



# A Closed Formula for the Interleaving Distance of Rectangle Persistence Modules

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**Abstract.** We give formulas for calculating the interleaving distance between rectangle persistence modules that depend solely on the geometry of the underlying rectangles. Moreover, we extend our results to calculate the bottleneck distance for rectangle decomposable persistence modules.

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**Keywords.** Interleaving distance, bottleneck distance, multiparameter persistence module, rectangle persistence module.

## 1. Introduction

Topological data analysis is a recently emerging and fast-growing field for analyzing complex data using geometry and topology. One of its main tools, persistent homology, has found a wide range of applications to biology [17], medicine [15], information networks [1], finance [13], etc. It works by performing a multiscale analysis of data structures. More precisely, persistent homology studies the topology of a multifiltration with respect to a poset  $P$  used as a parameter space. Taking homology, with coefficients over a field  $\mathbb{k}$  and for an arbitrary but fixed homology degree, of a multifiltration gives a persistence module.

Formally, a ( $P$ -indexed) persistence module  $\mathcal{M}$  is a functor  $\mathcal{M}: P \rightarrow \mathbf{Vec}$  where  $\mathbf{Vec}$  denotes the category of  $\mathbb{k}$ -vector spaces. Throughout the paper, we consider the poset of real numbers  $\mathbb{R}$  with its standard total order and the product poset  $\mathbb{R}^n$ . Modules of the form  $\mathbb{R}^n \rightarrow \mathbf{Vec}$  are known as  $n$ -parameter persistence modules.

In the 1-parameter setting ( $n = 1$ ), any pointwise finite-dimensional persistence module decomposes in an essentially unique way into simple summands called interval modules [11]. The decomposition is specified by a discrete invariant called a barcode. In contrast, the algebraic structure of a multiparameter ( $n \geq 2$ ) persistence module can be far more complex. While decomposition theorems exist, the underlying quivers are of wild-representation type, meaning that their sets of indecomposables are hard to classify [8].

Another useful property of 1-parameter persistence modules is that they are completely described up to isomorphism by the rank invariant [8]. Botnan et al. [6] prove that the rank invariant is complete on the class of rectangle-decomposable 2-parameter persistence modules and provide algorithms to determine efficiently whether a given 2-parameter persistence module belongs to this class.

Distances are tools for quantifying the (dis)similarity between persistence modules. They play an essential role in both theory and application. Different types of distances have been proposed on various types of persistence modules with values in  $\mathbf{Vec}$  [16]. Among the available distances, the interleaving distance, introduced by Chazal et al. [9], stands out because of its stability and universality properties. These properties can be extended to any number of parameters [14]. On the other hand, the bottleneck distance is the standard metric on barcodes; that is, it depends on the barcode decomposition of the given modules. For the 1-parameter persistence modules, it is now well known that these two distances are equal to each other [14]. Hence, the computation of the interleaving distance can be done by computing the bottleneck distance with efficient algorithms. Since a canonical definition of a barcode is unavailable for multiparameter persistence modules, the definition of bottleneck distance does not admit an immediate extension to the multidimensional setting. Nevertheless, it is still possible to define meaningful generalizations of the bottleneck distance between multiparameter persistence modules whose decompositions are given (see, for example, [4, Definition 2.9] for the interval decomposable modules). Unfortunately, the interleaving distance and the bottleneck distance are no longer equal. Still, the bottleneck distance is an upper bound for the interleaving distance since it can be defined as the supremum of the pairwise interleaving distances between indecomposables.

It is known that the interleaving distance is NP-hard to compute for persistence modules valued in  $\mathbf{Vec}$  [5]. So, it is natural to consider a subclass of persistence modules to obtain a formula for the interleaving distance. For a polynomial time algorithm that computes the bottleneck distance for 2-parameter interval decomposable modules, see [12]. Because of the results of [6] mentioned above, rectangle decomposable persistence modules are another interesting class.

In this paper, we give a closed formula to calculate the interleaving distance between rectangle persistence modules that depends solely on the geometry of the underlying rectangles.

The first main result of the paper, which we discuss in Sect. 3 in detail, shows that the interleaving distance is equal to the minimum of two quantities: one of them is related to the thickness of the underlying rectangles, and the other one is the Hausdorff distance between the underlying rectangles. Precisely, we prove:

**Theorem A.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two  $n$ -parameter rectangle persistence modules with underlying rectangles  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$ , respectively. We have*

$$d_I(\mathcal{M}, \mathcal{N}) = \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

We also want to note that by Proposition 2.20 and Corollary 2.21, our result holds regardless of whether the underlying rectangles are open, closed, or neither.

Furthermore, thanks to the fact that, if  $\mathcal{M}$  and  $\mathcal{N}$  are two interval persistence modules, then the interleaving distance is equal to the bottleneck distance (see Corollary 4.2), the formula above can be used to calculate the bottleneck distance between two rectangle persistence modules in essentially the same way as for the bottleneck distance in the 1-parameter case (see [16, p. 50]).

The second main result of the paper, which we discuss in Section 4 in detail, provides an alternative formula for the bottleneck distance of rectangle decomposable persistence modules using the geometry of underlying rectangles in the decompositions.

**Theorem B.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two  $n$ -parameter finitely rectangle decomposable persistence modules with decompositions*

$$\mathcal{M} = \bigoplus_{i=1}^m \mathcal{M}_i \text{ and } \mathcal{N} = \bigoplus_{j=1}^k \mathcal{N}_j$$

where we denote the underlying rectangle of each persistence module  $\mathcal{M}_i$  (respectively,  $\mathcal{N}_j$ ) by  $R_i$  (respectively by  $Q_j$ ). Let  $S$  be the set of all partial bijections between the barcodes  $B(\mathcal{M}) = \{R_i\}_{i=1}^m$  and  $B(\mathcal{N}) = \{Q_j\}_{j=1}^k$ . Let  $\sigma \in S$  and let  $\mathbf{I}'(\sigma) = B(\mathcal{M}) - \text{coim } \sigma$ ,  $\mathbf{J}'(\sigma) = B(\mathcal{N}) - \text{im } \sigma$ . Then, the bottleneck distance  $d_B(\mathcal{M}, \mathcal{N})$  is equal to

$$\min_{\sigma \in S} \max \left\{ \max_{R_{i'} \in \text{coim } \sigma} \left\{ d_I(\mathcal{M}_i, \sigma(\mathcal{M}_i)) \right\}, \max_{R_{i'} \in \mathbf{I}'(\sigma)} \left\{ \frac{\text{thick}(R_{i'})}{2} \right\}, \max_{Q_{j'} \in \mathbf{J}'(\sigma)} \left\{ \frac{\text{thick}(Q_{j'})}{2} \right\} \right\}$$

where the interleaving distance between  $\mathcal{M}_i$  and  $\sigma(\mathcal{M}_i)$  is equal to

$$\min \left\{ \max \left\{ \frac{\text{thick}(R_i)}{2}, \frac{\text{thick}(R_{\sigma(i)})}{2} \right\}, d_H(R_i, \sigma(R_i)) \right\}.$$

**Remark 1.1.** Note that Theorem B applies in particular to finitely presented rectangle decomposable modules.

## 2. Preliminaries

### 2.1. Notations and Conventions

For any integer  $n \geq 1$ , let us start with defining a partial order on  $\mathbb{R}^n$  by taking

$$u = (u_1, \dots, u_n) \preceq v = (v_1, \dots, v_n)$$

if and only if  $u_i \leq v_i$  for all  $1 \leq i \leq n, \dots, n$ , and  $u \prec v$  if and only if  $u_i < v_i$  for all  $1 \leq i \leq n, \dots, n$ . Note that  $u \succ v$  is not the negation of  $u \preceq v$ . Also, let us endow  $\mathbb{R}^n$  with the max-norm, that is

$$\|u\|_\infty \doteq \max_{1 \leq i \leq n} \{|u_i|\}$$

for all  $u \in \mathbb{R}^n$ . The metric induced by the max-norm is

$$d_\infty(u, v) \doteq \max \{|u_1 - v_1|, \dots, |u_n - v_n|\}$$

for all  $u, v \in \mathbb{R}^n$ . Let us define the **extended real line**, and more generally, the **extended space**, as  $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{-\infty, +\infty\}$  and  $\overline{\mathbb{R}}^n \doteq \prod_{i=1}^n \overline{\mathbb{R}}$ , respectively.

Throughout the paper, for any  $a \in \mathbb{R}$ , we follow the convention that

$$\begin{aligned} a + (\pm\infty) &= (\pm\infty) + a = \pm\infty, \\ -\infty &< a < +\infty, \end{aligned} \tag{1}$$

and, for any  $b \in \overline{\mathbb{R}}$ ,

$$-\infty \leq b \leq +\infty. \tag{2}$$

Moreover, we follow the convention that

$$\begin{aligned} (\pm\infty) - (\pm\infty) &= 0, \\ (\pm\infty) + (\pm\infty) &= \pm\infty, \\ |\pm\infty| &= +\infty. \end{aligned} \tag{3}$$

### 2.2. Multisets and Partial Multibijections

**Definition 2.1.** A **multiset** in the extended space  $\overline{\mathbb{R}}^n$  is a subset  $A$  of  $\overline{\mathbb{R}}^n$  such that each point  $a \in A$  is assigned a multiplicity  $\text{mult}_A(a) \in \mathbb{N}^+ \cup \{+\infty\}$ .

Basically, a multiset is a generalization of the concept of a set in which the multiplicity of elements matters.

**Definition 2.2.** A **multibijection**  $\sigma$  between two multisets  $(A, \text{mult}_A)$  and  $(B, \text{mult}_B)$  is a bijection

$$\sigma: \bigcup_{a \in A} \prod_{i=1}^{\text{mult}_A(a)} \{a\} \rightarrow \bigcup_{b \in B} \prod_{i=1}^{\text{mult}_B(b)} \{b\}.$$

We will use different symbols to denote different copies of the same element in a multiset.

