Journal of the American Statistical Association

ISSN: 0162-1459 (Print) 1537-274X (Online) Journal homepage: www.tandfonline.com/journals/uasa20

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To cite this article: Marco Avarucci, Maddalena Cavicchioli, Mario Forni & Paolo Zaffaroni (12 Jan 2026): Frequency-Band Estimation of the Number of Factors, Journal of the American Statistical Association, DOI: 10.1080/01621459.2025.2571246

To link to this article: https://doi.org/10.1080/01621459.2025.2571246

JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION
2025, VOL. 00, NO. 0, 1–13: Theory and Methods
https://doi.org/10.1080/01621459.2025.2571246

Frequency-Band Estimation of the Number of Factors

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ABSTRACT
We introduce consistent estimators for the number of shocks driving large-dimensional dynamic factor models. Our estimator can be applied to single frequencies and specific frequency bands, making it suitable for disentangling shocks affecting dynamic models with a factor model representation. Noticeably, our estimator requires the time-series and cross-section sizes to diverge simultaneously without any constraint and it is free of nuisance parameters, such as penalization terms. Our methodology appears ideal for macroeconomic analysis, as one can investigate how many shocks drive the business cycle or the long run, although the applicability of our methods is much wider, given the popularity of GDFMs in economics and finance. Its small-sample performance in simulations is excellent. We apply our estimator to the FRED-QD dataset, finding that the U.S. macroeconomy is driven by two shocks: an inflationary demand shock and a deflationary supply shock. Our methodology permits one to accurately estimate the number of shocks that drive medium-sized DSGE models despite their moderate cross-sectional size. Supplementary materials for this article are available online.

ARTICLE HISTORY
Received August 2023
Accepted September 2025

KEYWORDS
Business cycle; DSGE; Dynamic factors; Frequency bands; Generalized dynamic factor models

1. Introduction

Real Business Cycle (RBC) models postulate the existence of a single supply shock that drives variables related to real economic activity (Kydland and Prescott 1982). In contrast, modern Dynamic Stochastic General Equilibrium (DSGE) models feature several shocks (Smets and Wouters 2007). The task of choosing between these rival models raises two fundamental empirical questions: How many shocks are there in the economy overall? And, even more importantly, how many shocks drive the business cycle, and how many, instead, do so over the long run? Addressing these questions sheds light on which approach to macroeconomic modeling appears more appropriate.

Inspired by this empirical challenge, this article advances a methodological contribution proposing novel consistent estimators for the number of common shocks in the vein of Ahn and Horenstein (2013), within the realm of the Generalized Dynamic Factor Model (GDFM): the Dynamic Difference Ratio estimator (DDR), the Dynamic Eigenvalue Ratio estimator (DER), and the Dynamic Growth Rate estimator (DGR).

Static versus Dynamic Factors. To better introduce and motivate our work, let us recall briefly the main features of the static and the dynamic factor models to which we refer. In the static factor model the variables x_{it}, $i = 1, \dots n$, follow the relation

$$x_{it} = \phi_{it} + \epsilon_{it} = \sum_{j=1}^{r} \Lambda_{ik} F_{jt} + \epsilon_{it},$$

where the "idiosyncratic component" ϵ_{it} is orthogonal to the r latent "static factors" F_{jt} and therefore to the "common component" ϕ_{it}.[1] In this model the r largest eigenvalues of the variance covariance matrix of the first n observables $(x_{1t}, \cdots, x_{nt})$ go to infinity as n gets larger, whereas the other eigenvalues are bounded.[2] Hence, the ratio of the rth eigenvalue to the $r+1$th eigenvalue goes to infinity. Two prominent estimators for the number of static factors r, the Eigenvalue Ratio (ER) and the Growth Ratio (GR) estimators of Ahn and Horenstein (2013) (AH henceforth) are based on this property. The advantage of such estimators over alternative ones is that they perform well in small samples and, being free of nuisance parameters, do not require discretionary choices on the researcher's part.

In the dynamic factor model, the variables follow the relation

$$x_{it} = \chi_{it} + e_{it} = \sum_{k=1}^{q} \lambda_{ik}(L) f_{kt} + e_{it},$$

where L is the lag operator, the "dynamic factors" f_{kt} are serially and mutually uncorrelated and e_{it} is orthogonal to the q dynamic factors, and therefore to the common components χ_{it}, at all leads and lags.[3]

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[1]See Chamberlain and Rothschild (1983), Stock and Watson (2002), and Bai and Ng (2002).
[2]See Chamberlain and Rothschild (1983).
[3]See Forni et al. (2000) and Forni and Lippi (2001).

Supplementary materials for this article are available online. Please go to www.tandfonline.com/r/JASA.

The dynamic factor model may have a static representation.[4] In particular, Stock and Watson (2005), Bai and Ng (2007), Amengual and Watson (2007) and Forni et al. (2009) assume a static factor model where the factors F_{jt} follow a VAR whose residuals are the dynamic factors f_{kt}, that is $B(L)F_t = f_t$. In this case we have both a static and a dynamic factor representation, where $\phi_{it} = \chi_{it}$ and $\epsilon_{it} = e_{it}$. Hence, we have $\sum_{j=1}^{r} \Lambda_{ik} F_{jt} = \Lambda_i' B(L)^{-1} f_t = \sum_{k=1}^{q} \lambda_{ik}(L) f_{kt}$. It is easily seen that the rank of the covariance matrix of the common components $\chi_{it}, i = 1, \dots, n$, is r, whereas the rank of the spectral density matrix of the same common components (which is equal to the rank of the covariance matrix of the VAR residuals f_t) is $q \leq r$.

The dynamic model is well suited to represent macroeconomic variables: the dynamic factors can be interpreted as the structural shocks; the dynamic loadings $\lambda_{ik}(L)$ are the impulse-response functions and e_{it} is either a measurement error or a sectoral component. Hence, if we want to find how many structural shocks drive the macroeconomy we have to estimate the number of dynamic factors q.

Our Estimators of q. AH observes that "it might be interesting to investigate whether the ER and GR methods can be generalized to estimation of the number of dynamic factors" (p.210). Our estimators DER and DGR are precisely the dynamic analogues of ER and GR, respectively, based on the eigenvalues of the spectral density matrix, rather than the variance-covariance matrix, of the observable x's.

In addition, we introduce a third criterion, DDR, which is defined as the ratio of two consecutive eigenvalue differences.[5] DDR is close in spirit to DER and DGR, whereby all these estimators are designed to exploit the different asymptotic behavior of the q largest eigenvalues from that of the following ones and are free of nuisance parameters. The analytical relationship linking DDR to DER and DGR is discussed in the online supplement (Section S.1.6). DDR is our preferred estimator, because of its excellent small sample performances in our simulation exercises.

All our estimators can be applied to single frequencies and specific frequency bands of interest. This appears ideal for macroeconomic analysis, as one can investigate how many shocks drive the business cycle or the long run. However, the applicability of our methods is much wider, given the popularity of GDFMs in economics and finance (see, e.g., Giannone and Matheson 2007; Altissimo et al. 2010; Hallin and Liška 2011; Luciani 2014; Barigozzi and Hallin 2017, 2020).

Related Literature. This article belongs to the literature on methods for estimating the number of latent factors, initiated by Bai and Ng (2002) for the static setting and Hallin and Liška (2007) (hereafter HL) for the dynamic setting. In a sense, HL adapt the methodology of Bai and Ng (2002) to the GDFM, just like we adapt AH to the GDFM.

Bai and Ng (2007) and Amengual and Watson (2007) propose methods to determine the number of dynamic factors which are based on time domain techniques. Such methods rely on the existence of a static factor representation and require preliminary estimation of the number of static factors r. Working in the frequency domain is more general, in that the existence of a static representation is not needed; moreover, it enables us to apply our estimator to the full $(-\pi, \pi]$ band, to single frequencies and to frequency bands of interest.

On the other hand, HL and Onatski (2009) (hereafter O) are the main procedures to determine the number of factors in a dynamic factor model without assuming the static representation; whereas HL assume that n, T diverge simultaneously without any constraint, O's test requires that T diverges faster than n.[6] Being able to relax this condition is relevant as in most macroeconomic and financial dataset T and n have similar sizes. Just like AH, we do not need to show the convergence of the smoothed periodogram to the population spectral density matrix and, like them, require the condition $n/T \rightarrow c$ for a positive bounded constant c (see our key results in Lemmas S.3–S.5 of the online supplement). A basic advantage of our estimators over both HL and O is that, being essentially free of nuisance parameters, just like AH, they do not require discretionary choices which might influence the results.

Regarding the empirical results, our article is related to a scant literature concerning the number of shocks driving the macroeconomy. Our finding of two main shocks is in line with Sargent and Sims (1977), Giannone, Reichlin, and Sala (2005), Onatski (2009), Granese (2024), and Forni et al. (2025). Bai and Ng (2007) and HL, using their proposed criteria for the number of dynamic factors, find four shocks. Our article is also related to a wide array of literature about the role of demand and supply shocks. A few prominent papers are Kydland and Prescott (1982), Blanchard and Quah (1989), King et al. (1991), Galí (1999), Beaudry and Portier (2006), Bloom (2009), and Angeletos, Collard, and Dellas (2020). Given the strong analogies between GDFM and DSGE models, our paper contributes to the vast DSGE literature (see Smets and Wouters 2007; Justiniano, Primiceri, and Tambalotti 2010; Angeletos, Collard, and Dellas 2018; Onatski and Ruge-Murcia 2013, among many others), providing a sound way to estimate the number of shocks driving DSGE models, thereby providing a taxonomy in terms of their effect per frequency band.

