A lower bound for the complex flow number of a graph: A geometric approach / Mattiolo, D., Mazzuoccolo, G., Rajnik, J., Tabarelli, G.. - In: JOURNAL OF GRAPH THEORY. - ISSN 0364-9024. - 106:2(2024), pp. 239-256. [10.1002/jgt.23075]

A lower bound for the complex flow number of a graph: a geometric approach.

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October 3, 2023

Abstract

Let $r \geq 2$ be a real number. A complex nowhere-zero r-flow on a graph G is an orientation of G together with an assignment $\varphi\colon E(G) \to \mathbb{C}$ such that, for all $e \in E(G)$, the Euclidean norm of the complex number $\varphi(e)$ lies in the interval $[1, r-1]$ and, for every vertex, the incoming flow is equal to the outgoing flow. The complex flow number of a bridgeless graph G, denoted by $\phi_{\mathbb{C}}(G)$, is the minimum of the real numbers r such that G admits a complex nowhere-zero r-flow. The exact computation of $\phi_{\mathbb{C}}$ seems to be a hard task even for very small and symmetric graphs. In particular, the exact value of $\phi_{\mathbb{C}}$ is known only for families of graphs where a lower bound can be trivially proved. Here, we use geometric and combinatorial arguments to give a non trivial lower bound for $\phi_{\mathbb{C}}(G)$ in terms of the odd-girth of a cubic graph G (i.e. the length of a shortest odd cycle) and we show that this lower bound is tight. This result relies on the exact computation of the complex flow number of the wheel graph W_n. In particular, we show that for every odd n, the value of $\phi_{\mathbb{C}}(W_n)$ arises from one of three suitable configurations

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of points in the complex plane according to the congruence of n modulo 6.

Keywords: nowhere-zero flow; cubic graph; complex flow
MSC2020: 05C21.

1 Introduction

The theory of integer nowhere-zero flows on finite graphs represents a very active research area in graph theory (see for example [2], [6], [7], [8], [12], [13], [15], [17]). The generalization to real numbers is also well-studied (see for instance [3], [4], [5], [9], [11]), while very few is known in the complex case or, more in general, for flows taking values in $\mathbb{R}^d$ (see [14], [16], [18]). Let $r \geq 2$ be a real number. A d-dimensional nowhere-zero r-flow on a graph G, an (r, d)-NZF on G from now on, is an orientation of G together with an assignment $\varphi\colon E(G) \to \mathbb{R}^d$ such that, for all $e \in E(G)$, the (Euclidean) norm of $\varphi(e)$ lies in the interval $[1, r-1]$ and, for every vertex, the sum of the inflow and outflow is the zero element in $\mathbb{R}^d$. The d-dimensional flow number of a bridgeless graph G, denoted by $\phi_d(G)$, is defined as the infimum of the real numbers r such that G admits an (r, d)-NZF.

In this paper, we consider only the case $d = 2$. For this reason, in order to simplify the notation, we refer to an $(r, 2)$-NZF on a bridgeless graph G as a *complex nowhere-zero r-flow* on G, and to its 2-dimensional flow number $\phi_2(G)$ as its *complex flow number*, denoting it from now on by $\phi_{\mathbb{C}}(G)$.

It can be easily proved (see [10]) that $\phi_{\mathbb{C}}(G)$ (and more in general $\phi_d(G)$) is actually a minimum. So, given a bridgeless graph G, it always admits a complex $\phi_{\mathbb{C}}(G)$-flow, not necessarily unique. We will refer to a complex $\phi_{\mathbb{C}}(G)$-flow as an *optimal complex flow* of G.

A general upper bound for $\phi_{\mathbb{C}}(G)$, where G is a bridgeless graph, has been proposed by the authors in [10]. However, the exact value of $\phi_{\mathbb{C}}(G)$ is known only when G belongs to very specific classes of graphs. Indeed, establishing good lower bounds for $\phi_{\mathbb{C}}(G)$ remains in general the hardest task in the study of this parameter, even if we focus on the class of cubic graphs. Note that the restriction to the class of cubic graphs is standard in flow theory and it can be applied to complex flows as well.

In this paper, we use a combination of geometric and combinatorial arguments to prove a non-trivial lower bound for $\phi_{\mathbb{C}}(G)$ in terms of the length of a shortest odd cycle of a bridgeless cubic graph G.

Our main result is a straightforward consequence of the exact determination of the complex flow number for every wheel graph W_n of order $n+1$. In particular, we show that there exists an optimal complex flow of W_n which can be described by one of three suitable sequences of points in the complex plane, according to the congruence of n modulo 6.

2 Complex flow number of W_n

For every integer $n \geq 3$, let W_n be the wheel graph with $n+1$ vertices and consider the orientation of its edges as in Figure 1. More precisely, the n vertices of the external cycle of W_n are labeled with $v_0, v_1, ..., v_{n-1}$ and the central vertex with u. All edges uv_j and $v_{j-1}v_j$ in the chosen orientation of W_n are directed towards v_j (here and in what follows indices are taken modulo n).

Let φ be a $(\lambda+1,2)$-NZF of W_n. Set

$$\varphi(uv_j) = z_j \in \mathbb{C}, j \in \{0, ..., n-1\},$$

$$\varphi(v_jv_{j+1}) = p_j \in \mathbb{C}, j \in \{0, ..., n-1\}.$$

In particular, since φ is a $(\lambda+1,2)$-NZF of W_n, the norm of each flow value is a real number which lies in the interval $[1,\lambda]$, i.e. $1 \leq |p_j|, |z_j| \leq \lambda$ holds. Moreover, the relation

$$z_j = p_j - p_{j-1} \tag{1}$$

holds for every $j = 0, ..., n-1$. Relation (1) suggests that the knowledge of all values p_j is sufficient to reconstruct the entire flow. Hence, we can represent any $(\lambda+1,2)$-NZF of W_n as a cyclic sequence (i.e. the

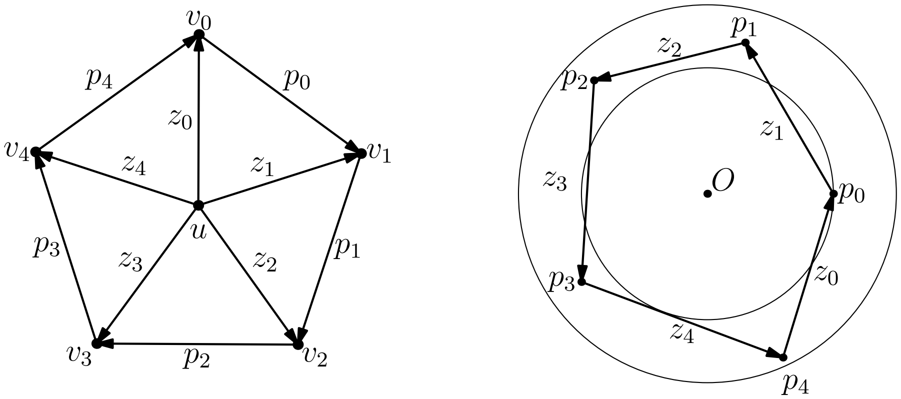

Figure 1: A representation of a complex flow of W_5.

first element of the sequence is considered to succeed the last one) of n points $(p_0, \ldots, p_{n-1})$ in the complex plane. We often need to refer to the vectors having ends in two consecutive points of the sequence. In particular, we denote by $p_{j-1}p_j$ the vector in the complex plane having its tail in the point p_{j-1} and its head in the point p_j. With a slight abuse of notation, we will sometimes denote the vector $p_{j-1}p_j$ by z_j, stressing, when necessary "vector z_j".