**Definition 2.3.** A **partial multibijection**  $\sigma: A \rightarrow B$  between multisets  $A$  and  $B$  is a multibijection  $\sigma: \tilde{A} \rightarrow \tilde{B}$  where  $\tilde{A} \doteq \text{coim } \sigma$  is a subset of  $A$  and  $\tilde{B} \doteq \text{im } \sigma$  is a subset of  $B$ .

### 2.3. Persistence Modules

**Definition 2.4.** An  $n$ -parameter **persistence module**  $\mathcal{M}$  over a field  $\mathbb{k}$  is a family of vector spaces  $\{\mathcal{M}_u\}_{u \in \mathbb{R}^n}$  over  $\mathbb{k}$  together with a family of linear maps called **transition maps**

$$\{\varphi_{\mathcal{M}}(u, v): \mathcal{M}_u \rightarrow \mathcal{M}_v, u \preceq v \in \mathbb{R}^n\}$$

such that for every  $u \preceq v \preceq w \in \mathbb{R}^n$ , we have

- (i)  $\varphi_{\mathcal{M}}(v, w) \circ \varphi_{\mathcal{M}}(u, v) = \varphi_{\mathcal{M}}(u, w)$ ,
- (ii)  $\varphi_{\mathcal{M}}(u, u) = \text{id}_{\mathcal{M}_u}$ .

We say that the  $n$ -parameter persistence module  $\mathcal{M}$  is the **zero persistence module** if  $\mathcal{M}_u$  is the zero vector space for all  $u \in \mathbb{R}^n$ .

**Definition 2.5.** A **morphism**  $f: \mathcal{M} \rightarrow \mathcal{N}$  between two  $n$ -parameter persistence modules  $\mathcal{M}$  and  $\mathcal{N}$  is a collection of linear maps  $\{f_u: \mathcal{M}_u \rightarrow \mathcal{N}_u\}$  such that the following diagram commutes for all  $u \preceq v \in \mathbb{R}^n$ .

$$\begin{array}{ccc} \mathcal{M}_u & \xrightarrow{\varphi_{\mathcal{M}}(u,v)} & \mathcal{M}_v \\ f_u \downarrow & & \downarrow f_v \\ \mathcal{N}_u & \xrightarrow{\varphi_{\mathcal{N}}(u,v)} & \mathcal{N}_v \end{array}$$

**Definition 2.6.** We say that the persistence modules  $\mathcal{M}$  and  $\mathcal{N}$  are isomorphic, denoted by  $\mathcal{M} \simeq \mathcal{N}$ , if there exist two morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}$  and  $g: \mathcal{N} \rightarrow \mathcal{M}$  such that  $f \circ g$  and  $g \circ f$  are identity maps.

**Definition 2.7.** Let  $\mathcal{M}$  be an  $n$ -parameter persistence module. Let  $\vec{\epsilon} = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$ . For any  $\vec{\epsilon} \in \mathbb{R}^n$  with  $\epsilon \geq 0$ , the  $\epsilon$ -**shift** of the persistence module  $\mathcal{M}$  is the  $n$ -parameter persistence module  $\mathcal{M}(\vec{\epsilon})$  defined as

- $\mathcal{M}(\vec{\epsilon})_u \doteq \mathcal{M}_{u+\vec{\epsilon}}$  for any  $u \in \mathbb{R}^n$ , and
- $\varphi_{\mathcal{M}(\vec{\epsilon})}(u, v) \doteq \varphi_{\mathcal{M}}(u + \vec{\epsilon}, v + \vec{\epsilon})$  for any  $u \preceq v \in \mathbb{R}^n$ .

**Definition 2.8.** Let  $\mathcal{M}$  be a persistence module, we call  $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$  a **direct sum** of the persistence modules  $\mathcal{M}_1$  and  $\mathcal{M}_2$  where

- $\mathcal{M}_u \simeq (\mathcal{M}_1)_u \oplus (\mathcal{M}_2)_u$  for any  $u \in \mathbb{R}^n$ , and
- $\varphi_{\mathcal{M}}(u, v) \doteq \begin{pmatrix} \varphi_{\mathcal{M}_1}(u, v) & 0 \\ 0 & \varphi_{\mathcal{M}_2}(u, v) \end{pmatrix}$  for any  $u \preceq v \in \mathbb{R}^n$ .

Analogously, we call  $\mathcal{M} = \bigoplus \mathcal{M}_i$  a **direct sum** of the persistence modules  $\mathcal{M}_i$  for some collection of persistence modules  $\{\mathcal{M}_i\}$  satisfying the same structure.

We say that  $\mathcal{M}$  is an **indecomposable persistence module** if  $\mathcal{M}$  is a non-zero persistence module and cannot be written as a direct sum of two non-zero persistence modules. By the Krull–Schmidt theorem [2], there exists an essentially unique (up to permutation and isomorphism) decomposition  $\mathcal{M} = \bigoplus \mathcal{M}_i$  with every  $\mathcal{M}_i$  being indecomposable.

**Definition 2.9** [4]. We say that  $I \subseteq \mathbb{R}^n$  is an  $n$ -parameter connected **interval** if

- $I$  is non-empty,
- If  $u, v \in I$  and  $u \preceq w \preceq v$ , then  $w \in I$ ,
- If  $u, v \in I$ , then for some  $m \in \mathbb{N}$  there exist  $u_1, u_2, \dots, u_{2m} \in I$  such that  $u \preceq u_1 \succcurlyeq u_2 \preceq \dots \succcurlyeq u_{2m} \preceq v$ .

Throughout the paper, we denote the topological closure of an interval  $I$  as a subset of  $\mathbb{R}^n$  by  $\bar{I}$ , and the interior of an interval  $I$  as  $I^\circ$ .

**Definition 2.10** [4]. We say that a persistence module  $\mathcal{M}$  is an **interval persistence module** if

- for some interval  $I \subseteq \mathbb{R}^n$ ,  $\mathcal{M}_u = \mathbb{k}$  for every  $u \in I$ , and  $\mathcal{M}_u = 0$  for every  $u \notin I$ .
- $\varphi_{\mathcal{M}}(u, v) = \text{id}_{\mathbb{k}}$  for every  $u \preceq v \in I$ .

Here,  $I$  is called the underlying interval of the persistence module  $\mathcal{M}$ .

We may denote an interval persistence module  $\mathcal{I}$  with underlying interval  $I$  by  $\mathcal{I}^I$ .

**Definition 2.11** [4]. If  $\mathcal{M} \simeq \bigoplus_{I \in B} \mathcal{I}^I$  for a multiset  $B$  of intervals of  $\mathbb{R}^n$ , we say that  $\mathcal{M}$  is an **interval decomposable persistence module** and the multiset  $B(\mathcal{M}) \doteq B$  is the **barcode** of  $\mathcal{M}$ . Moreover, we say that  $\mathcal{M}$  is a **finitely interval decomposable persistence module** if the barcode is finite.

If  $\mathcal{M} \simeq \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I$  is an interval decomposable persistence module, then

$\bar{\mathcal{M}}$  denotes the **closed interval decomposable persistence module**  $\bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{\bar{I}}$ .

Similarly,  $\mathcal{M}^\circ$  denotes the **open interval decomposable persistence module**  $\bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{I^\circ}$ .

**Remark 2.12.** Any interval persistence module  $\mathcal{M}$  is indecomposable [7, Proposition 2.2].

Now, let us define rectangles, rectangle persistence modules, and rectangle decomposable persistence modules.

**Definition 2.13.** We say that  $R \subseteq \mathbb{R}^n$  is an  $n$ -parameter **rectangle** if  $R = I_1 \times \dots \times I_n$  where each  $I_i$  is a 1-parameter interval. We say that  $\mathcal{M}$  is an  $n$ -parameter **rectangle persistence module** if it is an interval persistence module whose  $n$ -parameter underlying interval is an  $n$ -parameter rectangle  $R$ . We will call  $R$  as the underlying rectangle of the rectangle persistence module  $\mathcal{M}$ . We say that a persistence module is a **rectangle decomposable persistence module** if it decomposes into a direct sum of only rectangle persistence modules. If the sum consists of only finitely many such modules, we call it a **finitely rectangle decomposable persistence module**.

### 2.4. Interleaving Distance

In this subsection, we first define the interleaving distance for multiparameter persistence modules and then give some of its properties.

**Definition 2.14** [14]. Let  $\epsilon$  be a non-negative real number and let  $\vec{\epsilon} \doteq (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$ . An  $\epsilon$ -**interleaving** between  $n$ -parameter persistence modules  $\mathcal{M}$  and  $\mathcal{N}$  is a collection of morphisms  $f_u: \mathcal{M}_u \rightarrow \mathcal{N}_{u+\vec{\epsilon}}$  and  $g_u: \mathcal{N}_u \rightarrow \mathcal{M}_{u+\vec{\epsilon}}$  such that the four diagrams

$$\begin{array}{ccc}
 \mathcal{M}_u & \longrightarrow & \mathcal{M}_v & & \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \\
 \downarrow f_u & & \downarrow f_v & & \downarrow g_u & & \downarrow g_v \\
 \mathcal{N}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{N}_{v+\vec{\epsilon}} & & \mathcal{M}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{M}_{v+\vec{\epsilon}}
 \end{array} \tag{4}$$

$$\begin{array}{ccc}
 & & \mathcal{N}_{u+\vec{\epsilon}} & & & & \mathcal{M}_{u+\vec{\epsilon}} & & & \\
 & \nearrow f_u & & \downarrow g_{u+\vec{\epsilon}} & \searrow & & & \downarrow f_{u+\vec{\epsilon}} & & \\
 \mathcal{M}_u & & & & & \mathcal{N}_u & & & & \\
 & \searrow & & & \nearrow g_u & & & & & \\
 & & \mathcal{M}_{u+2\vec{\epsilon}} & & & & \mathcal{N}_{u+2\vec{\epsilon}} & & & 
 \end{array} \tag{5}$$

commute for all  $u \preceq v \in \mathbb{R}^n$ .

The non-labelled maps are the transition maps mentioned in Definition 2.4.

