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ON A NEW CONCEPT OF CONTROLLABILITY OF SECOND-ORDER SEMILINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper, the controllability of second-order problems in Banach spaces is investigated when the nonlinear term also depends on the first derivative. The main aim of the paper is to introduce the definition of controllability for second-order problems in Banach spaces that considers both the solution and its derivative at the final point using a unique control and to obtain sufficient conditions for such controllability. Our main results are derived by combining the Schauder fixed point theorem with the approximation solvability method and weak topology. This approach allows us to obtain results under easily verifiable and non-restrictive conditions imposed on the cosine family generated by the linear operator and on the right-hand side since any requirements for compactness are avoided. The paper concludes by applying the obtained results to a system governed by the one-dimensional Klein-Gordon equation.

1. Introduction.

In this paper, a new definition of controllability of second-order problems will be introduced and sufficient conditions for the controllability of the Cauchy problem for semilinear second-order differential equations in Banach spaces will be discussed. More precisely, let us consider the following control problem

$$\left.\begin{array}{r}\ddot{x}(t)=Ax(t)+f(t,x(t),\dot{x}(t))+Bu(t), \qquad \text{for a.a. } t\in[0,T],\\ x(0)=x_0,\dot{x}(0)=\overline{x}_0,\qquad\qquad\qquad\end{array}\right\} \tag{1}$$

where

- (i) E is a reflexive Banach space having a Schauder basis;
- (ii) $A\colon D(A)\subset E\to E$, $D(A)$ dense in E, is a closed, linear operator generating a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$;
- (iii) $f\colon[0,T]\times E\times E\to E$;
- (iv) $x_0,\overline{x}_0\in E$;
- (v) B is a linear, bounded operator from a uniformly convex Banach space U to E;

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(vi) the control function $u \in L^p([0,T],U)$, with $p \in (1,+\infty)$.

The notion of a solution will be understood in the paper in a mild sense. Namely, given $x_0 \in X = \{x \in E \,|\, C(\cdot)x \text{ is continuously differentiable}\}$, by a *mild solution* of (1), we mean a C^1-function $x\colon [0,T] \to E$ such that, for all $t \in [0,T]$,

$$x(t) = C(t)\,x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s)f(s,x(s),\dot{x}(s))\,ds + \int_0^t S(t-s)Bu(s)\,ds, \quad (2)$$

where $u \in L^p([0,T],U)$ and $\{S(t)\}_{t\in\mathbb{R}}$ is the strongly continuous sine family associated to the cosine family $\{C(t)\}_{t\in\mathbb{R}}$.

Notice that the function defined in Equation (2) is continuously differentiable (according to Lemma 2.4 below, to [15, Lemma II.4.1], and since $x_0 \in X$). Furthermore, for all $t \in [0,T]$, it holds that

$$\dot{x}(t) = AS(t)\,x_0 + C(t)\overline{x}_0 + \int_0^t C(t-s)f(s,x(s),\dot{x}(s))\,ds + \int_0^t C(t-s)Bu(s)\,ds.$$

The problem of the exact controllability for second-order differential systems in Banach spaces has received considerable attention recently, with the relevant theory still under development. The notion of controllability involves finding a control function u that guides the state variable from any fixed initial configuration to any fixed final configuration. Most authors assumed a definition of controllability for second-order equations that only ensures that the state function x reaches the target value (see, e.g., [4, 5, 22, 37], and the references therein). As pointed out in [23], these papers overlook the damping term in the definition of the exact controllability of the corresponding systems. This violates the controllability definition because $\dot{x}$ is also a state variable in second-order problems.

Since any second-order equation can always be reduced to the associated first-order system, and studying the controllability of such a first-order system involves naturally the derivative of the state function, neglecting the derivative at the final point seems unreasonable. This evidence stresses the need for a revised definition of controllability for second-order problems.

To address this, we propose a new definition of controllability involving both x and $\dot{x}$.

Definition 1.1. The system (1) is said to be *controllable on the interval* $[0,T]$ if, for every $x_0, x_1 \in X$, $\overline{x}_0, \overline{x}_1 \in E$, there exists a control $u \in L^p([0,T],U)$ such that the mild solution x of (1) satisfies

$$x(T) = x_1 \quad \text{and} \quad \dot{x}(T) = \overline{x}_1. \quad (3)$$

Definition 1.2. A pair (x,u) consisting of a mild solution x of (1) satisfying (3) and of the corresponding control $u \in L^p([0,T],U)$ is called a *solution of the controllability problem* (1).

Only a few authors have attempted to extend the definition of exact controllability in line with Definition 1.1 which requires the existence of a control function such both x and $\dot{x}$ reach the prescribed target values. The first effort in this direction was published in 2003 in [20]. This research, unfortunately, contained a contradiction among the assumptions necessary to ensure the existence of the control function. In fact, the authors required that the sine operator $S(t)$ is compact, for every $t \in \mathbb{R}$, and that the maps $W, Z : L^2([0,T],U) \to E$, defined by

$$Wu = \int_0^T S(T-s)Bu(s)\,ds, \quad (4)$$

and

$$Zu = \int_0^T C(T-s)Bu(s)\,ds, \tag{5}$$

are surjective. The authors then defined the control as

$$\begin{aligned} u = \ & \tfrac{1}{2}\Big\{W^{-1}\Big[x_1 - C(T)\,x_0 - S(T)\overline{x}_0 - \textstyle\int_0^T S(T-s)f(s,x(s))\,ds\Big] + \\ & Z^{-1}\Big[\overline{x}_1 - AS(T)\,x_0 - C(T)\overline{x}_0 - \textstyle\int_0^T C(T-s)f(s,x(s))\,ds\Big\}. \end{aligned}$$

In 2006, Balachandran and Kim in [3] proved that conditions in [20] imply the controllability of (1) only under the very strong assumption that $WZ^{-1} = ZW^{-1} = 0$ which was not addressed in [20]. Despite this fact, subsequent works such as [12] from 2006 and [11] from 2012 continued to repeat the same mistake dealing with regarding the missing assumption and the definition of the control function. Furthermore, following the well-known contradiction identified by Triggiani in 1977 (see [34, 35]), Balachandran and Kim pointed out that the compactness of the sine family and the findings in [16] imply that the surjective operator W from the Banach space $L^2([0,T],U)$ to the Banach space E is compact as well. The open mapping theorem then yields the fact that E in [20] must be finite-dimensional.

A weaker definition of exact controllability in $\mathbb{R}^n$ that takes into account both the solution and its derivative was recently introduced in [38]. However, this still violates the definition of controllability, since it does not guarantee the existence of the same control steering both the function and its derivative to fixed final configurations, instead it relies upon two different control functions.

To the best of our knowledge, there has not yet been correct research on the exact controllability for the second-order problems in Banach spaces considering both the target value for the solution and its derivative, while using one control function. Therefore, the main aim of the paper is to introduce a new definition of controllability for second-order problems in Banach spaces that takes into account both the solution and its derivative at the final point through a unique control function and to introduce relevant sufficient conditions for such controllability.

The proof of our main results will be based on the combination of the Schauder fixed point theorem with the approximation solvability method and the weak topology. If the fixed point theorem were applied directly without the use of the approximation solvability method, strong compactness conditions on the cosine family and the nonlinear term would be required. Since the approximation solvability method introduces a sequence of approximating problems with values in finite-dimensional spaces, its usage will allow us to avoid any compactness requirements in this paper. After applying this method, a limiting argument will lead to a solution to the original controllability problem. As a consequence of the used techniques, also the localization of this solution in a suitable bounded set will be obtained.

The structure of the paper is as follows. Section 2 reviews the basic facts related to cosine and sine families. This part also contains propositions and lemmas which are applied in the proofs of our main results. In Section 3, the main theorems are described and proven. Finally, the application of the proven theory to the system governed by the second-order one-dimensional Klein-Gordon equation is shown in Section 4.

2. **Preliminaries.** Let us consider an infinite-dimensional real Banach space E with the norm $\|\cdot\|$ and let us denote by E^* its dual. Throughout the paper, by E^ω, the space E endowed with the weak topology will be denoted.

In the paper, we will use also the following notation: the norms $||\cdot||_C$ and $||\cdot||_{C^1}$, the $C([0,T],E)$-norm and the $C^1([0,T],E)$-norm, respectively, will be (as usual) defined by

$$||x||_C = \max_{t\in[0,T]} ||x(t)||, \text{ for all } x \in C([0,T],E),$$

$$||x||_{C^1} = \max\{||x||_C, ||\dot{x}||_C\}, \text{ for all } x \in C^1([0,T],E).$$

Since the approximation solvability method will be applied in the proofs of main results, we will now briefly summarize the basic facts about a Schauder basis and natural projections.

Definition 2.1. A sequence $\{e_m\}_m$ of vectors in E is called a *Schauder basis* for E if, for every $x \in E$, there exists a unique sequence of real numbers $\alpha_m = \alpha_m(x)$, $m \in \mathbb{N}$, such that

$$\left\| x - \sum_{k=1}^{m} \alpha_k e_k \right\| \to 0, \qquad \text{as } m \to \infty.$$

It holds that $\alpha_m \in E^*$, for every $m \in \mathbb{N}$, (see [32, pp 18-20]).