On the other hand, an arbitrary cyclic sequence $(p_0, \ldots, p_{n-1})$ of n points represents a $(\lambda + 1, 2)$-NZF of W_n(see Figure 1) if:

- each point p_j belongs to the circular crown between circumferences centered in the origin and of radius 1 and λ, denoted by $\mathcal{C}_I$ and $\mathcal{C}_E$ respectively;
- the norm of each vector z_j lies in the interval $[1, \lambda]$.

In what follows, by using such a representation, we first exhibit a complex flow of W_n for each odd n and then we prove its optimality in Theorem 1. For even values of n, W_n has a $(3, 1)$-NZF. Therefore, by Proposition 1 in [14], $\phi_{\mathbb{C}}(W_n) = 2$.

2.1 An upper bound for the complex flow number of W_n

Let n be an odd number and set $t = \lfloor \frac{n}{6} \rfloor$. We distinguish three cases according to the congruence of n modulo 6. We furnish a geometric description of each case and then we formally give the sequences of points representing the flows. Figure 2 represents an example of the described sequences for each possible odd congruence class modulo 6.

For $n \equiv 5 \pmod 6$, we consider points p_j as the vertices of a regular star polygon $\{\frac{n}{t+1}\}$ (following the standard Schläfli notation, see [1]) inscribed in $\mathcal{C}_I$. The length of each side of the polygon is equal to $2\sin(\frac{\pi}{6} \cdot \frac{n+1}{n})$. For $n \equiv 1 \pmod 6$, we construct points p_j on $\mathcal{C}_I$ as follows: starting from p_0 and moving in clockwise direction, we have p_1 at distance 1 from p_0. All other points are obtained by moving on $\mathcal{C}_I$ in anticlockwise direction, each point at distance $2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1})$ from the previous one. The distance between p_{n-1} and p_0 results to be also $2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1})$. For $n \equiv 3 \pmod 6$, we construct points p_j on $\mathcal{C}_I$ except p_1 which belongs to $\mathcal{C}_E$. Starting from $p_0 \in \mathcal{C}_I$ and moving in clockwise direction, we have $p_1 \in \mathcal{C}_E$ at distance 1 from p_0. Then, $p_2 \in \mathcal{C}_I$ is at distance 1 from p_1, again in clockwise direction. All other points are obtained following $\mathcal{C}_I$ in anticlockwise direction, each at distance $2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1})$ from the previous one. Once again also the distance between p_{n-1} and p_0 is $2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1})$.

Hence, the sequences of points result to be the followings.

- if $n \equiv 5 \pmod 6$,

$$p_j = e^{ij\left(\frac{\pi}{3} \cdot \frac{n+1}{n}\right)}, \forall j : 0 \leq j \leq n-1,$$

- if $n \equiv 1 \pmod 6$,

$$p_0 = e^{i\frac{\pi}{3}} \text{ and } p_{j+1} = e^{ij\left(\frac{\pi}{3} \cdot \frac{n}{n-1}\right)}, \forall j : 0 \leq j \leq n-2,$$

- if $n \equiv 3 \pmod 6$,

$$p_0 = e^{2i\left(\frac{\pi}{6} \cdot \frac{2n-3}{n-1}\right)}, p_1 = 2\sin\left(\frac{\pi}{6} \cdot \frac{n}{n-1}\right) e^{i\left(\frac{\pi}{6} \cdot \frac{2n-3}{n-1}\right)} \text{ and}$$

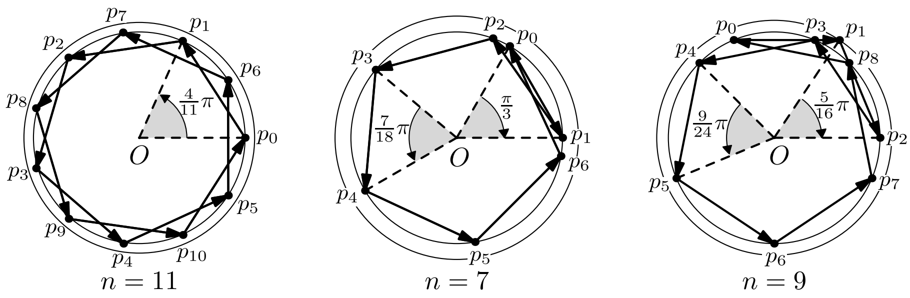

Figure 2: Three sequences corresponding to, from left to right, optimal flows of W_{11}, W_7 and W_9.

$$p_{j+2} = e^{ij\left(\frac{\pi}{3}\cdot\frac{n}{n-1}\right)}, \forall j : 0 \leq j \leq n-3.$$

For each n, we denote by λ^* the maximum between the distances of two consecutive points of the corresponding sequence and the norms of the points p_j's. Observe that, if $n \equiv 1, 3 \mod 6$, then $\lambda^* = 2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1})$, while if $n \equiv 5 \mod 6$, $\lambda^* = 2\sin(\frac{\pi}{6} \cdot \frac{n+1}{n})$. Hence, for each odd n, we have constructed a $(\lambda^* + 1, 2)$-NZF of W_n and $\lambda^* + 1$ gives an upper bound for $\phi_{\mathbb{C}}(W_n)$.

Remark 1. *For each* $n \geq 3$, $\phi_{\mathbb{C}}(W_n) \leq \phi_{\mathbb{C}}(W_3) \leq 1 + \sqrt{2}$.

We will make use of this remark along the proof of Theorem 1 in order to guarantee a general upper bound for $\phi_{\mathbb{C}}(W_n)$ which will be sufficiently small for our aims.

2.2 A lower bound for the complex flow number of W_n

In this section we prove that, for each odd n, the value of λ^* determined by the corresponding sequence described in Section 2.1, is indeed also a lower bound for $\phi_{\mathbb{C}}(W_n)$.

In other words, let φ be an optimal $(\lambda + 1, 2)$-NZF of W_n, n odd, then we already proved in Section 2.1 that $\lambda \leq \lambda^*$, and we want to show that $\lambda = \lambda^*$.