Note that, if  $\mathcal{M}$  and  $\mathcal{N}$  are  $\epsilon$ -interleaved, then they are  $\delta$ -interleaved for any  $\epsilon \leq \delta$ . Therefore, one can define the **interleaving distance** by

$$d_I(\mathcal{M}, \mathcal{N}) \doteq \inf\{\epsilon \in [0, +\infty) : \mathcal{M} \text{ and } \mathcal{N} \text{ are } \epsilon\text{-interleaved}\}.$$

If no such  $\epsilon$  exists, then we put  $d_I(\mathcal{M}, \mathcal{N}) \doteq +\infty$ .

**Remark 2.15.** Note that if the persistence modules  $\mathcal{M}$  and  $\mathcal{N}$  are 0-interleaved, then  $d_I(\mathcal{M}, \mathcal{N}) = 0$ . However, the following example shows that the converse is not always true.

**Example 2.16.** Let  $\mathcal{M}$  be a 1-parameter persistence module with  $\mathcal{M}_0 = \mathbb{k}$  and zero otherwise, and  $\mathcal{N}$  be the 1-parameter zero persistence module. Then,  $d_I(\mathcal{M}, \mathcal{N}) = 0$ , but  $\mathcal{M}$  and  $\mathcal{N}$  are not 0-interleaved (for more details, see [14, Example 2.3]).

**Definition 2.17.** Let  $\vec{\epsilon} = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$  with  $\epsilon \geq 0$ . An  $n$ -parameter persistence module  $\mathcal{M}$  is called  $\epsilon$ -**significant** if  $\varphi_{\mathcal{M}}(u, u + \vec{\epsilon}) \neq 0$  for some  $u \in \mathbb{R}^n$ , and  $\epsilon$ -**trivial** otherwise.

**Proposition 2.18.** Let  $\mathcal{M}$  be an  $n$ -parameter rectangle persistence module with underlying rectangle  $R_{\mathcal{M}} = (a_1, b_1) \times \dots \times (a_n, b_n)$ . Then,  $\mathcal{M}$  is  $\epsilon$ -trivial if and only if  $\epsilon \geq \min_{1 \leq i \leq n} \{b_i - a_i\}$ .

*Proof.* Let  $\vec{\epsilon} = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$  with  $\epsilon \geq 0$ . Suppose  $\mathcal{M}$  is  $\epsilon$ -trivial. By Definition 2.17,  $\varphi_{\mathcal{M}}(u, u + \vec{\epsilon}) = 0$  is zero for all  $u \in \mathbb{R}^n$ . Now, consider a point  $u \in R_{\mathcal{M}}$ . So,  $a \prec u \prec b$ . This implies that  $\mathcal{M}_u \neq 0$ , but  $\mathcal{M}_{u+\vec{\epsilon}} = 0$  as  $\varphi_{\mathcal{M}}(u, u + \vec{\epsilon}) = 0$ . Since  $\mathcal{M}_{u+\vec{\epsilon}} = 0$ ,  $u + \vec{\epsilon} \notin R_{\mathcal{M}}$ , and so  $u + \vec{\epsilon} \not\prec b$ . Thus, we have  $\vec{\epsilon} \not\prec b - u \prec b - a$  as  $a \prec u$ . This implies that there exists  $i = 1, \dots, n$  such that  $\epsilon \geq b_i - a_i$ . Hence,  $\epsilon \geq \min_{1 \leq i \leq n} \{b_i - a_i\}$ .

Conversely, assume that  $\epsilon \geq \min_{1 \leq i \leq n} \{b_i - a_i\}$ . This implies that there exists  $i = 1, \dots, n$  such that  $\epsilon \geq b_i - a_i$ . Take an arbitrary point  $u \in \mathbb{R}^n$ . If  $a \prec u$ , then we have  $b_i \leq a_i + \epsilon < u_i + \epsilon$ , and so  $\mathcal{M}_{u+\vec{\epsilon}} = 0$ . Thus,  $\varphi_{\mathcal{M}}(u, u + \vec{\epsilon}) : \mathcal{M}_u \rightarrow \mathcal{M}_{u+\vec{\epsilon}}$  is the zero map. If  $a \not\prec u$ , then  $\mathcal{M}_u = 0$ . Thus,  $\varphi_{\mathcal{M}}(u, u + \vec{\epsilon}) : \mathcal{M}_u \rightarrow \mathcal{M}_{u+\vec{\epsilon}}$  is the zero map, again. Hence,  $\varphi_{\mathcal{M}} : \mathcal{M}_u \rightarrow \mathcal{M}_{u+\vec{\epsilon}}$  is the zero map for every  $u \in \mathbb{R}^n$ , meaning that  $\mathcal{M}$  is  $\epsilon$ -trivial.  $\square$

**Proposition 2.19.** *An  $n$ -parameter persistence module  $\mathcal{M}$  is  $2\epsilon$ -trivial if and only if it is  $\epsilon$ -interleaved with the zero persistence module.*

*Proof.* Let  $\mathcal{M}$  be an  $n$ -parameter persistence module and let  $\mathcal{N}$  be the zero persistence module, that is  $\mathcal{N}_u = 0$  for all  $u \in \mathbb{R}^n$ . Consider the following diagrams.

$$\begin{array}{ccc}
 \mathcal{M}_u & \longrightarrow & \mathcal{M}_v & & 0 = \mathcal{N}_u & \longrightarrow & \mathcal{N}_v = 0 \\
 \downarrow f_u & & \downarrow f_v & & \downarrow g_u & & \downarrow g_v \\
 0 = \mathcal{N}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{N}_{v+\vec{\epsilon}} = 0 & & \mathcal{M}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{M}_{v+\vec{\epsilon}}
 \end{array} \tag{6}$$

$$\begin{array}{ccc}
 & & \mathcal{N}_{u+\vec{\epsilon}} = 0 & & & & \mathcal{M}_{u+\vec{\epsilon}} & & \\
 & \nearrow f_u & & \downarrow g_{u+\vec{\epsilon}} & & & \downarrow f_{u+\vec{\epsilon}} & & \\
 \mathcal{M}_u & & & & 0 = \mathcal{N}_u & \nearrow g_u & & & \\
 & \searrow & & & & \searrow & & & \\
 & & \mathcal{M}_{u+2\vec{\epsilon}} & & & & \mathcal{N}_{u+2\vec{\epsilon}} = 0 & & 
 \end{array} \tag{7}$$

Suppose the persistence module  $\mathcal{M}$  is  $2\epsilon$ -trivial. So, by Definition 2.17 the transition maps  $\varphi_{\mathcal{M}}(u, u + 2\epsilon) = 0$  for all  $u \in \mathbb{R}^n$ . This implies that all diagrams in (6) commute for all  $u \preceq v \in \mathbb{R}^n$  and all diagrams in (7) commute for all  $u \in \mathbb{R}^n$ . Hence, the persistence module  $\mathcal{M}$  is  $\epsilon$ -interleaved with the zero persistence module. Conversely, if the persistence module  $\mathcal{M}$  is  $\epsilon$ -interleaved with the zero persistence module, then all diagrams in (6) commute for all  $u \preceq v \in \mathbb{R}^n$  and all diagrams in (7) commute for all  $u \in \mathbb{R}^n$ . Thus, by commutativity of left triangle diagrams in (7) for all  $u \in \mathbb{R}^n$ , the transition maps  $\varphi_{\mathcal{M}}(u, u + 2\epsilon) = 0$  for all  $u \in \mathbb{R}^n$ . Hence, by Definition 2.17 the persistence module  $\mathcal{M}$  is  $2\epsilon$ -trivial.  $\square$

The following result [12, Proposition 15] will be used later.

**Proposition 2.20.** *Let  $\mathcal{M} = \mathcal{I}^I$  and  $\mathcal{N} = \mathcal{I}^J$  be two interval persistence modules with underlying intervals  $I$  and  $J$ , respectively. Then,  $d_I(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}, \mathcal{N})$  where  $\mathcal{M} \doteq \mathcal{I}^I$  and  $\mathcal{N} \doteq \mathcal{I}^J$ .*

From the definition of the interior of an interval, the following corollary is immediate.

**Corollary 2.21.** *Let  $\mathcal{M} = \mathcal{I}^I$  and  $\mathcal{N} = \mathcal{I}^J$  be two interval persistence modules. Then,  $d_I(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}^\circ, \mathcal{N}^\circ)$  where  $\mathcal{M}^\circ \doteq \mathcal{I}^{I^\circ}$  and  $\mathcal{N}^\circ \doteq \mathcal{I}^{J^\circ}$ .*

**2.5. Bottleneck Distance**

In this subsection, we briefly define the bottleneck distance. Later in Sect. 4, we extend our results to calculate the bottleneck distance for rectangle decomposable persistence modules.

**Definition 2.22** [4]. Let  $A$  and  $B$  be multisets of intervals. An  $\epsilon$ -**matching** between  $A$  and  $B$  is a partial multibijection  $\sigma : A \rightsquigarrow B$  such that

- (i) for all  $I \in A - \text{coim } \sigma$ ,  $\mathcal{I}^I$  is  $2\epsilon$ -trivial,
- (ii) for all  $I \in B - \text{im } \sigma$ ,  $\mathcal{I}^I$  is  $2\epsilon$ -trivial,
- (iii) for all  $I \in \text{coim } \sigma$ ,  $\mathcal{I}^I$  and  $\mathcal{I}^{\sigma(I)}$  are  $\epsilon$ -interleaved.

If there is an  $\epsilon$ -matching  $\sigma$  between the barcodes  $B(\mathcal{M})$  and  $B(\mathcal{N})$  of persistence modules  $\mathcal{M}$  and  $\mathcal{N}$ , then we say that  $\mathcal{M}$  and  $\mathcal{N}$  are  $\epsilon$ -**matched**.

Now, we can define the bottleneck distance between the barcodes of multiparameter interval decomposable persistence modules.