If $\{e_m\}_m$ is a Schauder basis for E and $E_m = \operatorname{span}\{e_1, \ldots, e_m\}$ the m-dimensional Banach space generated by the first m vectors of this basis, then the *natural projection* $P_m : E \to E_m$ of E onto E_m is defined by

$$P_m\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k=1}^{m} \alpha_k e_k.$$

It holds that the sequence $\{\|P_m\|\}_m$ is bounded, i.e. that $\|P_m(x)\| \le K^*\|x\|$, for all $m \in \mathbb{N}$ and $x \in E$ (see [24, Proposition 1.a.2]) for some $K^* \ge 1$. Furthermore, the Schauder basis $\{e_m\}_n$ is called *monotone* if $K^* = 1$, i.e. if $\|P_m\| = 1$, for every $m \in \mathbb{N}$.

Remark 2.2. For each bounded subset $\Omega \subset \mathbb{R}^m$ and p lying in the interval $(1;\infty)$, the space $L^p(\Omega,\mathbb{R})$ admits a monotone Schauder basis (see, e.g., [18, Chap. 1.3 and 1.4]) In particular, when $p = 2$, every ortonormal system is a monotone Schauder basis.

Crucial properties of the projection P_m that will be used in the paper are contained in the following lemma (see [6, Lemma 6], [7, Lemma 2.2] and [26, Proposition 7]).

Lemma 2.3. *The natural projection $P_m : E \to E_m$ satisfies the following properties:*

(a) $P_m : E^\omega \to E_m$ is continuous;
(b) if $x_m \to x$, then $P_m(x_m) \to x$;
(c) if $x_m \rightharpoonup x$, then $P_m x_m \rightharpoonup x$;
(d) if $f_m \rightharpoonup f$ in $L^1([0,T],E)$, then $P_m f_m \rightharpoonup f$ in $L^1([0,T],E)$,
(e) for every $x \in E$; $||P_m(x) - x|| \to 0$,
(f) if $x_k \rightharpoonup x$, then $P_m x_k \to P_m x$ as $k \to \infty$, for every $m \in \mathbb{N}$.

In the following sections of the paper, the cosine and sine families will be employed. For this purpose, their definitions and main properties will be mentioned now.

A one parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of linear, bounded operators mapping E into itself is called a *strongly continuous cosine family* if

- $C(t_1+t_2)+C(t_2-t_1)=2C(t_2)C(t_1)$, for all t_1, $t_2\in\mathbb{R}$;
- $C(0)=I$;
- the map $t\to C(t)x$ is continuous in $\mathbb{R}$, for each fixed $x\in E$.

It holds that, for all $t\in\mathbb{R}$,

$$\|C(t)\|\leq\gamma e^{\omega|t|},\tag{6}$$

where $\gamma\geq 1$ and $\omega\geq 0$.

Furthermore, the set

$$D(A)=\left\{x\in E:\exists\lim_{t\to 0^+}\frac{C(t)x-x}{t^2}\right\}$$

is dense in E. The closed, linear operator $A:D(A)\subset E\to E$ defined as

$$Ax=\frac{d^2}{dt^2}\Big[C(t)x\Big]_{t=0}=2\lim_{t\to 0^+}\frac{C(t)x-x}{t^2}$$

is called the *infinitesimal generator of the strongly continuous cosine family.*

In the following sections, we shall also make use of the following set that was already introduced in the Introduction:

$$X=\{x\in E\,|\,C(\cdot)x\text{ is continuously differentiable}\}.$$

The one parameter family $\{S(t)\}_{t\in\mathbb{R}}$ of linear, bounded operators mapping E into itself defined, for all $x\in E$ and $t\in\mathbb{R}$, by

$$S(t)x=\int_0^t C(s)x\,ds$$

is called the *strongly continuous sine family* associated to the cosine family $\{C(t)\}_{t\in\mathbb{R}}$.

The sine and cosine families possess several important properties summarized in the following lemma which will be used in our main results.

Lemma 2.4. *(see, e.g.,* [33, Propositions 2.1, 2.2]*) The one parameter families $\{C(t)\}_{t\in\mathbb{R}}$ and $\{S(t)\}_{t\in\mathbb{R}}$ satisfy the following:*

(a) for each fixed $x\in E$, the mapping $t\to S(t)x$ is continuous;

(b) for all $t_1,t_2\in\mathbb{R}$,

$$\|S(t_1)-S(t_2)\|\leq\gamma\left|\int_{t_2}^{t_1}\mathrm{e}^{\omega|s|}\,ds.\right|\tag{7}$$

(c) for every $t\in\mathbb{R}$ and $x\in E$, $S(t)x\in X$;

(d) for every $x\in X$ and $t\in\mathbb{R}$, $S(t)x\in D(A)$, $\lim\limits_{t\to 0}AS(t)x=0$, $\frac{d}{dt}C(t)x=AS(t)x$, and $\frac{d^2}{dt^2}S(t)x=AS(t)x$.

From the definition of S, we get that $S(0)=0$. Thus, by using (7), we can obtain

$$\|S(t)\|\leq\begin{cases}\gamma\frac{|e^{\omega|t|}-1|}{\omega} & \text{if }\omega\neq 0\\ \gamma|t| & \text{if }\omega=0.\end{cases}\tag{8}$$

In the following, we will also need the following results.

Lemma 2.5. *(see* [29, Lemma 3]*) Let E be a Banach space. Then the mappings $c : [0, T] \times E \to E, c' : [0, T] \times X \to E$ respectively defined as $c(t, x) = C(t)x$ and $c'(t, x) = C'(t)x$ are continuous.*

Proposition 2.6. *(see* [26, Proposition 1]*) Let E be a uniformly convex Banach space, F a normed vector space, and $V : E \to F$ a linear, bounded, and surjective operator. Then*

(i) the mapping $\mathcal{V} : E/\ker V \to F$ defined, for every $u \in E$, by $\mathcal{V}([u]) = V(u)$ is bounded, linear, one to one, and onto;

(ii) there exists a continuous mapping $\Pi : E/\ker V \to E$ such that

$$V(\Pi([u])) = V(u) \quad \textit{and} \quad \|\Pi([u])\| = \min\{\|v\| : V(u) = V(v)\};$$

(iii) the mapping $\tilde{V}^{-1} = \Pi \circ \mathcal{V}^{-1}$ is a continuous right inverse of V and

$$\|\tilde{V}^{-1}(w)\| = \min\left\{\|u\| : u \in V^{-1}(w)\right\}.$$

Furthermore, if E is a Hilbert space, then

(iv) $\tilde{V}^{-1}$ is linear.

Let us note that the previous result was proven in [26] for F Banach space, but it is valid also in this more general case.

3. A new concept of controllability of Second-order equations in Banach spaces.

In this section, we will first provide a detailed description of problems and mistakes appearing in the previous papers that dealt with the controllability of second-order equations, followed by a discussion of the conditions leading to the correct assumptions. As mentioned in the introduction, the sufficient conditions given in the existing literature which ensure exact controllability via a unique control function allowing the prescribed target value to be achieved for the first derivative as well, hold only if the space E is finite-dimensional. Furthermore, Banachandran and Kim in [3] proved that earlier proofs in [20] and subsequent research [11], [12] are valid only under very strong assumptions put on the operators W and Z.

To thoroughly understand the underlying reason for the contradiction appearing in the assumptions required in [20] to guarantee the controllability, let us consider the wave equation, i.e. the case when A is the Laplace operator

$$A : W^{2,q}([0, \pi]) \cap W_0^{1,q}([0, \pi]) \to L^q([0, \pi])$$

defined as

$$Au := u_{xx}.$$

Notice that, as proved in [33], the sine operator $S(t)$ generated by the Laplace operator is compact, for every $t \in \mathbb{R}$. Conversely, the controllability results for the wave equation in [14] were obtained by reducing the problem to the equivalent first-order system and assuming the surjectivity of the operator

$$V : L^p([0, T], U) \to [W^{1,q}([0, \pi]) \cap C_0([0, \pi])] \times L^q([0, \pi])$$

defined as

$$Vu = \int_0^T G(T - s)\mathcal{B}u(s)ds,$$

where G denotes the group defined by $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ corresponding to the Laplace operator and $\mathcal{B}u = (0, Bu)$ (see [14, Example VI - 8.10]). Notice that this is equivalent to requiring that the linear operator

$$N : L^p([0,T],U) \to [W^{1,q}(0,\pi]) \cap C_0([0,\pi])] \times L^q([0,\pi]),$$

defined as

$$Nu = \left(\int_0^T S(T-s)Bu(s)\,ds, \int_0^T C(T-s)Bu(s)\,ds\right),$$

is surjective. This implies that the compactness of the sine operator is not in contradiction with a certain surjectivity. We even stress that the surjectivity of N implies the surjectivity of W and Z, respectively, defined in (4) and (5), when taken on the proper domain. Therefore, motivated by the wave equation case, we aim to identify the correct assumptions guaranteeing the existence of a control function that satisfies Definition 1.1.

A crucial aspect in determining the proper assumptions is the observation that the surjectivity of operator W is not essentially a contradiction with the compactness of the sine operator. In fact, Lemma 2.4 implies that $S(T-s)Bu(s)$ belongs to the linear subspace X of E, for every $s \in [0,T]$. Thus, it is known that $Wu \in X$, for every $u \in L^p([0,T],U)$. If we assume that W defined from $L^p([0,T],U)$ to E is surjective, as in previous papers, this would imply that $X = E$, and hence leading us, by [33], to the strong assumption $D(A) = E$, i.e. to the fact that the generator A is bounded in the whole space. Therefore, W defined from $L^p([0,T],U)$ to E cannot be surjective when A is unbounded.