From now on we can assume $\lambda \leq \sqrt{2}$ due to Remark 1. Let $(p_0, \ldots, p_{n-1})$ be the associated cyclic sequence of points p_j. As already remarked, we denote the vector $p_{j-1}p_j$ by z_j for each $j \in \{0, \ldots, n-1\}$, where all indices are taken modulo n. In particular, we have $\max_j\{|z_j|, |p_j|\} = \lambda$ since φ is optimal. If $0 \neq \theta \in (-\pi, \pi)$ denotes the amplitude of an angle and $\theta > 0$ ($\theta < 0$), then the positive (negative) rotation is by definition in anticlockwise (clockwise) direction. Similarly, if $p_{j-1} = |p_{j-1}|e^{i\alpha_{j-1}}$ and $p_j = |p_j|e^{i\alpha_j}$ are two consecutive points in the cyclic sequence then the vector z_j is said to be positively (negatively) oriented, or simply positive (negative), if $\alpha_j - \alpha_{j-1}$ is positive (negative). Note that $\alpha_{j-1} \neq \alpha_j$ since $\lambda \leq \sqrt{2}$ (< 2) (see Remark 1).

First of all, let us define some geometric transformations of the cyclic sequence $(p_0, \ldots, p_{n-1})$ that will be largely used in what follows. Let $\theta \in \mathbb{R}$ and $h, k \in \{0, \ldots, n-1\}$. Define $\rho_{h,k}(\theta)$ as the transformation which rotates all points $p_h, p_{h+1}, \ldots, p_k$ around the origin by an angle of θ and fixes all the others. Note that we are considering a cyclic sequence, hence k could be less than h and $\rho_{h,k}(\theta) \neq \rho_{k,h}(\theta)$. Indeed, if $h < k$, we have

$$\begin{aligned}&\rho_{h,k}(\theta)(p_0, \ldots, p_h, \ldots, p_k, \ldots, p_{n-1}) = \\ &\qquad = (p_0, \ldots, p_{h-1}, p_h e^{i\theta}, \ldots, p_k e^{i\theta}, p_{k+1}, \ldots, p_{n-1}),\end{aligned}$$

and

$$\begin{aligned}&\rho_{k,h}(\theta)(p_0, \ldots, p_h, \ldots, p_k, \ldots, p_{n-1}) = \\ &\qquad = (p_0 e^{i\theta}, \ldots, p_h e^{i\theta}, p_{h+1}, \ldots, p_{k-1}, p_k e^{i\theta}, \ldots, p_{n-1} e^{i\theta}).\end{aligned}$$

Claim 1. *Let $(p_0, \ldots, p_{n-1})$ be a cyclic sequence of points representing a $(\lambda + 1, 2)$-NZF of W_n. Then, for any $h \neq k \in \{0, 1, \ldots, n-1\}$, there is an angle θ such that the sequence $\rho_{h,k-1}(\theta)(p_0, \ldots, p_{n-1}) = (p'_0, \ldots, p'_{n-1})$ satisfies $|p_j| = |p'_j|$ for all j and $|z_j| = |z'_j|$ for all $j \notin \{h, k\}$. Moreover, the angle θ can be chosen so that*

(a) If z_h and z_k have the same orientation, $|z_h| < \lambda$ and $|z_k| > 1$, then $\lambda > |z'_h| > |z_h|$ and $1 < |z'_k| < |z_k|$.

(b) If z_h and z_k have opposite orientations and $|z_h|, |z_k| < \lambda$, then $\lambda > |z'_h| > |z_h|$ and $\lambda > |z'_k| > |z_k|$.

(c) If z_h and z_k have opposite orientations and $|z_h|, |z_k| > 1$, then $1 < |z'_h| < |z_h|$ and $1 < |z'_k| < |z_k|$.

For $j \in \{h, k\}$, if $|z'_j| > |z_j|$ ($|z'_j| < |z_j|$), we say that the transformation lengthens (shortens) z_j (by an arbitrary small factor).

Proof. From the definition of $\rho_{h,k-1}(\theta)$, it follows directly that $|p_j| = |p'_j|$ for all j and $|z_j| = |z'_j|$ for all $j \notin \{h, k\}$. For Case *(a)* we choose θ positive or negative according to the common orientation of the vectors z_h and z_k. In Cases *(b)* and *(c)*, say that z_h is positive and z_k is negative, we choose θ positive or negative, respectively. In all the cases the absolute value of θ can be chosen arbitrary small to ensure arbitrary small scale factor and then $1 < |z'_h|, |z'_k| < \lambda$. □

For our aims, we also need to define $\sigma_{h,k}(\theta)$ as the transformation which rotates the point p_h around the point p_k by an angle θ and fixes any other point of the sequence.

$$\sigma_{h,k}(\theta)(p_0, \ldots, p_h, \ldots, p_{n-1}) = (p_0, \ldots, (p_h - p_k)e^{i\theta} + p_k, \ldots, p_{n-1}).$$

The main idea of the proof is choosing time by time an optimal flow φ of W_n satisfying additional minimality assumptions (explained later in details). We will show that if such a φ does not correspond to one of the three sequences (up to isometries) described in Section 2.1, then we can modify it to obtain a new sequence which contradicts the minimality assumptions on φ.

We need to distinguish two cases, corresponding to Section 2.2.1 and Section 2.2.2, respectively.

2.2.1 Each vector z_j has norm less than λ

The first case we consider is when $|z_j| < \lambda$ for every $j \in \{0, 1, ..., n-1\}$. In this case, consider the optimal 2-dimensional flows of W_n having the minimum number, say m_1, of vectors z_j with $|z_j| = 1$. Among them, choose φ as one with the minimum number, say m_2, of points p_j with

$|p_j| = \lambda$ (i.e. $p_j \in \mathcal{C}_E$). Moreover, without loss of generality, we can assume that φ has at least one of the vectors z_j which is positive, otherwise we can simply consider $-\varphi$.

First of all, we prove that by our choice of φ the relation $|z_j| = 1$ follows for every index j and that all vectors z_j are positive.

Suppose by contradiction that there exists an index h such that $|z_h| > 1$. By assumption $|z_h| < \lambda$. If $m_1 > 0$, then there exists k such that $|z_k| = 1$. According to Claim 1, we can lenghten z_k and shorten or lengthen z_h (according to its orientation) constructing a sequence of points having less than m_1 vectors of norm 1, a contradiction. Then, we can assume $m_1 = 0$. Note that since φ is optimal, there exists $l \in \{0, 1, ..., n-1\}$ such that $p_l \in \mathcal{C}_E$. Construct a new sequence by setting $p'_l = (1-\varepsilon)p_l$ and $p'_j = p_j$ for all $j \neq l$. It is possible to choose $\varepsilon > 0$ sufficiently small such that $1 < |p'_l|, |z'_l|, |z'_{l+1}| < \lambda$, so that the new sequence still represents a 2-dimensional flow of W_n. However, the new sequence has no vector z_j of norm 1 like the original sequence, but less than m_2 points belonging to $\mathcal{C}_E$, a contradiction again. Then, we have that $|z_j| = 1$, for every index j.

Assume there exist two indices h, k such that z_h is positive and z_k is negative. Applying Claim 1 we can lengthen both of them to reduce the number of vectors having norm 1, a contradiction with the choice of φ.