Finite-Sample Performance. We evaluate the finite-sample performance of DDR, DER, and DGR, and compare it with three existing criteria, namely the ones proposed by HL, O, and AH, using Monte Carlo simulations. Particularly, we run five experiments. For the first three, we evaluate our estimators, along with the competing estimators, on the whole interval $[0, \pi]$. We find that DDR performs better than, or comparably with, DER, DGR, HL, and O for all experiments. The comparison with ER and GR allows us to study the behavior of the static factors estimator in

[4]The relation between the static and the dynamic factor models is studied in detail in Gersing, Rust, and Deistler (2023). Large factor models, either in their static or dynamic representation, have been applied successfully to analyzing big macroeconomic and financial panels. Early theoretical contributions not cited above are Forni and Reichlin (1998), Forni et al. (2005), and Bai and Ng (2007). A partial list of early applications include forecasting in Stock and Watson (2002), and Boivin and Ng (2006), structural macroeconomic analysis in Bernanke and Boivin (2003), and Forni et al. (2009), as well as nowcasting and business cycle indicators in Forni and Lippi (2001), and the analysis of financial markets in Ludvigson and Ng (2009).

[5]DDR is closely related to the test statistic of Onatski (2009), although its use in our context is different.

[6]O assumes that $n/m \rightarrow c \in (0, \infty)$ and $m = o(T)$, where m is the bandwidth used to compute the smoothed periodogram.

our dynamic framework, making the use of dynamic estimators more compelling in a dynamic environment. In experiment four, the spectral density matrix of the common components has reduced rank $q-1$ at frequency zero. The economic interpretation is that when one considers only real activity variables and there is a shock, such as a demand or monetary policy shock, the latter has only transitory effects on all variables, inducing a rank reduction in the spectral density matrix of the χ's.[7] We show that DDR can detect such rank reduction.

We perform a further Monte Carlo exercise, tailored to assess the ability of our criteria to disentangle the number of structural shocks in medium-scale DSGE models, given the strong analogies between the GDFM and the reduced form of DSGE models, as they can both be represented as singular Vector Autoregressive models (VARs) of finite order. We show how the DDR criterion is extremely reliable in correctly identifying the number of structural shocks driving two popular DSGE models (see Justiniano, Primiceri, and Tambalotti (2010) and Angeletos, Collard, and Dellas (2018)), even if this number is relatively large. DDR performs better than ER and GR for the case of the DSGE model of Onatski and Ruge-Murcia (2013), although it is not always accurate due to the small degree of heterogeneity in the loadings to the common shocks. We conjecture that our methodology applies to any DSGE model that admits a singular VAR representation (around steady-state).

Empirical Application. Our analysis is based on a quarterly U.S. macroeconomic dataset: FRED-QD (McCracken and Ng 2020). DDR provides a clear-cut result: two dominant shocks drive the U.S. macroeconomy. This result holds *both* on specific frequency bands and the entire $[0, \pi]$ interval. Moreover, it holds *both* for the whole sample period and at several sub-periods. We find that two common shocks are sufficient to capture the bulk of the variance of the main macroeconomic aggregates, *both* at cyclical frequencies and in the long term. Full estimation of the model is not necessary to obtain the decomposition of the spectral density produced by the dynamic eigenvalues and the corresponding eigenvectors. This decomposition corresponds to one of the many possible identification schemes, namely the one in which the first shock maximizes the sum of the explained variances of all variables in the panel (Brillinger 1981). It turns out that this identification, although based on a statistical criterion, produces shocks that are *economically interpretable.* The first shock explains almost nothing of the long-run variance of GDP, consumption, investment, unemployment rate, and hours worked. Furthermore, it induces a positive covariance between GDP growth and inflation changes. Therefore, it has the salient features of a *demand* shock. This demand shock explains most of the *cyclical fluctuations* in GDP and other variables related to real economic activity. In addition, it explains most of the variance in inflation and the federal funds rate at the lower frequencies, including only the longer cyclical fluctuations (about four years and longer). The second shock has instead the typical traits of a *supply* shock, explaining the bulk of the *long-term* variance of the real activity variables and inducing negative covariance between GDP growth and inflation changes. The supply shock explains a minor, but not negligible, part of cyclical fluctuations of real activity variables, but it explains little about inflation and the federal funds rate at all frequencies. These results are in line with Blanchard and Quah (1989) and King et al. (1991) and in sharp contrast with the RBC model. Moreover, they confirm the finding of Angeletos, Collard, and Dellas (2020) that the transitory shock drive the bulk of business cycle fluctuations in real activity variables. On the other hand, the finding of Angeletos, Collard, and Dellas (2020) that the "Main Business Cycle" shock is disconnected from inflation is not confirmed by our results.

Exposition. The article is organized as follows. Section 2 presents the factor model setup along with our estimators of the number of shocks, our consistency result, and the required regularity assumptions. Section 3 presents a summary of the Monte Carlo exercises. Section 4 presents the empirical application based on the large-dimensional dataset of the U.S. economy. Section 5 concludes. The online supplement (Sections S.1–S.5) presents the proofs of main and auxiliary lemmas together with the proof of the Theorem (Section S.1), full details on the Monte Carlo experiments (Section S.2), and additional material related to the empirical application: some guidelines on how to calibrate the window size (Section S.3); the transformations of the observables used in the empirical application (Section S.4); the adopted variance decomposition in the frequency domain (Section S.5). The online material further includes the code to reproduce the empirical application and the simulation results.

2. Frequency-Band Estimation of the Number of Shocks

2.1. The GDFM Setup

The GDFM is a countably infinite set of observable stochastic processes x_{it} (Forni et al. 2000; Forni and Lippi 2001) with the following decomposition:

$$x_{it} = \chi_{it} + e_{it} = \lambda_{i1}(L)f_{1t} + \lambda_{i2}(L)f_{2t} + \cdots + \lambda_{iq}(L)f_{qt} + e_{it}, \quad i \in \mathbb{N},\ t \in \mathbb{Z}, \tag{2.1}$$

where $\mathbf{f}_t = (f_{1t} \cdots f_{qt})'$ is a q-dimensional orthonormal[8] unobservable vector and the impulse-response functions $\lambda_{ik}(L), i \in \mathbb{N},\ k = 1, \dots, q$, are absolutely summable functions in the lag operator L. Detailed assumptions on the *common component* χ_{it} and the *idiosyncratic component* e_{it} are given below. Let us only recall here that the *common shocks* f_{kt}, $k = 1, \dots, q$, often called *dynamic factors* in the literature and loaded through one-sided linear filters, that is, the impulse response functions (IRFs), $\lambda_{ij}(L)$, where L is the lag operator, and the e_{it} are uncorrelated

[7]Economists usually believe that demand shocks or monetary policy shocks have only transitory effects on macroeconomic variables such as GDP, consumption, investment, labor productivity, industrial production, etc. If one considers a dataset made up of variables like these (variables representing, say, real economic activity), transformed to achieve stationarity, we could then have the factor $(1-L)$, where L is the lag operator, in the IRFs of the macroeconomic variables to such demand shocks and thus have a rank reduction in the spectral density matrix of the observed variables at zero frequency.

[8]With "orthonormal" we mean that $\mathbb{E}(\mathbf{f}_t) = \mathbf{0}_q, \mathbb{V}\text{ar}(\mathbf{f}_t) = \boldsymbol{I}_q$, and $\mathbb{C}\text{ov}(\mathbf{f}_t, \mathbf{f}_s') = \mathbf{0}_{q\times q}$ for every $t \neq s$.

with the f_{jt} at any lead and lag. Moreover, the idiosyncratic components are weakly cross-correlated, uncorrelatedness (across units) being an extreme case. Weak cross-correlation essentially means that simple and weighted averages, such as $n^{-1}\sum_{i=1}^{n} e_{it}$, dissipate (in probability) when n becomes large.

A restriction that we do not impose in this article, but one often assumed in the literature is that the common components are *contemporaneous* linear combinations of $r \geq q$ unobservable variables F_{ht}, $h = 1, \ldots, r$, often called "static factors" (see Stock and Watson 2002; Bai and Ng 2002). In such a case, we say that the model admits a *static* factor representation; the dynamic nature of the model comes from the fact that the static factors have a dynamic representation in the common shocks.

The shocks and the corresponding impulse-response functions in (2.1) are not identified. This is best seen by writing the model in matrix form as

$$\mathbf{x}_{nt} = \boldsymbol{\chi}_{nt} + \mathbf{e}_{nt}, \qquad \boldsymbol{\chi}_{nt} = \Lambda_n(L)\mathbf{f}_t \tag{2.2}$$

where $\mathbf{x}_{nt} = [x_{1t} \cdots x_{nt}]'$, $\boldsymbol{\chi}_{nt} = [\chi_{1t} \cdots \chi_{nt}]'$, $\mathbf{e}_{nt} = [e_{1t} \cdots e_{nt}]'$, and $\Lambda_n(L)$ is the $n \times q$ matrix $[\lambda_{ik}(L) : i = 1, \ldots, n; k = 1, \ldots, q]$, and the prime ($'$) denotes transposition. It is easily seen that $\mathbf{x}_{nt}$ has the alternative representations

$$\mathbf{x}_{nt} = \Lambda_n(L)\mathbf{Q}\mathbf{Q}'\mathbf{f}_t + \mathbf{e}_{nt} = \Lambda_n^*(L)\mathbf{f}_t^* + \mathbf{e}_{nt}, \tag{2.3}$$

where $\Lambda_n^*(L) = \Lambda_n(L)\mathbf{Q}$ and $\mathbf{f}_t^* = \mathbf{Q}'\mathbf{f}_t$, $\mathbf{Q}$ being any orthogonal matrix (i.e., a matrix such that $\mathbf{Q}\mathbf{Q}' = \mathbf{I}_q$).