In the paper, we will prove the controllability of (1) under the assumption that

$$W : L^p([0,T],U) \to X$$

is surjective. Notice that X is not closed, because $D(A) \subset X \subset E$ and $\overline{D(A)} = E$. If X would be closed, we would again get the contradictory conclusion $X = E$. Therefore, X is not in our considerations a Banach space with the E-norm, and hence the result in [16] regarding the compactness of W when it is defined from $L^2([0,T],U)$ to E is not applicable, and the surjectivity of W onto X is not in contradiction with the compactness of the sine family generated by A.

We finally emphasise that the surjectivity of W onto the linear subspace X was essentially introduced by Ke and Obukhovskii in [21] in 2013, albeit in the context where only the state function x is steered to a fixed final configuration.

Furthermore, let us note that, to the best of our knowledge, in all previous papers concerning the controllability for the second-order problems, the control u belongs to a Hilbert space U and the functions W and/or Z are defined in $L^2([0,T],U)$. In this case, $L^2([0,T],U)$ is a Hilbert space, and thus, according to Proposition 2.6, W and Z have a well-defined, linear, and bounded right inverses W^{-1} and Z^{-1}. The linearity of right inverses is then heavily exploited in previous controllability studies by applying topological methods.

On the other hand, when U is an arbitrary Banach space, $L^p([0,T],U)$ is not a Hilbert space even if $p = 2$. However, by Proposition 2.6, if U is uniformly convex, it is still possible to define a right inverse of W and Z. But, in this case, $\tilde{W}^{-1}$ and $\tilde{Z}^{-1}$ are not necessarily linear as shown in [25]. For this reason, the previous investigations into controllability for second-order problems have involved control strategies belonging to Hilbert spaces, even if not in all cases explicitly stated.

Following the strategy implemented in [26] for first-order problems, we will now study the L^p-controllability of (1) with $p \in (1,+\infty)$ and a control u belonging to a uniformly convex Banach space $L^p([0,T],U)$.

Theorem 3.1. *Let us consider the Cauchy problem* (1), *where* $x_0 \in X$, *and assume that* $f : [0,T] \times E \times E \to E$ *fulfills the following assumptions:*

($f1$) *For every* $(x,y) \in E \times E$, $f(\cdot,x,y) : [0,T] \to E$ *is measurable w.r.t. the Lebesgue measure on* $[0,T]$ *and the Borel measure on* E;

($f2$) *For a.a.* $t \in [0,T]$, $f(t,\cdot,\cdot) : E^w \times E^w \to E^w$ *is sequentially continuous, i.e.*

$$\text{if } x_m \rightharpoonup x \text{ and } y_m \rightharpoonup y, \text{ then } f(t,x_m,y_m) \rightharpoonup f(t,x,y), \text{ for a.a. } t \in [0,T];$$

($f3$) *For every* $n \in \mathbb{N}$, *there exists* $\varphi_n \in L^1([0,T],\mathbb{R})$, *with*

$$\liminf_{n\to\infty} \frac{\|\varphi_n\|_{L^1}}{n} = 0,$$

such that

$$||f(t,x,y)|| \le \varphi_n(t),$$

for a.a. $t \in [0,T]$ *and every* $(x,y) \in nB \times nB$, *where* $B = \{x \in E : ||x|| \le 1\}$.

Furthermore, assume that

(WZ) *The linear operator* $N : L^p([0,T],U) \to X \times E$, *defined by*

$$Nu = \Big(\int_0^T S(T-s)Bu(s)\,ds, \int_0^T C(T-s)Bu(s)\,ds\Big),$$

is surjective.

Then the Cauchy problem (1) *is controllable on the interval* $[0,T]$.

Proof. Let $x_0, x_1 \in X$, $\overline{x}_0, \overline{x}_1 \in E$ be arbitrary and let us prove that there exists a control $u \in L^p([0,T],U)$ and a mild solution q of (1) such that $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$.

For simplicity, we will assume throughout the proof that the space E has a monotone Schauder basis, i.e., that, for every $m \in \mathbb{N}$, $\|P_m\| \le 1$. We remark that, with small changes, the proof also works in the general case.

To prove the existence of a solution to problem (1), we will apply the approximation solvability method. Thus, for each $m \in \mathbb{N}$, consider the map $g_m : [0,T] \times E_m \times E_m \to E_m$ defined as $g_m = P_m \circ f$ and the operator $\Sigma_m : C^1([0,T],E_m) \to C^1([0,T],E_m)$, defined by the formula

$$\begin{aligned}\Sigma_m(q)(t) = \quad & P_m C(t)\,x_0 + P_m S(t)\overline{x}_0 + \int_0^t P_m S(t-s) g_m(s,q(s),\dot{q}(s))\,ds \\ & + \int_0^t P_m S(t-s) B(\tilde{N}^{-1}(P_m(p_q))(s))\,ds,\end{aligned}$$

where

$$\begin{aligned}p_q = \quad & \Big(x_1 - C(T)\,x_0 - S(T)\overline{x}_0 - \int_0^T S(T-s) f(s,q(s),\dot{q}(s))\,ds, \\ & \overline{x}_1 - AS(T)\,x_0 - C(T)\overline{x}_0 - \int_0^T C(T-s) f(s,q(s),\dot{q}(s))\,ds\Big)\end{aligned}$$

and $\tilde{N}^{-1}$ is the continuous mapping defined in accordance with Proposition 2.6 satisfying $N \circ \tilde{N}^{-1} = id_{X\times E}$.

Notice that

$$\begin{aligned}\dot{\Sigma}_m(q)(t) = \quad & P_m AS(t)x_0 + P_m C(t)\overline{x}_0 + \int_0^t P_m C(t-s) g_m(s,q(s),\dot{q}(s))ds \\ & + \int_0^t P_m C(t-s) B(\tilde{N}^{-1}(P_m(p_q))(s))\,ds.\end{aligned}$$

The proof will consist of two steps:

Step 1 Proving that, for all $m \in \mathbb{N}$, Σ_m has a fixed point q_m;

Step 2 Proving that the sequence $\{q_m\}_m$ found in **Step 1** admits a subsequence pointwise weakly converging to a solution q of Problem (1) satisfying $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$.

In order to show in **Step 1** that Σ_m has a fixed point, we will prove that this map satisfies all assumptions of the Schauder fixed point theorem. For this reason, given $n \in \mathbb{N}$, we will use the following notation: nB_m will denote the closed, convex, and bounded subset of the Banach space $C^1([0,T],E_m)$ defined by

$$nB_m = \{q \in C^1([0,T],E_m) : \|q(t)\|, \|\dot{q}(t)\| \leq n, \text{ for every } t \in [0,T]\}$$

and L_0 will denote the constant given by

$$L_0 = \begin{cases} \gamma\frac{e^{\omega T}-1}{\omega} & \text{if } \omega \neq 0 \\ \gamma T & \text{if } \omega = 0, \end{cases}$$

where, according to Equations (6) and (8), γ is such that $||C(t)|| \leq \gamma e^{\omega T}$. Hence,

$$\|S(t)\| \leq L_0, \tag{9}$$

for every $t \in [0,T]$. Moreover, since $x_0 \in X$, denoted by

$$M = \max_{t\in[0,T]} \left\| \frac{d}{dt} C(t)x_0 \right\|,$$

we obtain from Lemma 2.5 that $||AS(t)x_0|| \leq M$ for every $t \in [0,T]$.

In order to apply in **Step 1** the Schauder fixed point theorem, we shall prove that, for every $m \in \mathbb{N}$,

(*a*) for all $n \in \mathbb{N}$, $\Sigma_m(nB_m)$ is a relatively compact subset of $C^1([0,T],E_m)$,
(*b*) for every $n \in \mathbb{N}$, $\Sigma_m : nB_m \to C^1([0,T],E_m)$ is continuous,
(*c*) $\Sigma_m(N_0B_m) \subset N_0B_m$ for some $N_0 \in \mathbb{N}$ (independent of m).

Step 1 (*a*) Proving that$\Sigma_m(nB_m)$ is a relatively compact subset of $C^1([0,T],E_m)$ for all $n \in \mathbb{N}$.

Fix $m, n \in \mathbb{N}$ and let us show that every sequence $\{\Sigma_m(q_k)\}_k$, $q_k \in nB_m$, for all $k \in \mathbb{N}$, admits a uniformly convergent subsequence. Let us set $f_k(\cdot) = f(\cdot, q_k(\cdot), \dot{q_k}(\cdot))$. Then, $(f3)$ and the monotonicity of the Schauder basis yield that the sequence $\{P_mf_k\}_k \subset L^1([0,T],E_m)$ is bounded and uniformly integrable, and, for a.a. $s \in [0,T]$, the sequence $\{P_mf_k(s)\}_k$ is bounded in E_m. Since E_m is finite-dimensional, we can apply the Dunford–Pettis Theorem (see [13], p. 294) and obtain the existence of a subsequence, denoted for the sake of simplicity as the sequence, and of a function f_0 such that $P_mf_k \rightharpoonup P_mf_0$ in $L^1([0,T],E_m)$, and hence also in $L^1([0,t],E_m)$, for every $t \in [0,T]$.

