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# Truths and Myths About the Equations of Electrodynamics

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**Abstract**

The purpose of this article is to collect in an organic and synthetic way a certain number of facts regarding the equations of electrodynamics. While it is not questioned the effectiveness of the models with respect to the huge number of practical implementations, there are theoretical situations for which clarification is needed. In these cases mathematics does not allow for doubts and the response is sometimes severe. Many arguments are often lightly justified and are sources of further inaccuracies. Here, I want to precisely distinguish the incontrovertible truths from the false myths that have accumulated over the centuries. These subjects have been discussed sporadically by many authors and particularly in my previous books. Many authoritative texts instead prefer to gloss over the topics, even if they hide embarrassing inconsistencies. I think it is time to present a comprehensive list of what is true and what is wrong, on the basis of irrefutable considerations. The most critical situations certainly require corrections. For this reason I believe that an appropriate revision of the models can lead to a better understanding of electromagnetic phenomena, and when applied to atomic and molecular structures, allows us to investigate the secrets of matter more deeply.

Keywords: Electrodynamics, Maxwell-Heaviside equations, vector potential, Hertz's solution, spherical wave, photon.

## 1 Introduction

J. C. Maxwell, at the end of the 19th century, merged the fundamental notions of electromagnetism into a set of equations that later became the famous fundamental laws that regulate electromagnetic phenomena. At the same time, he had a vision of the structure of matter which was not the one subsequently adopted by the scientific community, mainly interested to follow the principles of the quantum version. Moreover, Maxwell's equations are not exactly the

ones that are taken into account today. They have in fact gone through a certain number of reworkings. The interesting article [1] clearly collects Maxwell's various formulas [2, 3], providing an accurate explanation of their origin and meaning. Counting the vector relations as consisting of three components, the number of equations equals 20. I do not want to deal with historical issues here, so I will not take Maxwell's original papers into consideration. However, not everyone knows that, at the turn of the millennium, O. Heaviside took the responsibility of reformulating the model further, transforming it into the one in use today (see [4, 5] and the review paper [6]). Heaviside reduced the number of equations to 8. Here, for simplicity, only the $\vec{E}$ and $\vec{B}$ fields are taken into account (thus $\vec{D} = \vec{E}/\epsilon$ and $\vec{H} = \vec{B}/\mu$). In practice:

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{1}$$

$$\frac{\partial \vec{E}}{\partial t} = c^2 \vec{\nabla} \times \vec{B} + \vec{J} \qquad \vec{\nabla} \cdot \vec{E} = \rho \tag{2}$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ denotes the speed of light in vacuum. To facilitate notation, with the exception of $c$, I set all other constants to 1 which will also be considered dimensionless.

Heaviside's version is often confused with Maxwell's to the point that in Edinburgh (Scotland), at the foot of the Maxwell monument at St. Andrew square, there is a metal plaque attributing the modified equations to the wrong person.

The above can represent an excellent simplification from certain points of view, which can instead become a source of inconsistent facts in other specific situations. The purpose of this article is to list the strengths and weaknesses of the so-called Maxwell-Heaviside equations, precisely outlining their range of applicability and underlining when their role is ambiguous if not completely meaningless. To take stock of the situation it is more than sufficient to work in vacuum. In fact, it is already the simplest cases that can lead to serious inconsistencies, which then develop in a more subtle way in the more general context.

As previously observed by Maxwell, the setting hides the wave equations within itself. The speed of propagation of these waves is what made it possible to relate electromagnetic emissions to optical phenomena, although this aspect will soon be debunked in this paper, despite what is often believed. Without the presence of forcing terms, due for example to currents or stationary charges ($\vec{J} = 0$ and $\rho = 0$ in (2), so that $\vec{\nabla} \cdot \vec{E} = 0$), it is easy to carry out the calculation. It is just a matter of differentiating an using basic vector formulas.