Forni et al. (2015, 2017) show that the GDFM is tightly connected to VARs because, by assuming rational IRFs, the common components and the observables in (2.2) have the VAR representations

$$\mathbf{A}_n(L)\boldsymbol{\chi}_{nt} = \mathbf{R}_n\mathbf{f}_t \qquad \mathbf{A}_n(L)\mathbf{x}_{nt} = \mathbf{R}_n\mathbf{f}_t + \boldsymbol{\eta}_{nt}, \tag{2.4}$$

where $\mathbf{A}_n(L)$ is a block-diagonal polynomial matrix of *finite* order, $\mathbf{R}_n$ is a $n \times q$ matrix, and the components $\boldsymbol{\eta}_t = \mathbf{A}_n(L)\mathbf{e}_{nt}$ are idiosyncratic. Clearly the autoregressive filter in (2.4) and the moving average filter in (2.2) are related by $\Lambda_n(L) = \mathbf{A}_n^{-1}(L)\mathbf{R}_n$. As this paper focuses on estimation of the number of common shocks, we rely on representation (2.2), whereas identification of the *structural* shocks and estimation of the corresponding IRFs can be carried out by estimating the VAR representation (2.4), along the lines of Forni et al. (2015, 2017).

2.2. Estimators of the Number of Shocks in the Frequency-Domain

Define the periodogram-smoothing estimator of the spectral density matrix of the $\mathbf{x}_{nt}$

$$\widehat{\boldsymbol{\Sigma}}_n(\omega_\ell) \equiv \frac{1}{2M_T + 1}\sum_{j=-M_T}^{M_T} \widehat{\mathcal{I}}_n\left(\omega_{\ell+j}\right) \tag{2.5}$$

at the Fourier frequency $-\pi < \omega_\ell \equiv 2\pi\ell/T \leq \pi$, assuming T even without loss of generality (see Brillinger 1981, p. 132) where, hereafter, $\{M_T\}$ is a sequence of positive integers and $\mathcal{I}_n(\cdot)$ denotes the periodogram of the $\mathbf{x}_{nt}$,

$$\widehat{\mathcal{I}}_n(\omega) \equiv \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{x}_{nt}e^{-\mathrm{i}\omega t}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{x}_{nt}'e^{\mathrm{i}\omega t}\right], \tag{2.6}$$

with i in Roman font denoting the imaginary unit. Finally, let $\hat{\mu}_{nk}(\omega)$ denote the kth eigenvalue of $\widehat{\boldsymbol{\Sigma}}_n(\omega)$ in decreasing order of magnitude, with $1 \leq k \leq n$, here denominated as the *dynamic eigenvalues*, to distinguish them from the *static eigenvalues* (i.e., the eigenvalues of the variance-covariance matrix of x_t).

To construct our estimators, we need to evaluate $\widehat{\boldsymbol{\Sigma}}_n(\cdot)$ within any given frequency band $(\omega_{\underline{\ell}}, \omega_{\bar{\ell}})$, satisfying at minimum $-\pi \leq \omega_{\underline{\ell}} \leq \omega_{\bar{\ell}} < \pi$, yielding our DDR criterion[9]

$$\mathrm{DDR}_n^T(k) \equiv \frac{\sum_{\ell=\underline{\ell}}^{\bar{\ell}}\left(\hat{\mu}_{nk}(\omega_\ell) - \hat{\mu}_{n,k+1}(\omega_\ell)\right)}{\sum_{\ell=\underline{\ell}}^{\bar{\ell}}\left(\hat{\mu}_{n,k+1}(\omega_\ell) - \hat{\mu}_{n,k+2}(\omega_\ell)\right) \vee \sum_{\ell=\underline{\ell}}^{\bar{\ell}}\hat{\mu}_{n,m}(\omega_\ell)}, \tag{2.7}$$

setting $m \equiv 2M_T + 1$. Then, our DDR estimator of the number k of common shocks is

$$\widehat{k}_{\mathrm{DDR}} \equiv \underset{1 \leq k \leq q_{\max}}{\arg\max}\, \mathrm{DDR}_n^T(k), \tag{2.8}$$

where $q_{\max}$ is an upper bound for q chosen by the researcher. To provide an intuition for the DDR estimator, let us recall an important result in GDFM theory: a q-dimensional dynamic factor structure is characterized by the behavior of the eigenvalues of the spectral density matrix $\Sigma_n(\omega)$ of the first n variables, as $n \to \infty$.[10] The q largest eigenvalues diverge, whereas the others are bounded.[11] We show that, under suitable assumptions, similar properties hold for the sample analogue of such eigenvalues, that is $\hat{\mu}_{nk}(\omega)$.

Now view the sample eigenvalues $\hat{\mu}_{nk}$ as a function of k and the piecewise linear curve, often indicated as a polyline, linking the points $(k, \hat{\mu}_{nk})$ (dotted line, Figure 1). DDR is the ratio of the slopes of two adjacent segments; it measures the curvature of the polyline at $k + 1$. The larger DDR is, the smaller the angle above the polyline is at $k+1$. By maximizing DDR we identify the point $k + 1$ where the steep descent ends and the slight slope begins. The basic idea is that when $k = q$, the descent must be steep, because $\hat{\mu}_{nq}$ is large whereas $\hat{\mu}_{nq+1}$ is small. On the other hand, at $k = q + 1$ the slope becomes small because it is the difference between small eigenvalues.

Notice that the computation of DDR does not require discretionary preliminary choices of tuning parameters by the researcher, apart from the size of the spectral window (the bandwidth) used in the estimation of the spectral density matrix (this cannot be avoided when using nonparametric estimators). This is an advantage over both HL and O, which, besides the window size, require making additional choices, that may influence the results.[12]

[9] We correct the denominator of the DDR criterion by taking, for each k, the maximum between $\sum_{\ell=\underline{\ell}}^{\bar{\ell}}\left(\hat{\mu}_{n,k+1}(\omega_\ell) - \hat{\mu}_{n,k+2}(\omega_\ell)\right)$ and $\sum_{\ell=\underline{\ell}}^{\bar{\ell}}\hat{\mu}_{n,m}(\omega_\ell)$, to avoid the denominator being very close to zero. Notice that we allow the frequency band to reduce to a single frequency when $\omega_{\underline{\ell}} = \omega_{\bar{\ell}}$.

[10] See Forni et al. (2000) and Forni and Lippi (2001).

[11] Similarly, the existence of a *static* factor representation is linked to the behavior of the eigenvalues of the *covariance matrix* of the first n variables (Chamberlain and Rothschild 1983), and not of their spectral density matrix.

[12] More specifically, HL requires choosing the functional form for the penalty term and a grid $n_j, T_j, j = 1, \ldots, J$, which is needed for its calibration. O is a sequential-testing procedure, exhibiting a nonstandard distribution, and

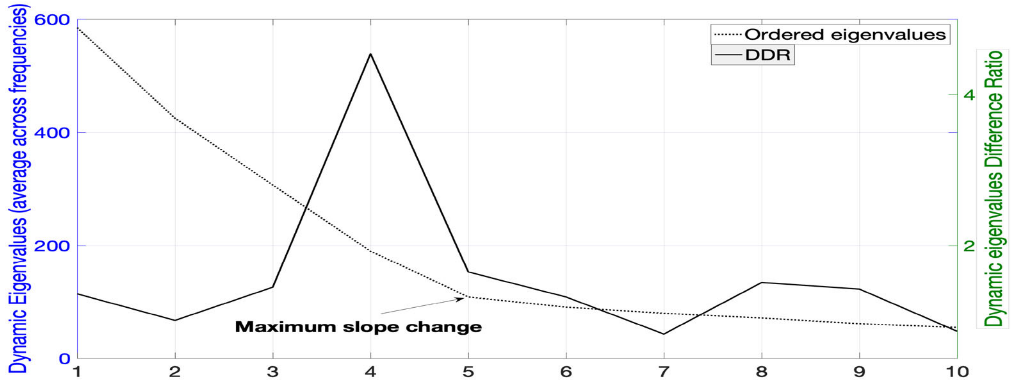

Figure 1. The dotted line is the plot of $\hat{\mu}_{nk}$ (left y axis) as a function of k (x axis). The solid line is DDR(k) (right y axis). Estimates are obtained from data generated with the ARMA specification of the third experiment in Section S.2.1 of the online supplement, $(n, T) = (100, 200)$, $q = 4$. $\text{DDR}_n^T(k)$ reaches its maximum in k^* when the maximum slope change of the dotted line is in $k^* + 1$.

