper semicontinuous in $(\overline{x}, \overline{y})$. From the arbitrariness of $(\overline{x}, \overline{y}) \in X \times X$ we conclude that **F3)$_\mathbf{S}$** holds.

In the sequel we achieve the following result where we obtain the existence of mild solutions for the Cauchy problem $(\mathbf{PC})_\mathbf{b}$ without assumptions on the values of the multimap F.

Theorem 4.3. *Let X be a Banach space having a Schauder basis, $I = [b, c] \subset \mathbb{R}_0^+$, $A : D(A) \subset X \to X$ be a infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, $y_1 \in E$, where the set E is defined in (8), and $y_2 \in X$. Let $F : I \times X \times X \to \mathcal{P}(X)$ be a multimap satisfying the following properties*

- **(l-SD)** *$\forall \varepsilon > 0$ there exists a compact $K_\varepsilon \subset I$ such that $\mu(I \setminus K_\varepsilon) < \varepsilon$ and $F_{|K_\varepsilon \times X\times X}$ is weakly lower semicontinuous;*
- **(M)** *for every weakly closed set $Z \subset I \times X \times X$ such that $F_{|Z}$ is weakly lower semicontinuous, there exists a weakly continuous selection of F on Z;*
- **F4)** *for every $n \in \mathbb{N}$, there exists $\varphi_n \in L_+^1(I)$ such that*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_1}{n} = 0 \tag{58}$$

and

$$\|F(t, \overline{B}_X(0, n), \overline{B}_X(0, n))\| \le \varphi_n(t), \text{ a.e. } t \in I;$$

- **F5)** *for every $t \in I$, the multimap $F(t, \cdot, \cdot)$ is weakly semi-compact.*

Then the Cauchy problem $(\mathbf{PC})_\mathbf{b}$ has at least one mild solution.

Proof. To begin, let us consider the topological Hausdorff spaces $(X \times X, \tau_w \times \tau_w)$ and (X, τ_w), where $\tau_w \times \tau_w$ and τ_w are respectively the weak topologies on the Banach spaces $X \times X$ and X. Since (X, τ_w) is a regular topological space, in virtue of **(l-SD)** and **(M)** we are in a position to apply Theorem 3.1 of [13], hence there exists a function $f : I \times X \times X \to X$ satisfying

$$\text{for every } t \in I,\ f(t, \cdot, \cdot) \text{ is weakly continuous;} \tag{59}$$

$$\text{for every } x, y \in X,\ f(\cdot, x, y) \text{ is } (\mathcal{M}(I), \mathcal{B}(X, \tau_w)) - \text{measurable;} \tag{60}$$

$$f(t, x, y) \in F(t, x, y),\ \text{a.e. } t \in I,\ \forall x, y \in X. \tag{61}$$

Now, X is separable, having a Schauder basis. Hence, by using Lemma 2.2, (60) implies

$$\text{for every } x, y \in X,\ f(\cdot, x, y) \text{ is B-measurable,} \tag{62}$$

and so $f(\cdot, x, y)$ is B-integrable (see **F4)**).
Now, by using Theorem 4.1, we can prove that the following Cauchy problem

$$\begin{cases} x''(t) \in Ax(t) + \tilde{F}(t, x(t), x'(t)), \text{ a.e. } t \in I \\ x(0) = y_1 \\ x'(0) = y_2, \end{cases} \tag{63}$$

where $\tilde{F} : I \times X \times X \to \mathcal{P}(X)$ is so defined

$$\tilde{F}(t, x, y) = \{f(t, x, y)\}, \ t \in I, \ x, y \in X, \tag{64}$$

has one mild solution.
To ensure clarity, we will use the notations $\tilde{\mathbf{F}}\mathbf{1}$), $\tilde{\mathbf{F}}\mathbf{2}$), $\tilde{\mathbf{F}}\mathbf{3}$), $\tilde{\mathbf{F}}\mathbf{4}$), and $\tilde{\mathbf{F}}\mathbf{5}$), to represent the assumptions of Theorem 4.1 regarding the multimap $\tilde{F}$.

First of all we note that, the multimap $\tilde{F}$ clearly satisfies $\tilde{\mathbf{F}}\mathbf{1}$) and $\tilde{\mathbf{F}}\mathbf{2}$) of Theorem 4.1 (see (64) and (62)). Moreover $\tilde{F}$ has also property $\tilde{\mathbf{F}}\mathbf{3}$) of Theorem 4.1.
Fixed $t \in I$, we consider $((x_n, y_n))_n$, $(x_n, y_n) \in X \times X$, and $(z_n)_n$, $z_n \in \tilde{F}(t, x_n, y_n)$, sequences such that

$$(x_n, y_n) \rightharpoonup (x, y) \tag{65}$$

and

$$z_n \rightharpoonup z \tag{66}$$

where $x, y, z \in X$. We have to show that $z \in \tilde{F}(t, x, y)$.
Now, being $z_n = f(t, x_n, y_n), \ n \in \mathbb{N}$ (see (64)), from (65) and (59) we have

$$z_n \rightharpoonup f(t, x, y). \tag{67}$$

Hence, by the uniqueness of the weak limit, (66) and (67) imply that $z = f(t, x, y) \in \tilde{F}(t, x, y)$. By the arbitrariness of the sequences, we conclude that $\tilde{\mathbf{F}}\mathbf{3}$) is true.
Next, to establish that $\tilde{F}$ has the property $\tilde{\mathbf{F}}\mathbf{4}$) of Theorem 4.1, let us consider a fixed value $t \in J \setminus N^*$, where N^* is the null measure set referred to (61) and **F4)**. Thanks to hypothesis **F4)** there exists $(\varphi_n)_n$, $\varphi_n \in L^1_+(I)$, satisfying (58) and such that (see (61) and (64))

$$\begin{aligned} \|\tilde{F}(t, \overline{B}_X(0, n), \overline{B}_X(0, n))\| &= \|f(t, \overline{B}_X(0, n), \overline{B}_X(0, n))\| \\ &\le \|F(t, \overline{B}_X(0, n), \overline{B}_X(0, n))\| \le \varphi_n(t), \ n \in \mathbb{N}. \end{aligned}$$

Therefore $\tilde{\mathbf{F}}\mathbf{4}$) holds.
Finally also $\tilde{\mathbf{F}}\mathbf{5}$) of Theorem 4.1 is satisfied. Indeed, put $\overline{N}$ the null measure set for which (61) and **F5)** hold, let us fix $t \in I \setminus \overline{N}$. Then, for every bounded set $B \subset X \times X$, we have (see (64))

$$\tilde{F}(t, B) = f(t, B) \subset F(t, B),$$

which implies the relative weak compactness of $\tilde{F}(t, B)$ (see **F5)**).

In conclusion, since all the hypotheses of Theorem 4.1 are fulfilled, there exists at least one mild solution for (63). Clearly, it is also a mild solution for the Cauchy problem $(\mathbf{PC})_\mathbf{b}$. □

Obviously Theorem 4.3 extends in a broad sense Theorem 3.5 of [36].

