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Profinite groups and the fixed points of coprime automorphisms ☆

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ABSTRACT

The main result of the paper is the following theorem. Let q be a prime and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$. Then the group G is locally nilpotent.

1. Introduction

Let A be a finite group acting on a finite group G. Many well-known results show that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has influence over the structure of G. The influence is especially strong if $(|A|, |G|) = 1$, that is, the action of A on G is coprime. Let $A^{\#}$ denote the set of non-identity elements of A. The following theorem was proved in [12].

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Theorem 1.1. *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ is nilpotent for each $a \in A^{\#}$. Then G is nilpotent.*

There are well-known examples that show that the above theorem fails if the order of A is q^2. Indeed, let p and r be odd primes and H and K the groups of order p and r respectively. Denote by $A = \langle a_1, a_2 \rangle$ the noncyclic group of order four with generators a_1, a_2 and by Y the semidirect product of K by A such that a_1 acts on K trivially and a_2 takes every element of K to its inverse. Let B be the base group of the wreath product $H \wr Y$ and note that $[B, a_1]$ is normal in $H \wr Y$. Set $G = [B, a_1]K$. The group G is naturally acted on by A and $C_G(A) = 1$. Therefore $C_G(a)$ is abelian for each $a \in A^{\#}$. But, of course, G is not nilpotent.

In [11] the situation of Theorem 1.1 was studied in greater detail and the following result was obtained.

Theorem 1.2. *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ is nilpotent of class at most c for each $a \in A^{\#}$. Then G is nilpotent and the class of G is bounded by a function depending only on q and c.*

Of course, the above results have a bearing on profinite groups. By an automorphism of a profinite group we always mean a continuous automorphism. A group A of automorphisms of a profinite group G is coprime if A has finite order while G is an inverse limit of finite groups whose orders are relatively prime to the order of A. Using the routine inverse limit argument it is easy to deduce from Theorem 1.1 and Theorem 1.2 that if G is a profinite group admitting a coprime group of automorphisms A of order q^3 such that $C_G(a)$ is pronilpotent for all $a \in A^{\#}$, then G is pronilpotent; and if $C_G(a)$ is nilpotent for all $a \in A^{\#}$, then G is nilpotent. Yet, certain results on fixed points in profinite groups cannot be deduced from corresponding results on finite groups. The purpose of the present paper is to establish the following theorem.

Theorem 1.3. *Let q be a prime and A an elementary abelian q-group of order at least q^3. Suppose that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$. Then G is locally nilpotent.*

Recall that a group is locally nilpotent if every finitely generated subgroup is nilpotent. Though Theorem 1.3 looks similar to Theorems 1.1 and 1.2, in fact it cannot be deduced directly from those results. Moreover, the proof of Theorem 1.3 is very much different from those of Theorems 1.1 and 1.2. In particular, unlike the other results, Theorem 1.3 relies heavily on the Lie-theoretical techniques created by Zelmanov in his solution of the restricted Burnside problem [14,15]. The general scheme of the proof of Theorem 1.3 is similar to that of the result in [5].

2. Preparatory work

Throughout the paper we use without special references the well-known properties of coprime actions:

Lemma 2.1. *If a group A acts coprimely on a finite group G, then $C_{G/N}(A) = C_G(A)N/N$ for any A-invariant normal subgroup N.*

Lemma 2.2. *If A is a noncyclic abelian group acting coprimely on a finite group G, then G is generated by the subgroups $C_G(B)$, where A/B is cyclic.*

The above results both easily extend to the case of coprime automorphisms of profinite groups (see for example [9, Lemma 3.2]). Let x, y be elements of a group, or a Lie algebra. We define inductively

$$[x, {}_0 y] = x \text{ and } [x, {}_n y] = [[x, {}_{n-1} y], y] \quad \text{for } n \geq 1.$$

Let L be a Lie algebra. An element $a \in L$ is called ad-nilpotent if there exists a positive integer n such that $[x, {}_n a] = 0$ for all $x \in L$. Let $X \subseteq L$ be any subset of L. By a commutator in elements of X we mean any element of L that can be obtained as a Lie product of elements of X with some system of brackets. The next theorem is due to Zelmanov (see [16] or [17]).

Theorem 2.3. *Let L be a Lie algebra generated by finitely many elements $a_1, a_2, \dots, a_m$ such that each commutator in these generators is ad-nilpotent. If L satisfies a polynomial identity, then L is nilpotent.*

An important criterion for a Lie algebra to satisfy a polynomial identity is the following theorem.