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -c^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = c^2 \Delta \vec{E} - c^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) = c^2 \Delta \vec{E} \tag{3}$$

$$\frac{\partial^2 \vec{B}}{\partial t^2} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = -c^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = c^2 \Delta \vec{B} - c^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) = c^2 \Delta \vec{B} \tag{4}$$

Of course, we assume hypotheses of good regularity in order to justify the differentiations. The two new equations for $\vec{E}$ and $\vec{B}$ appear independent, but a coupling can arise from appropriate boundary conditions. For the moment, let us examine only the case of $\vec{E}$. The fact of having used the free-divergence condition does not guarantee that it can be eliminated from the system. In truth, the set to consider is the following, which consists of 4 equations with 3 unknowns (the components of $\vec{E}$):

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 \Delta \vec{E} \qquad\qquad \vec{\nabla} \cdot \vec{E} = 0 \tag{5}$$

The presence of more relationships than unknowns in a linear system raises suspicion. Well-known theorems of Mathematical Analysis guarantee the existence and uniqueness of the solution of the vector wave equation under rather general assumptions regarding the geometry of the domain, as well as the initial and boundary conditions [7]. As $\vec{E}$ has been obtained, the probability that it also satisfies the free-divergence condition is practically zero. For this to happen you have to be very lucky. The typical attitude in the engineering sector is to not care at all: once you get $\vec{E}$ in some way, you somehow proceed with other successive calculations. There are interesting cases in which all relationships are magically satisfied. Many others show instead incompatibilities. Inspired by the successful outcomes, practitioners trust that everything can work out for the best even in the most complicated situations. However, the rigor of mathematics cannot be circumvented, so it is good to know which cases are allowed and which are not. In those situations where the numbers do not add up, it is not enough to take note and proceed without remorse. The correct way out is to understand the causes and modify the model accordingly. It is not fair to use mathematics when it is convenient, but to disdain it at the first difficulty.

Before concluding this brief introduction, I would also like to introduce the potentials:

$$\vec{B} = \frac{1}{c}(\vec{\nabla} \times \vec{A}) \qquad\qquad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \tag{6}$$

These were also used by Maxwell in his versions of the equations. In many contexts the 4 unknowns $\vec{A}$ and $\Phi$ are used instead of the 6 given by $\vec{E}$ and $\vec{B}$. The main reason is that the equations in (1) can be automatically eliminated by substituting into them the new expressions of $\vec{E}$ and $\vec{B}$. I reserve the right to discuss the remaining equations later, since this too is a crucial topic. It is very important to observe that the potentials are not uniquely determined. Indeed, by introducing the scalar $\Lambda$, it is possible to replace the potentials in the following way:

$$\vec{A} \longrightarrow \vec{A} + \vec{\nabla}\Lambda \qquad\qquad \Phi \longrightarrow \Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \tag{7}$$

without altering $\vec{E}$ and $\vec{B}$, as well as the equations in (1). Indeed, one has:

$$\vec{B} = \frac{1}{c}\Big[\vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda)\Big] = \frac{1}{c}(\vec{\nabla} \times \vec{A})$$

$$\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\Lambda) - \vec{\nabla}\left(\Phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \tag{8}$$

Thus, it can be stated formally that:

**Statement 1 -** *Whatever is* $\Lambda$, $\vec{\mathrm{E}}$ *and* $\vec{\mathrm{B}}$ *take always the same expression as a function of the potentials. Moreover, the equations* $\partial\vec{B}/\partial t = -\vec{\nabla} \times \vec{E}$ *and* $\vec{\nabla} \cdot \vec{B} = 0$ *are always satisfied.*

This may no longer be true for the other equations in (2), but many researchers tend to hide this fact, either for their convenience or because they were induced by wrong rules imparted in their training. It is important that this aspect is clarified here too.

## 2 A collection of good and bad examples

Let us start with some negative examples. As has been said, the system (5) has more equations than unknowns. It is therefore expected that there may be solutions of the wave equation that do not have zero divergence. A simple case is the following transverse wave, where the components of $\vec{E} = (E_1, E_2, E_3)$ are expressed in Cartesian coordinates:

$$\vec{E} = \Big(0,\ 0,\ \sin(\sqrt{3}ct - x)\ \sin y\ \sin z\Big) \tag{9}$$

with $-\infty < x < +\infty$, $\quad 0 \le y \le \pi$, $\quad 0 \le z \le \pi$. Vanishing conditions are here imposed at the boundary of a parallelepiped of infinite length in the $x$ direction. Each of the components of $\vec{E}$ satisfies the scalar wave equation, but the field is not divergence-free. There are infinite examples, generally approached with numerical techniques, where the wave equation is solved, but no one goes to check that in the end the divergence of the field is vanishing. This basically means that the charge is not preserved and therefore the problem is not addressed by considering all aspects of the physics. Nevertheless, by computing the magnetic field according to (1), one gets:

$$\vec{B} = \frac{1}{\sqrt{3}c}\Big(\cos(\sqrt{3}ct - x)\ \cos y\ \sin z,\ \ -\sin(\sqrt{3}ct - x)\ \sin y\ \sin z,\ \ 0\Big) \tag{10}$$

which, astonishingly, satisfies both the wave equation and the divergence-free condition. The equation $\partial\vec{E}/\partial t = c^2\vec{\nabla} \times \vec{B}$ is not satisfied instead. A typical approach is to stop at the first positive controls, assuming that everything else is verified. We have just seen that this way of proceeding could lead to erroneous conclusions. The number of articles where the numerical simulations focus only on the evolutionary part of the equations (thus neglecting the divergence-free conditions) fills almost all the available literature. Negligence multiplies without anyone making amends. I will not cite any of the works involved because they are countless and I do not want to express "preferences".

Let me also observe another important issue: the wave represented by (9) does not translate at the speed of light, but at a greater speed, i.e. $\sqrt{3}c$. I can therefore state:

**Statement 2 -** *The solutions of the vector wave equation do not necessarily travel at speed c.*

The same property will be observed other times later in this paper. The speed of propagation of an electromagnetic phenomenon in vacuum may therefore not be that of light. I understand that this is hard news to digest. Perhaps the solution in (9) does not represent any electromagnetic phenomenon, due to boundary conditions or some initial data not assigned correctly. This could be a possible explanation for the inconsistency of the solution from a physical point of view. It should be also paid attention to the fact that the wave equation is of second-order in time, so it requires an initial datum for both $\vec{E}$ and $\partial\vec{E}/\partial t$. However, it is hard to know in advance how to distinguish between good and bad initial or boundary conditions. The problem (5) is over-determined from the start and therefore we must move with caution.

If we apply the divergence operator to the equation $\partial\vec{E}/\partial t = c^2\vec{\nabla}\times\vec{B}$ we easily obtain that:

$$\frac{\partial\rho}{\partial t} = 0 \qquad \text{with} \quad \rho = \vec{\nabla}\cdot\vec{E} \tag{11}$$

This says that if the divergence is initially zero, it remains zero throughout evolution. Unfortunately, this is true if you are well away from the boundaries. In scattering phenomena, for example, the encounter of a wave with an obstacle can locally generate situations where the electric field is not divergence-free. However, almost no one checks that $\rho$ has remained unchanged. It is important to consider that the condition (11) cannot be directly recovered from $\partial^2\vec{E}/\partial t^2 = c^2\Delta\vec{E}$ , so the wave equation alone is not sufficient to guarantee that $\rho$ remains zero. A theoretical analysis of how incompatibilities can develop in the requirement to solve the wave equation, to enforce at the same time the divergence-free condition, and to impose certain boundary conditions, is provided in [8] for a special application.

Since the reader is certainly not yet convinced of my statements, it is convenient to provide further examples. Let me take:

$$\begin{aligned}\vec{E} &= \Big(\sin(ct-y),\ 0,\ 0\Big) + \Big(0,\ 2\sin(ct-x),\ 0\Big)\\ &= \Big(\sin(ct-y),\ 2\sin(ct-x),\ 0\Big)\end{aligned} \tag{12}$$

This is the sum of two differently polarized plane waves having the magnetic field equal to:

$$\vec{B} = \frac{1}{c}\Big(0,\ 0,\ -\sin(ct-y) + 2\sin(ct-x)\Big) \tag{13}$$

We are perfectly in the case where all Maxwell-Heaviside relations are fulfilled. So it is true that there are situations in which all the constrictions are compatible