We analyze two additional criteria that represent the dynamic equivalents of the criteria AH propose to estimate the number of static factors within a static factor model (i.e., the Eigenvalue Ratio and the Growth Ratio criteria). In analogy with these denominations, we call these new estimators the Dynamic Eigenvalue Ratio (DER) and the Dynamic Growth Ratio (DGR) estimators, defined as follows. Set

$$\text{DER}_n^T(k) \equiv \frac{\sum_{\ell=\underline{\ell}}^{\bar{\ell}} \hat{\mu}_{nk}(\omega_\ell)}{\sum_{\ell=\underline{\ell}}^{\bar{\ell}} \hat{\mu}_{n,k+1}(\omega_\ell)},$$

$$\text{DGR}_n^T(k) \equiv \frac{\ln\left[V_n^T(k-1)/V_n^T(k)\right]}{\ln\left[V_n^T(k)/V_n^T(k+1)\right]} = \frac{\ln(1+\hat{\mu}_{nk}^*)}{\ln(1+\hat{\mu}_{n,k+1}^*)},$$

where $V_n^T(k) \equiv L^{-1}\sum_{l=k+1}^{n}\sum_{\ell=\underline{\ell}}^{\bar{\ell}} \hat{\mu}_{nl}(\omega_\ell)$ and $\hat{\mu}_{nk}^* \equiv L^{-1}\sum_{\ell=\underline{\ell}}^{\bar{\ell}} \hat{\mu}_{nk}(\omega_\ell)/V_n^T(k)$. The corresponding estimators are

$$\widehat{k}_{\text{DER}} \equiv \underset{1\le k\le q_{\max}}{\arg\max}\ \text{DER}_n^T(k), \qquad (2.9)$$

$$\widehat{k}_{\text{DGR}} \equiv \underset{1\le k\le q_{\max}}{\arg\max}\ \text{DGR}_n^T(k)., \qquad (2.10)$$

2.3. Assumptions

We now state the following Assumptions for the model introduced in Section 2.1, required by the consistency result of Theorem 1.[13]

Let $\mathbf{A}$ be a $(n \times m)$ matrix, with complex or real entries a_{ij}; in short $\mathbf{A} = [a_{ij} : 1 \le i \le n, 1 \le j \le m]$. When $\mathbf{A}$ is Hermitian, $\mu_1(\mathbf{A}) \ge \mu_2(\mathbf{A}) \ge \cdots \ge \mu_n(\mathbf{A})$ denote the (real) eigenvalues of $\mathbf{A}$ in a decreasing order. The jth largest singular value of the matrix $\mathbf{A}$, $\sigma_j(\mathbf{A})$, is defined as $\sqrt{\mu_j(\mathbf{A}'\,\mathbf{A})}$. The prime attached to a complex-valued matrix denotes the conjugate-complex transpose of the matrix, or the transpose when the matrix has real entries. The spectral norm $\|\,\mathbf{A}\,\| = \sqrt{\mu_1(\mathbf{A}'\,\mathbf{A})}$ is the largest singular value of $\mathbf{A}$, whereas $\|\boldsymbol{a}\|$ is the Euclidean norm of the vector $\boldsymbol{a}$. We denote by $\mathbf{B} \equiv \text{diag}\{b_{11}, \ldots b_{nn}\}$ a $(n \times n)$ diagonal matrix. We use $a \vee b$ and $a \wedge b$ to denote the maximum and the minimum of a and b, respectively, for any scalars a and b.

For the double-indexed process $\{y_{it} : 1 \le i \le n, 1 \le t \le T\}$, we write $\mathbf{y}_{nt} \equiv [y_{1t}, \ldots, y_{nt}]'$. The discrete Fourier transform (DFT) of $\{\mathbf{y}_{n1}, \ldots, \mathbf{y}_{nT}\}$ at frequency ω_j is defined by $\widehat{\mathbf{y}}_{nj} \equiv T^{-1/2}\sum_{t=1}^{T} \mathbf{y}_t(n)e^{-\mathrm{i}\omega_j t}$ where $\omega_j = 2\pi j/T$ and T is assumed to be even without loss of generality.

Assumption 1. The process $\mathbf{f}_t$ is an orthonormal white noise.

Assumption 2. The coefficients of the filter $\Lambda_n(L) \equiv \sum_{u=0}^{\infty} \Lambda_n^{(u)} L^u$ are such that $\sum_{u=0}^{\infty} \|\Lambda_n^{(u)}\|(1+u) = O(n^{1/2})$.

Assumption 3. Let ε_{jt}, $j \in \mathbb{N}$, $t \in \mathbb{Z}$, be independent an identically distributed (iid) zero-mean random variables with unit variance and $\mathbb{E}\,|\varepsilon_{jt}|^4 < \infty$. Further, let $\mathbf{u}_{nt} = [u_{1t}, \ldots, u_{nt}]'$, with $u_{jt} \equiv C_j(L)\varepsilon_{jt} = \sum_{k=0}^{\infty} c_{jk}\varepsilon_{j,t-k}$ where $\sup_{j>0}\sum_{k=0}^{\infty} k|c_{jk}| < \infty$ and $\min_j |C_j(z)| > 0$ for $|z| \le 1$, with $C_j(z) = \sum_{k=0}^{\infty} c_{jk}z^k$ for $z \in \mathbb{C}$. We assume that the n-vector of the idiosyncratic components of the data is such that $\mathbf{e}_{nt} = \mathbf{A}_n\,\mathbf{u}_{nt}$,, where $\mathbf{A}_n$ is a nonsingular lower triangular matrix with $\|\,\mathbf{A}_n\,\|$ uniformly bounded in n.

Assumption 4. $\mathbb{C}\text{ov}(f_{jt}, e_{is}) = 0$ for any i, j, t, s.

Assumption 5. The bandwidth parameter M_T satisfies $n > (2M_T + 1)$, and is a function of T such that $1/M_T \to 0$ and $M_T^2/T = O(1)$ as $T \to \infty$. Moreover, $n/T \to y \in (0, \infty)$ as $n \to \infty$ and $T \to \infty$.

For a generic frequency ω_ℓ, $\underline{\ell} \le \ell \le \bar{\ell}$, we consider the index set

$$S(\ell) \equiv \{(\ell - M_T), (\ell - M_T + 1), \ldots, \ell, (\ell+1), \ldots (\ell + M_T)\} \qquad (2.11)$$

The cardinality of $S(\ell)$ is denoted by $m \equiv 2M_T + 1$. Let s_ℓ be a generic element of $S(\ell)$, $\underline{s}_\ell \equiv \ell - M_T$ and $\bar{s}_\ell \equiv \ell + M_T$. To simplify the notation, in the following, we will suppress the dependence on ℓ by setting $s \equiv s_\ell$ and $S \equiv S(\ell)$. Let $\widehat{\mathbf{F}} \equiv$

therefore it requires choosing the significance level; furthermore, it requires choosing a grid of frequencies and a criterion for weighting the potentially different results obtained for different frequencies. At any rate, both HL and O do provide concrete recommendations on how to implement their procedures.

[13]The proof of Theorem 1 is reported in the online supplement, together with several technical lemmas.

$\left[\hat{\mathbf{f}}_{\underline{s}}, \hat{\mathbf{f}}_{\underline{s}+1}, \ldots, \hat{\mathbf{f}}_{\bar{s}}\right]$, with $\hat{\mathbf{f}}_s \equiv \hat{\mathbf{f}}_{qs} = T^{-1/2}\sum_{t=1}^{T}\mathbf{f}_t e^{-i\omega_s t}$, and $\widehat{\mathbf{\Lambda}}_n(s) \equiv \mathbf{\Lambda}_n\left(e^{-i\omega_s}\right) = \sum_{u=1}^{\infty}\mathbf{\Lambda}_n^{(u)} e^{-i\omega_s u}$, for any $s \in S$.

Assumption 6. (i) For any $\underline{\ell} \leq \ell \leq \bar{\ell}$, let $\tilde{\nu}_{nk}^{\chi}(\omega_\ell) \equiv \mu_k\left(\widehat{\mathbf{\Lambda}}_n(\ell)'\widehat{\mathbf{\Lambda}}_n(\ell)/n)(\widehat{\mathbf{F}}\widehat{\mathbf{F}}'/m)\right)$, for $k = 1, \ldots, q$. Then, for each k, $\nu_k^{\chi}(\omega_\ell) \equiv \text{plim}_{n,T\to\infty}\tilde{\nu}_{nk}^{\chi}(\omega_\ell)$ and $0 < \nu_k^{\chi}(\omega_\ell) < \infty$. (ii) $\nu_k^{\chi}(\omega_\ell) > \nu_{k+1}^{\chi}(\omega_\ell)$, for all $\ell \in [\underline{\ell}, \bar{\ell}]$, and $k = 1, \ldots, q-1$. (iii) q is finite.

The first and the fourth assumptions are standard in the literature. Assumptions 2 and 3 are similar to Assumption 2(i) and 2(ii)(a) in O. They imply regularity of the DFT's of $\boldsymbol{\chi}_{nt}$ and $\mathbf{e}_{nt}$ local to the frequency of interest ω_ℓ.