5. Impulsive mild solutions on an unbounded interval

In this section we discuss the existence of mild solutions for the impulsive problem governed by a semilinear autonomous second order differential inclusion

$$\textbf{(PI)}\begin{cases} x''(t) \in Ax(t) + F(t, x(t), x'(t)), \text{ a.e. } t \in [0, \infty) \setminus \{t_k : \ k \in \mathbb{N}\} \\ x(0) = x_0 \\ x'(0) = \overline{x}_0 \\ x(t_k^+) = x(t_k) + I_k(x(t_k), x'(t_k)), \ k = 1, 2, ... \\ x'(t_k^+) = x'(t_k) + \tilde{I}_k(x(t_k), x'(t_k)), \ k = 1, 2, ... \end{cases}$$

where the jump points are an increasing sequence of times $(t_k)_{k=0}^{\infty}$ such that $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$, and, for every $k \in \mathbb{N}_0$, $x(t_k^+)$, $x'(t_k^+)$ $[x(t_k^-), \ x'(t_k^-)]$ denotes respectively $\lim_{t\to t_k^+} x(t_k)$ and $\lim_{t\to t_k^+} x'(t_k)$ $[\lim_{t\to t_k^-} x(t_k), \lim_{t\to t_k^-} x'(t_k)]$, and $I_k, \tilde{I}_k : X \times X \to X$ are the functions producing the impulses.

To introduce the concept of mild solution for the impulsive problem **(PI)**, we consider the following set

$$\Omega_{\mathcal{C}^1} = \{x : [0, \infty) \to X : x_{|[0,t_1]} \in \mathcal{C}^1([0, t_1]; X), \ x_{|]t_k,t_{k+1}]} \in \mathcal{C}^1(]t_k, t_{k+1}]; X), \ k = 1, 2, ... \\ \text{and there exist } x(t_k^+), x'(t_k^+) \in X, \ k = 1, 2, ...\}. \tag{68}$$

In the sequel, for simplicity, we use the symbols

$$I_0 = [0, t_1], \ I_k =]t_k, t_{k+1}], \ k = 1, 2, ...$$

and we put

$$x_{[k]} = x_{|I_k}, \ x'_{[k]} = x'_{|I_k}, \ k = 0, 1, 2...$$

Definition 5.1. A function $x : [0, \infty) \to X$ is said to be an *impulsive mild solution* for the problem **(PI)** if $x \in \Omega_{\mathcal{C}^1}$ and

$$x(t) = C(t)x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f(s)\,ds \\ + \sum_{0<t_k<t} [C(t - t_k)I_k(x(t_k), x'(t_k)) + S(t - t_k)\tilde{I}_k(x(t_k), x'(t_k))], \ t \in [0, \infty), \tag{69}$$

where

$$f \in S^{1,\text{loc}}_{F(\cdot,x(\cdot),x'(\cdot))} = \{f \in L^{1,\text{loc}}([0, \infty), X) : \ f(t) \in F(t, x(t), x'(t)), \text{ a.e. } t \in [0, \infty)\}.$$

If $t = 0$ we put $\sum_{0<t_k<t}[C(t - t_k)I_k(x(t_k), x'(t_k)) + S(t - t_k)\tilde{I}_k(x(t_k), x'(t_k))] = 0$

Thanks to the theorems achieved in the previous section without taking impulsive effects into account, we manage to establish the existence of mild solutions to the impulsive problem **(PI)**. These results are obtained without requiring either hypotheses of compactness or continuity on the functions that produce the impulses at the fixed instants on the half-line $[0, \infty)$.

Theorem 5.1. *Let X be a Banach space having a Schauder basis, $A : D(A) \subset X \to X$ be a infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, $x_0 \in E$, where the set E is defined in (8), and $\overline{x}_0 \in X$. Let $I_k : X \times X \to X$, $k = 0, 1, 2, ...$, be an impulse map such that*

IM) *$I_k(e, y) \in E$ for every $e \in E$, $y \in X$*

and $F : [0, \infty) \times X \times X \to \mathcal{P}(X)$ be a multimap satisfying ***F1)-F5)*** *of Theorem 4.1 for $I = [0, \infty)$. Then the problem* ***(PI)*** *has at least one impulsive mild solution $x \in \Omega_{\mathcal{C}^1}$.*

Proof. To obtain the thesis it is suitable to split the proof into three steps.
Step1. Existence of a mild solution for the non-impulsive Cauchy problem in $I_0 = [0, t_1]$

$$\begin{cases} x''(t) = Ax(t) + F(t, x(t), x'(t)), \text{ a.e. } t \in I_0 \\ x(0) = x_0 \\ x'(0) = \overline{x}_0. \end{cases} \tag{70}$$

Recalling that $x_0 \in E$, we are in a position to apply Theorem 4.1 in the desidered interval I_0. So, there exists at least a mild solution $x_{[0]} \in \mathcal{C}^1(I_0, X)$ for problem (70) satisfying

$$x_{[0]}(t) = C(t)x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f_{[0]}(s)\,ds,\ t \in [0, t_1] \tag{71}$$

and

$$x'_{[0]}(t) = AS(t)x_0 + C(t)\overline{x}_0 + \int_0^t C(t-s)f_{[0]}(s)\,ds,\ t \in [0, t_1], \tag{72}$$

where $f_{[0]} \in L^1(I_0, X)$ and $f_{[0]}(t) \in F(t, x_{[0]}(t), x'_{[0]}(t))$, a.e. $t \in [0, t_1]$.

Step2. Existence of a mild solution for the non-impulsive Cauchy problem in $\overline{I}_1 = [t_1, t_2]$

$$\begin{cases} x''(t) = Ax(t) + F(t, x(t), x'(t)), \text{ a.e. } t \in \overline{I}_1 \\ x(t_1) = x_{[0]}(t_1) + I_1(x_{[0]}(t_1), x'_{[0]}(t_1)) \\ x'(t_1) = x'_{[0]}(t_1) + \tilde{I}_1(x_{[0]}(t_1), x'_{[0]}(t_1)). \end{cases} \tag{73}$$

In order to apply again Theorem 4.1 in the interval $\overline{I}_1$, we note that all the hypotheses on the multimap F are clearly satisfied. Therefore, we have only to prove that $x(t_1) = x_{[0]}(t_1) + I_1(x_{[0]}(t_1), x'_{[0]}(t_1)) \in E$. Firstly, by (71) we can write

$$x_{[0]}(t_1) = C(t_1)x_0 + S(t_1)\overline{x}_0 + \int_0^{t_1} S(t_1 - s)f_{[0]}(s)\,ds. \tag{74}$$

Clearly $S(t_1)\overline{x}_0 \in E$ (see **h)** of Proposition 3.1). Moreover, we have also $C(t_1)x_0 \in E$. Indeed, since $x_0 \in E$, we can say that the map (see **C1)**)

$$C(\cdot)[C(t_1)x_0] = \frac{1}{2}[C(\cdot + t_1) + C(\cdot - t_1)]x_0$$

is twice continuously differentiable in $\mathbb{R}$, so recalling (8), we deduce that $C(t_1)x_0 \in E$.
Finally we have to prove that $\int_0^{t_1} S(t_1-s)f_{[0]}(s)\,ds \in E$.
For every $t \in \mathbb{R}$, taking into account that $C(t)$ is a bounded linear operator and of the properties of the Bochner integral, we write (see **d)** of Proposition 3.1)

$$C(t)\int_0^{t_1} S(t_1-s)f_{[0]}(s)\,ds = \int_0^{t_1} C(t)S(t_1-s)f_{[0]}(s)\,ds$$
$$= \frac{1}{2}\int_0^{t_1} S(t_1-s+t)f_{[0]}(s)\,ds + \frac{1}{2}\int_0^{t_1} S(t_1-s-t)f_{[0]}(s)\,ds. \tag{75}$$