(and there will be others later). Nevertheless, we must recognize that statement 2 is still valid. As a matter of fact, when considering the restriction $x = y$, the corresponding information moves along the direction $(1, 1, 0)$ at a speed $\sqrt{2}c > c$. Here we are also violating a further dogma of electrodynamics. Indeed, the message transported is the vector $(1, 2, 0)\sin(ct - \xi)$, where $\xi = x = y$, which is not orthogonal to the direction of propagation. In other words, the wave in (12) is not transverse. This leads us to establish:

**Statement 3 -** *Even in vacuum, the algebric sum of two (or more) transversal electromagnetic plane waves is generally not a transversal electromagnetic wave.*

I am sure the reader is more impressed now. Of course, there is a cure for all ills, so the most informed experts can invoke the principles of Einsteinian relativity to get out of the impasse. In this context, the sum of the two plane waves in (12) must not be done in a linear manner but according to the axioms of special relativity, where suitable nonlinear rules apply to the sum of velocities (see, e.g. [9], section 10). I wonder, however, how many engineering papers apply such relativistic considerations in their calculations.

Many texts on optics assert that almost any type of electromagnetic wave can be obtained with an appropriate "combination" or "superposition" (and here the terms should be better clarified) of monochromatic plane waves. This is true when the corresponding Poynting's vectors are all lined up. Instead, the claim: *any three-dimensional wave can be expressed as a combination of plane waves* (see [10], p. 27, or [11], p. 478), has little foundation and it is easy to find appropriate counterexamples. As a matter of fact, people tend to forget that planes are huge bulky objects. They cannot be cut without generating discontinuities and the information written on them cannot be concentrated in some areas without affecting the divergence of the fields. It is unclear then how these unmanageable objects can be used to "approximate" curved fronts, both in terms of their geometry and the signal written on them. The fact that these considerations are also made by some authors in the case of photons makes the construction even less credible. Unfortunately, in order to justify many aspects of quantum mechanics it is necessary to sacrifice many others of electrodynamics, and to resort for example to a "localization" generally not allowed by the classical equations [12]. The original mistake is that many results are obtained by reasoning along the direction of evolution of point-wise signals, as if an entire solid wave was composed by a beam of rays. In other words, even in the classical analysis there is a tendency to attribute to individual points the capacity to carry an isolated piece of information (see, e.g. [13], chapter 2), as happens with the photon which is in fact the "carrier" of electromagnetic information. In this fashion, the three-dimensionality of the problem is lost, although the classical model requires the evaluation of partial derivatives in all directions.

If the material provided so far is not yet sufficient, I still have plenty of ammunition to shoot. The upcoming one should be pretty convincing (see also my paper [14]). The domain is the external one to a sphere of radius

$R$. I now work in the spherical coordinate system $(r, \theta, \phi)$, assuming that the fields do not depend on both variables $\theta$ and $\phi$. This ensures a perfect central symmetry. Time-dependent boundary conditions are imposed on the ball surface. These are homogeneously distributed in such a way that the electric field $\vec{E} = (E_1, 0, 0)$ is of radial type. During the evolution, outside the ball, the function $E_1$ will continue to depend only on $t$ and $r$, since no magnetic field is going to be generated in this circumstance. The boundary conditions are of the form $E_1(R, t) = f(t)/R^2$, where the function $f$ is given. Let us see a possible way of operating. The divergence is required to be zero:

$$\rho = \vec{\nabla} \cdot \vec{E} = \frac{\partial E_1}{\partial r} + \frac{2E_1}{r} = \frac{1}{r^2} \frac{\partial (E_1 r^2)}{\partial r} = 0 \tag{14}$$

The solution is trivially $E_1(r, t) = f(t)/r^2$, for $r \geq R$. Unfortunately, this expression does not satisfy the wave equation in spherical coordinates. Furthermore, we have a signal that propagates throughout space instantaneously, and this is not very in tune with the theory of electromagnetism. We are starting to get into trouble, but we can still rely on Maxwell-Heaviside equations. The magnetic field is zero, so the Ampère law in vacuum says: $\partial \vec{E}/\partial t = c^2 \vec{\nabla} \times \vec{B} = 0$, implying that only stationary electric field can be taken into account. This situation does not agree with the setting of our original problem. We are completely stuck at this point. The change in charge of a ball (or its motion) produces an electromagnetic field which is not compatible with the known laws. Yet it was not a difficult exercise!