Then, all vectors z_j are positive and with $|z_j| = 1$. Now we show that, for each odd n, a sequence of points p_j with such properties corresponds to a $(\lambda+1, 2)$-flow having $\lambda > \lambda^*$. This leads to a contradiction since φ is chosen to be optimal. Indeed, let $\alpha_j > 0$ be the angle subtended by the vector z_j. It holds that $\sum_{j=0}^{n-1} \alpha_j = 2a\pi$ for some positive integer a. Moreover, since $\lambda \leq \sqrt{2} < \Phi$ (Golden Ratio) the angle α_j is at least $2\arcsin(\frac{1}{2\lambda})$ which is the angle obtained with $p_{j-1}, p_j \in \mathcal{C}_E$. Hence, we have

$$\arcsin\left(\frac{1}{2\lambda}\right) \leq \frac{a}{n}\pi.$$

We look for the minimum possible λ which realizes previous inequality. It is clearly obtained when the equality holds. Moreover, since $\lambda > 1$, $\arcsin(\frac{1}{2\lambda}) < \frac{\pi}{6}$ holds, that is $a < \frac{n}{6}$. So, λ is minimum and larger than 1 for $a = \lfloor \frac{n}{6} \rfloor$. Hence, if $n = 6t + h$, we have that $\arcsin(\frac{1}{2\lambda}) = \frac{t}{n}\pi$ and so $\lambda = \frac{1}{2\sin(\frac{t}{n}\pi)}$.

For $n > 1$, it follows that the relations

$$\lambda \geq \frac{1}{2\sin(\frac{n-1}{n} \cdot \frac{\pi}{6})}$$

and

$$\lambda^* \leq 2\sin\left(\frac{n}{n-1} \cdot \frac{\pi}{6}\right)$$

hold. Moreover, direct computations show that the real function

$$\sin\left(\frac{\pi x}{6}\right)\sin\left(\frac{\pi}{6x}\right)$$

has maximum equal to $\frac{1}{4}$ reached only for $x = 1$. Then, $\lambda > \lambda^*$ holds for every odd $n > 1$, a contradiction.

2.2.2 There is a vector z_j of norm λ

Now we can assume that for some $k \in \{0, ..., n-1\}$, it holds $|z_k| = \lambda$.

Without loss of generality assume that the vector z_k is positive. Consider the set of optimal complex flows of W_n having the minimum number, say $m_1 > 0$, of vectors z_j with $|z_j| = \lambda$. Among all such optimal flows, we choose φ in such a way that it has the minimum number, say m_2, of points p_j with $|p_j| = \lambda$, that is with the minimum number of points which belong to $\mathcal{C}_E$.

Claim 2. $|z_j|, |p_j| \in \{1, \lambda\}$ *for every* $j \in \{0, \ldots, n-1\}$. *In particular,* $|z_j| = \lambda$ *if and only if* z_j *is positive (and then* $|z_j| = 1$ *if and only if* z_j *is negative).*

Proof. First we prove that if the vector z_j is positive then $|z_j| = \lambda$. By contradiction suppose there exists $h \in \{0, \ldots, n-1\}$ such that $|z_h| < \lambda$ and z_h is positive. By Claim 1 we can shorten z_k and lengthen z_h yielding an optimal flow having less than m_1 vectors with norm λ, a contradiction.

In a similar way we prove that if the vector z_j is negative then $|z_j| = 1$. By contradiction assume there exists $h \in \{0, \ldots, n-1\}$ such that $|z_h| > 1$ and z_h is negative. Following Claim 1 we shorten z_k and z_h, obtaining again a contradiction as in the previous case on the choice of φ.

Hence, for every z_j we have $|z_j| = \lambda$ if z_j is positive, while $|z_j| = 1$ if z_j is negative.

We complete the proof of the claim by showing that there is no index h such that $1 < |p_h| < \lambda$. If this is the case, then we will construct a new sequence of points p'_j by applying a suitable transformation of the original sequence which leads to a contradiction. If z_h and z_{h+1} are both positive, we set $p'_h = (1-\varepsilon)p_h$ and $p'_j = p_j$ for all $j \neq h$, where $\varepsilon > 0$ is chosen sufficiently small in such a way that $1 < |p'_{h-1}p'_h| < \lambda$ and $1 < |p'_h p'_{h+1}| < \lambda$. The new sequence corresponds to an optimal flow with $m_1 - 2$ vectors z'_j with norm λ, a contradiction. If z_h and z_{h+1} are both negative, we set $p'_h = (1+\varepsilon)p_h$ and $p'_j = p_j$ for all $j \neq h$, where $\varepsilon > 0$ is chosen sufficiently small in such a way that $1 < |p'_{h-1}p'_h| < \lambda$ and $1 < |p'_h p'_{h+1}| < \lambda$. Then we shorten z'_k and z'_h as in Claim 1 and we obtain an optimal flow with $m_1 - 1$ vectors with norm λ, a contradiction. If z_h is positive and z_{h+1} is negative, then we transform the original sequence by using $\sigma_{h,h+1}(\theta)$, where θ is sufficiently small and it is positive (resp. negative) if the angle $\angle p_{h-1}p_h p_{h+1}$ is non-negative (resp. negative). Vice versa, if z_h is negative and z_{h+1} is positive, then we transform the original sequence by using $\sigma_{h,h-1}(\theta)$, where θ is sufficiently small and it is negative (resp. positive) if the angle $\angle p_{h-1}p_h p_{h+1}$ is non-negative (resp. negative). In all cases, the resulting sequence of points defines a complex nowhere-zero flow on W_n with less than m_1 vectors having norm λ, a contradiction. This completes the proof of Claim 2. □

By Claim 2 we have only eight different types of vectors z_j in φ. Indeed, z_j is completely defined up to rotations once we have its direction (and then its norm by Claim 2) and the norm of p_{j-1} and p_j, which is in $\{1, \lambda\}$. Then, a vector z_j can be denoted by XY^*(see Figure 3), where $X, Y \in \{I, E\}$ and $* \in \{+, -\}$ are chosen in the following way.

- $X = I$ if $|p_{j-1}| = 1$ and $X = E$ if $|p_{j-1}| = \lambda$;
- $Y = I$ if $|p_j| = 1$ and $Y = E$ if $|p_j| = \lambda$;
- $* = +$ or $* = -$ if z_j is positive or negative, respectively.

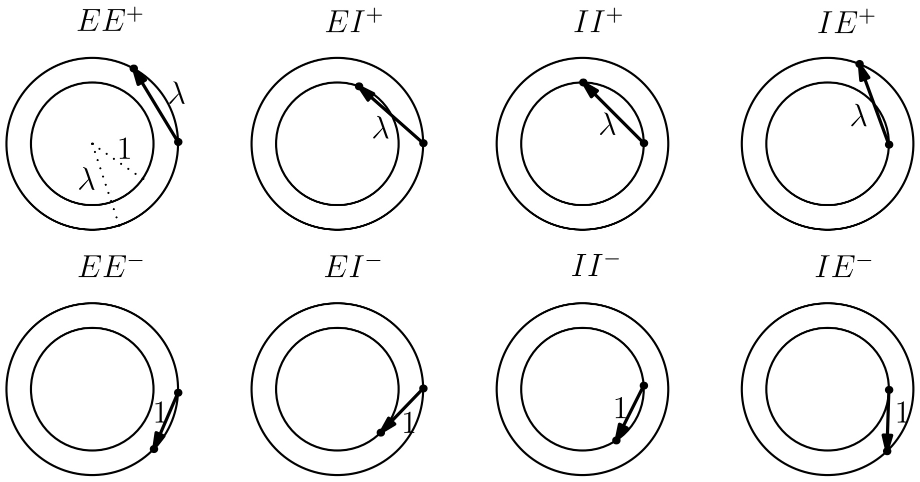

Figure 3: The eight types of vectors z_j in a representation of the chosen optimal flow φ of W_n.