Let us show that the map $g : I_0 \times I_0 \to X$ so defined

$$g(t,s) = S(t_1-s+t)f_{[0]}(s),\ t,s \in I_0$$

verifies all the hypotheses of Proposition 2.4.
First of all we note that, for every $t \in I_0$, $g(t,\cdot)$ is B-measurable. This is a consequence of [26], Theorem 1.3.5, since we can consider $g(t,\cdot) = p(\cdot, f_{[0]}(\cdot))$, where $p : I_0 \times X \to X$, $p(s,x) = S(t_1-s+t)x$. Then, taking into account **c)** of Proposition 3.1, boundedness and linearity of the operators $S(t_1+t-s)$, $s \in I_0$, and B-measurability of the function $f_{[0]}$, we can conclude that g satisfies **i)** of Proposition 2.4.
On the other hand, for a.e. $s \in I_0$, the map $g(\cdot,s) = S(t_1-s+\cdot)f_{[0]}(s) \in \mathcal{C}^1(I_0;X)$ (see (12)), i.e. **ii)** of Proposition 2.4 is true.
Finally, **iii)** of Proposition 2.4 is satisfied, indeed we can write (see (11) and (13))

$$\|g(t,s)\|_X + \left\|\frac{\partial}{\partial t}g(t,s)\right\|_X \le L_1^{I_0}\|f_{[0]}(s)\|_X + L_2^{I_0}\|f_{[0]}(s)\|_X := \varphi(s),$$

where $\varphi \in L^1_+(I_0)$.
Therefore, Proposition 2.4 implies that the map $t \mapsto \int_0^{t_1} S(t_1-s+t)f_{[0]}(s)\,ds$ is continuously differentiable in I_0.
Analogously the map $t \mapsto \int_0^{t_1} S(t_1-s-t)f_{[0]}(s)\,ds$ is continuously differentiable in I_0 too.
Hence, from (75) we have $C(\cdot)\int_0^{t_1} S(t_1-s)f_{[0]}(s)\,ds \in \mathcal{C}^1(I_0;X)$.
Thus $\int_0^{t_1} S(t_1-s)f_{[0]}(s)\,ds \in E$ (see (8)).
Then, taking into account that E is a linear subspace of X, we obtain that

$$x_{[0]}(t_1) = C(t_1)x_0 + S(t_1)\overline{x}_0 + \int_0^{t_1} S(t_1-s)f_{[0]}(s)\,ds \in E.$$

Now **IM)** implies that $I_1(x_{[0]}(t_1), x'_{[0]}(t_1)) \in E$, so $x_{[0]}(t_1) + I_1(x_{[0]}(t_1), x'_{[0]}(t_1)) \in E$.
Reasoning as in **Step1.**, there exists a mild solution of (73) $x_{[1]} \in \mathcal{C}^1(\overline{I}_1, X)$ defined as

$$x_{[1]}(t) = C(t-t_1)[x_{[0]}(t_1) + I_1(x_{[0]}(t_1), x'_{[0]}(t_1))]$$
$$+ S(t-t_1)[x'_{[0]}(t_1) + \tilde{I}_1(x_{[0]}(t_1), x'_{[0]}(t_1))] + \int_{t_1}^{t} S(t-s)f_{[1]}(s)\,ds,\ t \in [t_1,t_2], \tag{76}$$

where $f_{[1]} \in L^1(\overline{I}_1, X)$ and $f_{[1]}(t) \in F(t, x_{[1]}(t), x'_{[1]}(t))$, a.e. $t \in \overline{I}_1$.
Moreover, we note that (see (72))

$$x'_{[0]}(t_1) = AS(t_1)x_0 + C(t_1)\overline{x}_0 + \int_0^{t_1} C(t_1 - s) f_{[0]}(s)\, ds. \tag{77}$$

Now, taking into account (74) and (77), from (76) we can write

$$\begin{aligned}
x_{[1]}(t) &= C(t-t_1)[C(t_1)x_0 + S(t_1)\overline{x}_0 + \int_0^{t_1} S(t_1 - s) f_{[0]}(s)\, ds + I_1(x_{[0]}(t_1), x'_{[0]}(t_1))] \\
&+ S(t-t_1)[AS(t_1)x_0 + C(t_1)\overline{x}_0 + \int_0^{t_1} C(t_1 - s) f_{[0]}(s)\, ds + \tilde{I}_1(x_{[0]}(t_1), x'_{[0]}(t_1))] + \int_{t_1}^{t} S(t-s) f_{[1]}(s)\, ds \\
&= [C(t-t_1)C(t_1) + S(t-t_1)AS(t_1)]x_0 + [C(t-t_1)S(t_1) + S(t-t_1)C(t_1)]\overline{x}_0 \\
&+ C(t-t_1)\left[\int_0^{t_1} S(t_1 - s) f_{[0]}(s)\, ds\right] + S(t-t_1)\left[\int_0^{t_1} C(t_1 - s) f_{[0]}(s)\, ds\right] \\
&+ \int_{t_1}^{t} S(t-s) f_{[1]}(s)\, ds + C(t-t_1)[I_1(x_{[0]}(t_1), x'_{[0]}(t_1))] + S(t-t_1)[\tilde{I}_1(x_{[0]}(t_1), x'_{[0]}(t_1))].
\end{aligned}$$

Now, recalling that $S(t_1)x_0 \in D(A)$ (see **i)** of Proposition 3.1 and $x_0 \in E$), by using **C1)** and **a)**, **j)**, **g)**, **k)**, **e)** of Proposition 3.1 we have

$$\begin{aligned}
x_{[1]}(t) &= \frac{1}{2}[C(t) + C(t-2t_1)]x_0 + [AS(t-t_1)S(t_1)]x_0 + [S(t_1)C(t-t_1) + S(t-t_1)C(t_1)]\overline{x}_0 \\
&+ \int_0^{t_1} [C(t-t_1)S(t_1-s) + S(t-t_1)C(t_1-s)] f_{[0]}(s)\, ds + \int_{t_1}^{t} S(t-s) f_{[1]}(s)\, ds \\
&+ C(t-t_1)[I_1(x_{[0]}(t_1), x'_{[0]}(t_1))] + S(t-t_1)[\tilde{I}_1(x_{[0]}(t_1), x'_{[0]}(t_1))] \\
&= \frac{1}{2}[C(t) + C(t-2t_1)]x_0 + \frac{1}{2}[C(t)x_0 - C(t-2t_1)]x_0 \\
&+ S(t)\overline{x}_0 + \int_0^{t_1} S(t-s) f_{[0]}(s)\, ds + \int_{t_1}^{t} S(t-s) f_{[1]}(s)\, ds \\
&+ C(t-t_1)[I_1(x_{[0]}(t_1), x'_{[0]}(t_1))] + S(t-t_1)[\tilde{I}_1(x_{[0]}(t_1), x'_{[0]}(t_1))].
\end{aligned}$$

Then, denoting by $f_{|[0,t_2]} : [0, t_2] \to X$ the map so defined

$$f_{|[0,t_2]}(t) = \begin{cases} f_{[0]}(t), & t \in [0, t_1] \\ f_{[1]}(t), & t \in (t_1, t_2] \end{cases}$$

we are in a position to introduce an impulsive mild solution x on the interval $[0, t_2]$ as

$$x_{|[0,t_2]}(t) = \begin{cases} x_{[0]}(t), & t \in [0, t_1] \\ x_{[1]}(t), & t \in (t_1, t_2]. \end{cases}$$

Step3. Existence of a mild solution for the impulsive Cauchy problem **(PI)**.