Reasoning of this kind certainly heated up the discussion at the very beginning of the last century regarding the type of waves transmitted by a Tesla coil generator [15, 16, 17, 18]. Locally, the signal emitted has the nature of a longitudinal wave, whereas the Maxwellian theory is only expected to handle (in vacuum) Hertzian transversal waves. In the end, the faction that supports the transversality of electromagnetic radiations won, but these examples continue to call into question the basic principles on which today's theory of electrodynamics and optics is based [19] (see also my observations at the concluding section of this paper). I add to this context the difficulty of explaining the charge of a capacitor through the Maxwell-Heaviside model [8] and related wave-propagation problems [20]. Not to forget the famous "capacitor paradox" ([21], vol. 2, p. 684), in which it is difficult (if not impossible) to combine the dynamics of the fields with charge and energy conservation.

Many researchers are oriented towards the formulation that exploits the potentials. They believe they can get around the obstacles, but bad news is coming from that sector too. This is to be expected concerning the last example given above, since the reduction from 6 to 4 unknowns is nothing more than a rewrite leading to similar inconsistencies. The adoption of the potentials $\vec{A}$ and $\Phi$ has the effect of eliminating the two equations $\partial \vec{B}/\partial t = -\vec{\nabla} \times \vec{E}$ and $\vec{\nabla} \cdot \vec{B} = 0$. For the pulsating charged ball (under conditions of perfect symmetry) the magnetic field does not develop, therefore from (6) it follows that $\vec{A}$ is irrotational so that $\vec{E}$ is a pure gradient field, i.e.: $\vec{E} = \vec{\nabla} P$, where $P$ is a scalar function

depending on $r$ and $t$. By imposing that $\vec{\nabla} \cdot \vec{E} = 0$, we get $\Delta P = 0$ for any time $t$. Compatibly with the boundary conditions, this brings us back to the solution $P = -f(t)/r$ corresponding to $\vec{E}(r,t) = (f(t)/r^2, 0, 0)$, that was already discarded before. Similar conclusions were reached in [22] for a non-point-wise moving charged body, where the solution is represented through integral equations expressing the so called retarded potential. Other considerations in this direction are provided in [23], where a possible reformulation of the model is also proposed.

The use of $\vec{A}$ and $\Phi$ is often related to the Liénard-Wiechert potential. A classical example describes the electromagnetic field produced by a point-wise charge moving at velocity $\vec{v}$. I provide a brief introduction to the problem here below. Details can be found in many textbooks (see, e.g., [24], section 4.1). In the Cartesian system of coordinates $\vec{x} = (x_1, x_2, x_3)$, the couple $(\vec{x}, t)$ denotes the observation point, whereas $(\vec{x}_r, t_r)$ denotes the observed point at the retarded time $t_r = t - |\vec{x} - \vec{x}_r|/c$. Up to dimensional constant, the potential $\Phi$ takes the form:

$$\Phi(\vec{x}, t) = \frac{1}{(1 - \vec{n} \cdot \vec{\beta})|\vec{x} - \vec{x}_r|} \tag{15}$$

where:

$$\vec{n} = (n_1, n_2, n_3) = \frac{\vec{x} - \vec{x}_r}{|\vec{x} - \vec{x}_r|} \qquad \vec{\beta} = (\beta_1, \beta_2, \beta_3) = \vec{v}/c \tag{16}$$

and the velocity $\vec{v} = d\vec{x}_r/dt$ is evaluated at $t = t_r$. This construction is made on purpose to get the non-homogeneous wave equation:

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \Delta \Phi + c\rho \tag{17}$$

with $\rho = c\ \delta_0(\vec{x} - \vec{x}_r)$ denoting the Dirac delta positioned in $\vec{x} = \vec{x}_r$. As far the vector potential is concerned, one sets:

$$\vec{A}(\vec{x}, t) = \vec{\beta}(t_r)\ \Phi(\vec{x}, t) \qquad \vec{J}(\vec{x}, t) = c\ \vec{\beta}(t_r)\ \rho(\vec{x}, t) \tag{18}$$

Such a construction is compatible with the relation:

$$\frac{\partial^2 \vec{A}}{\partial t^2} = c^2 \Delta \vec{A} + \vec{J} \tag{19}$$