Claim 3. *There exists an index* $j \in \{0, \dots, n-1\}$ *such that* $|p_j| = 1$.

Proof. By contradiction, if $|p_j| = \lambda$ for every index j, then all vectors z_j are of type either EE^+ or EE^-. Moreover, z_k is of type EE^+ and at least one of them is of type EE^-. Otherwise, $|p_j| = \lambda$ and $|z_j| = \lambda$ for all j, that is impossible in an optimal flow. In particular, there must be an index h with z_h of type EE^+ and z_{h+1} of type EE^-. The sequence is cyclic so we surely find the sequence EE^+, EE^-. Hence we can construct a sequence of points p'_j which defines a flow with less than m_1 vectors of maximum length λ by applying the transformation $\sigma_{h,h+1}(\theta)$, for a sufficiently small $\theta > 0$. $\square$

By Claim 3, without loss of generality, we can assume $|p_0| = 1$ from now on.

Claim 4. *All positive vectors z_j are of type II^+.*

Proof. First we prove that φ has no vector z_j of type IE^+. By contradiction, assume z_h is of type IE^+. There are four possibilities for the vector z_{h+1}, namely EE^+, EE^-, EI^+ and EI^-. Since we have $\lambda \leq \sqrt{2} < \Phi$ for every odd n, the mutual position of the two vectors z_h and z_{h+1} in each case is like the ones represented in Figure 4. In all these cases, by applying $\sigma_{h,h-1}(\theta)$ for a sufficiently small $\theta > 0$ we obtain a new sequence of points p'_j which corresponds to an optimal flow with either less than m_1 vectors of norm λ (if z_{h+1} is positive) or m_1 vectors of norm λ but less than m_2 points on $\mathcal{C}_E$ (if z_{h+1} is negative), a contradiction in both cases.

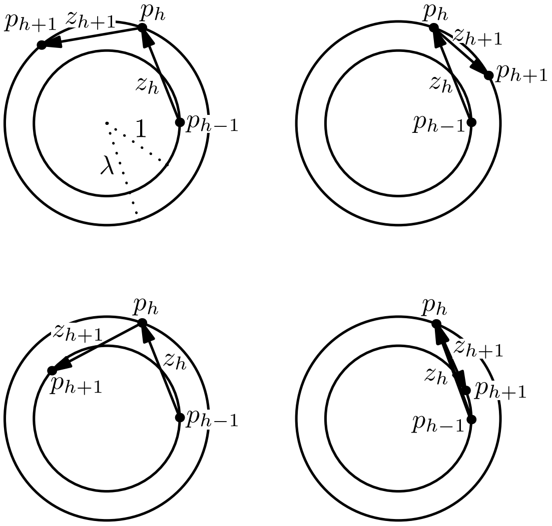

Figure 4: Mutual position of z_h and z_{h+1}.

Moreover, φ has no vector of type EE^+. Indeed, note that the angle subtended at the centre by a vector of type EE^+ on $\mathcal{C}_E$ is equal to $\frac{\pi}{3}$. Then, it is the same angle subtended at the centre by a vector of type II^+ on $\mathcal{C}_I$. If z_h is of type EE^+ (note that $h \geq 2$ since $p_0 \in \mathcal{C}_I$), then we can construct a new sequence of points $(p'_0, ..., p'_{n-1})$ in the following way:

- $p'_0 = p_0$

- $|p'_0 p'_1| = 1$ and $p'_1 \in \mathcal{C}_I$
- $p'_{j-1}p'_j$ is of the same type as $p_{j-2}p_{j-1}$ for $2 \leq j \leq h$
- $p'_{j-1}p'_j$ is of the same type as $p_{j-1}p_j$ for $h < j \leq n-1$.

Since we replaced a vector $p_{h-1}p_h$ of type EE^+ with a vector $p'_0p'_1$ of type II^+ which subtends the same angle, while maintaining all the other vectors of the same type, the new sequence of points has less than m_1 vectors having norm λ, a contradiction.

Finally, we prove that φ has no vectors of type EI^+. Indeed, if z_h is of type EI^+ with $h > 0$, then z_{h-1} cannot be positive because both vectors of type EE^+ and IE^+ are already excluded. Then, it could be either of type IE^- or EE^-. Since $\lambda \leq \sqrt{2}$ the mutual position of the points p_{h-2}, p_{h-1} and p_h is like the ones in Figure 5.

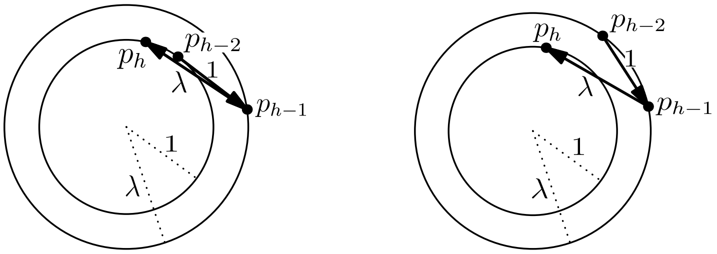

Figure 5: Mutual position of the points p_{h-2}, p_{h-1} and p_h.

In both these cases, by applying $\sigma_{h-1,h-2}(\theta)$ for a sufficiently small $\theta < 0$, we obtain a new configuration of points p'_j which corresponds to an optimal flow with less than m_1 vectors z'_j of norm λ, a contradiction. This completes the proof of Claim 4. $\square$

In what follows we will make use of the measure of some angles depicted in Figure 6. We denote by 2α and 2β the angles subtended at the centre by a chord of length 1 on $\mathcal{C}_E$ and of length λ on $\mathcal{C}_I$, respectively. The following relations hold.

$$\alpha = \arcsin\left(\frac{1}{2\lambda}\right), \quad \beta = \arcsin\left(\frac{\lambda}{2}\right).$$