Recalling that $I_0 = [0, t_1]$ and $I_k =]t_k, t_{k+1}]$, $k = 1, 2, ...$, proceeding iteratively in I_k, $k \geq 2$, we can construct the functions $x : [0, \infty) \to X$ and $f : [0, \infty) \to X$ so defined

$$x(t) = x_{[0]}(t)\chi_{I_0}(t) + \sum_{k=1}^{\infty} x_{[k]}(t)\chi_{I_k}(t),\ t \in [0, \infty)$$

and

$$f(t) = f_{[0]}(t)\chi_{I_0}(t) + \sum_{k=1}^{\infty} f_{[k]}(t)\chi_{I_k}(t),\ t \in [0, \infty),$$

where χ_{I_k} is the characteristic function for the interval I_k, $k \in \mathbb{N}_0$.
By the construction we have that $f \in S^{1,\text{loc}}_{F(\cdot,x(\cdot),x'(\cdot))}$ and x satisfies equation (69).
Therefore $x : [0, \infty) \to X$ is a mild solution for the impulsive Cauchy problem **(PI)**. □

Reasoning as in the proof of Theorem 5.1, but using Theorem 4.3 instead of Theorem 4.1, we also have the following result

Theorem 5.2. *Let X be a Banach space having a Schauder basis, $A : D(A) \subset X \to X$ be a infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$, $x_0 \in E$, where the set E is defined in (8), and $\overline{x}_0 \in X$. Let $I_k : X \times X \to X$, $k = 0, 1, 2, ...$, be an impulse map such that*

IM) *$I_k(e, y) \in E$ for every $e \in E$, $y \in X$*

and $F : I \times X \times X \to \mathcal{P}(X)$ be a multimap satisfying **F4)**, **F5)**, **(M)** *of Theorem 4.3 on $I = [0, \infty)$, and having the following property*

(l-SD)$_{\text{loc}}$ *for every $k \in N_0$ and every $\varepsilon > 0$ there exists a compact set $K_{\varepsilon,k} \subset \overline{I_k}$ such that $\mu(\overline{I_k} \setminus K_{\varepsilon,k}) < \varepsilon$ and $F_{|K_{\varepsilon,k}\times X\times X}$ is weakly lower semicontinuous.*

Then the impulsive problem ***(PI)*** *has at least one impulsive mild solution $x \in \Omega_{\mathcal{C}^1}$.*

Remark 5.3. Let us note that in the theorems presented above the continuity property on the impulsive functions is omitted, unlike [6], [8], [23], and [36] where this assumption is required.

6. On a generalization of the telegraph equation with impulses

6.1. Existence of admissible trajectory-control pairs

In this section we study the existence of admissible trajectory-control pairs for the following impulsive problem

$$
\textbf{(CP-V)} \begin{cases} w''_{tt}(t,\xi) = \sum_{i,j=1}^n a_{ij}(\xi)\frac{\partial^2 w}{\partial \xi_i \partial \xi_j}(t,\xi) + \sum_{i=1}^n b_i(\xi)\frac{\partial w}{\partial \xi_i}(t,\xi) + c(\xi)w(t,\xi) \\ \qquad +d(t,\xi)f\left(\int_\Omega K(\xi,s)w'_t(t,s)\,ds\right) + v(t,\xi),\ t \in [0,\infty)\setminus\{t_k\}_{k\in\mathbb{N}},\ \text{a.e. } \xi \in \Omega \\ w(0,\xi) = w_0(\xi),\ \text{a.e. } \xi \in \Omega \\ w'_t(0,\xi) = \overline{w}_0(\xi),\ \text{a.e. } \xi \in \Omega \\ w(t,\xi) = 0,\ t \in [0,\infty),\ \text{a.e. } \xi \in \partial\Omega \\ w(t_k^+,\xi) = w(t_k,\xi) + p_k w(t_k,\xi),\ k \in \mathbb{N},\ \text{a.e. } \xi \in \Omega \\ w'_t(t_k^+,\xi) = w'_t(t_k,\xi) + g_k(w(t_k,\xi)) + h_k(w'_t(t_k,\xi)),\ k \in \mathbb{N},\ \text{a.e. } \xi \in \Omega \\ v(t,\xi) \in V(t),\ t \in [0,\infty),\ \text{a.e. } \xi \in \Omega, \end{cases}
$$

where the jump points are an increasing sequence of time $(t_k)_{k=0}^\infty$ such that $t_0 = 0$ and $\lim_{k\to\infty} t_k = \infty$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $a_{ij} : \overline{\Omega} \to \mathbb{R}$, $b_i, c : \Omega \to \mathbb{R}$, $i, j \in \{1, ..., n\}$, $d : [0,\infty) \times \Omega \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, $K : \Omega \times \Omega \to \mathbb{R}$, $V : [0,\infty) \to \mathcal{P}(\mathbb{R})$, $p_k \in \mathbb{R}$, $g_k, h_k : \mathbb{R} \to \mathbb{R}$ are bounded and continuous, $k \in \mathbb{N}$, $w_0 \in W^{1,p}(\Omega) \cap \mathcal{C}_0(\Omega)$, and $\overline{w}_0 \in L^p(\Omega)$, $1 < p < \infty$, being $W^{1,p}(\Omega) = \{u \in L^p(\Omega) :$ there exists $\frac{du}{d\xi} \in L^p(\Omega)\}$ and $\mathcal{C}_0(\Omega) = \{u \in \mathcal{C}(\Omega) :\ \textit{supp}\ u$ is compact$\}$.
Let us start by rewriting problem **(CP-V)** in the abstract form **(PI)**, defining in an appropriate way the Banach space X, the set E presented in (8), the operator A, and the multivalued term F.

First of all we put $X = L^p(\Omega)$. Further we consider the operator $A : D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ defined, for every $u \in D(A)$, as follows

$$
Au(\xi) = \sum_{i,j=1}^n a_{ij}(\xi)\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(\xi) + \sum_{i=1}^n b_i(\xi)\frac{\partial u}{\partial \xi_i}(\xi) + c(\xi)u(\xi),\ \text{a.e. } \xi \in \Omega, \tag{78}
$$

where the functions $a_{ij} : \overline{\Omega} \to \mathbb{R}$ and $b_i, c : \Omega \to \mathbb{R}$ satisfy the following assumptions

a1) $a_{ij} \in \mathcal{C}(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $a_{ij}(\xi) = a_{ji}(\xi)$, for every $\xi \in \Omega$ and $i, j = 1, ..., n$;
a2) there exists $H > 0$ such that $\sum_{i,j=1}^n a_{ij}(\xi)z_i z_k \ge H\|z\|^2_{\mathbb{R}^n}$, for every $z = (z_1, ..., z_n) \in \mathbb{R}^n$ and a.e. $\xi \in \overline{\Omega}$.

From **a1)** and **a2)** we have that the operator A is symmetric strongly elliptic (see [37], p. 214), it generates a cosine family $\{C(t)\}_{t\in\mathbb{R}}$ (see [36] and, if $p = 2$, [17], Section IV.8), and the set presented in (8) is $E = W^{1,p}(\Omega) \cap \mathcal{C}_0(\Omega)$.