At this point, using the definitions in (6), the full set of equations (1) and (2) is satisfied. Moreover, the following relation known as the Lorenz gauge, holds:

$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0 \tag{20}$$

The above is quite an important relation, since it is a relativistic invariant. It practically comes for free and arises naturally from the context of the setting

(15) and (18). In accordance with (7), the equation (20) is compatible with the homogeneous wave equation for $\Lambda$, i.e.:

$$\frac{\partial^2 \Lambda}{\partial t^2} = c^2 \Delta \Lambda \tag{21}$$

The discussion ends here in the most elegant way possible. However, if one examines things better, some problems arise. As observed in the Statement 1, the scalar $\Lambda$ is not a new unknown and should not in any way influence the solutions. Nevertheless, by modifying the gauge (and consequently $\Lambda$), new fields can be created that are not compatible with those previously found. This is what was discovered in [25, 26, 27], where new results were obtained by working with the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, corresponding to requiring $\Delta\Lambda = 0$. This leads us to a set of equations that appears overdetermined, and therefore at risk of not admitting solutions:

$$\frac{\partial^2 \vec{A}}{\partial t^2} = c^2 \Delta \vec{A} + \vec{J} - c\vec{\nabla}\left(\frac{\partial \Phi}{\partial t}\right) \qquad \vec{\nabla} \cdot \vec{A} = 0 \qquad \Delta\Phi = 0 \tag{22}$$

The authors recover $\Phi$ from Laplace's equation and then use it as an extra forcing term in the wave equation for $\vec{A}$. In the new context the choice of the gauge appears too restrictive. For the sake of simplicity, I will not report here the technical details of these computations.

The above conclusions make us reflect on the possible anomalies hidden in the Maxwell-Heaviside model. If the fields may depend on the gauge (when they shouldn't), it is no longer clear what their real evolution is in the presence of a moving charge, thus calling into question the deepest beliefs. Some explanations are provided in [26]. I would like to add one as an alternative. The Liénard-Wiechert potential is built on the possibility of reconstructing the solution of a hyperbolic wave equation by following the characteristic curves backwards. Information propagates from $(\vec{x}_r, t_r)$ through spherical balls and is observed after a retarded time at the point $(\vec{x}, t)$. As will be seen shortly, spherical waves carrying vector messages do not exist in the current context, invalidating the entire construction process. I am sure this last observation left the reader disconcerted. However, this is what raw mathematics teaches us. Let us see why.

Let us work again in spherical coordinates $(r, \theta, \phi)$ and take the following fields, distributed on the tangent planes of spherical wave-fronts:

$$\begin{aligned} \vec{E} &= \frac{1}{r}\Big(0,\ E_2(\theta,\phi),\ E_3(\theta,\phi)\Big) g(r-ct) \\ \vec{B} &= \frac{1}{r}\Big(0,\ B_2(\theta,\phi),\ B_3(\theta,\phi)\Big) g(r-ct) \end{aligned} \tag{23}$$

where $g$ is an arbitrary smooth function modulating the wave. Because of the multiplicative factor $1/r$, the energy density $|\vec{E}|^2 + |c\vec{B}|^2$ remains constant when integrated over a spherical surface growing linearly in time. Using notions

of calculus in spherical coordinates, it follows from $\partial\vec{E}/\partial t = \vec{\nabla}\times\vec{B}$ and $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}$ that:

$$E_3 = -cB_2 \quad E_2 = cB_3 \quad \frac{\partial}{\partial\theta}(B_3\sin\theta) = \frac{\partial B_2}{\partial\phi} \quad \frac{\partial}{\partial\theta}(E_3\sin\theta) = \frac{\partial E_2}{\partial\phi} \tag{24}$$

Note that the result has been obtained without enforcing the extra conditions $\vec{\nabla}\cdot\vec{E} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$, which however can follow by manipulating the relations in (24), i.e.:

$$\frac{\partial}{\partial\theta}(B_2\sin\theta) = -\frac{\partial B_3}{\partial\phi} \qquad \frac{\partial}{\partial\theta}(E_2\sin\theta) = -\frac{\partial E_3}{\partial\phi} \tag{25}$$