**Definition 2.23** [4]. Let  $\mathcal{M}$  and  $\mathcal{N}$  be two interval decomposable persistence modules and  $S$  be the set of all partial multibijections between the barcodes  $B(\mathcal{M})$  and  $B(\mathcal{N})$ . Then, the **bottleneck distance** between interval decomposable persistence modules is defined as

$$d_B(\mathcal{M}, \mathcal{N}) \doteq \inf\{\epsilon \in [0, +\infty) : \mathcal{M} \text{ and } \mathcal{N} \text{ are } \epsilon\text{-matched}\}.$$

If there is no such an  $\epsilon$ , we put  $d_B(\mathcal{M}, \mathcal{N}) \doteq +\infty$ .

Note that, this is also equivalent to setting  $d_B(\mathcal{M}, \mathcal{N}) = \inf_{\sigma \in S} \text{cost}(\sigma)$ .

Also, bear in mind that if  $B(\mathcal{M})$  and  $B(\mathcal{N})$  are finite barcodes, then  $S$  is a finite set. Thus, one can use minimum in the definition of the bottleneck distance instead of infimum since there always exists a partial multibijection  $\bar{\sigma} : B(\mathcal{M}) \rightsquigarrow B(\mathcal{N})$  such that  $d_B(\mathcal{M}, \mathcal{N}) = \text{cost}(\bar{\sigma})$ . Following [10, Definition 4], we call the partial multibijection  $\bar{\sigma}$  **optimal**.

**3. Computing the Interleaving Distance for Rectangle Persistence Modules**

In this section, we show that the interleaving distance on rectangle persistence modules can be computed by a closed formula using the geometry of underlying rectangles.

In the following, we have results for open rectangle persistence modules. Fortunately, by Proposition 2.20 and by Corollary 2.21, we know that, for any interval persistence modules  $\mathcal{M} = \mathcal{I}^I$  and  $\mathcal{N} = \mathcal{I}^J$ ,  $d_I(\mathcal{M}, \mathcal{N}) = d_I(\overline{\mathcal{M}}, \overline{\mathcal{N}}) = d_I(\mathcal{M}^\circ, \mathcal{N}^\circ)$  where  $\overline{\mathcal{M}} \doteq \overline{\mathcal{I}^I}$  and  $\overline{\mathcal{N}} \doteq \overline{\mathcal{I}^J}$ , and  $\mathcal{M}^\circ \doteq \mathcal{I}^{I^\circ}$  and  $\mathcal{N}^\circ \doteq \mathcal{I}^{J^\circ}$ . Thus, after proving one for open rectangles, our results are also valid for closed, non-open, and non-closed rectangle persistence modules.

Throughout this section,  $\mathcal{M}$  and  $\mathcal{N}$  always denote two  $n$ -parameter rectangle persistence modules with underlying rectangles  $R_{\mathcal{M}} = (a_1, b_1) \times \dots \times (a_n, b_n)$  and  $R_{\mathcal{N}} = (c_1, d_1) \times \dots \times (c_n, d_n)$ . Also, we always assume that  $a_i, c_i \in \mathbb{R} \cup \{-\infty\}$ ,  $b_i, d_i \in \mathbb{R} \cup \{+\infty\}$ , and  $a_i < b_i$ ,  $c_i < d_i$  for  $i = 1, \dots, n$ .

Setting  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$ ,  $c = (c_1, \dots, c_n)$  and  $d = (d_1, \dots, d_n)$ , respectively, we observe that  $R_{\mathcal{M}} = \{x \in \mathbb{R}^n : a \prec x \prec b\}$  and  $R_{\mathcal{N}} = \{x \in \mathbb{R}^n : c \prec x \prec d\}$ , respectively. We say  $a$  and  $b$  are the **minimal** and the **maximal** points of the closure of  $R_{\mathcal{M}}$ , respectively, and  $c$  and  $d$  are the **minimal** and the **maximal** points of the closure of  $R_{\mathcal{N}}$ , respectively. Moreover, we define and denote **thickness** of these rectangles as  $\text{thick}(R_{\mathcal{M}}) \doteq \min_{1 \leq i \leq n} \{b_i - a_i\}$  and  $\text{thick}(R_{\mathcal{N}}) \doteq \min_{1 \leq i \leq n} \{d_i - c_i\}$ .

In the next result, we show that we can define a non-trivial morphism  $f: \mathcal{M} \rightarrow \mathcal{N}$  if  $R_{\mathcal{M}}$  overlaps with  $R_{\mathcal{N}}$  from above.

**Lemma 3.1.** *Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. The set of non-trivial morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}$  is non-empty if and only if  $c \preceq a \prec d \preceq b$ . Moreover, if such set is non-empty, it contains the morphism  $f: \mathcal{M} \rightarrow \mathcal{N}$  such that  $f_u$  is the identity map for all  $u \in R_{\mathcal{M}} \cap R_{\mathcal{N}}$ , and the zero map for all  $u \notin R_{\mathcal{M}} \cap R_{\mathcal{N}}$ .*

*Proof.* Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a non-trivial morphism. So, there is  $u \in \mathbb{R}^n$  such that  $f_u \neq 0$ . Thus,  $a \prec u \prec b$  and  $c \prec u \prec d$ , and so  $a \prec u \prec d$ . Hence, it is enough to show that  $c \preceq a$  and  $d \preceq b$ .

We will first show that  $c \preceq a$ . Suppose on the contrary that  $c \not\preceq a$ , i.e., there exists at least one  $i \in \{1, \dots, n\}$  for which  $c_i > a_i$ . Without loss of generality, let  $c_1 > a_1$  so that  $c_1 > -\infty$ . Now, if  $u = (u_1, \dots, u_n) \notin R_{\mathcal{M}}$ , then  $f_u = 0$ . If  $u = (u_1, \dots, u_n) \in R_{\mathcal{M}}$ , then  $a_j < u_j$  for every  $j = 2, \dots, n$ . Thus, there exists  $a_2^+, \dots, a_n^+ \in \mathbb{R}$  such that  $a_j < a_j^+ \leq u_j$  for every  $j = 2, \dots, n$ . Now, if  $u_1 < c_1$ , then  $u \notin R_{\mathcal{N}}$ , and so  $f_u = 0$ . Otherwise, if  $u_1 \geq c_1$ , since we are assuming that  $a_1 < c_1$ , there exists  $a_1^+ \in \mathbb{R}$  such that  $a_1 < a_1^+ \leq c_1 \leq u_1$ . This implies that  $a^+ = (a_1^+, \dots, a_n^+) \notin R_{\mathcal{N}}$  and  $a \prec a^+ \preceq u$ . So,  $a^+ \in R_{\mathcal{M}}$ . Now, consider the following diagram, where the horizontal maps are the transition morphisms of  $\mathcal{M}$  and  $\mathcal{N}$ , respectively.

$$\begin{array}{ccc} \mathbb{k} = \mathcal{M}_{a^+} & \xrightarrow{\text{id}_{\mathbb{k}}} & \mathcal{M}_u = \mathbb{k} \\ \downarrow 0=f_{a^+} & & \downarrow f_u \\ 0 = \mathcal{N}_{a^+} & \longrightarrow & \mathcal{N}_u \end{array} \quad \text{The diagram must be commutative since}$$

$f: \mathcal{M} \rightarrow \mathcal{N}$  is a morphism. This implies that  $f_u = 0$  also for the case  $u_1 \geq c_1$ . Hence,  $f_u = 0$  for all  $u \in \mathbb{R}^n$ . So, it contradicts  $f: \mathcal{M} \rightarrow \mathcal{N}$  being a non-trivial morphism. Hence,  $c \preceq a$ .

Still assuming that  $f: \mathcal{M} \rightarrow \mathcal{N}$  is a non-trivial morphism, we will show that  $d \preceq b$ . Suppose on the contrary that  $d \not\preceq b$ , i.e. there exists at least one  $i \in \{1, \dots, n\}$  for which  $d_i > b_i$ . Without loss of generality, let  $d_1 > b_1$  so that  $b_1 < +\infty$ . Now, if  $u = (u_1, \dots, u_n) \notin R_{\mathcal{N}}$ , then  $f_u = 0$ . If  $u = (u_1, \dots, u_n) \in R_{\mathcal{N}}$ , then  $u_j < d_j$  for every  $j = 2, \dots, n$ . Thus, there exists  $d_2^-, \dots, d_n^- \in \mathbb{R}$  such that  $u_j < d_j^- \leq d_j$  for every  $j = 2, \dots, n$ . Now, if  $b_1 < u_1$ , then  $u \notin R_{\mathcal{M}}$ , and so  $f_u = 0$ . Otherwise, if  $b_1 \geq u_1$ , since we are assuming that  $b_1 < d_1$ , there exists  $d_1^- \in \mathbb{R}$  such that  $u_1 < b_1 \leq d_1^- < d_1$ . This implies that

$d^- = (d_1^-, \dots, d_n^-) \notin R_{\mathcal{M}}$  and  $u \preceq d^- \prec d$ , and so  $d^- \in R_{\mathcal{N}}$ . Now, consider the following diagram, where the horizontal maps are the interval morphisms of  $\mathcal{M}$  and  $\mathcal{N}$ , respectively.

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_{d^-} = 0 \\ \downarrow f_u & & \downarrow f_{d^-} = 0 \\ \mathbb{k} = \mathcal{N}_u & \xrightarrow{\text{id}_{\mathbb{k}}} & \mathcal{N}_{d^-} = \mathbb{k} \end{array}$$

The diagram must be commutative since  $f: \mathcal{M} \rightarrow \mathcal{N}$  is a morphism. This implies that  $f_u = 0$  also for the case  $b_1 \geq u_1$ . Hence,  $f_u = 0$  for all  $u \in \mathbb{R}^n$ . So, it contradicts  $f: \mathcal{M} \rightarrow \mathcal{N}$  being a non-trivial morphism. Hence,  $d \preceq b$ .

For the proof of the converse statement, assuming  $c \preceq a \prec d \preceq b$ , we can define a non-trivial map  $f: \mathcal{M} \rightarrow \mathcal{N}$  as follows:

$$f_u = \begin{cases} \text{id}_{\mathbb{k}} & \text{if } a \prec u \prec d, \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

for any  $u \in \mathbb{R}^n$ .