Theorem 2.4 *(Bahturin–Linchenko–Zaicev). Assume that a finite group A acts on a Lie algebra L by automorphisms in such a manner that $C_L(A)$, the subalgebra formed by fixed elements, satisfies a polynomial identity. Assume further that the characteristic of the ground field is either 0 or prime to the order of A. Then L satisfies a polynomial identity.*

The above theorem was first proved by Bahturin and Zaicev in the case where A is soluble [1] and later extended by Linchenko to the general case [7]. In the present paper we only require the case where A is abelian.

Let G be a (profinite) group. A series of subgroups

$$G = G_1 \geq G_2 \geq \dots \tag{$*$}$$

is called an N-series if it satisfies $[G_i, G_j] \le G_{i+j}$ for all $i, j \ge 1$. Here and throughout the paper when dealing with a profinite group we consider only closed subgroups. Obviously any N-series is central, i.e. $G_i/G_{i+1} \le Z(G/G_{i+1})$ for any i. Let p be a prime. An N-series is called N_p-series if $G_i^p \le G_{pi}$ for all i. Given an N-series (∗), let $L^*(G)$ be the direct sum of the abelian groups $L_i^* = G_i/G_{i+1}$, written additively. Commutation in G induces a binary operation $[,]$ in L. For homogeneous elements $xG_{i+1} \in L_i^*, yG_{j+1} \in L_j^*$ the operation is defined by

$$[xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in L_{i+j}^*$$

and extended to arbitrary elements of $L^*(G)$ by linearity. It is easy to check that the operation is well-defined and that $L^*(G)$ with the operations $+$ and $[,]$ is a Lie ring. If all quotients G_i/G_{i+1} of an N-series (∗) have prime exponent p then $L^*(G)$ can be viewed as a Lie algebra over $\mathbb{F}_p$, the field with p elements. In the important case where the series (∗) is the p-dimension central series (also known under the name of Zassenhaus–Jennings–Lazard series) of G we write $L_p(G)$ for the subalgebra generated by the first homogeneous component G_1/G_2 in the associated Lie algebra over the field with p elements. Observe that the p-dimension central series is an N_p-series (see [3, p. 250] for details).

Any automorphism of G in the natural way induces an automorphism of $L^*(G)$. If G is profinite and α is a coprime automorphism of G, then the subring (subalgebra) of fixed points of α in $L^*(G)$ is isomorphic with the Lie ring associated to the group $C_G(\alpha)$ via the series formed by intersections of $C_G(\alpha)$ with the terms of the series (∗) (see [10] for more details).

Let $w = w(x_1, x_2, \dots, x_k)$ be a group-word. Let H be a subgroup of a group G and $g_1, g_2, \dots, g_k \in G$. We say that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \dots, g_kH$ if $w(g_1h_1, g_2h_2, \dots, g_kh_k) = 1$ for all $h_1, h_2, \dots, h_k \in H$. Wilson and Zelmanov showed in [13] that if a profinite group G has an open subgroup H and elements $g_1, g_2, \dots, g_k$ such that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \dots, g_kH$, then $L_p(G)$ satisfies a polynomial identity for each prime p. More precisely, the proof in [13] shows that whenever a profinite group G has an open subgroup H and elements $g_1, g_2, \dots, g_k$ such that the law $w \equiv 1$ is satisfied on the cosets $g_1H, g_2H, \dots, g_kH$, the Lie algebra $L^*(G)$ satisfies a multilinear polynomial identity for any prime p and any N_p-series (∗) in G.

Lemma 2.5. *For any locally nilpotent profinite group G there exist a positive integer n, elements $g_1, g_2 \in G$ and an open subgroup $H \le G$ such that the law $[x, {}_n y] \equiv 1$ is satisfied on the cosets g_1H, g_2H.*

Proof. Since any finitely generated subgroup of G is nilpotent, for every pair of elements g_1, g_2 there exists a positive number j such that $[g_1, {}_j g_2] = 1$. For each integer i we set

$$S_i = \{(x, y) \in G \times G : [x, {}_i y] = 1\}.$$

Since the sets S_i are closed in $G \times G$ and have union $G \times G$, by Baire category theorem [4, p. 200] at least one of these sets has a non-empty interior. Therefore we can find an open subgroup H in G, elements $g_1, g_2 \in G$ and an integer n with the required property. $\square$

The following proposition is now straightforward.