Assumption 2 entails that, for n, T large, $n^{-1/2}\widehat{\boldsymbol{\chi}}_{ns}$ is well approximated by $n^{-1/2}\widehat{\mathbf{\Lambda}}_n(\ell)\hat{\mathbf{f}}_s$ (see Lemma S.1). This implies that Assumption 6 could be restated in terms of the eigenvalues of $\widehat{\boldsymbol{\chi}}'\widehat{\boldsymbol{\chi}}/n$, with $\widehat{\boldsymbol{\chi}} = [\widehat{\boldsymbol{\chi}}_{n\underline{s}}, \ldots, \widehat{\boldsymbol{\chi}}_{n\bar{s}}]$. We find the current formulation more intuitive and easily comparable with Assumption A in AH.[14] Moreover, it suggests that $\widehat{\mathbf{X}}\widehat{\mathbf{X}}'/n$, with $\widehat{\mathbf{X}} = [\hat{\mathbf{x}}_{n\underline{s}}, \ldots, \hat{\mathbf{x}}_{n\bar{s}}]$, can be *interpreted* as a sample covariance matrix, and the q dynamic factors in the data correspond to q static factors in the $\widehat{\mathbf{X}}$. We show that largest q eigenvalues of $\widehat{\mathbf{X}}\widehat{\mathbf{X}}'/n$ diverge, as $n, T \to \infty$, and the remaining $(2M_T+1)-q$ largest eigenvalues staying bounded and bounded away from zero.

For any $s \in S$, $\widehat{\mathbf{E}} \equiv \left[\hat{\mathbf{e}}_{n\underline{s}}, \hat{\mathbf{e}}_{n\underline{s}+1}, \ldots, \hat{\mathbf{e}}_{n\bar{s}}\right]$, with $\hat{\mathbf{e}}_{ns} \equiv T^{-1/2}\sum_{t=1}^{T}\mathbf{e}_t e^{-i\omega_s t}$ and define $\tilde{\mathbf{E}} \equiv \mathbf{A}_n\widehat{\mathbf{C}}_n(\ell)\widehat{\boldsymbol{\mathcal{E}}}$, where $\widehat{\mathbf{C}}_n(s) \equiv \text{diag}\left(\sum_{u=0}^{\infty} c_{1u}e^{-i\omega_s u}, \ldots, \sum_{u=0}^{\infty} c_{nu}e^{-i\omega_s u}\right)$ is a $n \times n$ diagonal matrix and $\widehat{\boldsymbol{\mathcal{E}}} \equiv [\hat{\boldsymbol{\varepsilon}}_{ns}, \hat{\boldsymbol{\varepsilon}}_{n,\underline{s}+1}, \ldots, \hat{\boldsymbol{\varepsilon}}_{n\bar{s}}]$ is a $n \times m$ matrix. Assumption 3, which can be seen as the dynamic counterpart of Assumption C in AH, guarantees that, for n, T large $n^{-1/2}\widehat{\mathbf{E}}$ is well approximated by $n^{-1/2}\tilde{\mathbf{E}}$ (see Lemma S.3). We show that the first q eigenvalues of $\tilde{\mathbf{E}}'\tilde{\mathbf{E}}/n$ are bounded and bounded away from zero (see Lemma S.4). The latter result is obtained by using results from the large dimensional random matrix theory (see Lemma S.17), requiring that $n/T \to y$, as postulated in Assumption 5 (see also Assumption C(i) in AH).

Assumption 6 implies that the $\mathbf{f}_t$ are *strong* dynamic factors, as opposed to the case of weak (i.e., the eigenvalues of $\mathbf{\Sigma}_n(\omega_\ell)$ are uniformly bounded) and semi-strong ((i.e., the eigenvalues of $\mathbf{\Sigma}_n(\omega_\ell)$ are $o(n)$). The case of weak factors is attracting attention in the literature (see, e.g., Onatski 2012; Lettau and Pelger 2020; Gersing, Rust, and Deistler 2023 among others). We do not consider the case of weak factors here.

2.4. Main Result

We are now ready to state our main theoretical result.

Theorem 1. Suppose that Assumptions (1)–(6) hold for some $q \geq 1$. Then,

$$\lim_{n,T\to\infty} \Pr\left(\widehat{k}_{\text{DER}} = q\right) = 1, \tag{2.12}$$

$$\lim_{n,T\to\infty} \Pr\left(\widehat{k}_{\text{DGR}} = q\right) = 1, \tag{2.13}$$

$$\lim_{n,T\to\infty} \Pr\left(\widehat{k}_{\text{DDR}} = q\right) = 1. \tag{2.14}$$

for any $q_{\max} \in (q, 2M_T - q - 1]$.

Proof. See the online supplement, Section S.1.5. □

3. Monte Carlo Simulations

In this section, we provide a summary of the thoughtful Monte Carlo experiments described in Section S.2 of the online supplement in detail, to which we refer, including tables reporting the numerical results.

3.1. Simulations: Econometric Models

We evaluate the finite-sample performance of our estimators, DDR, DER, and DGR, and we compare them with that of HL and O. We also consider the eigenvalue ratio (ER) and growth ratio (GR) estimators proposed by AH to show that estimators designed for the static model do not work properly within a dynamic setting. We do not consider the methods proposed by Bai and Ng (2007) and Amengual and Watson (2007), in part because of space constraints and assume a factor model that can be written in the static form, which is a restriction that we do not impose here, and in part because they are analyzed in HL and O.

We consider several data-generating processes (DGPs) to demonstrate the validity of our methodology. We run four experiments. In the first three, we evaluate the competing estimators on the whole interval $[0, \pi]$. The fourth experiment is devoted to checking the ability of DDR to detect a rank reduction in the spectral density matrix at specific frequencies of interest (see below).

Experiments one and two are based on DGPs already proposed in the econometric literature, namely in HL and in O. Experiment three is our own DGP. In experiment four we focus on DDR. The DGP is such that the spectral density matrix of the variables has reduced rank at frequency zero. One of the shocks is a supply shock and has permanent effects on several variables. The other is a demand shock having transitory effects on all variables, which are supposed to be real activity variables. DDR should detect two shocks when evaluated on large frequency bands and just one shock when evaluated at frequency zero.

To compute DER, DGR, and DDR, we use the periodogram smoothing estimator (2.5) with the bandwidth parameter $M_T = [0.75\sqrt{T}]$, and we take the average of the eigenvalues evaluated in the frequency grid $\omega_\ell = 2\pi\ell/T$, $\ell = -\tau, \ldots, \tau$, $\tau = \lfloor (T-1)/2 \rfloor$.[15] Concerning HL, we use the log information criterion $IC_{2;n}^{T}$ with penalty $p_1(n, T)$ and the Bartlett lag window with

[14] From a technical standpoint, the spectral techniques used here are significantly different from those used in the static framework of Ahn and Horenstein (2013). A key step here achieved is to bound the smaller and largest eigenvalues of the estimated spectral density matrix of the idiosyncratic component deploying results of the random matrix literature, whilst imposing realistic restrictions on the relative speed of divergence of n and T.

[15] Although the results are fairly robust to the choice of the M_T, we explain in detail how we select the bandwidth for the empirical application (see Section S.3 of the online supplement).

parameter $M_T = [0.75\sqrt{T}]$, which yields the best performance in the authors' simulations. The method requires evaluation of the loss function over a grid $n_j, T_j, j = 1, \ldots, J$; we stick to the one proposed by the authors (i.e., $n_j = n - 10j$, $T_j = T - 10j$, $j = 0, 1, 2, 3$). When dealing with O, we use the procedure described in sec. 5.3 of O. We find that the results are sensitive to the choice of the bandwidth m. For the second experiment, we stick to O's choice, which is very effective ($m = 30$ for $(n, T) = (70, 70)$, $m = 40$ for $(n, T) = (100, 120)$, $m = 65$ for $(n, T) = (150, 500)$). For the first DGP, we use $m = 15$; for the third experiment, we use $m = 15, 20, 30$ for $T = 80, 240, 480$, respectively. These values produce better results than the larger ones suggested in O. For all the experiments and all estimators, we set $q_{\max} = 8$ and we generate 500 artificial datasets. We evaluate the results in terms of the percentage of times we find the correct number of shocks.

The results are the following. For experiment one, DDR and HL perform similarly and dominate the other estimators. For experiment two, with MA loadings, O ranks first among the estimators for dynamic factors, and does slightly better than DDR; with AR loadings, DDR dominates all other estimators. Somewhat surprisingly, both the ER and the GR estimators show excellent performance in the second experiment.[16] The comparison with the results in the first experiment (MA loadings) and the third experiment suggests that the AH's estimator performs well in a dynamic framework only if the coefficients of the filter $\lambda_{ik}(L)$ are mildly heterogeneous across "i". For experiment three, in the version with small idiosyncratic components, DDR and DGR perform similarly and dominate the other estimators; in the version with large idiosyncratic components, DDR outperforms the other estimators. Finally, in experiment four, we show that with DDR, when evaluated at a single frequency (or small frequency bands around this frequency) in which the spectral density matrix of the variables has reduced rank $q - 1$, the number of shocks at this frequency is $q-1$. Overall, we conclude that DDR is an excellent alternative to existing criteria in finite samples across different DGPs.

3.2. Simulations: Macroeconomic Models (DSGE)

Strong analogies exist between certain DSGE models and the GDFM, particularly for the DSGE models of Angeletos, Collard, and Dellas (2018) (ACD henceforth), of Justiniano, Primiceri, and Tambalotti (2010) (JPT henceforth) and Bouakez, Cardia, and Ruge-Murcia (2014) (BCR henceforth), which is a multi-sectoral DSGE model studied in Onatski and Ruge-Murcia (2013). In fact, these DSGE models and GDFMs admit a finite-order VAR representation with a singular covariance matrix of the error term, implied when the number of shocks is smaller than the number n of observables.[17] Given this commonality, we explore how our methodology can dissect (by frequency) the number of structural shocks driving DSGE models using a Monte Carlo experiment.