Let us now check that it is a valid morphism  $\mathcal{M} \rightarrow \mathcal{N}$ . For  $a \prec u \preceq v \prec d$ , we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{k} = \mathcal{M}_u & \xrightarrow{\text{id}_{\mathbb{k}}} & \mathcal{M}_v = \mathbb{k} \\ \downarrow \text{id}_{\mathbb{k}} & & \downarrow \text{id}_{\mathbb{k}} \\ \mathbb{k} = \mathcal{N}_u & \xrightarrow{\text{id}_{\mathbb{k}}} & \mathcal{N}_v = \mathbb{k} \end{array}$$

For  $a \not\prec u$  and  $u \preceq v$ , observe that the following diagram commutes.

$$\begin{array}{ccc} 0 = \mathcal{M}_u & \xrightarrow{0} & \mathcal{M}_v \\ \downarrow 0 & & \downarrow \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

For  $a \prec u \preceq v$  and  $v \not\prec d$ , observe that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow & & \downarrow 0 \\ \mathcal{N}_u & \xrightarrow{0} & \mathcal{N}_v = 0 \end{array}$$

s Therefore, the following diagram commutes for all  $u \preceq v \in \mathbb{R}^n$ .

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow f_u & & \downarrow f_v \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

Hence,  $f: \mathcal{M} \rightarrow \mathcal{N}$  as in (8) is the desired non-trivial morphism (Fig. 1).

□

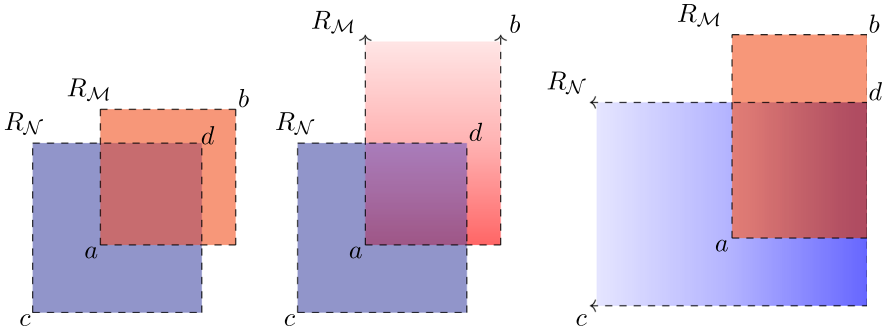


Figure 1. Some rectangles configurations in the 2-parameter case where we can define a non-trivial morphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  as in (8)

**Corollary 3.2.** Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. Let  $\vec{\epsilon} = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$  with  $\epsilon \geq 0$  and let  $\mathcal{N}(\vec{\epsilon})$  be the  $\epsilon$ -shift of the persistence module  $\mathcal{N}$ . There exists a non-trivial morphism  $f : \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$ , among which the one defined by

$$f_u = \begin{cases} \text{id}_{\mathbb{k}} & \text{if } u \in R_{\mathcal{M}} \cap R_{\mathcal{N}(\vec{\epsilon})}, \\ 0 & \text{otherwise} \end{cases},$$

if and only if

$$\max \left\{ \max_{1 \leq i \leq n} \{c_i - a_i\}, \max_{1 \leq i \leq n} \{d_i - b_i\} \right\} \leq \epsilon < \min_{1 \leq i \leq n} \{d_i - a_i\}.$$

*Proof.* By Lemma 3.1, there is a non-trivial morphism  $f : \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$  if and only if  $c - \vec{\epsilon} \preceq a \prec d - \vec{\epsilon} \preceq b$ . We note that:

- $c - \vec{\epsilon} \preceq a$  if and only if  $c_i - \epsilon \leq a_i$  for  $1 \leq i \leq n$ , or equivalently  $\epsilon \geq c_i - a_i$  for  $1 \leq i \leq n$ ;
- $a \prec d - \vec{\epsilon}$  if and only if  $a_i < d_i - \epsilon$  for  $1 \leq i \leq n$ , or equivalently  $\epsilon < d_i - a_i$  for  $1 \leq i \leq n$ ;
- $d - \vec{\epsilon} \preceq b$  if and only if  $d_i - \epsilon \leq b_i$ , for  $1 \leq i \leq n$ , or equivalently  $\epsilon \geq d_i - b_i$  for  $1 \leq i \leq n$ .

Hence, there is a non-trivial morphism  $f : \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$  if and only if

$$\max \left\{ \max_{1 \leq i \leq n} \{c_i - a_i\}, \max_{1 \leq i \leq n} \{d_i - b_i\} \right\} \leq \epsilon < \min_{1 \leq i \leq n} \{d_i - a_i\}.$$

□

Next, we prove that the interleaving distance is always less than or equal to half the maximum of the thicknesses of  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$ .

**Lemma 3.3.** Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. We have

$$d_I(\mathcal{M}, \mathcal{N}) \leq \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}.$$

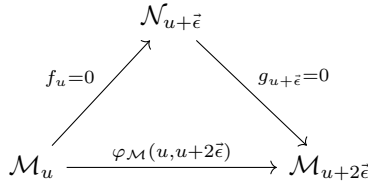
*Proof.* If

$$\max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\} = \max \left\{ \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}, \min_{1 \leq i \leq n} \frac{d_i - c_i}{2} \right\} = +\infty,$$

then there is nothing to prove. Thus, let

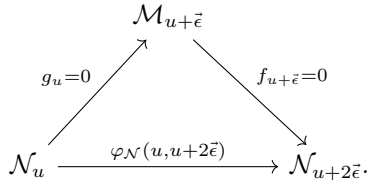
$$\epsilon \doteq \max \left\{ \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}, \min_{1 \leq i \leq n} \frac{d_i - c_i}{2} \right\} < +\infty. \tag{9}$$

We note that  $\epsilon > 0$ . So, we can consider  $\mathcal{M}(\vec{\epsilon})$  and  $\mathcal{N}(\vec{\epsilon})$ . Let us take the maps  $f: \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$  and  $g: \mathcal{N} \rightarrow \mathcal{M}(\vec{\epsilon})$  to be trivial. Thus, all square diagrams (4) in the Definition 2.14 are commutative. Now consider the following diagram.



By Eq. 9, we have  $\epsilon \geq \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}$ . By Proposition 2.18, it follows that  $\mathcal{M}$  is  $2\epsilon$ -trivial. Thus,  $\varphi_{\mathcal{M}}(u, u + 2\vec{\epsilon}) = 0$  for every point  $u \in \mathbb{R}^n$ , and hence the diagram above is commutative.

Similarly, one can show that the following diagram is also commutative for all points  $u \in \mathbb{R}^n$  by showing  $\varphi_{\mathcal{N}}(u, u + 2\vec{\epsilon}) = 0$  for every point  $u \in \mathbb{R}^n$ .



Therefore, the trivial morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$  and  $g: \mathcal{N} \rightarrow \mathcal{M}(\vec{\epsilon})$  are  $\epsilon$ -interleaving morphisms such that all diagrams in Definition 2.14 are commutative for every point in  $\mathbb{R}^n$ . Hence, the rectangle persistence modules  $\mathcal{M}$  and  $\mathcal{N}$  are  $\epsilon$ -interleaved, and thus, we can conclude that

$$d_I(\mathcal{M}, \mathcal{N}) \leq \epsilon = \max \left\{ \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}, \min_{1 \leq i \leq n} \frac{d_i - c_i}{2} \right\}.$$

□

We now review the definition of the well-known Hausdorff distance for subsets of  $\mathbb{R}^n$ .

**Definition 3.4.** Let  $X$  and  $Y$  be two subsets of  $\mathbb{R}^n$  and  $\vec{\epsilon} = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$  with  $\epsilon \geq 0$ . The **Hausdorff distance** between these sets is defined and denoted as

$$d_H(X, Y) \doteq \inf \{ \epsilon : X \subseteq Y(\vec{\epsilon}) \text{ and } Y \subseteq X(\vec{\epsilon}) \}$$

where  $X(\vec{\epsilon}) \doteq \bigcup_{x \in X} \{z \in \mathbb{R}^n : \|z - x\|_\infty \leq \epsilon\}$ , and  $Y(\vec{\epsilon})$  is defined analogously.

**Proposition 3.5.** *The Hausdorff distance between the two  $n$ -parameter rectangles  $R = \{u \in \mathbb{R}^n : a \prec u \prec b\}$  and  $R' = \{u' \in \mathbb{R}^n : c \prec u' \prec d\}$  is equal to*

$$\max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\}.$$

*Proof.* Let  $\vec{\epsilon} = (\epsilon, \dots, \epsilon) \in \mathbb{R}^n$  with  $\epsilon \geq 0$ . Observe that  $R \subseteq R'(\vec{\epsilon})$  and  $R' \subseteq R(\vec{\epsilon})$  if and only if  $b \preceq d + \vec{\epsilon}$ ,  $c - \vec{\epsilon} \preceq a$ , and  $d \preceq b + \vec{\epsilon}$ ,  $a - \vec{\epsilon} \preceq c$ , respectively. Equivalently, we have  $-\vec{\epsilon} \preceq b - d \preceq \vec{\epsilon}$  and  $-\vec{\epsilon} \preceq c - a \preceq \vec{\epsilon}$ . This is also equivalent to having  $\|d - b\|_\infty \leq \epsilon$  and  $\|c - a\|_\infty \leq \epsilon$ . Therefore,  $\max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\} \leq \epsilon$  if and only if  $d_H(R, R') \leq \epsilon$ . Hence, we get

$$d_H(R, R') = \max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\}.$$

□

The following two results show that the interleaving distance between the rectangle persistence modules is also bounded by the Hausdorff distance of their underlying rectangles.