Proposition 2.6. *Assume that a finite group A acts coprimely on a profinite group G in such a manner that $C_G(A)$ is locally nilpotent. Then for each prime p the Lie algebra $L_p(G)$ satisfies a multilinear polynomial identity.*

Proof. Let $L = L_p(G)$. In view of Theorem 2.4 it is sufficient to show that $C_L(A)$ satisfies a polynomial identity. We know that $C_L(A)$ is isomorphic with the Lie algebra associated with the central series of $C_G(A)$ obtained by intersecting $C_G(A)$ with the p-dimension central series of G. Since $C_G(A)$ is locally nilpotent, Lemma 2.5 applies. Thus, the Wilson–Zelmanov result [13, Theorem 1] tells us that $C_L(A)$ satisfies a polynomial identity. $\square$

We will also require the following lemma that essentially is due to Wilson and Zelmanov (cf. [13, Lemma in Section 3]).

Lemma 2.7. *Let G be a profinite group and $g \in G$ an element such that for any $x \in G$ there exists a positive n with the property that $[x, {}_n g] = 1$. Let $L^*(G)$ be the Lie algebra associated with G using an N_p-series (∗) for some prime p. Then the image of g in $L^*(G)$ is ad-nilpotent.*

Finally, we quote a useful lemma from [5].

Lemma 2.8. *Let L be a Lie algebra and H a subalgebra of L generated by m elements $h_1, \ldots, h_m$ such that all commutators in the generators h_i are ad-nilpotent in L. If H is nilpotent, then we have $[L, \underbrace{H, \ldots, H}_{d}] = 0$ for some number d.*

3. Proof

As usual, for a profinite group G we denote by $\pi(G)$ the set of prime divisors of the orders of finite continuous homomorphic images of G. We say that G is a π-group if $\pi(G) \subseteq \pi$ and G is a π'-group if $\pi(G) \cap \pi = \emptyset$. If m is an integer, we denote by $\pi(m)$ the set of prime divisors of m. If π is a set of primes, we denote by $O_\pi(G)$ the maximal normal π-subgroup of G and by $O_{\pi'}(G)$ the maximal normal π'-subgroup.

We are ready to embark on the proof of Theorem 1.3.

Proof of Theorem 1.3. Recall that q is a prime and A an elementary abelian group of order q^3 acting coprimely on a profinite group G in such a manner that $C_G(a)$ is locally nilpotent for all $a \in A^{\#}$. We wish to show that G is locally nilpotent. In view of Ward's Theorem 1.1 the group G is pronilpotent and therefore G is the Cartesian product of its Sylow subgroups.

Choose $a \in A^{\#}$. By Lemma 2.5 $C_G(a)$ contains an open subgroup H and elements u, v such that for some n the law $[x,{}_n y] \equiv 1$ is satisfied on the cosets uH, vH. Let $[C_G(a) : H] = m$ and let $\pi_1 = \pi(m)$. Denote $O_{\pi_1'}(C_G(a))$ by T. Since T is isomorphic to the image of H in $C_G(a)/O_{\pi_1}(C_G(a))$, it is easy to see that T satisfies the law $[x,{}_n y] \equiv 1$, that is, T is n-Engel. By the result of Burns and Medvedev [2] the subgroup T has a nilpotent normal subgroup U such that T/U has finite exponent, say e. Set $\pi_2 = \pi(e)$. Of course, the sets π_1 and π_2 depend on the choice of $a \in A^{\#}$ so strictly speaking they should be denoted by $\pi_1(a)$ and $\pi_2(a)$. For each such choice let $\pi_a = \pi_1(a) \cup \pi_2(a)$.

We repeat this argument for every $a \in A^{\#}$. Set $\pi = \cup_{a \in A^{\#}} \pi_a$ and $K = O_{\pi'}(G)$. Since all sets $\pi_1(a)$ and $\pi_2(a)$ are finite, so is π. The choice of the set π guarantees that $C_K(a)$ is nilpotent for every $a \in A^{\#}$. Thus, by Theorem 1.2, the subgroup K is nilpotent. Let $p_1, p_2, \ldots, p_r$ be the finitely many primes in π and let $P_1, P_2, \ldots, P_r$ be the corresponding Sylow subgroups of G. Then $G = P_1 \times P_2 \times \cdots \times P_r \times K$ and therefore it is sufficient to show that each subgroup P_i is locally nilpotent. Thus, from now on without loss of generality we assume that G is a pro-p group for some prime p. Since every finite subset of G is contained in a finitely generated A-invariant subgroup, we can further assume that G is finitely generated.

Let $A_1, A_2, \ldots, A_s$ be the distinct maximal subgroups of A. We denote by $D_j = D_j(G)$ the terms of the p-dimension central series of G. Set $L = L_p(G)$ and $L_j = L \cap (D_j/D_{j+1})$, so that $L = \oplus L_j$. The group A naturally acts on L. Since each subgroup A_i is noncyclic, by Lemma 2.2 we have $L = \sum_{a \in A_i^{\#}} C_L(a)$ for every $i \le s$.