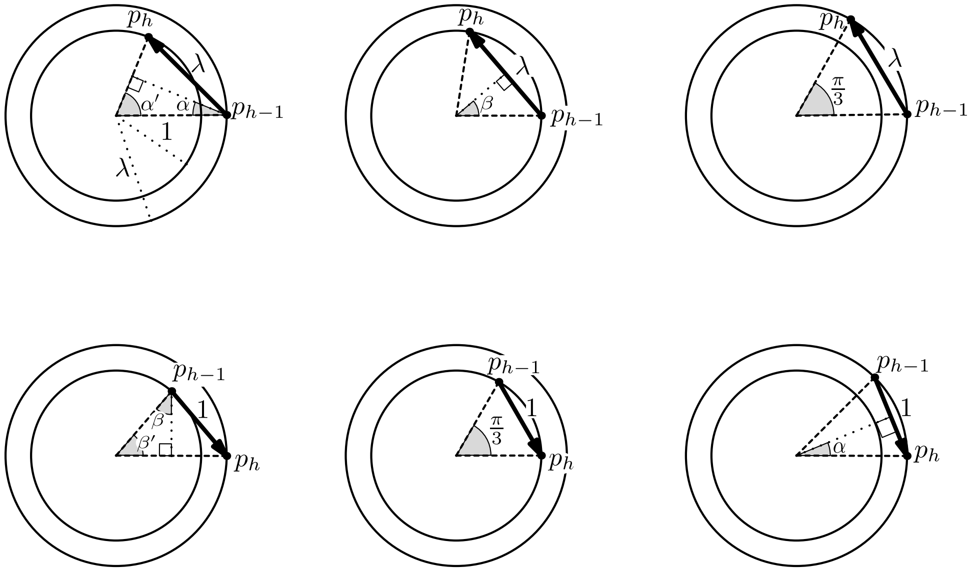

Figure 6: The angles subtended at the centre by all different types of vectors having norm 1 and λ.

Since $1 < \lambda \leq \sqrt{2}$, it follows

$$\arcsin\left(\frac{\sqrt{2}}{4}\right) \leq \alpha < \frac{\pi}{6}, \quad \frac{\pi}{6} < \beta \leq \frac{\pi}{4}.$$

Moreover, we prove the inequality $\alpha + \beta > \frac{\pi}{3}$ which will be used in what follows.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \sqrt{\left(1-\frac{1}{4\lambda^2}\right)\left(1-\frac{\lambda^2}{4}\right)} - \frac{1}{4}.$$

Since $\alpha + \beta$ is not larger than $\frac{5\pi}{3}$, then $\alpha + \beta > \frac{\pi}{3}$ if and only if $\sqrt{\left(1-\frac{1}{4\lambda^2}\right)\left(1-\frac{\lambda^2}{4}\right)} - \frac{1}{4} < \frac{1}{2}$. This inequality easily leads to $4(\lambda^2 - 1)^2 > 0$ which is always satisfied.

Finally, we denote by $\alpha' = \frac{\pi}{2} - \alpha$ and $\beta' = \frac{\pi}{2} - \beta$ the complement angles of α and β, respectively.

Claim 5. *No vector z_j is of type EE^-.*

Proof. By Claim 4 and since $p_0 \in \mathcal{C}_I$, to prove this claim it suffices to show that the two ordered sequences of three consecutive vectors of types IE^-, EE^-, EI^- and IE^-, EE^-, EE^- cannot appear in φ.

We first prove that a subsequence of type IE^-, EE^-, EI^- cannot appear. Assume that the points p_j corresponding to the subsequence IE^-, EE^-, EI^- are p_j, p_{j+1}, p_{j+2} and p_{j+3} as in Figure 7. Observe that the angle subtended at the centre by p_jp_{j+3} is

$$\beta' + 2\alpha + \beta' = 2\beta' + 2\alpha = 2\left(\frac{\pi}{2} - \beta\right) + 2\alpha = \pi - 2(\beta - \alpha) < \pi \quad (2)$$

where the last inequality holds since $\beta > \alpha$ for every $\lambda \in [1, \sqrt{2}]$.

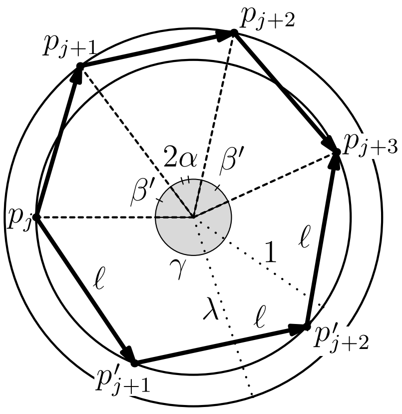

Figure 7: Configuration of points p_j, p_{j+1}, p_{j+2} and p_{j+3} corresponding to the subsequence of types IE^-, EE^-, EI^-.

Replace p_{j+1} and p_{j+2} by $p'_{j+1}, p'_{j+2} \in \mathcal{C}_I$, respectively, in such a way that $p_jp'_{j+1}, p'_{j+1}p'_{j+2}$ and $p'_{j+2}p_{j+3}$ are all positive vectors with $|p_jp'_{j+1}| = |p'_{j+1}p'_{j+2}| = |p_{j+2}p'_{j+3}| = \ell$ (see Figure 7). Let us prove that $1 < \ell < \lambda$. Indeed, we have

$$\gamma = 2\pi - (2\alpha + 2\beta') = \pi - 2\alpha + 2\beta$$

Recalling that $\alpha + \beta > \frac{\pi}{3}$ and $\beta > \frac{\pi}{6}$, we obtain $2\alpha + 4\beta > \pi$, that is $6\beta > \pi - 2\alpha + 2\beta = \gamma$ and so $\frac{\gamma}{3} < 2\beta$. Hence, $\ell < \lambda$. Moreover, since $\gamma > \pi$, and so $\frac{\gamma}{3} > \frac{\pi}{3}$, by (2), we have also that $\ell > 1$. Hence, the new sequence of points corresponds to an optimal flow with the same number m_1 of vectors of norm λ, but less than m_2 points on $\mathcal{C}_E$, a contradiction.

In a very similar way we prove that the subsequence of types IE^-, EE^-,EE^- cannot appear in a representation of φ. Again, assume that the points p_j corresponding to the subsequence IE^-, EE^-, EE^- are p_j, p_{j+1}, p_{j+2} and p_{j+3}.

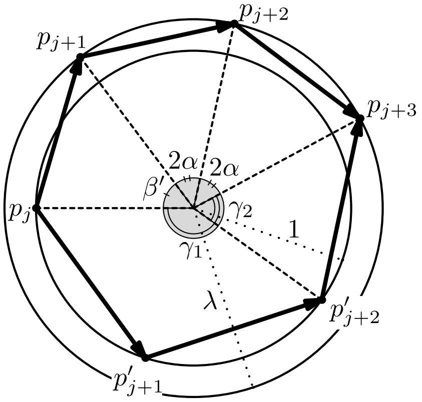

Figure 8: Configuration of points p_j, p_{j+1}, p_{j+2} and p_{j+3} corresponding to the subsequence of types IE^-, EE^-, EE^-.

Again we replace the points p_{j+1} and p_{j+2} of the sequence by two new points $p'_{j+1}, p'_{j+2} \in \mathcal{C}_I$ in such a way that $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p'_{j+2} p_{j+3}| = \ell$, as shown in Figure 8. We prove that $1 < \ell < \lambda$. Denote by γ_1 the angle subtended at the centre by $p_j p'_{j+2}$ and by γ_2 the angle subtended by $p'_{j+2} p_{j+3}$. Then, ℓ is such that $\beta' + 4\alpha + \gamma_1 + \gamma_2 = 2\pi$ holds.