**Lemma 3.6.** *Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. If*

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) < \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}$$

*then*

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) < \min \left\{ \min_{1 \leq i \leq n} \{d_i - a_i\}, \min_{1 \leq i \leq n} \{b_i - c_i\} \right\}.$$

*Proof.* Without loss of generality, let us suppose that

$$\min \left\{ \min_{1 \leq i \leq n} \{d_i - a_i\}, \min_{1 \leq i \leq n} \{b_i - c_i\} \right\} = d_1 - a_1.$$

Now, let  $\epsilon \doteq \max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\}$ , and suppose on the contrary that  $d_1 - a_1 \leq \epsilon$ . Note that  $\frac{d_1 - c_1}{2} \leq \frac{d_1 - a_1 + \epsilon}{2}$  since  $\|c - a\|_\infty \leq \epsilon$  by definition of  $\epsilon$ . Thus, we have  $\frac{d_1 - c_1}{2} \leq \epsilon$  as  $d_1 - a_1 \leq \epsilon$  by assumption. Similarly, note that  $\frac{b_1 - a_1}{2} \leq \frac{d_1 - a_1 + \epsilon}{2}$  since  $\|d - b\|_\infty \leq \epsilon$  by definition of  $\epsilon$ . Thus, we have  $\frac{b_1 - a_1}{2} \leq \epsilon$  as  $d_1 - a_1 \leq \epsilon$  by assumption. Therefore, we have

$$\min_{1 \leq i \leq n} \frac{b_i - a_i}{2} \leq \epsilon \quad \text{and} \quad \min_{1 \leq i \leq n} \frac{d_i - c_i}{2} \leq \epsilon.$$

Hence, we have

$$\max \left\{ \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}, \min_{1 \leq i \leq n} \frac{d_i - c_i}{2} \right\} \leq \epsilon,$$

which contradicts the assumption. □

**Lemma 3.7.** *Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. If*

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) < \min \left\{ \min_{1 \leq i \leq n} \{d_i - a_i\}, \min_{1 \leq i \leq n} \{b_i - c_i\} \right\},$$

*then*

$$d_I(\mathcal{M}, \mathcal{N}) \leq d_H(R_{\mathcal{M}}, R_{\mathcal{N}}).$$

*Proof.* If

$$\max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\} = +\infty,$$

then there is nothing to prove. Thus, let

$$0 \leq \epsilon \doteq \max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\} < +\infty.$$

Hence,

$$\epsilon \geq \max \left\{ \max_{1 \leq i \leq n} \{c_i - a_i\}, \max_{1 \leq i \leq n} \{d_i - b_i\} \right\}$$

and, by assumption,

$$\epsilon < \min \left\{ \min_{1 \leq i \leq n} \{d_i - a_i\}, \min_{1 \leq i \leq n} \{b_i - c_i\} \right\} \leq \min_{1 \leq i \leq n} \{d_i - a_i\}.$$

Analogously,

$$\epsilon \geq \max \left\{ \max_{1 \leq i \leq n} \{a_i - c_i\}, \max_{1 \leq i \leq n} \{b_i - d_i\} \right\}$$

and, by assumption,

$$\epsilon < \min \left\{ \min_{1 \leq i \leq n} \{d_i - a_i\}, \min_{1 \leq i \leq n} \{b_i - c_i\} \right\} \leq \min_{1 \leq i \leq n} \{b_i - c_i\}.$$

By Corollary 3.2, we can take morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$  and  $g: \mathcal{N} \rightarrow \mathcal{M}(\vec{\epsilon})$  to be non-trivial and precisely  $\text{id}_{\mathbb{k}}: \mathbb{k} \rightarrow \mathbb{k}$  at all points  $u \in \mathbb{R}^n$  such that domain and codomain of  $f_u$  (respectively,  $g_u$ ) are both non-trivial.

Let us show that they make the triangle diagram

$$\begin{array}{ccc}
 & \mathcal{N}_{u+\vec{\epsilon}} & \\
 f_u \nearrow & & \searrow g_{u+\vec{\epsilon}} \\
 \mathcal{M}_u & \xrightarrow{\varphi_{\mathcal{M}}(u, u+2\vec{\epsilon})} & \mathcal{M}_{u+2\vec{\epsilon}}
 \end{array}
 \quad \text{commute for every } u \in \mathbb{R}^n.$$

If  $\mathcal{M}_u = 0$  or  $\mathcal{M}_{u+2\vec{\epsilon}} = 0$ , then the triangle diagram is clearly commutative. So, suppose that  $\mathcal{M}_u = \mathbb{k}$  and  $\mathcal{M}_{u+2\vec{\epsilon}} = \mathbb{k}$ . Thus,  $\varphi_{\mathcal{M}}(u, u+2\vec{\epsilon}) = \text{id}_{\mathbb{k}}$  and  $a \prec u \prec b - 2\vec{\epsilon}$ . Thus, we have

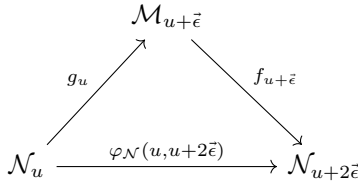
$$u + \vec{\epsilon} \prec b - \vec{\epsilon} \preceq b - \|d - b\|_\infty \cdot (1, \dots, 1) \preceq d$$

since  $\|d - b\|_\infty \leq \epsilon$  by definition of  $\epsilon$ . Also, we have

$$c \prec a + \|c - a\|_\infty \cdot (1, \dots, 1) \preceq a + \vec{\epsilon} \preceq u + \vec{\epsilon}$$

since  $\|c - a\|_\infty \leq \epsilon$  by definition of  $\epsilon$ . These two inequalities together imply that  $c \prec u + \vec{\epsilon} \prec d$ . It follows that  $\mathcal{N}_{u+\vec{\epsilon}} = \mathbb{k}$ , and hence  $f_u = \text{id}_{\mathbb{k}}$  and  $g_{u+\vec{\epsilon}} = \text{id}_{\mathbb{k}}$  make the diagram commute since we are assuming that  $\mathcal{M}_u = \mathbb{k}$  and  $\mathcal{M}_{u+2\vec{\epsilon}} = \mathbb{k}$ . Hence, the triangle diagram above commutes for all points

$u \in \mathbb{R}^n$ . Analogously, the triangle diagram below is commutative for all points  $u \in \mathbb{R}^n$ .



Therefore,  $f, g$  form an  $\epsilon$ -interleaving pair, implying that

$$d_I(\mathcal{M}, \mathcal{N}) \leq \epsilon = \max \left\{ \|c - a\|_\infty, \|d - b\|_\infty \right\}.$$

□

The following statement shows that the interleaving distance is always less than or equal to the minimum of two quantities. One of them is the maximum of half the thicknesses of  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$ , and the other one is the Hausdorff distance between  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$ .

**Theorem 3.8.** *Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. We have*

$$d_I(\mathcal{M}, \mathcal{N}) \leq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

*Proof.* First, suppose that

$$\max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\} \leq d_H(R_{\mathcal{M}}, R_{\mathcal{N}}).$$

By Lemma 3.3, we know that

$$d_I(\mathcal{M}, \mathcal{N}) \leq \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}.$$

Hence, the result follows in this case.

Now, suppose that

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) < \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}.$$

Then, by Lemma 3.6, we know that

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) < \min \left\{ \min_{1 \leq i \leq n} \{d_i - a_i\}, \min_{1 \leq i \leq n} \{b_i - c_i\} \right\}.$$

Therefore, by Lemma 3.7,

$$d_I(\mathcal{M}, \mathcal{N}) \leq d_H(R_{\mathcal{M}}, R_{\mathcal{N}}).$$

Hence, in any case

$$d_I(\mathcal{M}, \mathcal{N}) \leq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

□

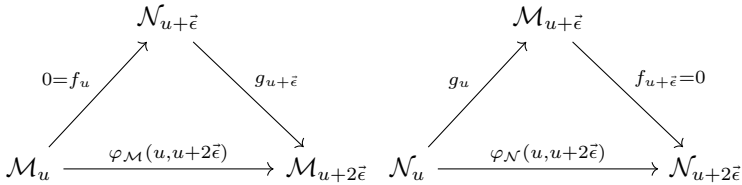
Next, we prove the converse of the inequality of Theorem 3.8. That is,

$$d_I(\mathcal{M}, \mathcal{N}) \geq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

**Lemma 3.9.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $\epsilon$ -interleaved persistence modules with two interleaving morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}(\bar{\epsilon})$  and  $g: \mathcal{N} \rightarrow \mathcal{M}(\bar{\epsilon})$ . If  $f$  or  $g$  is a trivial morphism, then*

$$\epsilon \geq \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}.$$

*Proof.* Without loss of generality, suppose that  $f: \mathcal{M} \rightarrow \mathcal{N}(\bar{\epsilon})$  is a trivial morphism. Thus,  $f_u: \mathcal{M}_u \rightarrow \mathcal{N}_{u+\bar{\epsilon}}$  is a zero map for every  $u \in \mathbb{R}^n$ . By assumption, we know that  $\mathcal{M}$  and  $\mathcal{N}$  are  $\epsilon$ -interleaved. Hence, both of the diagrams below must be commutative for every  $u \in \mathbb{R}^n$ .



As a result, both  $\varphi_{\mathcal{M}}(u, u + 2\bar{\epsilon})$  and  $\varphi_{\mathcal{N}}(u, u + 2\bar{\epsilon})$  are the zero maps for every  $u \in \mathbb{R}^n$ . Therefore, both  $\mathcal{M}$  and  $\mathcal{N}$  are  $2\epsilon$ -trivial persistence modules. By Proposition 2.18, it follows that  $2\epsilon \geq \min_{1 \leq i \leq n} \{b_i - a_i\}$  and  $2\epsilon \geq \min_{1 \leq i \leq n} \{d_i - c_i\}$ . Hence, we can conclude that

$$\epsilon \geq \max \left\{ \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}, \min_{1 \leq i \leq n} \frac{d_i - c_i}{2} \right\}.$$

□

**Lemma 3.10.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be  $\epsilon$ -interleaved persistence modules with two interleaving morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}(\bar{\epsilon})$  and  $g: \mathcal{N} \rightarrow \mathcal{M}(\bar{\epsilon})$ . If  $f$  and  $g$  are non-trivial morphisms, then*

$$\epsilon \geq d_H(R_{\mathcal{M}}, R_{\mathcal{N}}).$$

*Proof.* From Corollary 3.2, since  $f: \mathcal{M} \rightarrow \mathcal{N}(\bar{\epsilon})$  is a non-trivial morphism, we get

$$\epsilon \geq \max \left\{ \max_{1 \leq i \leq n} \{c_i - a_i\}, \max_{1 \leq i \leq n} \{d_i - b_i\} \right\}.$$

Analogously, since  $g: \mathcal{N} \rightarrow \mathcal{M}(\bar{\epsilon})$  is a non-trivial morphism, we have

$$\epsilon \geq \max \left\{ \max_{1 \leq i \leq n} \{a_i - c_i\}, \max_{1 \leq i \leq n} \{b_i - d_i\} \right\}.$$

Hence, we can conclude that

$$\epsilon \geq \max \left\{ \|c - a\|_{\infty}, \|d - b\|_{\infty} \right\}.$$

□

We present our first main result, showing that the interleaving distance equals the smaller of the two quantities.