Let $L_{ij} = C_{L_j}(A_i)$. Again by Lemma 2.2, for any j we have

$$L_j = \sum_{1 \le i \le s} L_{ij}.$$

In view of Lemma 2.1 for any $l \in L_{ij}$ there exists $x \in D_j \cap C_G(A_i)$ such that $l = xD_{j+1}$. Therefore, by Lemma 2.7, the element l is ad-nilpotent in $C_L(a)$ for every $a \in A_i^{\#}$. Since $L = \sum_{a \in A_i^{\#}} C_L(a)$, we conclude that

$$\text{any element } l \text{ in } L_{ij} \text{ is ad-nilpotent in } L. \qquad (**)$$

Let ω be a primitive qth root of unity and $\overline{L} = L \otimes \mathbb{F}_p[\omega]$. We can view $\overline{L}$ both as a Lie algebra over $\mathbb{F}_p$ and that over $\mathbb{F}_p[\omega]$. It is natural to identify L with the $\mathbb{F}_p$-subalgebra $L \otimes 1$ of $\overline{L}$. We note that if an element $x \in L$ is ad-nilpotent of index r, say, then the "same" element $x \otimes 1$ is ad-nilpotent in $\overline{L}$ of the same index r. Put $\overline{L_j} = L_j \otimes \mathbb{F}_p[\omega]$; then $\overline{L} = \langle \overline{L_1} \rangle$, since $L = \langle L_1 \rangle$, and $\overline{L}$ is the direct sum of the homogeneous components $\overline{L_j}$.

The group A acts naturally on $\overline{L}$, and we have $\overline{L_{ij}} = C_{\overline{L_j}}(A_i)$, where $\overline{L_{ij}} = L_{ij} \otimes \mathbb{F}_p[\omega]$. Let us show that

$$\text{any element } y \in \overline{L_{ij}} \text{ is ad-nilpotent in } \overline{L}. \tag{$***$}$$

Since $\overline{L_{ij}} = L_{ij} \otimes \mathbb{F}_p[\omega]$, we can write

$$y = x_0 + \omega x_1 + \omega^2 x_2 + \cdots + \omega^{q-2} x_{q-2}$$

for some $x_0, x_1, x_2, \ldots, x_{q-2} \in L_{ij}$, so that each of the summands $\omega^t x_t$ is ad-nilpotent by $(**)$. Set $J = \langle x_0, \omega x_1, \ldots, \omega^{q-2} x_{q-2} \rangle$. This is the subalgebra generated by $x_0, \omega x_1, \ldots, \omega^{q-2} x_{q-2}$. Note that $J \subseteq C_{\overline{L}}(A_i)$. A commutator of weight k in the elements $\omega^t x_t$ has the form $\omega^s x$ for some x that belongs to L_{im}, where $m = kj$. By $(**)$ the element x is ad-nilpotent and so such a commutator must be ad-nilpotent.

Proposition 2.6 tells us that the Lie algebra L satisfies a multilinear polynomial identity. The multilinear identity is also satisfied in $\overline{L}$ and so it is satisfied in J. Hence by Theorem 2.3 J is nilpotent. Lemma 2.8 now says that $[\overline{L}, \underbrace{J, \ldots, J}_{d}] = 0$ for some d. This establishes $(***)$.

Since A is abelian and the ground field is now a splitting field for A, every $\overline{L_j}$ decomposes in the direct sum of common eigenspaces for A. In particular, $\overline{L_1}$ is spanned by finitely many common eigenvectors for A. Hence $\overline{L}$ is generated by finitely many common eigenvectors for A from $\overline{L_1}$. Every common eigenspace is contained in the centralizer $C_{\overline{L}}(A_i)$ for some $i \leq s$, since A is of order q^3. We also note that any commutator in common eigenvectors is again a common eigenvector. Thus, if $l_1, \ldots, l_r \in \overline{L_1}$ are common eigenvectors for A generating $\overline{L}$ then any commutator in these generators belongs to some $\overline{L_{ij}}$ and therefore, by $(***)$, is ad-nilpotent.

As we have seen, $\overline{L}$ satisfies a polynomial identity. It follows from Theorem 2.3 that $\overline{L}$ is nilpotent. We now deduce that L is nilpotent as well.

According to Lazard [6] the nilpotency of L is equivalent to G being p-adic analytic. The Lubotzky–Mann theory [8] now tells us that G is of finite rank, that is, all closed subgroups of G are finitely generated. In particular, we conclude that $C_G(a)$ is finitely generated for every $a \in A^{\#}$. It follows that the centralizers $C_G(a)$ are nilpotent. Theorem 1.2 now tells us that G is nilpotent. The proof is complete. □

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