Given $\phi \in E^*$ and $t \in [0,T]$, let us consider the operator $\Phi : L^1([0,t],E) \to \mathbb{R}$ defined by

$$\Phi(p) := \phi\left(\int_0^t S(t-s)p(s)\,ds\right).$$

Since $S(t-s)$ is linear and bounded, for every $t,s \in [0,T]$, according to (9), it follows that Φ is linear and bounded, too. Therefore, we have that

$$\begin{aligned} \phi\bigg(\int_0^t S(t-s)P_mf_k(s)\,ds\bigg) &= \Phi(P_mf_k) \to \Phi(P_mf_0) \\ &= \phi\bigg(\int_0^t S(t-s)P_mf_0(s)\,ds\bigg). \end{aligned}$$

Since ϕ is arbitrary, reasoning like in [19, Corollary 5.1.1], we can conclude that

$$\int_0^t S(t-s)P_m f_k(s)\,ds \rightharpoonup \int_0^t S(t-s)P_m f_0(s)\,ds.$$

Since P_m is a bounded and linear operator with values in the finite-dimensional space E_m, we finally get that

$$\int_0^t P_m S(t-s)P_m f_k(s)\,ds \to \int_0^t P_m S(t-s)P_m f_0(s)\,ds \tag{10}$$

uniformly in $[0,T]$.

By the same way, it is possible to show that

$$\int_0^t P_m C(t-s)P_m f_k(s)\,ds \to \int_0^t P_m C(t-s)P_m f_0(s)\,ds, \tag{11}$$

$$\int_0^t S(t-s)f_k(s)\,ds \rightharpoonup \int_0^t S(t-s)f_0(s)\,ds$$

and

$$\int_0^t C(t-s)f_k(s)\,ds \rightharpoonup \int_0^t C(t-s)f_0(s)\,ds$$

uniformly. Therefore,

$$\begin{aligned} p_{q_k} \rightharpoonup p_0 = \; & \Big(x_1 - C(T)\,x_0 - S(T)\overline{x}_0 - \int_0^T S(T-s)f_0(s)\,ds, \\ & \overline{x}_1 - AS(T)\,x_0 - C(T)\overline{x}_0 - \int_0^T C(T-s)f_0(s)\,ds\Big) \end{aligned}$$

in E, and $P_m p_{q_k} \to P_m p_0$ in E, according to Lemma 2.3 (f).

By the continuity of P_m and $\tilde{N}^{-1}$, we have that

$$\alpha_k = \tilde{N}^{-1}(P_m(p_{q_k})) \to \tilde{N}^{-1}(P_m(p_0)) = \alpha_0$$

in $L^p([0,T],E)$. So, using the Hölder inequality, we obtain that

$$\begin{aligned} & \left\| \int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s))ds - \int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_0))(s))ds \right\| \\ & = \left\| \int_0^t P_m S(t-s)B(\alpha_k(s)-\alpha_0(s))\,ds \right\| \\ & \le \int_0^t \|P_m S(t-s)B(\alpha_k(s)-\alpha_0(s))\|\,ds \\ & \le \int_0^t L_0\|B\|\|\alpha_k(s)-\alpha_0(s)\|_U\,ds \le L_0\|B\|T^{1-\frac{1}{p}}\|\alpha_k-\alpha_0\|_{L_p}, \end{aligned}$$

for every $t \in [0,T]$.

By the similar way, it is possible to prove that, for every $t \in [0,T]$,

$$\begin{aligned} & \left\| \int_0^t P_m C(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s))ds - \int_0^t P_m C(t-s)B(\tilde{N}^{-1}(P_m(p_0))(s))ds \right\| \\ & \le \; \gamma e^{\omega T}\,\|B\|\,T^{1-\frac{1}{p}}\,\|\alpha_k-\alpha_0\|_{L_p}. \end{aligned}$$

Hence,

$$\int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s))\,ds \to \int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_0))(s))\,ds \tag{12}$$

and

$$\int_0^t P_m C(t-s) B(\tilde{N}^{-1}(P_m(p_{q_k}))(s))\, ds \to \int_0^t P_m C(t-s) B(\tilde{N}^{-1}(P_m(p_0))(s))\, ds \tag{13}$$

in $C([0,T],E_m)$.

Relations (10), (11), (12), and (13) then imply that $\{\Sigma_m(q_k)\}_k$, $q_k \in nB_m$, admits a uniformly convergent subsequence.

Step 1 (b) Proving that, for every $n \in \mathbb{N}$, $\Sigma_m : nB_m \to C^1([0,T],E_m)$ is continuous.

Let $\{q_k\}_k$ be a sequence in nB_m convergent to $q \in nB_m$ in $C^1([0,T],E_m)$. We will prove that $\Sigma_m(q_k) \to \Sigma_m(q)$ as $k \to \infty$ in $C^1([0,T],E_m)$. To obtain this, according to **Step 1** (a), it is sufficient to prove that $\Sigma_m(q_k)(t) \to \Sigma_m(q)(t)$ and $\dot{\Sigma}_m(q_k)(t) \to \dot{\Sigma}_m(q)(t)$ as $k \to \infty$, for every $t \in [0,T]$.

Fix $t \in [0,T]$. Since $q_k \to q$ in $C^1([0,T],E_m)$, $q_k \rightharpoonup q$ in $C^1([0,T],E)$ and $q_k(s) \rightharpoonup q(s)$, $\dot{q}_k(s) \rightharpoonup \dot{q}(s)$, for every $s \in [0,t]$ (see [8, Theorem 4.3]). Therefore, by $(f2)$ and setting $f_k(\cdot) = f(\cdot, q_k(\cdot), \dot{q}_k(\cdot))$, $f_0(\cdot) = f(\cdot, q(\cdot), \dot{q}(\cdot))$, we get that

$$f_k(s) \rightharpoonup f_0(s), \text{ for a.a. } s \in [0,t].$$

Since P_m is a bounded operator taking values in the finite-dimensional space E_m,

$$P_m f_k(s) \to P_m f_0(s), \text{ for a.a. } s \in [0,t].$$

The boundedness of S and C then yields that

$$P_m S(t-s) P_m f_k(s) \to P_m S(t-s) P_m f_0(s), \text{ for a.a. } s \in [0,t]$$

and

$$P_m C(t-s) P_m f_k(s) \to P_m C(t-s) P_m f_0(s), \text{ for a.a. } s \in [0,t].$$

By $(f3)$ and properties of S and C, we subsequently obtain that

$$\|P_m S(t-s) P_m f_k(s)\| \le L_0 \varphi_n(s)$$

and

$$\|P_m C(t-s) P_m f_k(s)\| \le \gamma e^{\omega T} \varphi_n(s),$$

for a.a. $s \in [0,t]$, which implies, using the dominated convergence theorem, that

$$\int_0^t P_m S(t-s) P_m f_k(s)\, ds \to \int_0^t P_m S(t-s) P_m f_0(s)\, ds \tag{14}$$

and

$$\int_0^t P_m C(t-s) P_m f_k(s)\, ds \to \int_0^t P_m C(t-s) P_m f_0(s)\, ds \tag{15}$$

as $k \to \infty$.

Moreover,

$$\int_0^T P_m S(T-s) f_k(s)\, ds \to \int_0^T P_m S(T-s) f_0(s)\, ds$$

and

$$\int_0^T P_m C(T-s) f_k(s)\, ds \to \int_0^T P_m C(T-s) f_0(s)\, ds$$

as $k \to \infty$. Therefore, $P_m(p_{q_k}) \to P_m(p_q)$ as $k \to \infty$.

Furthermore, if $\mathcal{N} : L^p([0,T],U)/\ker N \to E$ is the linear, bounded and surjective map defined in accordance with Proposition 2.6, then also $\mathcal{N}^{-1}$ is linear and bounded and

$$\mathcal{N}^{-1}(P_m(p_{q_k})) \to \mathcal{N}^{-1}(P_m(p_q)) \text{ in } L^p([0,T],U)/\ker N$$

as $k \to \infty$. By Proposition 2.6, $\Pi : L^p([0,T],U)/\ker N \to L^p([0,T],U)$ is continuous, and so

$$\tilde{N}^{-1}(P_m(p_{q_k})) = \Pi(\mathcal{N}^{-1}(P_m(p_{q_k})) \to \Pi(\mathcal{N}^{-1}(P_m(p_q)) = \tilde{N}^{-1}(P_m(p_q)) \tag{16}$$

in $L^p([0,T],U)$ as $k \to \infty$. Finally, by (16) and the Hölder inequality, we obtain that

$$\begin{aligned}
&\left\| \int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s))ds - \int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_q))(s))ds \right\| \\
&= \left\| \int_0^t P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s)) - \tilde{N}^{-1}(P_m(p_q))(s)))ds \right\| \\
&\leq \int_0^t \|P_m S(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s)) - \tilde{N}^{-1}(P_m(p_q))(s))\|ds \\
&\leq \int_0^t L_0\|B\|\|\tilde{N}^{-1}(P_m(p_{q_k}))(s) - \tilde{N}^{-1}(P_m(p_q))(s)\|ds \\
&\leq L_0\|B\|T^{1-\frac{1}{p}}\|\tilde{N}^{-1}(P_m(p_{q_k})) - \tilde{N}^{-1}(P_m(p_q))\|_{L_p} \to 0
\end{aligned} \tag{17}$$

as $k \to \infty$.

By the similar way, it is possible to prove that

$$\begin{aligned}
&\left\| \int_0^t P_m C(t-s)B(\tilde{N}^{-1}(P_m(p_{q_k}))(s))ds - \int_0^t P_m C(t-s)B(\tilde{N}^{-1}(P_m(p_q))(s))ds \right\| \\
&\leq \gamma e^{\omega T}\|B\|T^{1-\frac{1}{p}}\|\tilde{N}^{-1}(P_m(p_{q_k})) - \tilde{N}^{-1}(P_m(p_q))\|_{L_p} \to 0
\end{aligned} \tag{18}$$

as $k \to \infty$.