The aim of this experiment is twofold: test our estimators with a DGP from a macroeconomic model, and verify whether our criteria capture a relatively large value of q. The ACD model has $q = 8$ shocks and $n_1 = 10$ variables; the JPT has $q = 7$ shocks and $n_1 = 11$ variables. In contrast, the ORM has only $q = 3$ shocks although the challenge here is related to the (low) degree of heterogeneity of the loadings to the common shocks. This exercise is developed in a Monte Carlo setting, with the DGPs for these variables reported in Section S.2.3 of the online supplement.

We evaluate the results in terms of the percentage of times we find the correct number of shocks.

Regarding ACD and JPT models, for almost all configurations DGR and DDR dominate DER, with no clear ranking between the DGR and DDR. When measurement errors are relatively small (1/11 of total variance, about 9%) both criteria capture the correct number of shocks in most cases, even if there are seven or eight shocks. With $T = 240$ both criteria detect the correct number of shocks even when the errors are large (1/6 of total variance, about 17%). The BCR case differs because, despite being characterized by a small q ($q = 3$), estimating the correct number of shocks remains challenging for the low degree of heterogeneity in the corresponding loadings (see also (Onatski and Ruge-Murcia 2013, sec 5.)); DDR comes closer to the correct result than DGR and DER, detecting 2 factors in most replications.

Summarizing, DDR and DGR detect correctly the number of shocks driving DSGE models even if the number of shocks is relatively large (seven and eight in the ACD and JPT models, respectively) when compared with the cross-sectional size.

4. Empirical Application: Dissecting the U.S. Economy

How many shocks drive the economy? And, more specifically, how many shocks affect the macroeconomic variables in the long run, and how many, instead, are the ones driving the business cycle? Finally, do demand or supply shocks dominate the economy?

The scant available evidence on the number of shocks is mixed. Sargent and Sims (1977), using a small-dimensional dynamic factor model, find that two shocks fit U.S. macroeconomic data reasonably well. Giannone, Reichlin, and Sala (2005) argue informally in favor of two shocks, based on the explained variances of the principal-component series of a large factor model. Bai and Ng (2007), using their information criteria, find seven static factors and four shocks. On the other hand, Stock and Watson (2005), using Bai and Ng (2007) criteria, finds seven shocks, suggesting lack of robustness of Bai and Ng method with actual data. Amengual and Watson (2007) find seven static factors and seven shocks. HL fails to find a clear-cut result: four shocks, but perhaps only one. Forni et al. (2009) and Forni and Gambetti (2021), by using HL, find a range of shocks between 2 and 5, confirming that the method may produce ambiguous results.[18]

[16]Indeed, with MA loadings, the model has 6 static factors, but both estimators detect just two factors.

[17]For the DSGE models this is given by their reduced form, obtained by the standard practice of log-linearizing the models around their steady state (see (S.35)–(S.36) and (S.37)–(S.38) in Section S.2.3 of the online supplement whereas for GDFMs the VAR representation (2.4) holds under suitable assumptions on the IRFs (see Forni et al. 2015).

[18]Hallin and Liška (2007) and Onatski (2009) demonstrate the fragility of Bai and Ng (2007) with several Monte Carlo experiments for a variety of data-

For our empirical analysis, we use the U.S. quarterly macroeconomic dataset recently developed by McCracken and Ng (2020) (FRED-QD). Of this dataset, we consider the $n = 216$ series starting from the first quarter of 1960. The final date of the sample is the first quarter of 2020. As for the transformations, we deviate from those McCracken and Ng (2020) suggest for the interest rates, which we take in levels rather than in differences; furthermore, we take prices and other nominal variables in log-differences, rather than in double differences of the logs. The reason is that we want to avoid a possible over-differentiation, which enhances high frequencies, of little interest for macroeconomic analysis. The complete list of variables and transformations is in Section S.4 of the online supplement. After the transformations, the number of observations over time is $T = 240$. Section S.3 explains how to calibrate the window size M_T required to compute our estimators. For robustness, we also use the FRED-MD monthly dataset (see McCracken and Ng 2016). Precisely, we use the 122 series starting from 1960M1. The final date is 2020M3. The treatment is the same as used for the quarterly dataset.

We find a clear-cut result: *two* common shocks drive the U.S. economy. These two shocks explain the bulk of the variance in the main macroeconomic variables, both at business-cycle frequencies and in the long term. We also find that the first dynamic factor behaves like a demand shock and the second one like a supply shock.

Finding (i): The Number of Shocks Driving the U.S. Economy. We consider the entire quarterly sample and nine sub-samples: the five 40-year sub-samples 1960Q2-2000Q1, 1965Q2-2005Q1, 1970Q2-2010Q1, 1975Q2-2015Q1, and 1980Q2-2020Q1, and the four 30-year sub-samples 1960Q2-1990Q1, 1970Q2-2000Q1, 1980Q2-2010Q1, and 1990Q2-2020Q1. To estimate q, we compute DDR on three frequency bands: the entire $[0, \pi]$ band, the $[0, 2\pi/6]$ band, excluding fluctuations of less than 18 months, which are of little interest for macroeconomic analysis, and the $[2\pi/32,\ 2\pi/6]$ cyclical band, which includes waves ranging from 18 months to eight years. The DDR estimator for the $[0, 2\pi/6]$ band is named DDRa; the one relating to the cyclical band is named DDRbc. For comparison, we consider also DER, DGR, HL, and O (only on the whole $[0, \pi]$ band).[19] For DER, DGR, and DDR we set the bandwidth parameter $M_T = [0.75\sqrt{T}]$ for the whole sample and $M_T = [\sqrt{T}]$ for all subsamples, according to the results of the simulation exercise reported in Section S.3. For all estimators, we set $q_{\max} = 8$.

Results are reported in Table 1. Our main estimator DDR selects *two* shocks; this finding is reasonably consistent across sub-samples. The same holds for DDRa, which excludes the short-run frequencies, and DDRbc. DER and DGR behave similarly, except for the sub-samples 1960Q2-2000Q1 and 1965Q2-2005Q1, for which they select one dynamic factor only. By contrast, HL selects five factors, whereas O is in favor of three factors. As for the sub-samples, HL oscillates between two and four factors, with a prevalence of four factors, and O varies between two and five, with a prevalence of two factors. Overall, DDR, DDRa, and DDRbc are more parsimonious than HL and O, and more consistent across sub-samples.

Table 1. Number of factors detected by the competing criteria for the whole sample and nine sub-samples (quarterly data) and the FRED-MD sample (monthly data).

Sample	DDR	DDRa	DDRbc	DER	DGR	HL	O
FRED-QD							
1960Q2-2020Q1	2	2	2	2	2	5	3
1960Q2-2000Q1	2	2	2	1	1	4	2
1965Q2-2005Q1	2	2	2	1	1	4	2
1970Q2-2010Q1	2	2	2	2	2	4	2
1975Q2-2015Q1	2	2	2	2	2	4	5
1980Q2-2020Q1	2	2	2	2	2	4	2
1960Q2-1990Q1	1	2	2	1	1	3	5
1970Q2-2000Q1	1	1	1	1	1	4	4
1980Q2-2010Q1	2	3	3	2	2	2	2
1990Q2-2020Q1	2	2	2	2	2	3	3
FRED-MD							
1960M1-2020M3	2	2	2	2	2	2	4

NOTE: DDR: Dynamic Difference Ratio Estimator. DDRa: Dynamic Difference Ratio Estimator evaluated on the $[0, 2\pi/6]$ frequency band. DDRbc: Dynamic Difference Ratio Estimator evaluated on the cyclical band $[2\pi/32, 2\pi/6]$. DER: Dynamic Eigenvalue Ratio estimator. DGR: Dynamic Growth Ratio estimator. HL: Hallin and Liška (2007) estimator. O: Onatski (2009) estimator.

The last row of Table 1 reports results for the FRED-MD monthly dataset.[20] All criteria point to two factors, except for O, which selects 4 factors. Hence, DDR, DER, and DGR produce the same estimate for the quarterly and the monthly datasets, whereas HL and O do not.

Why do DDR and HL differ so much? One possible explanation is that there are two large factors and three smaller factors and that the latter are elusive to DDR but not to HL. This interpretation contrasts, however, with the simulation of Table S.1, reported in the online supplement, Section S.2, where there are factors of different variance and DDR has a performance similar to that of HL.

For the subsequent empirical analysis, we focus solely on our main estimator, DDR. Figure 2 shows the number of estimated factors by DDR for each single Fourier frequency in the interval $[0, \pi/4]$ and each one of the frequency bands $[0, 2\pi/80]$ (long-run), $[2\pi/80, 2\pi/32]$ (long cycles), $[2\pi/32, 2\pi/6]$ (business cycle), and $[2\pi/6, \pi]$ (short-run). At frequency-zero DDR selects two factors; the estimator is somewhat unstable when evaluated at single frequencies, fluctuating between one and four factors. Its value, however, stabilizes at two when averaging the eigenvalues on the four bands above.

Figure 3 shows the number of factors estimated with DDR, DDRa, and DDRbc with a rolling sample of 40 years starting in 1960Q2, 1960Q3,…,1980Q2. We see that the number of estimated factors is reasonably stable across time, especially for DDRa and DDRbc.