Moreover we assume the following hypotheses on the functions $K : \Omega \times \Omega \to \mathbb{R}$, $d : [0,\infty) \times \Omega \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, and on the multimap $V : [0,\infty) \to \mathcal{P}(\mathbb{R})$

k) $K(\cdot, s)$ is continuous, for a.e. $s \in \Omega$, $K(\xi, \cdot)$ is measurable for every $\xi \in \Omega$, and there exists $\overline{K} > 0$: $|K(\xi, s)| \le \overline{K}$, for every $\xi \in \Omega$, a.e. $s \in \Omega$;
f1) $f \in \mathcal{C}(\mathbb{R})$;
f2) there exists $\beta : [0,\infty) \to [0,\infty)$ not decreasing with

$$
\lim_{t\to\infty} \frac{\beta(t)}{t} = 0
$$

and

$$
|f(t)| \le \beta(|t|),\ t \in \mathbb{R};
$$

d1) $d : [0, \infty) \times \Omega \to \mathbb{R}$ is measurable;

d2) there exist $\alpha : [0, \infty) \to [0, \infty)$ $\mathcal{L}$-integrable and $\gamma \in L^p_+(\Omega)$ such that $|d(t, \xi)| \le \alpha(t)\gamma(\xi)$, $t \in [0, \infty)$, a.e. $\xi \in \Omega$;

V1) $V(t)$ is closed and convex, for every $t \in [0, \infty)$;

V2) for every $y \in L^p(\Omega)$, the function $t \mapsto \inf_{z \in V(t)} \left(\int_\Omega \left| y(\xi) - \frac{z}{e^t} \right|^p d\xi \right)^{\frac{1}{p}}$ is $(\mathcal{M}([0, \infty)), \mathcal{B}(\mathbb{R}))$-measurable;

V3) there exists $\varrho \in L^1_+([0, \infty))$ such that $\|V(t)\| \le \varrho(t)$, a.e. $t \in [0, \infty)$.

We want to prove the existence of an admissible mild pair $\{w, v\}$ for problem **(CP-V)**, where $w, v : [0, \infty) \times \Omega \to \mathbb{R}$ are such that $w(t, \cdot), v(t, \cdot) \in L^p(\Omega)$, $t \in [0, \infty)$. Let us note that, since we want to establish the existence of a mild solution, it is not necessary to have $w(t, \cdot) \in D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. To rewrite problem **(CP-V)** in the abstract form we reinterpret the maps w, v as two functions $\tilde{w} : [0, \infty) \to L^p(\Omega)$ and $\tilde{v} : [0, \infty) \to L^p(\Omega)$ respectively so defined

$$\begin{aligned} \tilde{w}(t)(\xi) &= w(t, \xi), \text{ a.e. } \xi \in \Omega \\ \tilde{v}(t)(\xi) &= v(t, \xi), \text{ a.e. } \xi \in \Omega, \end{aligned} \tag{79}$$

for every $t \in [0, \infty)$.

Moreover, we define the multimap $\tilde{V} : [0, \infty) \to \mathcal{P}(L^p(\Omega))$ as

$$\tilde{V}(t) = \bigcup_{z \in V(t)} \{u : \Omega \to \mathbb{R} : \ u(\xi) = \frac{z}{e^t}, \text{ a.e. } \xi \in \Omega\}, \ t \in [0, \infty). \tag{80}$$

Clearly $\tilde{V}$ is well posed, since $\tilde{V}(t) \subset L^p(\Omega)$, $t \in [0, \infty)$. Thanks to **V1)** we obtain that $\tilde{V}(t)$ is convex and closed for every $t \in [0, \infty)$.

Additionally, from **V2)** and from the separability of the space $L^p(\Omega)$, Proposition 4.2.4 of [16] allows us to conclude that $\tilde{V}$ is a measurable multimap.

Now we consider the multimap $F : [0, \infty) \times L^p(\Omega) \times L^p(\Omega) \to \mathcal{P}(L^p(\Omega))$ defined as

$$F(t, y, z)(\xi) = \left\{ d(t, \xi) f \left(\int_\Omega K(\xi, s) z(s)\, ds \right) \right\} + \tilde{V}(t)(\xi), \text{ a.e. } \xi \in \Omega \tag{81}$$

for every $t \in [0, \infty)$, $y, z \in L^p(\Omega)$.

Let us note that F is well defined. First of all, fixed $t \in [0, \infty)$, $y, z \in L^p(\Omega)$, for every $\xi \in \Omega$, $K(\xi, \cdot)z(\cdot) \in L^1(\Omega)$ (see **k)**, the boundedness of Ω, and $z \in L^p(\Omega)$). Moreover, since $\tilde{V}(t) \subset L^p(\Omega)$, to say that $F(t, y, z) \in L^p(\Omega)$ it is sufficient to prove that $d(t, \cdot) f \left(\int_\Omega K(\cdot, s) z(s)\, ds \right) \in L^p(\Omega)$. Taking into account **d2)**, **f2)**, and **k)**, the Hölder inequality implies for a.e. $\xi \in \Omega$ (see **d2)**)

$$\left| d(t, \xi) f \left(\int_\Omega K(\xi, s) z(s)\, ds \right) \right| \le \alpha(t)\gamma(\xi)\beta \left(\overline{K} \int_\Omega \|z(s)\|_{\mathbb{R}^n}\, ds \right) \le \alpha(t)\gamma(\xi)\beta \left(\overline{K} \mu(\Omega)^{1 - \frac{1}{p}} \|z\|_p \right). \tag{82}$$

Now by Proposition 2.3 we have the continuity on Ω of the map $\xi \mapsto \int_\Omega K(\xi, s) z(s)\, ds$ (see **k)**), hence by **f1)** and **d1)** we deduce the measurability of the function $\xi \mapsto d(t, \xi) f \left(\int_\Omega K(\xi, s) z(s)\, ds \right)$. Therefore, from (82) we conclude that $F(t, y, z) \in \mathcal{P}(L^p(\Omega))$.

Finally, for $k \in \mathbb{N}$, we introduce the maps $\tilde{g}_k, \tilde{h}_k : L^p(\Omega) \to L^p(\Omega)$ respectively so defined

$$\tilde{g}_k(y)(\xi) = g_k(y(\xi)), \ \tilde{h}_k(y)(\xi) = h_k(y(\xi)), \text{ a.e. } \xi \in \Omega, \tag{83}$$

for every $y \in L^p(\Omega)$.

Now, recalling (78), (79), (80), and (81) problem **(CP-V)** can be rewritten in the abstract form

$$\begin{cases} \tilde{w}''(t) \in A\tilde{w}(t) + F(t, \tilde{w}(t), \tilde{w}'(t)), \text{ a.e. } t \in [0, \infty) \setminus \{t_k\}_k \\ \tilde{w}(0) = w_0 \\ \tilde{w}'(0) = \overline{w}_0 \\ \tilde{w}(t_k^+) = \tilde{w}(t_k) + p_k \tilde{w}(t_k), \ k \in \mathbb{N} \\ \tilde{w}'(t_k^+) = \tilde{w}'(t_k) + \tilde{g}_k(\tilde{w}(t_k)) + \tilde{h}_k(\tilde{w}'(t_k)), \ k \in \mathbb{N}. \end{cases}$$

Our goal is to prove that all the assumptions of Theorem 5.1 are satisfied.
First of all we note that $X = L^p(\Omega)$, $1 < p < \infty$ has a Schauder basis (see [10], p. 146), $w_0 \in E = W^{1,p}(\Omega) \cap \mathcal{C}_0(\Omega)$ and $\overline{w}_0 \in X$. Moreover the operator A generates a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ (see [17], Section IV.8).
Now, let us show that hypotheses **F1)-F5)** hold.
Obviously, since $\tilde{V}$ has convex values, **F1)** of Theorem 5.1 is satisfied.
Next we prove that, fixed $(y, z) \in L^p(\Omega) \times L^p(\Omega)$, $F(\cdot, y, z)$ has a B-measurable selection. At first, the multimap $\tilde{V}$ has a B-measurable selection. Indeed $\tilde{V}$ is measurable and assumes closed values in the Polish space $L^p(\Omega)$, hence there exists $u : [0, \infty) \to L^p(\Omega)$ $(\mathcal{M}([0, \infty)), \mathcal{B}(L^p(\Omega)))$-measurable such that $u(t) \in \tilde{V}(t)$ for a.e. $t \in [0, \infty)$ (see Theorem 2.6). Then, thanks to the separability of $L^p(\Omega)$, we have the B-measurability of the selection u. So $h_{(y,z)} : [0, \infty) \to L^p(\Omega)$ defined as