Thus, (14), (15), (17), and (18) imply that $\Sigma_m(q_k)(t) \to \Sigma_m(q)(t)$ and $\dot{\Sigma}_m(q_k)(t) \to \dot{\Sigma}_m(q)(t)$ as $k \to \infty$, for every $t \in [0,T]$.

Step 1 (c) Showing that there exists $N_0 \in \mathbb{N}$ independent of m such that $\Sigma_m(N_0B_m) \subset N_0B_m$.

Let m, $n \in \mathbb{N}$, let $q \in nB_m$, and let $h = \Sigma_m(q)$. Then

$$\begin{aligned}
\|P_m p_q\| \leq\ & ||x_1|| + ||\overline{x}_1|| + \gamma e^{\omega T}\left(||x_0|| + ||\overline{x}_0||\right) + L_0||\overline{x}_0|| + M||x_0|| \\
& + ||\varphi_n||_{L^1}\left(L_0 + \gamma e^{\omega T}\right)
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{N}^{-1}P_m(p_q)\|_{L^p} =\ & \left\|\Pi\mathcal{N}^{-1}(P_m(p_q))\right\|_{L^p} \leq \left\|\mathcal{N}^{-1}\right\| \|P_m(p_q))\| \\
\leq\ & \left\|\mathcal{N}^{-1}\right\| [||x_1|| + ||\overline{x}_1|| + \gamma e^{\omega T}\left(||x_0|| + ||\overline{x}_0||\right) \\
& + L_0||\overline{x}_0|| + M||x_0|| + ||\varphi_n||_{L^1}\left(L_0 + \gamma e^{\omega T}\right)].
\end{aligned}$$

Therefore, by the Hölder inequality, for every $t \in [0,T]$,

$$\begin{aligned}
||\Sigma_m(q)(t)|| \leq\ & \gamma e^{\omega T}||x_0|| + L_0||\overline{x}_0|| + L_0||\varphi_n||_{L^1} \\
& + L_0\,\|B\|\,T^{1-\frac{1}{p}}\|\tilde{N}^{-1}P_m(p_q)\|_{L^p} \leq C_1 + C_2||\varphi_n||_{L^1},
\end{aligned}$$

where

$$\begin{aligned}
C_1 =\ & \gamma e^{\omega T}||x_0|| + L_0||\overline{x}_0|| + L_0\,\|B\|\,T^{1-\frac{1}{p}}\left\|\mathcal{N}^{-1}\right\|\left(||x_1|| + ||\overline{x}_1||\right. \\
& \left. + \gamma e^{\omega T}\left(||x_0|| + ||\overline{x}_0||\right) + L_0||\overline{x}_0|| + M||x_0||\right)
\end{aligned}$$

and

$$C_2 = L_0 + L_0 \|B\| T^{1-\frac{1}{p}} \left\|\mathcal{N}^{-1}\right\| \left(L_0 + \gamma e^{\omega T}\right).$$

Furthermore, for every $t \in [0, T]$,

$$\begin{aligned} ||\dot{\Sigma}_m(q)(t)|| \leq \quad & M||x_0|| + \gamma e^{\omega T}||\overline{x}_0|| + \gamma e^{\omega T}||\varphi_n||_{L^1} \\ & + \gamma e^{\omega T} \|B\| T^{1-\frac{1}{p}} \|\tilde{N}^{-1} P_m(p_q)\|_{L^p} \leq D_1 + D_2||\varphi_n||_{L^1}, \end{aligned}$$

where

$$\begin{aligned} D_1 = \quad & M||x_0|| + \gamma e^{\omega T}||\overline{x}_0|| \\ & + \gamma e^{\omega T} \|B\| T^{1-\frac{1}{p}} \left\|\mathcal{N}^{-1}\right\| (||x_1|| + ||\overline{x}_1|| + \gamma e^{\omega T} (||x_0|| + ||\overline{x}_0||) \\ & + L_0||\overline{x}_0|| + M||x_0||) \end{aligned}$$

and

$$D_2 = \gamma e^{\omega T} + \gamma e^{\omega T} \|B\| T^{1-\frac{1}{p}} \left\|\mathcal{N}^{-1}\right\| \left(L_0 + \gamma e^{\omega T}\right).$$

Consequently, if

$$L_1 = \max\{C_1,\, D_1\} \text{ and } L_2 = \max\{C_2,\, D_2\}, \tag{19}$$

we obtain that

$$\|h\|_{C^1} \leq L_1 + L_2\|\varphi_n\|_{L^1}. \tag{20}$$

According to $(f3)$, there exists a subsequence, denoted for the sake of simplicity as the sequence, such that

$$\lim_{n\to\infty} \frac{L_1 + L_2\|\varphi_n\|_{L^1}}{n} = 0.$$

Therefore, there exists $N_0 > 0$ such that

$$\frac{L_1 + L_2\|\varphi_{N_0}\|_{L^1}}{N_0} < 1,$$

which, combined with Equation (20), implies that

$$\frac{1}{N_0}\|h\|_{C^1} < 1,$$

i.e., that $h \in N_0 B_m$, for every $m \in \mathbb{N}$, and the claim is proven.

Subsequently, by applying the Schauder fixed point theorem, we get that, for all $m \in \mathbb{N}$, the operator Σ_m has a fixed point q_m. Furthermore, due to the technique used, we can deduce that the fixed points lie in the set

$$N_0 B = \left\{q \in C^1([0, T], E) : \|q(t)\|, \|\dot{q}(t)\| \leq N_0, \text{ for every } t \in [0, T]\right\}.$$

Step 2. Limiting procedure.

The sequence $\{q_m\}_m$ found in **Step 1** satisfies, for all $m \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} q_m(t) = \quad & P_m C(t)\, x_0 + P_m S(t)\overline{x}_0 + \int_0^t P_m S(t-s) P_m f(s, q_m(s), \dot{q}_m(s))\, ds \\ & + \int_0^t P_m S(t-s) B(\tilde{N}^{-1}(P_m(p_{q_m}))(s))\, ds \end{aligned} \tag{21}$$

where

$$\begin{aligned} p_{q_m} = \quad & \Big(x_1 - C(T)\, x_0 - S(T)\overline{x}_0 - \int_0^T S(T-s) f(s, q_m(s), \dot{q}_m(s))\, ds, \\ & \overline{x}_1 - AS(T)\, x_0 - C(T)\overline{x}_0 - \int_0^T C(T-s) f(s, q_m(s), \dot{q}_m(s))\, ds\Big). \end{aligned} \tag{22}$$

Furthermore,

$$\begin{aligned}\dot{q}_m(t) = \quad & P_m AS(t)x_0 + P_m C(t)\overline{x}_0 + \int_0^t P_m C(t-s) f(s, q_m(s), \dot{q}_m(s))\, ds \\ & + \int_0^t P_m C(t-s) B(\tilde{N}^{-1}(P_m(p_{q_m}))(s))\, ds,\end{aligned} \tag{23}$$

where p_{q_m} is defined by (22).

Let us show now that the sequence $\{q_m\}_m$ admits a subsequence that pointwise weakly converges to a function $q \in C^1([0,T],E)$ which is a solution of Problem (1) satisfying $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$.

Let us set

$$f_m(\cdot) = f(\cdot, q_m(\cdot), \dot{q_m}(\cdot)) \text{ and } g_m(\cdot) = P_m f_m(\cdot),$$

for all $m \in \mathbb{N}$. Since $q_m \in N_0 B$ for every m, we then obtain from $(f3)$ that

$$\|f_m(s)\| \le \varphi_{N_0}(s),$$

for a.e. $s \in [0,T]$. Therefore, $\{f_m\}_m$ is uniformly integrable, bounded and $\{f_m(s)\}_m$ is bounded, for a.a $s \in [0,T]$. Since E is reflexive, we can apply the Dunford–Pettis Theorem and get the existence of a subsequence, denoted for the sake of simplicity as the sequence, and of a function g such that $f_m \rightharpoonup g$ in $L^1([0,T],E)$. Using Lemma 2.3 (d), we then also get that $g_m \rightharpoonup g$ in $L^1([0,T],E)$.

Reasoning like in **Step 1**(a) it easily follows that

$$\int_0^t S(t-s) g_m(s)\, ds \rightharpoonup \int_0^t S(t-s) g(s)\, ds, \text{ for every } t \in [0,T].$$

By Lemma 2.3 (c),

$$\int_0^t P_m S(t-s) g_m(s)\, ds \rightharpoonup \int_0^t S(t-s) g(s)\, ds, \text{ for every } t \in [0,T]. \tag{24}$$

Similarly, we can obtain that

$$\int_0^t P_m C(t-s) g_m(s)\, ds \rightharpoonup \int_0^t C(t-s) g(s)\, ds, \text{ for every } t \in [0,T]. \tag{25}$$

Since $q_m \in N_0 B$, for all $m \in \mathbb{N}$, reasoning like in **Step 1**(c), it is possible to prove, for every m, that

$$\|P_m(p_{q_m})\| \le L,$$

where

$$\begin{aligned} L = \quad & ||x_1|| + ||\overline{x}_1|| + \gamma e^{\omega T}\left(||x_0|| + ||\overline{x}_0||\right) + L_0||\overline{x}_0|| + M||x_0|| \\ & + ||\varphi_{N_0}||_{L^1}\left(L_0 + \gamma e^{\omega T}\right).\end{aligned}$$

Therefore,

$$\|\tilde{N}^{-1}(P_m(p_{q_m}))\|_{L^p} = \|\Pi(\mathcal{N}^{-1}(P_m(p_{q_m}))\| \le \|\mathcal{N}^{-1}\|\|P_m(p_{q_m})\| \le \|\mathcal{N}^{-1}\| L. \tag{26}$$

Since U is uniformly convex, $L^p([0\,,T]\,,U)$, $1 < p < \infty$, is uniformly convex as well, and hence reflexive. Therefore, using (26), there exists a subsequence, still denoted as the sequence, such that $\tilde{N}^{-1}(P_m(p_{q_m}))$ weakly converges to u in $L^p([0\,,T]\,,U)$.