All of this implies the elliptic equation:

$$-\left[\frac{\partial}{\partial\theta}\left(\frac{\partial G}{\partial\theta}\sin\theta\right) + \frac{1}{\sin\theta}\frac{\partial^2 G}{\partial\phi^2}\right] = 0 \tag{26}$$

where $G$ can be anyone of the following functions: $E_1\sin\theta$, $E_2\sin\theta$, $B_1\sin\theta$, $B_2\sin\theta$. By standard arguments of Analysis for PDEs (see [7]), one finds out that the differential operator in (26) is positive-definite (multiply by $G\sin\theta$, integrate on a spherical surface and use Green's formulas). Leaving aside the details, it can be deduced that the only bounded solutions are those for which $G = 0$. As a consequence, it follows that the only possible solutions, having bounded electromagnetic fields laying on tangent planes of spherical fronts, are identically zero. This is true despite common belief. Note that one of the allowed solutions is $E_2 = 1/\sin\theta$, $E_3 = 0$, $B_2 = 0$, $B_3 = 1/c\sin\theta$, which is clearly unbounded and not compatible with any physical set up.

A more elegant explanation can also be given in the framework of Complex Analysis. Indeed, (24) and (25) represent the Cauchy-Riemann conditions for the complex functions $(E_2 - iE_3)\sin\theta$ and $(B_2 - iB_3)\sin\theta$ , where $i$ denotes the imaginary unit. They turn out to be holomorphic on the Riemann sphere, where a local orthogonal basis is defined by $(d\theta,\ d\phi/\sin\theta)$. This last property implies that $\vec{E}$ and $\vec{B}$ are bounded if and only if they are zero (see, e.g., [28] or [29]). The same arguments apply to any closed, bounded, compact, oriented surface.

The choice $E_2 = \sin\theta$, $E_3 = 0$, $B_2 = 0$, $B_3 = \sin\theta/c$, often reported in engineering texts, solves the scalar wave equation but not the vector one. The reader must not be fooled by a consolidated approach, which presents spherical waves as solutions of a scalar equation ([13], section 2.9). Considering that, in spherical coordinates, the Laplacian of a vector is not the Laplacian of its components, the vector wave equation contains extra terms that do not make it compatible with the scalar version. It is exactly for this reason that things become complicated, to the point that there is no solution to the problem.

These observations also raise doubts about the framework of the theory of special relativity, when we talk about luminous information, produced by an object, which propagates along wave fronts following the rules of geometrical

optics, or about luminous rays coming from a source. If the equations of electromagnetism are the basis of the phenomena that describe the propagation of light, it becomes difficult to think that such an interpretation actually corresponds to the geometric one. Returning to (22), the equations should describe the evolution of a vector information distributed on spherical wave-fronts centered on a moving infinitesimal charge. Based on the latest arguments we understand how this problem can generate perplexities about its effective resolution.

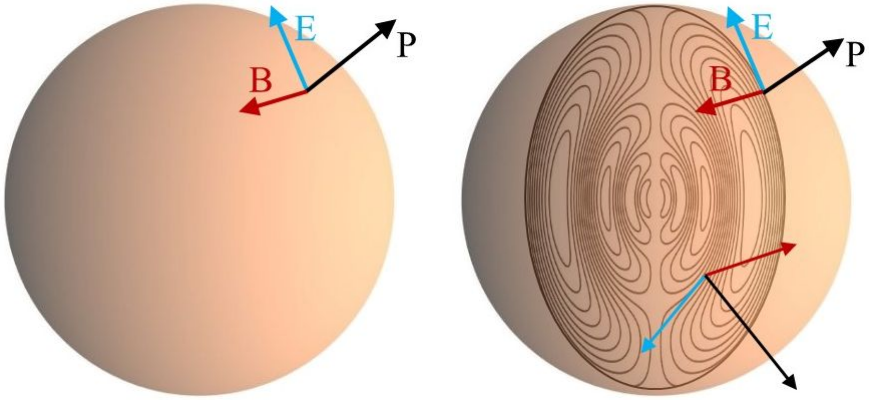


Figure 1: In a transversal spherical wave (left) the fields proceed so as to remain tangent to the surface. A typical case is that in which the electric field follows the meridians while the magnetic field follows the parallels. The Poynting vector $\vec{P} = \vec{E} \times \vec{B}$ is orthogonal to the surface and indicates the direction of the moving front. For the Hertz solution (right), the situation can be considered similar, but the topology is completely different. The envelope of the fields represents toroidal surfaces that expand over time intersecting each other. The Poynting vectors no longer indicate the direction of the energy flow and the rules of geometrical optics are disregarded.