We conclude that two major shocks are driving the U.S. economy. This is true both for the whole band $[0, \pi]$ and at business cycle frequencies. The finding of two shocks is in line

generating processes, and Stock and Watson (2005) echo a similar finding in empirical applications.

[19] HL is calculated as in the simulations, with $M_T = [0.75\sqrt{T}]$ (see the online supplement, Section S.2.1 for details). For O, we use $m = 20$ for the whole sample and for the 40-year sub-samples, and we use $m = 15$ for the 30-year sub-samples.

[20] With monthly data the cyclical band is defined as the interval $[2\pi/96,\ 2\pi/18]$ and the band excluding high frequencies, that is, the long-run frequencies, is defined as the interval $[0, 2\pi/18]$. For HL, DER, DGR, and DDR the bandwidth parameter is $M_T = [0.75\sqrt{T}]$. For O, we use $m = 20$.

Table 2. Percentage of variance explained by the first two dynamic factors and the following three for a few selected variables, by frequency band.

	PC series	All freq.	Long run > 20 years	Long cycles 8–20 years	Medium cycles 4–8 years	Short cycles 1.5–4 years	Bus. cycle 1.5–8 years
GDP	first 2	73.1	84.6	85.7	84.5	82.3	83.1
	next 3	16.4	8.1	5.9	7.1	9.3	8.6
Cons.	first 2	56.8	70.9	73.8	72.6	66.7	68.7
	next 3	16.6	15.1	13.3	14.9	11.1	12.9
Inv.	first 2	73.1	74.7	78.6	87.1	85.3	85.7
	next 3	8.8	16.7	11.2	4.6	3.0	3.7
U rate	ffirst 2	80.7	80.3	79.1	86.7	91.7	89.7
	next 3	8.5	14.0	15.4	8.9	2.3	4.7
Hours	first 2	77.3	84.7	85.3	91.4	91.2	91.1
	next 3	8.6	10.0	10.3	5.5	2.0	3.4
Inflation	first 2	84.5	91.6	91.1	91.3	63.2	86.0
	next 3	7.9	5.7	6.2	6.1	18.3	8.5
FFR	first 2	78.6	78.9	79.2	81.8	66.5	79.7
	next 3	16.1	18.1	17.2	13.6	20.1	14.1

NOTE: All frequencies: $[0, \pi]$; Long run: $[0, 2\pi/80)$; Long cycles: $[2\pi/80, 2\pi/32)$; Medium cycles: $[2\pi/32, 2\pi/16)$; Short cycles: $[2\pi/16, 2\pi/6)$; Business cycle: $[2\pi/32, 2\pi/6)$.

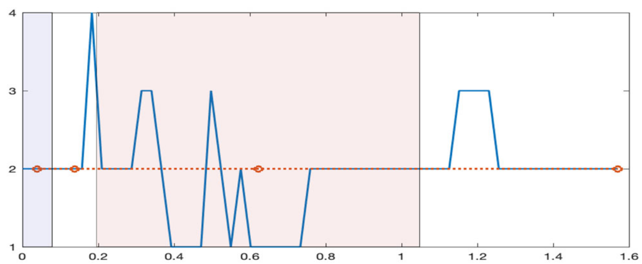

Figure 2. Number of factors estimated by DDR when evaluated at each frequency (blue solid line) and on four frequency bands (red dotted line with circles).

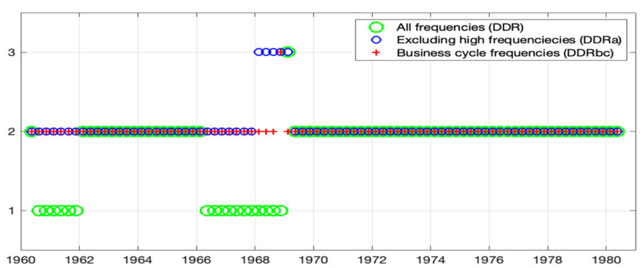

Figure 3. Number of factors estimated by DDR, DDRa, and DDRbc with a rolling sample of 40 years from 1960Q2 to 1980Q2.

with Sargent and Sims (1977), Giannone, Reichlin, and Sala (2005), and O, as cited above.

Finding (ii): Common Variance. In this paragraph, we use variance decomposition in the frequency domain, outlined in Section S.5 of the online supplement, to estimate, for seven key macroeconomic variables, the variance explained by the first and second dynamic factors along with the variance explained by the subsequent three factors (from the third to the fifth factor). The aim is to quantify the size of the explained variance for these variables when retaining only two factors, and the size of the variance that is not accounted for when $q = 5$ suggested by HL. Table 2 shows the result. From the table, one can see that on the bands of macroeconomic interest, namely the long run, the long waves, and the business cycle, two factors capture about 85% of GDP growth fluctuations, 70% of consumption, 75%–85% of investment, 80%–90% of the unemployment rate variation, 85%–90% of hours worked, 85%–90% of inflation, and 80% of the federal funds rate. The variance not accounted for by selecting $q = 2$ instead of $q = 5$ is not negligible, particularly for consumption and the interest rate. Nevertheless, we conclude that two factors are enough to capture the bulk of the variance in the main macroeconomic aggregates on the frequency bands of the main macroeconomic interest.

Finding (iii): Cyclical Shock versus Long-Run Shock. We now investigate how much the first and second shocks explain the variance in the above variables for each frequency band, based on the variance decomposition in the frequency domain, outlined in Section S.5 of the online supplement.

The explained variances are shown in Table 3. The first shock accounts for almost nothing of the variance of GDP in the long run and, more generally, over the long cycles band, which is instead explained by the second shock. The same result applies to all real activity variables. We, therefore, find ourselves, without having imposed it, in front of an identification *à la* Blanchard and Quah (1989): the first factor is a *transitory* shock, while the second one is a *permanent* shock. It is very much tempting to interpret the transitory shock as a *demand shock* and the long-run shock as a *supply shock*. To confirm this interpretation, we look at the covariances of GDP growth and inflation changes induced by the two shocks and find that, in fact, such covariance is *positive* for the transitory shock, which therefore has the features of a demand shock, and it is *negative* for the permanent shock, which can then be regarded as a supply shock (see the lower-right panel of Figure 4).[21]

In the last column of the table, we report the explained variances at business cycle frequencies. The demand shock is the most important cyclical shock for real activity variables. It accounts for 62% of GDP growth fluctuations, 42% of consumption, 63% of investment, 71% of unemployment, and 68% of

[21]The variance decomposition reported in Table 3 is reminiscent of results presented by Mark Watson in his comment on Giannone, Reichlin, and Sala (2005). Watson showed similar findings for Sargent and Sims (1977) and Giannone, Reichlin, and Sala (2005).

Table 3. Percentage of variance explained by the first and second factor for a few selected variables, by frequency band.

	Shocks	All freq.	Long run > 20 years	Long cycles 8–20 years	Medium cycles 4–8 years	Short cycles 1.5–4 years	Bus. cycle 1.5–8 years
GDP	1st	47.2	1.4	3.3	34.1	77.4	62.2
	2nd	25.9	83.2	82.5	50.3	4.9	20.8
Cons.	1st	29.5	1.2	1.9	24.8	57.0	42.3
	2nd	27.3	69.7	71.9	47.8	9.7	26.4
Inv.	1st	43.2	4.0	5.2	35.7	79.4	63.1
	2nd	29.9	70.7	73.3	51.4	5.9	22.6
U rate	1st	51.8	9.9	11.8	40.6	87.6	70.8
	2nd	28.9	70.4	67.3	46.1	4.1	18.9
Hours	1st	44.2	0.8	4.8	37.3	88.3	68.5
	2nd	33.1	83.9	80.4	54.0	3.0	22.6
Inflation	1st	78.3	91.3	89.7	81.8	30.9	72.8
	2nd	6.2	0.3	1.4	9.5	32.3	13.2
FFR	1st	75.2	78.7	78.6	76.4	52.7	72.9
	2nd	3.4	0.2	0.6	5.4	13.8	6.8

NOTE: All frequencies: $[0, \pi]$; Long run: $[0, 2\pi/80)$; Long cycles: $[2\pi/80, 2\pi/32)$; Medium cycles: $[2\pi/32, 2\pi/16)$; Short cycles: $[2\pi/16, 2\pi/6)$; Business cycle: $[2\pi/32, 2\pi/6)$.

hours worked. The contribution of the permanent supply shock is still sizeable, particularly for consumption, but much smaller; for real activity variables, it varies between 19% (unemployment) and 26% (consumption).

Figure 4 illustrates the same points by showing the spectral density of the seven variables above, along with the spectra of the common components driven by the two shocks. The upper-left panel refers to GDP growth. The supply shock accounts for the long run and the long- medium cycles but explains almost nothing of short cycles and short-run frequencies as if it were cut by a low-pass filter canceling waves of periodicity shorter than five years. By contrast, the demand shock explains almost all cycles of four years or fewer and almost nothing of longer cycles. A similar result holds for the other variables related to real activity: consumption, investment, unemployment, and hours worked.

Demand shocks, both at business cycle frequencies and in the long run, almost exclusively explain the interest rate. The demand shock is therefore closely related to monetary policy, whereas the supply shock is not. A possible explanation is that an expansionary supply shock reduces inflation so that the Fed does not react to it.