Furthermore, since B is continuous, reasoning like in **Step 1**(a), we get that for every $\phi \in E^*$ and $t \in [0,T]$, the operator $\Phi : L^p([0,t],U) \to E$ defined as

$$\Phi(q) = \phi\left(\int_0^t S(t-s) B(q(s))\, ds\right)$$

is linear and bounded, and hence that

$$\int_0^t P_m S(t-s) B\left(\tilde{N}^{-1}(P_m(p_{q_m}))(s)\right) ds \rightharpoonup \int_0^t S(t-s) B\left(u(s)\right) ds. \tag{27}$$

Similarly, we can obtain that

$$\int_0^t P_m C(t-s) B\left(\tilde{N}^{-1}(P_m(p_{q_m}))(s)\right) ds \rightharpoonup \int_0^t C(t-s) B\left(u(s)\right) ds, \tag{28}$$

due to Lemma 2.3 (c), for every $t \in [0, T]$.

By (24), (25), (27), and (28), we have that, for every $t \in [0, T]$,

$$q_m(t) \rightharpoonup q(t) = C(t)\, x_0 + S(t)\overline{x}_0 + \int_0^t S(t-s) g(s)\, ds + \int_0^t S(t-s) B(u(s))\, ds$$

and

$$\dot{q}_m(t) \rightharpoonup \dot{q}(t) = AS(t)\, x_0 + C(t)\overline{x}_0 + \int_0^t C(t-s) g(s)\, ds + \int_0^t C(t-s) B(u(s))\, ds.$$

Hence, by $(f2)$, for a.a. $t \in [0, T]$,

$$f(t, q_m(t), \dot{q}_m(t)) \rightharpoonup f(t, q(t), \dot{q}(t)),$$

and then

$$g_m(t) = P_m f(t, q_m(t), \dot{q}_m(t)) \rightharpoonup f(t, q(t), \dot{q}(t)),$$

by Lemma 2.3 (c). Therefore, $g(t) = f(t, q(t), \dot{q}(t))$.

It remains to prove that $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$. We observe that, according to (4), condition (WZ), (22) and the definition of $\tilde{N}^{-1}$, denoted by $\pi_1 : X \times E \to X$ the map defined as $\pi_1(u_1, u_2) = u_1$, it follows that

$$\begin{aligned} \int_0^T & S(T-s) B(\tilde{N}^{-1}(P_m(p_{q_m}))(s))\, ds \\ = \;& W(\tilde{N}^{-1}(P_m(p_{q_m})) = \pi_1(N(\tilde{N}^{-1}(P_m(p_{q_m})))) = \pi_1(P_m(p_{q_m})) \\ = \;& P_m\left(x_1 - C(T)\, x_0 - S(T)\overline{x}_0 - \int_0^T S(T-s) f(s, q_m(s), \dot{q}_m(s))\, ds\right). \end{aligned}$$

Thus, by (21), we obtain, for all $m \in \mathbb{N}$, that

$$\begin{aligned} q_m(T) = \;& P_m C(T)\, x_0 + P_m S(T)\overline{x}_0 + \int_0^T P_m S(T-s) P_m f(s, q_m(s), \dot{q}_m(s))\, ds \\ & + \int_0^T P_m S(T-s) B(\tilde{N}^{-1}(P_m(p_{q_m}))(s))\, ds \\ = \;& P_m C(T)\, x_0 + P_m S(T)\overline{x}_0 + \int_0^T P_m S(T-s) P_m f(s, q_m(s), \dot{q}_m(s))\, ds \\ & + P_m\left(x_1 - C(T)\, x_0 - S(T)\overline{x}_0 - \int_0^T S(T-s) f(s, q_m(s), \dot{q}_m(s)) ds\right) \\ = \;& P_m x_1 \end{aligned} \tag{29}$$

Similarly, by (5) and (22), we obtain that

$$\begin{aligned}\dot{q}_m(T) = \;& P_mAS(T)x_0 + P_mC(T)\overline{x}_0 + \int_0^T P_mC(T-s)P_mf(s,q_m(s),\dot{q}_m(s))ds \\ & + \int_0^T P_mC(T-s)B(\tilde{N}^{-1}(P_m(p_{q_m}))(s))\,ds \\ = \;& P_mAS(T)x_0 + P_mC(T)\overline{x}_0 + \int_0^T P_mC(T-s)P_mf(s,q_m(s),\dot{q}_m(s))ds \\ & + P_m\Big(\overline{x}_1 - AS(T)\,x_0 - C(T)\overline{x}_0 - \int_0^T C(T-s)f(s,q_m(s),\dot{q}_m(s))ds\Big) \\ = \;& P_m\overline{x}_1.\end{aligned} \tag{30}$$

Passing to the weak limit in (29) and (30), we obtain that $q(T) = x_1$ and $\dot{q}(T) = \overline{x}_1$. □

In the following theorem, the growth condition $(f3)$ is replaced by the growth condition $(f3^{'})$; the comparison between $(f3)$ and $(f3^{'})$ has been studied in detail in [26].

The sketch of the proof of this controllability result can be seen as a generalization of the method applied in [10] for second-order problems with the nonlinear term that does not contain the first derivative.

Theorem 3.2. *Consider the Cauchy problem* (1)*, where* $x_0 \in X$ *and* $f : [0,T] \times E \times E \to E$ *satisfies conditions* $(f1)-(f2)$*. Moreover, assume that condition* (WZ) *is fulfilled and that the following assumption hold:*

$(f3')$ *There exist* $\alpha, \beta \in L^1([0,T],\mathbb{R})$ *such that, for a.a.* $t \in [0,T]$ *and all* x, $y \in E$,

$$||F(t,x,y)|| \leq \alpha(t)\max\{||x||,||y||\} + \beta(t). \tag{31}$$

Then the Cauchy problem (1) *is controllable in* $[0,T]$.

Proof. The proof proceed by analogy with Theorem 3.1 when changing $\varphi_n(t)$ for $n\alpha(t)+\beta(t)$ and subsequently modifying the proof. **Step 1** (c) is the only point that would be significantly different. Therefore, we will focus on this point now and show that there exists a bounded, closed, and convex set H_m, such that $\Sigma_m(H_m) \subset H_m$, for all $m \in \mathbb{N}$.

For this purpose, define, for every $j \in \mathbb{N}$,

$$q_j = \max_{t\in[0,T]} \int_0^T e^{-j(t-s)}\chi_{[0,t]}(s)\alpha(s)\,ds;$$

its existence is guaranteed by continuity. Furthermore, let, for every $j \in \mathbb{N}$, t_j be the point where the maximum is reached. Since $\{t_j\}_j \subset [0,T]$, there exists $\overline{t}$ such that $t_j \to \overline{t}$ (eventually passing to a subsequence). Therefore, the sequence $\{\phi_j\}_j \subset L^1([0,T],E)$ defined by $\phi_j(s) = e^{-j(t_j-s)}\chi_{[0,t_j]}(s)\alpha(s)$ converges pointwise to 0. Since the convergence is dominated, $\phi_j \to 0$ in $L^1([0,T],E)$. In particular, there exists a subsequence, for the sake of simplicity denoted as the sequence, such that $q_j \to 0$. Let us take $R_0 \in \mathbb{R}$ and $\overline{j} \in \mathbb{N}$ such that $1 - L_2q_{\overline{j}} > 0$, and

$$R_0 > \frac{L_1 + L_2\|\beta\|_{L^1}}{1 - L_2q_{\overline{j}}},$$

where L_1 and L_2 are constants introduced in (19). Moreover, let us consider the bounded, closed, and convex set

$$H_m = \{x \in C^1([0,T], E_m) : \max_{t\in[0,T]} (e^{-\overline{j}t} \max\{\|x(t)\|, \|\dot{x}(t)\|\}) \leq R_0\}.$$

Now, using the estimates and notations from the proof of Theorem 3.1, we have that, for every $q \in H_m, t \in [0,T]$,

$$\begin{aligned} e^{-\overline{j}t}\|\Sigma_m(q)(t)\| \leq \;& e^{-\overline{j}t}C_1 \\ & +e^{-\overline{j}t}C_2\|\beta\|_{L^1} + C_2 e^{-\overline{j}t}\textstyle\int_0^t \alpha(s)\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\ \leq \;& e^{-\overline{j}t}[C_1 + C_2\|\beta\|_{L^1}] \\ & +C_2 \textstyle\int_0^t e^{-\overline{j}(t-s)}\alpha(s)e^{-\overline{j}s}\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\ \leq \;& e^{-\overline{j}t}[C_1 + C_2\|\beta\|_{L^1}] + C_2 R_0 \int_0^T e^{-\overline{j}(t-s)}\chi_{[0,t]}(s)\alpha(s)ds \\ \leq \;& C_1 + C_2\|\beta\|_{L^1} + C_2 R_0 q_{\overline{j}} < R_0, \end{aligned}$$

due to the definition of R_0 and since $L_i = \max\{C_i,\, D_i\}$, $i = 1,\, 2$.