**Theorem A.** *Let  $R_{\mathcal{M}}$  and  $R_{\mathcal{N}}$  be as above. We have*

$$d_I(\mathcal{M}, \mathcal{N}) = \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

*Proof.* First, note that by Theorem 3.8, we already have

$$d_I(\mathcal{M}, \mathcal{N}) \leq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

For the remaining part of the proof, let  $\mathcal{M}$  and  $\mathcal{N}$  be two  $\epsilon$ -interleaved persistence modules with interleaving morphisms  $f: \mathcal{M} \rightarrow \mathcal{N}(\vec{\epsilon})$  and  $g: \mathcal{N} \rightarrow \mathcal{M}(\vec{\epsilon})$ . Suppose first that  $f$  or  $g$  is a trivial morphism. By Lemma 3.9, we know that

$$\epsilon \geq \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}.$$

Thus, we have

$$\epsilon \geq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

Now, suppose that  $f$  and  $g$  are non-trivial morphisms. By Lemma 3.10, we know that

$$\epsilon \geq d_H(R_{\mathcal{M}}, R_{\mathcal{N}}).$$

Thus, again, we have

$$\epsilon \geq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

Therefore, in both cases, we have

$$\epsilon \geq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

Hence, we can immediately conclude that

$$d_I(\mathcal{M}, \mathcal{N}) \geq \min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\},$$

which finishes the proof. □

Now, we present two examples in the 2-parameter case where the interleaving distance is directly computed using the closed formula of Theorem A. This distance is either equal to the maximum of half the thicknesses, or the maximum distance between the minimal and maximal elements, i.e., the Hausdorff distance.

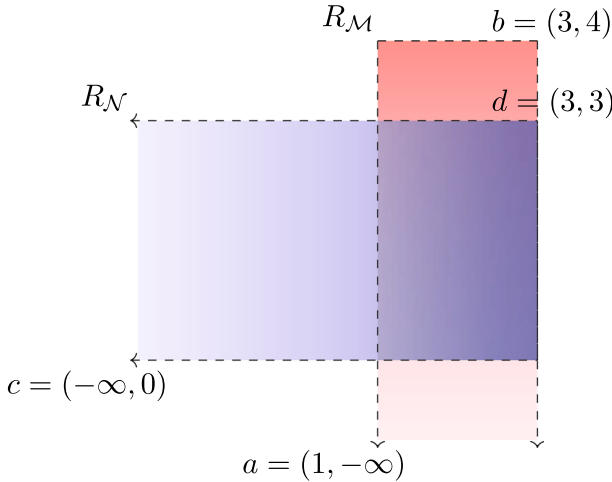


Figure 2. Two rectangle persistence modules whose interleaving distance is equal to half the maximum of the thicknesses of the underlying rectangles

**Example 3.11.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be two 2-parameter rectangle persistence modules given as in Fig. 2. By Theorem A, we know that the interleaving distance is equal to

$$\min \left\{ \max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\}, d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) \right\}.$$

Since

$$\max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\} = \max \left\{ 1, \frac{3}{2} \right\} = \frac{3}{2}$$

and

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) = \max \left\{ \|c - a\|_{\infty}, \|d - b\|_{\infty} \right\} = \max \left\{ +\infty, 1 \right\} = +\infty,$$

we get  $d_I(\mathcal{M}, \mathcal{N}) = \min \left\{ \frac{3}{2}, +\infty \right\} = \frac{3}{2}$ .

In the next example, the interleaving distance is equal to the Hausdorff distance.

**Example 3.12.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be two 2-parameter rectangle persistence modules given as in Fig. 3. Since

$$\max \left\{ \frac{\text{thick}(R_{\mathcal{M}})}{2}, \frac{\text{thick}(R_{\mathcal{N}})}{2} \right\} = \max \left\{ 1, \frac{3}{2} \right\} = \frac{3}{2}$$

and

$$d_H(R_{\mathcal{M}}, R_{\mathcal{N}}) = \max \left\{ \|c - a\|_{\infty}, \|d - b\|_{\infty} \right\} = \max \left\{ 1, 1 \right\} = 1,$$

we get  $d_I(\mathcal{M}, \mathcal{N}) = \min \left\{ \frac{3}{2}, 1 \right\} = 1$ .

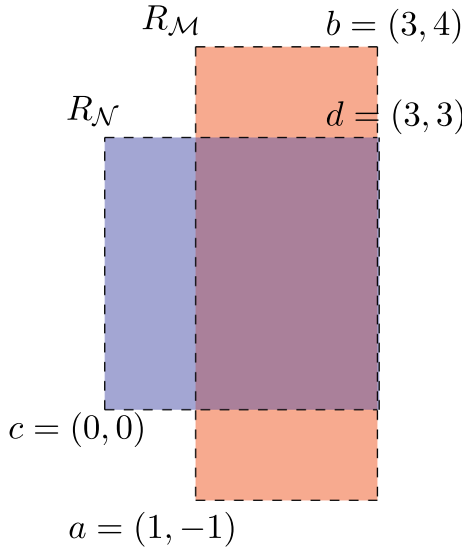


Figure 3. Two rectangle persistence modules whose interleaving distance is equal to the Hausdorff distance

**Remark 3.13.** Thanks to Proposition 2.20 and Corollary 2.21, Theorem A holds independently of whether the underlying rectangles are open, closed, or neither.

### 4. Computing the Bottleneck Distance for Rectangle Decomposable Persistence Modules

In this section, we first give an equivalent expression of Definition 2.22 for the calculation of the bottleneck distance for  $n$ -parameter interval decomposable persistence modules. Then, we present our result for the calculation of the bottleneck distance for  $n$ -parameter rectangle decomposable persistence modules.

**Proposition 4.1.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two finitely interval decomposable persistence modules with given decompositions*

$$\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I \text{ and } \mathcal{N} = \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^J$$

where summands  $\mathcal{I}^I$  and  $\mathcal{I}^J$  are interval persistence modules with underlying intervals  $I$  and  $J$ , respectively. Let  $S$  be the set of all partial multibijections between the barcodes  $B(\mathcal{M})$  and  $B(\mathcal{N})$ . Let  $\sigma \in S$  and let  $\mathbf{I}'(\sigma) = B(\mathcal{M}) - \text{coim } \sigma$ ,  $\mathbf{J}'(\sigma) = B(\mathcal{N}) - \text{im } \sigma$ . Then, the bottleneck distance  $d_B(\mathcal{M}, \mathcal{N})$  is equal to

$$\min_{\sigma \in S} \max \left\{ \max_{I \in \text{coim } \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'(\sigma)} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'(\sigma)} \{d_I(0, \mathcal{I}^J)\} \right\}. \tag{10}$$

*Proof.* Since  $\mathcal{M}$  and  $\mathcal{N}$  are finitely interval decomposable persistence modules, the barcodes  $B(\mathcal{M})$  and  $B(\mathcal{N})$  are finite, and hence  $S$  is also a finite set. If  $d_B(\mathcal{M}, \mathcal{N}) = +\infty$ , then the quantity in (10) is less than or equal to  $d_B(\mathcal{M}, \mathcal{N})$ . If  $d_B(\mathcal{M}, \mathcal{N}) < +\infty$ , consider a partial multibijection  $\bar{\sigma}: B(\mathcal{M}) \rightsquigarrow B(\mathcal{N})$  between the barcodes  $B(\mathcal{M})$  and  $B(\mathcal{N})$  such that  $\bar{\sigma}$  is an  $\epsilon$ -matching. Therefore,

- (i) for all  $I \in \mathbf{I}'(\bar{\sigma})$ ,  $\mathcal{I}^I$  is  $2\epsilon$ -trivial,
- (ii) for all  $J \in \mathbf{J}'(\bar{\sigma})$ ,  $\mathcal{I}^J$  is  $2\epsilon$ -trivial,
- (iii) for all  $I \in \text{coim } \bar{\sigma}$ ,  $\mathcal{I}^I$  and  $\mathcal{I}^{\bar{\sigma}(I)}$  are  $\epsilon$ -interleaved.

Now, by Proposition 2.19, the interval persistence module  $\mathcal{I}^I$  is  $2\epsilon$ -trivial if and only if  $\mathcal{I}^I$  is  $\epsilon$ -interleaved with the zero persistence module for all  $I \in \mathbf{I}'(\bar{\sigma})$ . Thus,  $\epsilon \geq d_I(\mathcal{I}^I, 0)$  for all  $I \in \mathbf{I}'(\bar{\sigma})$ .

Similarly, by Proposition 2.19, the interval persistence module  $\mathcal{I}^J$  is  $2\epsilon$ -trivial if and only if  $\mathcal{I}^J$  is  $\epsilon$ -interleaved with the zero persistence module for all  $J \in \mathbf{J}'(\bar{\sigma})$ . Thus,  $\epsilon \geq d_I(0, \mathcal{I}^J)$  for all  $J \in \mathbf{J}'(\bar{\sigma})$ .