$$h_{(y,z)}(t) = d(t, \cdot) f\left(\int_\Omega K(\cdot, s) z(s)\, ds\right) + u(t)$$

is a sum of two B-measurable maps and $h_{(y,z)}(t) \in F(t, y, z)$ for a.e. $t \in [0, \infty)$. Therefore **F2)** of Theorem 5.1 holds.
Now, to achieve that **F3)** of Theorem 5.1 is satisfied, we fix $t \in [0, \infty)$ and the sequences $(y_n)_n$, $(z_n)_n$, $(h_n)_n$ in $L^p(\Omega)$ such that $y_n \rightharpoonup y$, $z_n \rightharpoonup z$, $h_n \rightharpoonup h$, $y, z, h \in L^p(\Omega)$, with $h_n \in F(t, y_n, z_n)$, $n \in \mathbb{N}$. We want to prove that $h \in F(t, y, z)$. From (81), for every $n \in \mathbb{N}$, there exists $u_n \in \tilde{V}(t)$ such that

$$h_n = d(t, \cdot) f\left(\int_\Omega K(\cdot, s) z_n(s)\, ds\right) + u_n. \tag{84}$$

Now, we prove that the weak convergence of $(z_n)_n$ to z implies that

$$f\left(\int_\Omega K(\xi, s) z_n(s)\, ds\right) \to f\left(\int_\Omega K(\xi, s) z(s)\, ds\right), \ \xi \in \Omega. \tag{85}$$

To this aim, we fix $\xi \in \Omega$ and we note that the linear operator $l_\xi : L^p(\Omega) \to \mathbb{R}$ so defined

$$l_\xi(u) = \int_\Omega K(\xi, s) u(s)\, ds, \ u \in L^p(\Omega)$$

is also bounded (see **k)**). So, being $l_\xi \in (L^p(\Omega))'$, we have $l_\xi(z_n) \to l_\xi(z)$. Then the continuity of f (see **f1)**) implies (85). So we have

$$d(t,\xi)f\left(\int_\Omega K(\xi,s)z_n(s)\,ds\right) \to d(t,\xi)f\left(\int_\Omega K(\xi,s)z(s)\,ds\right),\ \xi\in\Omega.$$

Next, from the weak convergence of the sequence $(z_n)_n$, there exists $L>0$ such that $\|z_n\|_p \le L$, for every $n\in\mathbb{N}$. Hence, recalling (82), the following inequality holds

$$\left|d(t,\xi)f\left(\int_\Omega K(\xi,s)z_n(s)\,ds\right)\right| \le \alpha(t)\gamma(\xi)\beta(\overline{K}L\mu(\Omega)^{1-\frac{1}{p}}),\ \text{a.e. } \xi\in\Omega,\ n\in\mathbb{N},$$

with $\alpha(t)\gamma(\cdot)\beta(\overline{K}L\mu(\Omega)^{1-\frac{1}{p}}) \in L^p(\Omega)$ (see **d2)**).
Hence, by the Dominate Convergence Theorem (see [11], Proposition 3.36) we deduce

$$d(t,\cdot)f\left(\int_\Omega K(\cdot,s)z_n(s)\,ds\right) \to d(t,\cdot)f\left(\int_\Omega K(\cdot,s)z(s)\right)\ \text{ in } L^p(\Omega). \tag{86}$$

Now, from (86) and recalling that $h_n \rightharpoonup h$, the sequence $(u_n)_n$ satisfies (see (84))

$$u_n \rightharpoonup h - d(t,\cdot)f\left(\int_\Omega K(\cdot,s)z(s)\,ds\right) := u \in L^p(\Omega).$$

Finally, since u_n belongs to the weakly closed set $\tilde{V}(t)$, $n\in\mathbb{N}$, we deduce that $u\in\tilde{V}(t)$. Therefore $h(\cdot) = d(t,\cdot)f\left(\int_\Omega K(\cdot,s)z(s)\,ds\right) + u(\cdot) \in F(t,y,z)$. Then **F3)** is true.
Next we prove that the multimap F satisfies also **F4)**.
First of all, put $t\in[0,\infty)\setminus N^*$, where N^* is the null measure set for which **V3)** holds, $n\in\mathbb{N}$, $y,z\in \overline{B}_{L^p(\Omega)}(0,n)$, we fix $h\in F(t,y,z)$. Then, there exists $u\in\tilde{V}(t)$ such that $h(\cdot) = d(t,\cdot)f\left(\int_\Omega K(\cdot,s)z(s)\,ds\right) + u(\cdot)$. Denoting $\omega\in V(t)$ such that $u(\xi) = \frac{\omega}{e^t}$, a.e. $\xi\in\Omega$ (see (80)), we can write (see (82), **f2)** and **V3)**)

$$\begin{aligned}
\|h\|_p &\le \left\|d(t,\cdot)f\left(\int_\Omega K(\cdot,s)z(s)\,ds\right)\right\|_p + \|u\|_p \\
&= \left(\int_\Omega \left|d(t,\xi)f\left(\int_\Omega K(\xi,s)z(s)\,ds\right)\right|^p d\xi\right)^{\frac{1}{p}} + \left(\int_\Omega \left|\frac{\omega}{e^t}\right|^p d\xi\right)^{\frac{1}{p}} \\
&\le \left(\int_\Omega \left[\alpha(t)\gamma(\xi)\beta\left(\overline{K}\mu(\Omega)^{1-\frac{1}{p}}\|z\|_p\right)\right]^p d\xi\right)^{\frac{1}{p}} + \frac{|\omega|}{e^t}\mu(\Omega)^{\frac{1}{p}} \\
&\le \alpha(t)\beta\left(\overline{K}\mu(\Omega)^{1-\frac{1}{p}}n\right)\|\gamma\|_p + \frac{\varrho(t)}{e^t}\mu(\Omega)^{\frac{1}{p}} =: \varphi_n(t),
\end{aligned}$$

where $\varphi_n : [0,\infty)\to\mathbb{R}_0^+$ is an L^1_+-map (see **d2)**).
By the arbitrariness of $h\in F(t,y,z)$ and $y,z\in\overline{B}_{L^p(\Omega)}(0,n)$, we deduce

$$\|F(t,\overline{B}_{L^p(\Omega)}(0,n),\overline{B}_{L^p(\Omega)}(0,n))\| \le \varphi_n(t),\ t\in[0,\infty).$$

Moreover, from **f2)** we also have

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_1}{n} = \liminf_{n\to\infty} \frac{\beta\left(\overline{K}\mu(\Omega)^{1-\frac{1}{p}}n\right)\|\gamma\|_p\|\alpha\|_1 + \varrho(t)\mu(\Omega)^{\frac{1}{p}}}{n} = 0.$$

Therefore **F4)** holds.

Clearly, taking into account the reflexivity of $L^p(\Omega)$ and of the property **F4)**, recalling Remark 4.2 we can conclude that **F5)** is satisfied too.