Similarly,

$$\begin{aligned} e^{-\overline{j}t}\|\dot{\Sigma}_m(q)(t)\| \leq \;& e^{-\overline{j}t}D_1 + e^{-\overline{j}t}D_2\|\beta\|_{L^1} \\ & +D_2 e^{-\overline{j}t}\textstyle\int_0^t \alpha(s)\max\{\|x(s)\|, \|\dot{x}(s)\|\}ds \\ \leq \;& D_1 + D_2\|\beta\|_{L^1} + D_2 R_0 q_{\overline{j}} < R_0. \end{aligned}$$

Therefore, $\Sigma_m(q) \in H_m$. Since H_m is a subset of the bounded set

$$H = \{x \in C^1([0,T], E) : \max_{t\in[0,T]} (e^{-\overline{j}t} \max\{\|x(t)\|, \|\dot{x}(t)\|\}) \leq R_0\},$$

we can proceed similarly like in the proof of Theorem 3.1 in order to obtain the conclusion. □

4. Controllability of the Klein-Gordon equation.

In the last section of the paper, we study the controllability problem for the system governed by the second-order one-dimensional Klein-Gordon equation modeled, for $0 \leq \xi \leq \pi$, $0 \leq t \leq T$, by the following hyperbolic integro-differential equation

$$\begin{cases} z_{tt} = z_{\xi\xi} - a^2 z(t,\xi) + g\left(t, \xi, \displaystyle\int_0^\pi h(\xi,\tau) z'_t(t,\tau)d\tau\right) + b(\xi)u(t,\xi) \\ z(t,0) = z(t,\pi) = 0 \quad 0 \leq t \leq T, \\ z(0,\xi) = z_0(\xi),\, z_t(0,\xi) = \overline{z_0}(\xi) \quad 0 \leq \xi \leq \pi. \end{cases} \tag{32}$$

The equation in (32) models the quantum physics wave function of elementary free massive particles, providing a relativistic description of the electron and neutrino states (see, e.g., [28]). In this context, the interaction between the waves and the propagation medium generates energy dissipation, referred to as damping. In equation (32), we consider both a material damping term with constant coefficient a^2 and a viscous damping term of nonlinear Balakrishnan–Taylor-type $g\left(t, \xi, \int_0^\pi h(\xi,\tau)z'_t(t,\tau)d\tau\right)$ (see [1, 2, 17, 27, 30]).

We look for a solution $z \in \mathcal{C}^1([0,T], L^2([0,\pi]))$ associated to the control $u \in L^2([0,T], L^2([0,\pi]))$.

We assume the following hypotheses:

(i) for all $c \in \mathbb{R}$, $g(\cdot,\cdot,c) : [0,T] \times [0,\pi] \to \mathbb{R}$ is measurable;
(ii) for a.a. $(t,\xi) \in [0,T] \times [0,\pi]$, $g(t,\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous;

(iii) there exist $\delta \in L^1([0,T]), \gamma \in L^2([0,\pi])$ such that, for every $c \in \mathbb{R}, t \in [0,T]$ and a.a. $\xi \in [0,\pi]$, $|g(t,\xi,c)| \leq \delta(t)\gamma(\xi)|c|$;
(iv) h is measurable and there exists $\psi \in L^2([0,\pi])$ such that, for every ξ and a.e. $\tau \in [0,\pi]$, $|h(\xi,\tau)| \leq \psi(\tau)$;
(v) $b \in L^\infty([0,\pi])$ and, for a.a. $\xi \in [0,\pi]$, $|b(\xi)| \geq \overline{b} > 0$;
(vi) $z_0 \in W^{1,2}([0,\pi]) \cap C_0([0,\pi]), \overline{z_0} \in L^2([0,\pi])$.

In order to rewrite problem (32) in abstract form, we identify z and u respectively with functions $t \mapsto z(t,\cdot)$ and $t \mapsto u(t,\cdot)$. Using the notation introduced in Section 3, we consider the Hilbert spaces $U = E = L^2([0,\pi])$ and rewrite Problem (32) as

$$\begin{cases} \ddot{z}(t) = Az(t) + f(t, z(t), \dot{z}(t)) + Bu(t), \; t \in [0,T], \\ z(0) = z_0; \; \dot{z}(0) = \overline{z_0}, \end{cases} \tag{33}$$

where $A : D(A) = \{y \in W^{2,2}([0,\pi]) : y(0) = y(\pi) = 0\} \subset E \to E$ is the operator

$$Ay = y'' - a^2 y,$$

$f : [0,T] \times E \times E \to E$ is defined as

$$f(t,y,w)(\xi) = g\left(t, \xi, \int_0^\pi h(\xi,\tau)w(\tau)d\tau\right),$$

and $B : U \to E$ is defined by

$$By(\xi) = b(\xi)y(\xi).$$

We will now prove that problem (33) satisfies all the assumptions of Theorem 3.2 which will imply the controllability of (32).

The operator A generates a cosine family (see [15, Lemma III.4.1]). Moreover, the space X associated to the cosine family generated by A is $X = W^{1,2}([0,\pi]) \cap C_0([0,\pi]) = H_0^1([0,\pi])$.

We claim that f is well defined, i.e. that $f(t,y,w)$ belongs to $L^2([0,\pi])$ for every $t \in [0,T]$ and $y, w \in L^2([0,\pi])$. We notice, first of all, that, according to Tonelli's theorem and condition (iv), the function $h(\xi,\cdot)w(\cdot)$ is integrable in $[0,\pi]$ for every $\xi \in [0,\pi]$ and every $w \in L^2([0,\pi])$, and that the map $\xi \to \int_0^\pi h(\xi,\tau)w(\tau)d\tau$ is measurable in $[0,\pi]$. Conditions (i) and (ii) yield that g is a Carathéodory map, thus the map $\xi \to g(t,\xi,\int_0^\pi h(\xi,\tau)w(\tau)d\tau)$ is measurable, for every $w \in L^2([0,\pi])$. We now observe that, for every $\xi \in [0,\pi]$ and $w \in L^2([0,\pi])$, assumption (iv) implies the following estimate

$$\left|\int_0^\pi h(\xi,\tau)w(\tau)d\tau\right| \leq \int_0^\pi \psi(\tau)|w(\tau)|d\tau \leq \|\psi\|_2\|w\|_2.$$

Therefore, for every $t \in [0,T], y, w \in L^2([0,\pi])$, and a.a. $\xi \in [0,\pi]$,

$$|f(t,y,w)(\xi)| \leq \left|g\left(t,\xi,\int_0^\pi h(\xi.\tau)w(\tau)d\tau\right)\right| \leq \delta(t)\gamma(\xi)\|\psi\|_2\|w\|_2, \tag{34}$$

i.e. f is well defined and

$$\|f(t,y,w)\|_2 \leq \delta(t)\|\gamma\|_2\|\psi\|_2\|w\|_2.$$

Moreover, the measurability of f follows by the Pettis measurability theorem (see [31, p. 278]), the separability of $L^2([0,\pi])$ and conditions (i) and (ii) (see [19, Corollary 1.3.1]). Given now $t \in [0,T]$ satisfying (ii), take $y_n \rightharpoonup y$ and $w_n \rightharpoonup w$ in $L^2([0,\pi])$. Then, for every $\xi \in [0,\pi]$, we have that

$$\int_0^\pi h(\xi,\tau)w_n(\tau)d\tau \to \int_0^\pi h(\xi,\tau)w(\tau)d\tau.$$

Thus, condition (ii) implies, for a.a. $\xi \in [0,\pi]$,

$$g\left(t,\xi,\int_0^\pi h(\xi,\tau)w_n(\tau)d\tau\right) \to g\left(t,\xi,\int_0^\pi h(\xi,\tau)w(\tau)d\tau\right).$$

Since the weak convergence of $\{w_n\}_n$ yields its boundedness, condition (34) and the Lebesgue Convergence Theorem imply that

$$g\left(t,\cdot,\int_0^\pi h(\cdot,\tau)w_n(\tau)d\tau\right) \to g\left(t,\cdot,\int_0^\pi h(\cdot,\tau)w(\tau)d\tau\right),$$

i.e. that $f(t,y_n,w_n) \to f(t,y,w)$ in $L^2([0,\pi])$.

It remains to prove that (WZ) holds as well. First of all, notice that, according to (v), for a.a. $\xi \in [0,\pi]$, $|By(\xi)| \le \|b\|_\infty |y(\xi)|$, i.e. B is well defined, linear and bounded from $L^2([0,\pi])$ to $L^2([0,\pi])$. By the definition, $N : L^2([0,T],L^2([0,\pi])) \to H_0^1([0,\pi]) \times L^2([0,\pi])$, is defined by

$$\begin{aligned} Ny(\xi) = {} & \left(\int_0^T S(T-s)b(\xi)y(s)(\xi)\,ds, \int_0^T C(T-s)b(\xi)y(s)(\xi)\,ds\right) \\ = {} & b(\xi)\left(\int_0^T S(T-s)y(s)(\xi)ds, \int_0^T C(T-s)y(s)(\xi)ds\right) \end{aligned}$$

and (v) implies that N is surjective if and only if the map

$$My = \left(\int_0^T S(T-s)y(s)ds, \int_0^T C(T-s)y(s)ds\right)$$

is surjective. In fact, (v) implies that $\frac{1}{b} \in L^\infty([0,\pi])$ as well, thus if $My = v$, then the function $\tilde{y}(t)(\xi) = \frac{y(t)(\xi)}{b(\xi)}$ belongs to $L^2([0,T],L^2([0,\pi]))$ and $N\tilde{y} = v$.