Also,  $\mathcal{I}^I$  and  $\mathcal{I}^{\bar{\sigma}(I)}$  are  $\epsilon$ -interleaved for all  $I \in \text{coim } \bar{\sigma}$ . Thus,  $\epsilon \geq d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})$  for all  $I \in \text{coim } \bar{\sigma}$ .

Therefore,

$$\epsilon \geq \max \left\{ \max_{I \in \text{coim } \bar{\sigma}} \{d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})\}, \max_{I \in \mathbf{I}'(\bar{\sigma})} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'(\bar{\sigma})} \{d_I(0, \mathcal{I}^J)\} \right\}.$$

It follows that

$$\epsilon \geq \min_{\sigma \in S} \left( \max \left\{ \max_{I \in \text{coim } \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'(\sigma)} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'(\sigma)} \{d_I(0, \mathcal{I}^J)\} \right\} \right).$$

By the arbitrariness of  $\epsilon$ , we deduce that

$$d_B(\mathcal{M}, \mathcal{N}) \geq \min_{\sigma \in S} \max \left\{ \max_{I \in \text{coim } \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'(\sigma)} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'(\sigma)} \{d_I(0, \mathcal{I}^J)\} \right\}.$$

Conversely, let us set

$$\epsilon = \min_{\sigma \in S} \max \left\{ \max_{I \in \text{coim } \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'(\sigma)} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'(\sigma)} \{d_I(0, \mathcal{I}^J)\} \right\}.$$

Let  $\bar{\sigma}$  be a partial multibijection such that

$$\epsilon = \max \left\{ \max_{I \in \text{coim } \bar{\sigma}} \{d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})\}, \max_{I \in \mathbf{I}'(\bar{\sigma})} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'(\bar{\sigma})} \{d_I(0, \mathcal{I}^J)\} \right\}.$$

It follows that  $\epsilon \geq d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})$  for all  $I \in \text{coim } \bar{\sigma}$ . Similarly,  $\epsilon \geq d_I(\mathcal{I}^I, 0)$  for all  $I \in \mathbf{I}'(\bar{\sigma})$  and  $\epsilon \geq d_I(0, \mathcal{I}^J)$  for all  $J \in \mathbf{J}'(\bar{\sigma})$ . Clearly,  $\mathcal{I}^I$  is  $\epsilon$ -interleaved with the zero persistence module for each  $I \in \mathbf{I}'(\bar{\sigma})$  and  $\mathcal{I}^J$  is  $\epsilon$ -interleaved with the zero persistence module for each  $J \in \mathbf{J}'(\bar{\sigma})$ . So, by Proposition 2.19,  $\mathcal{I}^I$  is  $2\epsilon$ -trivial for each  $I \in \mathbf{I}'(\bar{\sigma})$  and  $\mathcal{I}^J$  is  $2\epsilon$ -trivial for each  $J \in \mathbf{J}'(\bar{\sigma})$ . So, we have

- (i) for all  $I \in \mathbf{I}'(\bar{\sigma})$ ,  $\mathcal{I}^I$  is  $2\epsilon$ -trivial,
- (ii) for all  $J \in \mathbf{J}'(\bar{\sigma})$ ,  $\mathcal{I}^J$  is  $2\epsilon$ -trivial,
- (iii) for all  $I \in \text{coim } \bar{\sigma}$ ,  $\mathcal{I}^I$  and  $\mathcal{I}^{\bar{\sigma}(I)}$  are  $\epsilon$ -interleaved.

Thus,  $\bar{\sigma}$  is an  $\epsilon$ -matching. So, by Definition 2.23,  $\epsilon \geq d_B(\mathcal{M}, \mathcal{N})$ . Hence, the claim is proved.  $\square$

The following result shows that the bottleneck distance is equal to the interleaving distance if  $\mathcal{M}$  and  $\mathcal{N}$  are two  $n$ -parameter interval persistence modules (for more details of the proof, we refer to the reader to [3, Proposition 4.3.1]).

**Corollary 4.2.** *If  $\mathcal{M} = \mathcal{I}^I$  and  $\mathcal{N} = \mathcal{I}^J$  are two  $n$ -parameter interval persistence modules, then  $d_B(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}, \mathcal{N})$ .*

*Proof.* It is well known that  $d_I(\mathcal{M}, \mathcal{N}) \leq d_B(\mathcal{M}, \mathcal{N})$ . Thus, it is enough to show that  $d_B(\mathcal{M}, \mathcal{N}) \leq d_I(\mathcal{M}, \mathcal{N})$ . Suppose that the interval persistence modules  $\mathcal{M} = \mathcal{I}^I$  and  $\mathcal{N} = \mathcal{I}^J$  are  $\epsilon$ -interleaved. Now, consider the bijection  $\sigma: \{I\} \rightarrow \{J\}$  with  $\text{im } \sigma = \{J\}$  and  $\text{coim } \sigma = \{I\}$ . Therefore,  $B(\mathcal{M}) - \text{coim } \sigma = \emptyset$  and  $B(\mathcal{N}) - \text{im } \sigma = \emptyset$ . Now, by Definition 2.22, the bijection  $\sigma: \{I\} \rightarrow \{J\}$  is an  $\epsilon$ -matching since  $\mathcal{I}^I$  and  $\mathcal{I}^J$  are  $\epsilon$ -interleaved. In particular, we have  $d_B(\mathcal{M}, \mathcal{N}) \leq d_I(\mathcal{M}, \mathcal{N})$ .  $\square$

Now, we are ready to give an exact computation of the bottleneck distance between  $n$ -parameter rectangle decomposable persistence modules.

Let  $\mathcal{M}$  and  $\mathcal{N}$  be two finitely  $n$ -parameter rectangle decomposable persistence modules with decompositions

$$\mathcal{M} = \bigoplus_{i=1}^m \mathcal{M}_i \text{ and } \mathcal{N} = \bigoplus_{j=1}^k \mathcal{N}_j.$$

Let us denote the underlying rectangle of the persistence module  $\mathcal{M}_i$  by  $R_i = (a_1^i, b_1^i) \times \dots \times (a_n^i, b_n^i)$  and  $\mathcal{N}_j$  by  $Q_j = (c_1^j, d_1^j) \times \dots \times (c_n^j, d_n^j)$  where  $a^i = (a_1^i, \dots, a_n^i)$ ,  $b^i = (b_1^i, \dots, b_n^i)$ ,  $c^j = (c_1^j, \dots, c_n^j)$  and  $d^j = (d_1^j, \dots, d_n^j)$ .

**Theorem B.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two finitely rectangle decomposable persistence modules given as above. Let  $S$  be the set of all partial multibijections between the barcodes  $B(\mathcal{M})$  and  $B(\mathcal{N})$ . Let  $\sigma \in S$  and let  $\mathbf{I}'(\sigma) = B(\mathcal{M}) - \text{coim } \sigma$ ,  $\mathbf{J}'(\sigma) = B(\mathcal{N}) - \text{im } \sigma$ . Then, the bottleneck distance  $d_B(\mathcal{M}, \mathcal{N})$  is equal to*

$$\min_{\sigma \in S} \max \left\{ \max_{R_i \in \text{coim } \sigma} \left\{ d_I(\mathcal{M}_i, \sigma(\mathcal{M}_i)) \right\}, \max_{R_{i'} \in \mathbf{I}'(\sigma)} \left\{ \frac{\text{thick}(R_{i'})}{2} \right\}, \max_{Q_{j'} \in \mathbf{J}'(\sigma)} \left\{ \frac{\text{thick}(Q_{j'})}{2} \right\} \right\},$$

where the interleaving distance between  $\mathcal{M}_i$  and  $\sigma(\mathcal{M}_i)$  is equal to

$$\min \left\{ \max \left\{ \frac{\text{thick}(R_i)}{2}, \frac{\text{thick}(R_{\sigma(i)})}{2} \right\}, d_H(R_i, \sigma(R_i)) \right\}.$$

*Proof.* By Proposition 4.1, the bottleneck distance between two finitely rectangle decomposable persistence modules  $\mathcal{M}$  and  $\mathcal{N}$  with decompositions

$$\mathcal{M} = \bigoplus_{i=1}^m \mathcal{M}_i \text{ and } \mathcal{N} = \bigoplus_{j=1}^k \mathcal{N}_j$$

can be computed as

$$\min_{\sigma \in S} \max \left\{ \max_{R_i \in \text{coim } \sigma} \left\{ d_I(\mathcal{M}_i, \sigma(\mathcal{M}_i)) \right\}, \max_{R_{i'} \in \mathbf{I}'(\sigma)} \left\{ d_I(\mathcal{M}_{i'}, 0) \right\}, \max_{Q_{j'} \in \mathbf{J}'(\sigma)} \left\{ d_I(0, \mathcal{N}_{j'}) \right\} \right\}.$$

Propositions 2.18 and 2.19 together imply

$$d_I(\mathcal{M}_{i'}, 0) = \min_{1 \leq s \leq n} \frac{b_s^{i'} - a_s^{i'}}{2} = \frac{\text{thick}(R_{i'})}{2}$$

and

$$d_I(0, \mathcal{N}_{j'}) = \min_{1 \leq s \leq n} \frac{d_s^{j'} - c_s^{j'}}{2} = \frac{\text{thick}(Q_{j'})}{2}.$$

Finally, by Theorem A, we know that the interleaving distance between  $\mathcal{M}_i$  and  $\sigma(\mathcal{M}_i)$  is equal to

$$\min \left\{ \max \left\{ \frac{\text{thick}(R_i)}{2}, \frac{\text{thick}(R_{\sigma(i)})}{2} \right\}, d_H(R_i, \sigma(R_i)) \right\}.$$

□

The next stage of our ongoing research is to get an analogous formula for the case of interval modules. This is not a straightforward task: An underlying rectangle can be represented by just two corner points, but this is no longer true and is more complicated for an underlying interval of an interval persistence module. Moreover, two rectangles or their shiftings intersect along a single component, which is again a rectangle, but for intervals or their shiftings, the number of intersection components can be more than one.

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## Declarations

**Conflict of interest** The authors declare that there is no Conflict of interest.

**Ethical approval** Not applicable.

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