Demand shocks explain most of the inflation fluctuations, especially in the long run. This result, coupled with the previous one, explains why our criterion detects two shocks around frequency zero. If we want to explain long-run fluctuations of real activity variables, the supply shock is sufficient; but if we want to explain long-run fluctuations of all variables, including nominal variables and interest rates, we need two shocks.

The bottom-right panel of Figure 4 shows the co-spectra of GDP growth and inflation changes relative to the transitory shock (red line) and the permanent shock (blue line). As anticipated, the transitory shock induces a positive covariance between GDP growth and inflation changes, whereas the opposite is true for the permanent shock.[22]

The above results are in line with Blanchard and Quah (1989) and King et al. (1991) and in sharp contrast with the RBC model. Moreover, they confirm the finding of Angeletos, Collard, and Dellas (2020) that the long-run shock does not drive the bulk of business cycle fluctuations. On the other hand, the hypothesis put forward by Angeletos, Collard, and Dellas (2020) that there is just one non-inflationary demand shock affecting real activity variables at business cycle frequencies is not in line with our results, since the demand shock is closely related to inflation. The implication for macroeconomic modeling is that our analysis does not rule out standard demand shocks of the textbook type.

The finding of one demand shock and one supply shock is in line with Giannone, Reichlin, and Sala (2005), Granese (2024), and Forni et al. (2025). In these papers the result is based on informal consideration rather than a statistical criterion. The present paper can be viewed as complementary to Forni et al. (2025) in that it provides a statistical justification for the identification of demand and supply shocks through economic criteria, which is carried out in that paper.

The fact that the macro economy is well described by just two shocks is not necessarily in contrast with the existence of a plurality of sources of fluctuations, such as the ones extensively analyzed in the literature: monetary and fiscal shocks, financial shocks, technology shocks, news shocks, noise shocks, uncertainty shocks, to cite a few of the most important ones. Rather, we think that such shocks can be grouped into the broader supply and demand categories: for instance, technology, and news shocks are supply shock, whereas uncertainty, noise and credit shocks are best seen as demand shocks. Our idea is that shocks having different nature but belonging to the same group, demand or supply, do exhibit similar propagation dynamics for most macroeconomic aggregates, so that they cannot be easily distinguished from one another on the basis of our dataset.[23]

[22]We consider inflation changes in place of inflation because the latter exhibits a large negative co-spectrum with GDP growth for all of the first five dynamic factors, owing to the 1970s and the early 1980s, characterized by low growth and high inflation.

[23]Of course, a researcher interested in a specific shock should not reduce the number of shocks but estimate a standard structural VAR, a CCSVAR as in Forni et al. (2020), or a large VAR as in Banbura, Giannone, and Reichlin (2010).

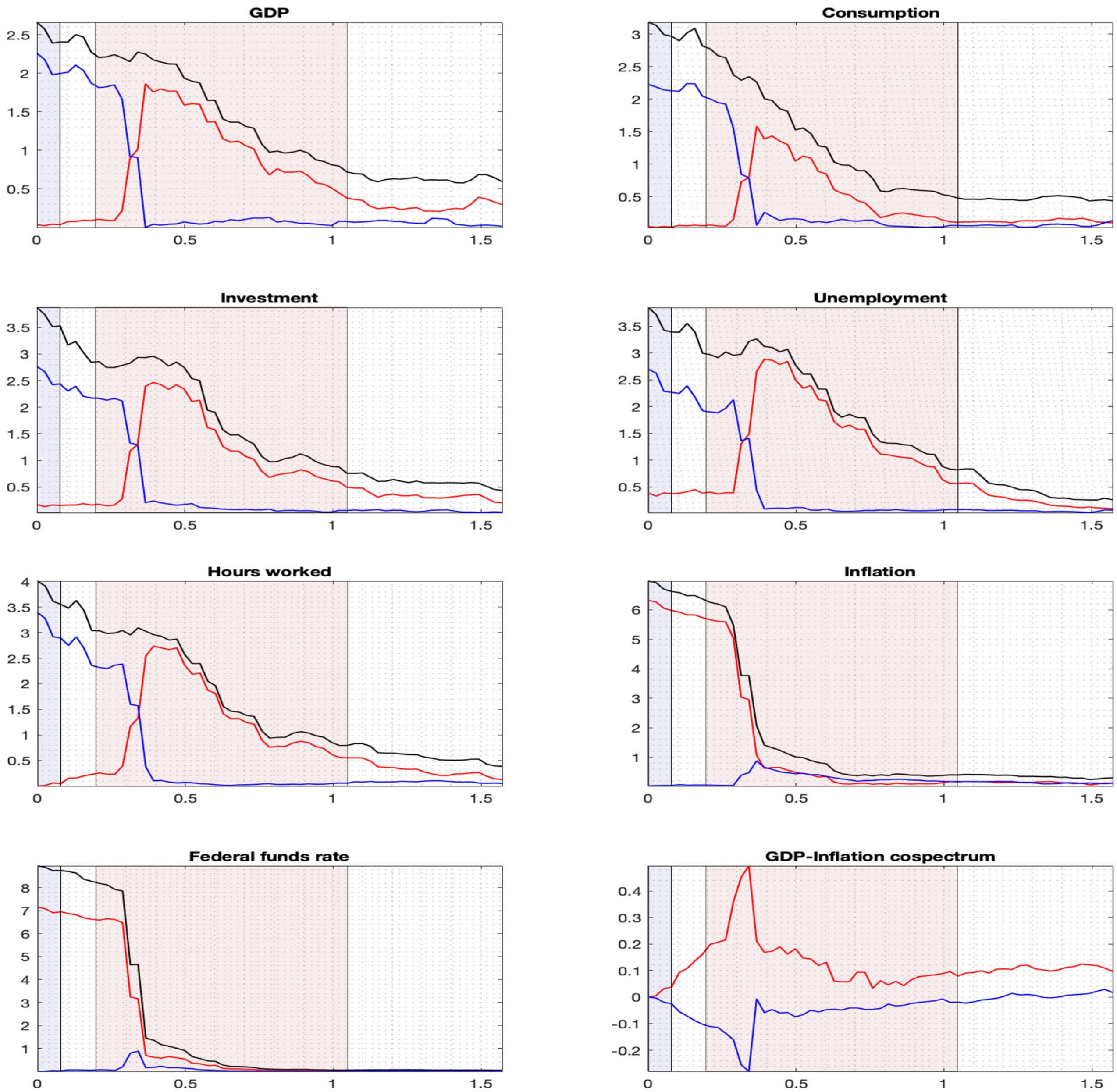

Figure 4. The estimated spectral density functions of seven variables (black line) and the components are driven by the first factor (red line) and the second factor (blue line). The variables are GDP growth, consumption growth, investment growth, unemployment rate changes, hours worked changes, GDP deflator inflation, and federal funds rate. Bottom-right panel: co-spectrum of GDP growth and inflation changes produced by the first factor (red line) and the second factor (blue line). Lilac shadowed area: long-run frequencies; pink shadowed area: business cycle frequencies.

5. Conclusions

Determining the number of different sources of fluctuations affecting the economy is a fundamental quest for macroeconomic modeling and the business cycle debate. However, existing methods to estimate the number of shocks are not entirely satisfactory, mainly because they do not focus on specific frequencies and cycles. Motivated by this empirical challenge, in this article, we study a novel criterion for single frequencies and selected frequency bands of interest. Our estimator does not require pre-selecting penalty functions and nuisance parameters.

We establish the consistency of this estimator and evaluate its small-sample performance with Monte Carlo exercises. Our estimator works even when applied to data stemming from DSGE models where the number of shocks is relatively large. Our methodology appears relevant for a variety of applications in economics and finance when dissecting the effect of economic shocks for specific frequency bands (i.e., for specific economic cycles).

We apply DDR to the FRED-QD database and find a clear-cut result: two major shocks drive the U.S. economy. They account for the bulk of the variance in the main macroeconomic aggregates at both business-cycle and long-run frequencies. The first factor appears to be a demand shock and the second factor a supply shock. The demand shock explains most of the business cycle fluctuations in real activity variables, as well as inflation and interest rate volatility.

Supplementary Materials

The Supplemental Material provides the code and data necessary to replicate the simulation studies and the empirical analysis. It also includes the proofs of the main and auxiliary lemmas, the proof of Theorem 1, details of the Monte Carlo experiments, additional material related to the empirical application, and the variance decomposition in the frequency domain.

Acknowledgments

We are grateful to the Editor (Annie Qu) and three anonymous referees for comments that greatly improved the article. We also thank Seung (Min) Ahn, Georges-Marios Angeletos, Juhee Bae, Fabrice Collard, Luca Gambetti, Alex Horenstein, Marc Hallin, Paymon Khorrami, Marco Lippi, Roman Liška, Emi Nakamura, Alexei Onatski, Luca Sala, and Stefano Soccorsi for sharing codes and providing useful suggestions. A special thanks goes to Georges-Marios Angeletos, whose suggestions were very helpful in improving the empirical part of the article; to Fabrice Collard who, in addition to sharing his codes, patiently explained to us how to use them for our purposes; and to Alex Horenstein, who provided useful comments about the motivation and the intuition of our preferred estimator.

Disclosure Statement

The authors report there are no competing interests to declare.

Funding

Forni acknowledges financial support from the PRIN-MIUR Grant 20174ESLX7.

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