On this aim it is sufficient to prove that there exists $\rho > 0$ such that, for every $z^* \in \{H_0^1([0,\pi]) \times L^2([0,\pi])\}^*$,

$$\|z^*\| \le \rho \|M^* z^*\|$$

(see [14, Lemma B.13]). Trivially, $z^* \in \{H_0^1([0,\pi]) \times L^2([0,\pi])\}^*$ if and only if

$$z^* = (y_1, y_2)$$

with $y_1 \in H_0^1([0,\pi])^* = H^{-1}([0,\pi])$ and $y_2 \in L^2([0,\pi])$.

Denoted $M_1 : L^2([0,T],L^2([0,\pi])) \to H_0^1([0,\pi])$ and $M_2 : L^2([0,T],L^2([0,\pi])) \to L^2([0,\pi])$, respectively, defined as

$$M_1 y = \int_0^T S(T-s)y(s)ds,$$

$$M_2 y = \int_0^T C(T-s)y(s)ds,$$

it clearly holds that

$$M^*(y_1,y_2) = M_1^* y_1 + M_2^* y_2.$$

Moreover, $M_1^* : H^{-1}([0,\pi]) \to L^2([0,T],L^2([0,\pi]))$ and $M_2^* : L^2([0,\pi]) \to L^2([0,T],L^2([0,\pi]))$ are, respectively, defined as

$$M_1^* = S(T-\cdot)^*$$

and

$$M_2^* = C(T-\cdot)^*$$

(see [14, Lemma 8.7]). Hence, the surjectivity of M is equivalent to prove that there exists $\rho > 0$ such that for every $y_1 \in H^{-1}([0,\pi]), y_2 \in L^2([0,\pi])$,

$$\begin{aligned}\big[\max\{\|y_1\|_{H^{-1}}, \|y_2\|_2\}\big]^2 \leq\ & \rho\big[\|S(T-\cdot)^* y_1 + C(T-\cdot)^* y_2\|_2\big]^2 \\ =\ & \rho\Big[\int_0^T \|S(T-s)^* y_1\|_2^2 ds + \int_0^T \|C(T-s)^* y_2\|_2^2 ds \\ & + 2\int_0^T (S(T-s)^* y_1, C(T-s)^* y_2) ds\Big].\end{aligned}$$

We now recall (see [9, Theorem VIII.20]) that there exists an orthonormal basis $\{e_n\}_n$ of $L^2([0,\pi])$ composed by eigenvectors of $-A$. It easily follows that $e_n = \sqrt{\frac{2}{\pi}}\sin(\sqrt{n^2+a^2}x)$, and hence, since A is self-adjoint, that

$$Ay = -\sum_{n=1}^{+\infty}(n^2+a^2)(y,e_n)e_n,$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2([0,\pi])$. Recalling the definitions of $C(t)$ and $S(t)$, by easily computations, we then get that

$$C(t)y = \sum_{n=1}^{+\infty}\cos(\sqrt{n^2+a^2}t)(y,e_n)e_n \tag{35}$$

and

$$S(t)y = \sum_{n=1}^{+\infty}\frac{\sin(\sqrt{n^2+a^2}t)}{\sqrt{n^2+a^2}}(y,e_n)e_n. \tag{36}$$

The operator defined in (35) is clearly self-adjoint. According to the definition of the sine family, this implies that the operator defined in (36) is also self-adjoint. Thus the boundedness of the cosine and sine functions and the Lebesgue convergence theorem imply that

$$\begin{aligned}\int_0^T \|C(T-s)^* y_2\|_2^2 ds =\ & \int_0^T \|C(T-s) y_2\|_2^2 ds \\ =\ & \int_0^T \Big[\sum_{n=1}^{+\infty}\cos^2(\sqrt{n^2+a^2}(T-s))(y_2,e_n)^2\Big] ds \\ =\ & \sum_{n=1}^{+\infty}(y_2,e_n)^2\int_0^T \cos^2(\sqrt{n^2+a^2}(T-s)) ds \\ =\ & \sum_{n=1}^{+\infty}\Big(\frac{T}{2} + \frac{\sin(\sqrt{n^2+a^2}T)}{4\sqrt{n^2+a^2}}\Big)(y_2,e_n)^2\end{aligned} \tag{37}$$

and similarly that

$$\int_0^T \|S(T-s)^* y_1\|_2^2 ds = \sum_{n=1}^{+\infty}\frac{1}{n^2+a^2}\Big(\frac{T}{2} - \frac{\sin(\sqrt{n^2+a^2}T)}{4\sqrt{n^2+a^2}}\Big)(y_1,e_n)^2 \tag{38}$$

and, recalling that $2\alpha\beta \leq \alpha^2 + \beta^2$ for every $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
&2\int_0^T (S(T-s)^*y_1, C(T-s)^*y_2])ds \\
&= -2\sum_{n=1}^{+\infty} \frac{1-\cos(2\sqrt{n^2+a^2}T)}{2(n^2+a^2)}(y_1,e_n)(y_2,e_n) \\
&\geq -\sum_{n=1}^{+\infty} \frac{1-\cos(2\sqrt{n^2+a^2}T)}{2(n^2+a^2)}(y_1,e_n)^2 \\
&\quad -\sum_{n=1}^{+\infty} \frac{1-\cos(2\sqrt{n^2+a^2}T)}{2(n^2+a^2)}(y_2,e_n)^2.
\end{aligned} \tag{39}$$

Therefore from (37), (38) and (39), we conclude that

$$\begin{aligned}
&\int_0^T \|S(T-s)^*y_1\|_2^2 ds + \int_0^T \|C(T-s)^*y_2\|_2^2 ds \\
&+2\int_0^T [S(T-s)^*y_1][C(T-s)^*y_2]ds \\
&\geq \sum_{n=1}^{+\infty} \frac{1}{n^2+a^2}\left(\frac{T}{2} - \frac{\sin(\sqrt{n^2+a^2}T)}{4\sqrt{n^2+a^2}} - \frac{1-\cos(2\sqrt{n^2+a^2}T)}{2}\right)(y_1,e_n)^2 \\
&+\sum_{n=1}^{+\infty} \left(\frac{T}{2} + \frac{\sin(\sqrt{n^2+a^2}T)}{4\sqrt{n^2+a^2}} - \frac{1-\cos(2\sqrt{n^2+a^2}T)}{2(n^2+a^2)}\right)(y_2,e_n)^2 \\
&\geq \sum_{n=1}^{+\infty} \frac{1}{n^2+a^2}\left(\frac{T}{2} - \frac{1}{4\sqrt{1+a^2}} - 1\right)(y_1,e_n)^2 \\
&+\sum_{n=1}^{+\infty} \left(\frac{T}{2} - \frac{1}{4\sqrt{1+a^2}} - \frac{1}{1+a^2}\right)(y_2,e_n)^2.
\end{aligned}$$

On the other hand, $-A$ is an isomorphism between $H_0^1([0,\pi])$ and $H^{-1}([0,\pi])$ (see [36, Corollary 1.1.5]) and

$$\|y\|^2_{H_0^1} = (-Ay, y). \tag{40}$$

(see [9, Page 216]). Moreover, since the map $y \to -y''$ is self-adjoint and positive definite (see [14, page 455]), $-A$ is self-adjoint and positive definite as well. Thus, it is possible to define a unique positive definite square root $(-A)^{\frac{1}{2}} : H_0^1([0,\pi]) \to H^{-1}([0,\pi])$ and its inverse $(-A)^{-\frac{1}{2}} : H^{-1}([0,\pi]) \to H_0^1([0,\pi])$ respectively defined as

$$(-A)^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} \sqrt{n^2+a^2}(y,e_n)e_n$$

and

$$(-A)^{-\frac{1}{2}}y = \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+a^2}}(y,e_n)e_n.$$

Hence, from (40), we get that

$$\|y\|^2_{H_0^1} = ((-A)^{\frac{1}{2}}y, (-A)^{\frac{1}{2}}y) = \|(-A)^{\frac{1}{2}}y\|^2_{H^{-1}},$$

i.e. that

$$\|y_1\|^2_{H^{-1}} = \|A^{-\frac{1}{2}}y_1\|^2_{H^1_0} = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2}(y_1, e_n)^2.$$

Since $\|y_2\|_2^2 = \sum_{n=1}^{\infty}(y_2, e_n)^2$, we conclude that

$$\left[\max\{\|y_1\|_{H^{-1}}, \|y_2\|_2\}\right]^2 = \max\left\{\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}(y_1,e_n)^2, \sum_{n=1}^{\infty}(y_2,e_n)^2\right\}$$

and the surjectivity of M follows assuming

$$T > \frac{1}{2\sqrt{1+a^2}} + 2.$$

Remark 4.1. The controllability result just proved extends the one contained in [21, Section 4] to the case when $a \neq 0$ and the nonlinear term depends also on z'_t.

5. Conclusions.

This paper introduces a new definition of controllability for second-order problems in Banach spaces which takes into account both the solution and its derivative at the final point using a unique control. Subsequently, the conditions sufficient for such controllability are studied using the Schauder fixed point theorem together with the approximation solvability method and the weak topology. The paper concludes by applying the obtained result to the system governed by the one-dimensional Klein-Gordon equation. The main advantage of our research is that, to the best of our knowledge, there has been no correct research on the exact controllability for the second-order problems in Banach spaces considering one control and not only the target value for the solution but also for its first derivative.

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