In order to prove $\ell < \lambda$, it suffices to show that the sum of the angles obtained with $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p'_{j+2} p_{j+3}| = \lambda$, that is $\beta' + 4\alpha + 4\beta + \alpha'$, is strictly larger than 2π. Since

$$\beta' + 4\alpha + 4\beta + \alpha' = \frac{\pi}{2} - \beta + 4\alpha + 4\beta + \frac{\pi}{2} - \alpha = \pi + 3(\alpha + \beta),$$

it follows that $\beta' + 4\alpha + 4\beta + \alpha' > 2\pi$ if and only if $\alpha + \beta > \frac{\pi}{3}$, which is already proved to be satisfied.

In order to prove $\ell > 1$, it suffices to show that the sum of the angles obtained with $|p_j p'_{j+1}| = |p'_{j+1} p'_{j+2}| = |p'_{j+2} p_{j+3}| = 1$, that is $\beta' + 4\alpha + 2\frac{\pi}{3} + \beta'$, is strictly smaller than 2π. Since

$$\beta' + 4\alpha + 2\frac{\pi}{3} + \beta' = \pi - 2\beta + 4\alpha + \frac{2}{3}\pi,$$

it follows that $\beta' + 4\alpha + 2\frac{\pi}{3} + \beta' < 2\pi$ if and only if $2\alpha - \beta < \frac{\pi}{6}$.

Recalling that $\alpha + \beta > \frac{\pi}{3}$ and $\alpha < \frac{\pi}{6}$, we have $2\alpha - \beta = 3\alpha - (\alpha + \beta) < \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$.

Hence, the new sequence of points corresponds to an optimal flow having $m_1 - 1$ vectors of norm λ, a contradiction once again. This completes the proof of Claim 5.

□

Now we are going to rotate some points of the sequence $(p_0, ..., p_{n-1})$ associated with φ to obtain a new sequence denoted by $(q_0, ..., q_{n-1})$, in such a way that for every vector $z_h = p_{h-1}p_h$ there exists a unique vector $w_k = q_{k-1}q_k$ such that w_k is obtained by a suitable rotation of the vector z_h around the origin. By definition, the sequence of types of the vectors w_j is a permutation of the sequence of the types of the vectors z_j. Hence, the values of m_1 and m_2 do not change for this new sequence.

By previous claims such a sequence can contain only the following four types of vectors: IE^-, EI^-, II^- and II^+. Moreover, if a vector of type IE^- appears in the sequence, then it is necessarily followed by a vector of type EI^-. We choose $(q_0, ..., q_{n-1})$ in such a way that $p_0 \equiv q_0$ and all pairs IE^-, EI^-, if present, appear at the beginning of the sequence. They are followed by all vectors of type II^-, if present, and finally by all vectors of type II^+ (note that at least one positive vector appears by our assumptions). The sequence of types can be described in general by the following ordered sequence.

$$(IE^-, EI^-, ..., IE^-, EI^-, II^-, \ldots, II^-, II^+, ..., II^+)$$

Now we prove that some specific subsequences cannot appear in the sequence associated to $(q_0, \ldots, q_{n-1})$.

Claim 6. *The subsequences of consecutive vectors of types*

(a) IE^-, EI^-, IE^-

(b) IE^-EI^-, II^-

(c) II^-, II^-

cannot appear in the ordered sequence of types associated to $(q_0, ..., q_n)$.

Proof. (a) We argue similarly to what we did in the proof of Claim 5. Assume that the four consecutive points corresponding to the subsequence IE^-, EI^-, IE^- are q_j, q_{j+1}, q_{j+2} and q_{j+3}. We obtain a new sequence by replacing the two points q_{j+1}, q_{j+2} by the points $q'_{j+1}, q'_{j+2} \in \mathcal{C}_I$ such that $|q_j q'_{j+1}| = |q'_{j+1} q'_{j+2}|$ and $1 < |q'_{j+2} q_{j+3}| < \lambda$ (see Figure 9). Let us show that such a choice is admissible.

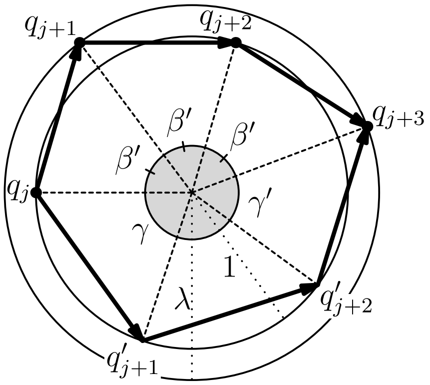

Figure 9: Configuration of points q_j, q_{j+1}, q_{j+2} and q_{j+3} corresponding to the subsequence of types IE^-, EI^-, IE^-.

Denote by γ and γ' the angles subtended at the centre by $q_j q'_{j+2}$ and $q'_{j+2} q_{j+3}$, respectively. Let $|q'_{j+2} q_{j+3}| > 1$, which is possible since $3\beta' < \pi$, so that $\gamma' > \beta'$. Hence, $\gamma = 2\pi - 3\beta' - \gamma' < 2\pi - 4\beta' = 4\arcsin(\frac{\lambda}{2}) = 4\beta$.

Since $\beta' < \pi/3$, $2\pi - 4\beta' > \frac{2}{3}\pi$. Note that, $\gamma' = \beta' + \varepsilon$, for a certain $\varepsilon > 0$. We choose ε sufficiently small in such a way that $\gamma =$

$2\pi - 3\beta' - \gamma' = 2\pi - 4\beta' - \varepsilon > \frac{2}{3}\pi$. So $|q_j q'_{j+1}| > 1$. Moreover, $\gamma = 2\pi - 3\beta' - \gamma' < 2\pi - 4\beta' = 4\beta$ implies that $|q_j q'_{j+1}| < \lambda$.

The new sequence of points has m_1 vectors of norm λ, but less than m_2 points which belongs to $\mathcal{C}_E$, a contradiction.

(b) Assume that the four consecutive points corresponding to the subsequence IE^-, EI^-, II^- are q_j, q_{j+1}, q_{j+2} and q_{j+3}. Denote by γ the explement angle of the angle subtended at the centre by $q_j q_{j+3}$. We obtain a new sequence by replacing the two points q_{j+1}, q_{j+2} by the points $q'_{j+1}, q'_{j+2} \in \mathcal{C}_I$ such that $|q_j q'_{j+1}| = |q'_{j+1} q'_{j+2}| = |q'_{j+2} q_{j+3}|$ (see Figure 10). Since $\beta' < \frac{\pi}{3}$, we have $2\beta' + \frac{\pi}{3} < \pi$, then $|q_j q'_{j+1}| > 1$.

Moreover, since $\lambda > 1$, $\gamma = 2\pi - (\frac{\pi}{3} + 2\beta') = \frac{2}{3}\pi + 2\beta < 6\beta$. Hence, $|q_j q'_{j+1}| < \lambda$.

The new sequence of points corresponds to an optimal flow having m_1 vectors of maximum norm λ, but less than m_2 points q_i on $\mathcal{C}_E$, a contradiction.