Finally, for every $k \in \mathbb{N}$, we define $I_k, \tilde{I}_k : L^p(\Omega) \times L^p(\Omega) \to L^p(\Omega)$ respectively as

$$I_k(y, z) = p_k y, \;\; \tilde{I}_k(y, z) = \tilde{g}_k(y) + \tilde{h}_k(z).$$

Let us note that, for every $k \in \mathbb{N}$ and $(y, z) \in L^p(\Omega) \times L^p(\Omega)$, $\tilde{I}_k(y, z) \in L^p(\Omega)$ since, by using the continuity of g_k and h_k, the map $\tilde{g}_k(y)+\tilde{h}_k(z)$ is measurable and, being g_k and h_k bounded, $\|\tilde{g}_k(y)+\tilde{h}_k(z)\|_p^p \le C_k\mu(\Omega)$, where $C_k = \sup_{t\in\mathbb{R}} |g_k(t) + h_k(t)| < \infty$ (see (83)). On the other hand, it is obvious that $I_k(y, z) \in E = W^{1,p}(\Omega) \cap \mathcal{C}_0(\Omega)$, for every $y \in E$, $z \in L^p(\Omega)$, so **IM)** of Theorem 5.1 is satisfied.

Taking into account the arguments above presented, we are in a position to apply Theorem 5.1 and so there exists at least one impulsive mild solution $\hat{w} : [0, \infty) \to L^p(\Omega)$, $\hat{w}$ in $\Omega_{\mathcal{C}^1}$ (see (68)), such that

$$\hat{w}(t) = C(t)w_0 + S(t)\overline{w}_0 + \int_0^t S(t-s)\hat{g}(s)\, ds + \sum_{0<t_k<t} [C(t-t_k)I_k(\hat{w}(t_k), \hat{w}'(t_k)) + S(t-t_k)\tilde{I}_k(\hat{w}(t_k), \hat{w}'(t_k))], \;\; t \in [0, \infty), \tag{87}$$

where $\hat{g} \in S^{1,\mathrm{loc}}_{F(\cdot,\hat{w}(\cdot),\hat{w}'(\cdot))} = \{g \in L^{1,\mathrm{loc}}([0, \infty), L^p(\Omega)) : \;\; g(t) \in F(t, \hat{w}(t), \hat{w}'(t)), \;\text{a.e. } t \in [0, \infty)\}$.
In particular, from (87), we have (see **C1)** and (9))

$$\hat{w}(0) = C(0)w_0 + S(0)\overline{w}_0 = w_0 \tag{88}$$

and there exists $\hat{w}'(0)$ (see (68)) such that (see **i)** of Proposition 3.1, (9), (13), and **C2)**)

$$\hat{w}'(0) = \frac{d}{dt}C(t)w_0\,|_{t=0} + \frac{d}{dt}S(t)\overline{w}_0\,|_{t=0} = AS(0)w_0 + C(0)\overline{w}_0 = \overline{w}_0 \tag{89}$$

being $w_0 \in E$.
Now we define the B-measurable map $u_{\hat{w}} : [0, \infty) \to L^p(\Omega)$ as follows

$$u_{\hat{w}}(t)(\cdot) = \hat{g}(t)(\cdot) - d(t, \cdot)f\left(\int_\Omega K(\cdot, s)\frac{d}{dt}\hat{w}(t)(s)\, ds\right), \;\; t \in [0, \infty) \tag{90}$$

and we show that $u_{\hat{w}}(t) \in \tilde{V}(t)$, a.e. $t \in [0, \infty)$.
To this aim let us fix $t \in [0, \infty) \setminus N$, where N is the null measure set such that $\hat{g}(t) \in F(t, \hat{w}(t), \hat{w}'(t))$.
Then, from (81), there exists $u_t \in \tilde{V}(t)$ so characterized

$$\hat{g}(t)(\cdot) = d(t, \cdot)f\left(\int_\Omega K(\cdot, s)\frac{d}{dt}\hat{w}(t)(s)\, ds\right) + u_t(\cdot).$$

On the other hand, from (90) we have

$$\hat{g}(t)(\cdot) = d(t,\cdot)f\left(\int_{\Omega} K(\cdot,s)\frac{d}{dt}\hat{w}(s)\,ds\right) + u_{\hat{w}}(t)(\cdot),\ t\in[0,\infty).$$

Therefore $u_{\hat{w}}(t) = u_t$ for a.e. $t\in[0,\infty)$. Hence $u_{\hat{w}}(t)\in\tilde{V}(t)$, a.e. $t\in[0,\infty)$. Then by (80) we have that

$$u_{\hat{w}}(t)(\xi)e^t \in V(t),\ \text{a.e. } t\in[0,\infty),\ \text{a.e. } \xi\in\Omega. \tag{91}$$

At this point, we can consider the functions w and v defined respectively

$$w(t,\xi) = \hat{w}(t)(\xi),\ v(t,\xi) = u_{\hat{w}}(t)(\xi)e^t,\ t\in[0,\infty),\ \text{a.e. } \xi\in\Omega. \tag{92}$$

From (91) we have

$$v(t,\xi)\in V(t),\ \text{a.e. } t\in[0,\infty),\ \text{a.e. } \xi\in\Omega. \tag{93}$$

Moreover, let us note that

$$v(\cdot,\xi)\text{ is B-measurable, a.e. } \xi\in\Omega. \tag{94}$$

Indeed, from the B-measurability of $u_{\hat{w}}$ there exists a sequence of simple functions $(s_n)_n$ converging to $u_{\hat{w}}$ in $[0,\infty)$. So, w.l.o.g. $s_n(t)(\xi)\to u_{\hat{w}}(t)(\xi)$, a.e. $\xi\in\Omega$. Then, for a.e. $\xi\in\Omega$ the sequence of simple functions $(s_n(\cdot)(\xi))_n$ converges to $u_{\hat{w}}(\cdot)(\xi)$ in $[0,\infty)$. Therefore (94) holds (see (92)).
Hence taking into account (87) we can conclude that $\{w,v\}$ is an admissible pair for **(CP-V)**.

What has been proven leads the following existence result (see (87), (88), (89), (94), and (93)).

Theorem 6.1. *In the framework above described, there exists an admissible trajectory-control pair $\{w,v\}$ for **(CP-V)**, i.e. there exist $w,v:[0,\infty)\times\Omega\to\mathbb{R}$ having the following properties*

(w1) *for every $t\in[0,\infty)$, $w(t,\cdot)\in L^p(\Omega)$;*
(w2) *for every $t\in[0,\infty)$, there exists $w'_t(t^-,\cdot)\in L^p(\Omega)$;*
(w3) *for a.e. $\xi\in\Omega$, $w(0,\xi)=w_0(\xi)$;*
(w4) *for a.e. $\xi\in\Omega$, $\frac{\partial w}{\partial t}(0,\xi)=\overline{w}_0(\xi)$;*
(v1) *for every $t\in[0,\infty)$, $v(t,\cdot)\in L^p(\Omega)$;*
(v2) *for a.e. $\xi\in\Omega$, $v(\cdot,\xi)$ is B-measurable;*
(v3) *for a.e. $t\in[0,\infty)$, a.e. $\xi\in\Omega$, $v(t,\xi)\in V(t)$,*

and satisfying the equality

$$\begin{aligned} w(t,\xi) &= [C(t)w_0 + S(t)\overline{w}_0](\xi) + \left[\int_0^t S(t-s)g(s,\cdot)\,ds\right](\xi) \\ &+ \sum_{0<t_k<t}\{C(t-t_k)[p_k w(t_k,\cdot)] + S(t-t_k)[g_k(w(t_k,\cdot)) + h_k(w'_t(t_k,\cdot))]\}(\xi), \\ & t\in[0,\infty),\ \text{a.e. } \xi\in\Omega, \end{aligned}$$

where $g:[0,\infty)\times\Omega\to\mathbb{R}$ is so defined

$$g(t,\xi) = d(t,\xi)f\left(\int_{\Omega} K(\xi,s)w'_t(t,s)\,ds\right) + v(t,\xi),\ t\in[0,\infty)\setminus\{t_k:\ k\in\mathbb{N}\},\ \text{a.e. } \xi\in\Omega.$$

6.2. Instant-controllability relatively to a suitable functional

Now we are in a position to study the instant-controllability, introduced in the sequel in Definitions 6.1-6.2, for the impulsive problem **(CP-W)** relatively to a functional $I : L^p(\Omega) \times L^p(\Omega) \to \mathbb{R}$ and to a control-multimap $W : [0, \infty) \to \mathcal{P}(\mathbb{R})$ satisfying the following assumptions

(W1) $W(t)$ is closed, for every $t \in [0, \infty)$;
(W2) W is measurable;
(W3) there exists $\varrho \in L^1_+([0, \infty))$ such that $\|W(t)\| \leq \varrho(t)$, a.e. $t \in [0, \infty)$.