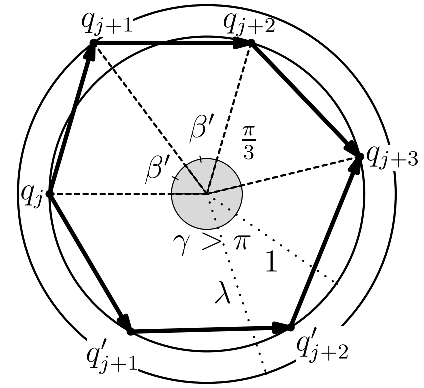

Figure 10: Configuration of points q_j, q_{j+1}, q_{j+2} and q_{j+3} corresponding to the subsequence of types IE^-, EI^-, II^-.

(c) Let w_j and w_{j+1} be the last two vectors of type II^- in the sequence, that is w_{j+2} is of type II^+. We obtain a new sequence by replacing the two points $q_{j+1}, q_{j+2} \in \mathcal{C}_I$ by the points q'_{j+1}, q'_{j+2} such that $q'_{j+1} = (1+\varepsilon)q_{j+1}$ and $q'_{j+2} = q_{j+2}e^{i\theta}$, where $\varepsilon > 0$ and $\theta > 0$ are chosen sufficiently small and in such a way that $|q'_{j+1} q'_{j+2}| = 1$ and $1 < |q'_{j+2} q_{j+3}| < \lambda$. The new sequence of points corresponds to

an optimal flow having less than m_1 vectors of maximum norm λ, a contradiction. This completes the proof of Claim 6. $\square$

By previous claims there exists an optimal flow of W_n such that the types of its vectors respect one of the following three sequences.

(i) $(IE^-, EI^-, II^+, ..., II^+)$

(ii) $(II^-, II^+, ..., II^+)$

(iii) $(II^+, ..., II^+)$

For each given n odd, the maximum among all values $|z_j|$ and $|p_j|$ is completely determined once we know which of the three sequences we are considering. The exact values in each of these cases are summarized in Table 1. We give here an example of direct computation of the values in the last column of the table. The remaining values are computed similarly. In this particular sequence, every vector z_j has the same norm λ, while all points p_j belong to $\mathcal{C}_I$. Hence, the angle subtended at the centre by each z_j on $\mathcal{C}_I$ is exactly $2\beta = 2\arcsin(\frac{\lambda}{2})$. Hence, for some integer $k > 0$, we have $2n \arcsin \frac{\lambda}{2} = 2k\pi$, that is

$$\lambda = 2\sin\left(\frac{k}{n}\pi\right).$$

We are looking for the minimum possible $\lambda > 1$. Then, k is chosen as the smallest integer such that $\frac{k}{n}\pi > \frac{\pi}{6}$, that is $k = \lceil \frac{n}{6} \rceil$. Set $n = 6t + h$, for $h = 1, 3, 5$. We obtain $\lambda = 2\sin\left(\frac{\pi(t+1)}{6t+h}\right) = 2\sin\left(\frac{\pi}{6} \cdot \frac{6t+6}{6t+h}\right) = 2\sin\left(\frac{\pi}{6} \cdot \frac{n+(6-h)}{n}\right)$.

Finally, comparing for each congruence of n the three possible values for λ, it turns out that the minimum λ is obtained with configuration (ii) if $n \equiv 1 \mod 6$, configuration (i) if $n \equiv 3 \mod 6$ and configuration (iii) if $n \equiv 5 \mod 6$. Observe that such optimal configurations are exactly the ones presented in Section 2.1.

Recall that, if n is even, then $\phi_{\mathbb{C}}(W_n) = 2$. Therefore, the following theorem follows.

	$IE^-, EI^-, II^+, ..., II^+$	$II^-, II^+, ..., II^+$	$II^+, ..., II^+$
$n \equiv 1 \mod 6$	$2\sin\left(\frac{\pi}{6} \cdot \frac{n+2}{n-1}\right)$	$\boldsymbol{2\sin\left(\frac{\pi}{6} \cdot \frac{n}{n-1}\right)}$	$2\sin\left(\frac{\pi}{6} \cdot \frac{n+5}{n}\right)$
$n \equiv 3 \mod 6$	$\boldsymbol{2\sin\left(\frac{\pi}{6} \cdot \frac{n}{n-1}\right)}$	$2\sin\left(\frac{\pi}{6} \cdot \frac{n+4}{n-1}\right)$	$2\sin\left(\frac{\pi}{6} \cdot \frac{n+3}{n}\right)$
$n \equiv 5 \mod 6$	$2\sin\left(\frac{\pi}{6} \cdot \frac{n+4}{n-1}\right)$	$2\sin\left(\frac{\pi}{6} \cdot \frac{n+2}{n-1}\right)$	$\boldsymbol{2\sin\left(\frac{\pi}{6} \cdot \frac{n+1}{n}\right)}$

Table 1: Exact values for the maximum norm in configurations $(i), (ii)$ and (iii), according to the congruence of n modulo 6. In bold the value λ^* for each of the three cases.

Theorem 1. *Let W_n be the wheel graph of order $n+1$, for $n \geq 3$. Then,*

$$\phi_{\mathbb{C}}(W_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1}) & \text{if } n \equiv 1, 3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{n+1}{n}) & \text{if } n \equiv 5 \mod 6. \end{cases}$$

3 A general lower bound for $\phi_{\mathbb{C}}$

The value $\phi_{\mathbb{C}}(W_n)$, for each odd n, gives a general non-trivial lower bound for $\phi_{\mathbb{C}}(G)$, where G is a bridgeless cubic graph, in terms of its odd-girth (the length of a shortest odd cycle of G). This is a straightforward consequence of the following standard observation. If C is an odd cycle of minimum length in G, then C is chordless. Contract all vertices of G not belonging to C to a unique vertex, thus obtaining a wheel graph whose complex flow number cannot be more than the complex flow number of G. Then, by Theorem 1 we deduce the following general result.

Theorem 2. *Let G be a non-bipartite cubic graph and let g be its odd-girth. Then,*

$$\phi_{\mathbb{C}}(G) \geq \begin{cases} 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g}{g-1}) & \text{if } g \equiv 1, 3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{g+1}{g}) & \text{if } g \equiv 5 \mod 6. \end{cases}$$

Let us remark that lower bounds in Theorem 2 are tight due to the prism graph P_n of order $2n$. Indeed, it is easy to see that each complex $\phi_{\mathbb{C}}(W_n)$-flow on W_n can be extended to a complex $\phi_{\mathbb{C}}(W_n)$-flow on P_n by a symmetry argument. Hence, the following holds.

Corollary 3. *Let P_n be the prism graph of order $2n$, $n \geq 3$. Then,*

$$\phi_{\mathbb{C}}(P_n) = \begin{cases} 2 & \textit{if } n \textit{ even,} \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{n}{n-1}) & \textit{if } n \equiv 1, 3 \mod 6, \\ 1 + 2\sin(\frac{\pi}{6} \cdot \frac{n+1}{n}) & \textit{if } n \equiv 5 \mod 6. \end{cases}$$

4 Acknowledgments

Davide Mattiolo is supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO), project number 1268323N. Jozef Rajník is supported by the research grants APVV-19-0308, VEGA 1/0743/21 and VEGA 1/0727/22.

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