In our opinion the study of this type of controllability can be helpful to monitor the impulses. For example, when the model describes the telegraph behaviour, the instant-controllability can be useful to achieve a minimization of the signal distortion or the maximization of the distance reached by the signal, depending on the meaning given to the functional I.
First of all let us note that, for every $(\mathcal{M}([0, \infty)), \mathcal{B}(\mathbb{R}))$-measurable selection $\overline{v} : [0, \infty) \to \mathbb{R}$ of W, which there exists by virtue of Theorem 2.6, we can consider $V_{\overline{v}} : [0, \infty) \to \mathcal{P}(\mathbb{R})$ defined as

$$V_{\overline{v}}(t) = \{\overline{v}(t)\}, \ t \in [0, \infty).$$

Clearly $V_{\overline{v}}$ has the property **V1)** and **V3)** (see **(W3)**). On the other hand, for every $y \in L^p(\Omega)$, the map

$$t \mapsto inf_{z \in V_{\overline{v}}(t)} \left(\int_{\Omega} \left| y(\xi) - \frac{z}{e^t} \right|^p \, d\xi \right)^{\frac{1}{p}} = \left\| y(\cdot) - \frac{\overline{v}(t)}{e^t} \right\|_p \tag{95}$$

is $(\mathcal{M}([0, \infty)), \mathcal{B}(\mathbb{R}))$-measurable. Indeed $p_y : [0, \infty) \to L^p(\Omega)$, $p_y(t) = y(\cdot) - \frac{\overline{v}(t)}{e^t}$, is $\mathcal{L}$-measurable, being $\overline{v}$ a $\mathcal{L}$-measurable map (see [16], Corollary 3.10.5). So there exists a sequence $(s_n)_n$ of simple functions, $s_n : [0, \infty) \to L^p(\Omega)$, such that $s_n(t) \to p_y(t)$, a.e. $t \in [0, \infty)$. Then, taking into account the continuity of the norm $\| \cdot \|_p$, we deduce

$$\|s_n(t)\|_p \to \|p_y(t)\|_p \text{ a.e. } t \in [0, \infty),$$

where, for every $n \in \mathbb{N}$, $t \mapsto \|s_n(t)\|_p$ is a simple function. Therefore the map defined in (95) is $\mathcal{L}$-measurable too. Since the space $L^p(\Omega)$ is separable we obtain the desired $(\mathcal{M}([0, \infty)), \mathcal{B}(\mathbb{R}))$-measurability of (95) (see [16], Corollary 3.10.5).
Now, if the maps $a_{ij} : \overline{\Omega} \to \mathbb{R}$, $b_i, c : \Omega \to \mathbb{R}$, $K : \Omega \times \Omega \to \mathbb{R}$, $d : [0, \infty) \times \Omega \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ satisfy the same assumptions presented in the previous subsection 6.1, from Theorem 6.1 there exists an admissible trajectory-control pair $\{w_{\overline{v}}, v_{\overline{v}}\}$ for $(\mathbf{CP} - \mathbf{V}_{\overline{\mathbf{v}}})$, having the properties *(w1)-(v3)*.
Next, for every $t \in [0, \infty)$, we define the set

$$\mathcal{U}_t = \{(w_{\overline{v}}(t, \cdot), v_{\overline{v}}(t, \cdot)) : \ \overline{v} \text{ is } (\mathcal{M}([0, \infty)), \mathcal{B}(\mathbb{R}))\text{-measurable selection of } W\}, \tag{96}$$

where $\{w_{\overline{v}}, v_{\overline{v}}\}$ is an admissible trajectory-control pair for $(\mathbf{CP} - \mathbf{V}_{\overline{\mathbf{v}}})$.
From the previous arguments we can claim that the set $\mathcal{U}_t$ is a nonempty subset of $L^p(\Omega) \times L^p(\Omega)$.
Hence we are in a position to introduce the definitions

Definition 6.1. The impulsive problem **(CP-W)** is said to be *instantly controllable from below* by I, if for any $t \in [0, \infty)$, there exists an admissible trajectory-control pair $\{w^*, v^*\}$ such that

$$\inf_{(w(t,\cdot),v(t,\cdot)) \in \mathcal{U}_t} I(w(t, \cdot), v(t, \cdot)) = I(w^*(t, \cdot), v^*(t, \cdot)).$$

Definition 6.2. The impulsive problem **(CP-W)** is said to be *instantly controllable from above* by I if, for any $t \in [0,\infty)$, there exists an admissible trajectory-control pair $\{w^*, v^*\}$ such that

$$\sup_{(w(t,\cdot),v(t,\cdot))\in\mathcal{U}_t} I(w(t,\cdot),v(t,\cdot)) = I(w^*(t,\cdot),v^*(t,\cdot)).$$

Let us note that the instant-controllability by I for **(CP-W)** is well posed.

Now, we assume on the functional $I : L^p(\Omega)\times L^p(\Omega) \to \mathbb{R}$ the following properties

i) for every $t \in [0,\infty)$, $I_{|\mathcal{U}_t}$ is coercive and lower semicontinuous,

where $\mathcal{U}_t$ is a topological space endowed with the induced topology of $L^p(\Omega)\times L^p(\Omega)$.
Next, fixed $t \in [0,\infty)$, thanks to Theorem 1.3.19 of [16], applied to $I_{|\mathcal{U}_t}$, we obtain that there exists an admissible trajectory-control pair $\{w^*, v^*\}$ such that

$$\inf_{(w(t,\cdot),v(t,\cdot))\in\mathcal{U}_t} I(w(t,\cdot),v(t,\cdot)) = I(w^*(t,\cdot),v^*(t,\cdot)).$$

So **(CP-W)** is instantly controllable from below by I.

Clearly, if the functional $-I$ satisfies **i)**, we have that **(CP-W)** is instantly controllable from above by I.

Remark 6.2. Let us note that if $\mathcal{F}$ is a family of $(\mathcal{M}([0,\infty)),\mathcal{B}(\mathbb{R}))$-measurable functions then we can consider the problem $(\mathbf{CP}-\mathcal{F})$, where is present the condition

$$v \in \mathcal{F},$$

instead of $v(t,\xi) \in W(t)$ a.e. $t \in [0,\infty)$, a.e. $\xi \in \Omega$. In this case with the same arguments we can achieve the instant-controllability for the problem $(\mathbf{CP}-\mathcal{F})$ relatively to I.

7. Conclusions

The novelty of our paper is the study of instant-controllability for a multidimensional differential equation, on an unbounded interval, subjected to a damping term and impulses. Since the study of controllability has a great relevance in the applied sciences and it is important taking into account the damping term to describe many models, such as the “spillover” model, we expect that our research may stimulate further investigations.

CRediT authorship contribution statement

Writing-original draft preparation, T.C. and G.D.
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Ethics approval

The authors declare that they accept principles of ethical and professional conduct required by the Journal.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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Data availability

No data was used for the research described in the article.

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