Non trivial solutions (for example, the well-known Hertz solution) are instead possible by allowing the fields to have radial components $E_1$ and $B_1$. In this way, however, the Poynting vector $\vec{E} \times \vec{B}$ is not lined up with the (radial) direction of motion. Indeed, this new displacement does not comply with the rules of geometrical optics (see figure 1). The theoretical expression simulates the electromagnetic field generated by an infinitesimal dipole and satisfies all the Maxwell-Heaviside equations. The explicit formula is based on Bessel's functions (see (32)). The so produced wave seems to evolve on spherical surfaces, but from a more careful examination, it follows that the envelope of the fields corresponds to toroidal surfaces, whose section resembles the shape of a banana. Thus, the waves of the Hertz solution are not perfectly transversal (see [30], chapter 1, and [31], where the author examines the wandering behavior of the Poynting vector in a Hertzian antenna). This suggests to take this new ruling into consideration:

**Statement 4 -** *Even in vacuum, the solutions of the Maxwell-Heaviside system of equations may display an electric field partly longitudinal to the direction of propagation of the signal.*

By approaching a Hertzian wave while remaining on the plane of the equator ($\theta = \pi/2$), we can approximate the local behavior through plane waves. However, this is the only link that connects the Hertzian "spherical" wave with geometrical optics and which perhaps generated the erroneous intuition that each wave could be obtained as an envelope of plane waves. In order not to go too far off the beaten path of this article, I will not continue further with this type of investigation. A discussion on the limits of geometrical optics is provided for example in [32], section 3.1.4.

Transverse electromagnetic waves laying on parallel flat wave-fronts are completely prohibited, unless they are constant fields on entire planes, and therefore with infinite energy. This property should be well known, even if today's textbooks do not appear to report it. A simple proof of this fact follows the same path as that discussed above and it is given here below. In the Cartesian reference frame $(x, y, z)$, take the following solution candidates:

$$\vec{E} = \Big(E_1(x,y),\ E_2(x,y),\ 0\Big)g(z-ct)$$

$$\vec{B} = \Big(B_1(x,y),\ B_2(x,y),\ 0\Big)g(z-ct) \tag{27}$$

representing two field distributions modulated in time by the function $g$. The fields are transverse and lay on parallel surfaces shifting in the direction of the $z$-axis at the speed of light $c$. The functions $E_1$, $E_2$, $B_1$ and $B_2$ are supposed to be smooth. They will be asked to be zero at the boundary of a domain of finite measure. The function $g$ is also supposed to be smooth. As in (24), one gets:

$$E_1 = cB_2 \qquad E_2 = -cB_1 \qquad \frac{\partial B_2}{\partial x} = \frac{\partial B_1}{\partial y} \qquad \frac{\partial E_2}{\partial x} = \frac{\partial E_1}{\partial y} \tag{28}$$

It turns out that $\vec{E}$ and $\vec{B}$ are orthogonal with $|\vec{E}| = |c\vec{B}|$. By differentiating and combining the various relations, it is not difficult to get:

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \tag{29}$$

where $G$ can be anyone of the following functions: $E_1$, $E_2$, $B_1$, $B_2$. The divergence-free conditions also follow from similar manipulations.

Due to the homogeneous boundary conditions, bounded solutions are only admissible with $G = 0$ (see, e.g. [7]), implying the trivial choice $\vec{E} = 0$ and $\vec{B} = 0$. Taking a different approach, analogous conclusions are deduced from observing that the functions $E_1 - iE_2$ and $B_1 - iB_2$ are holomorphic, so that their constancy follows from Liouville's theorem [29]. The acceptable case of an entire plane wave carrying constant fields, thus of of infinite energy, falls as a topology among the toroidal waves, by noting that the lines of force close

at infinity. Therefore, in the Maxwellian world there are no transverse waves contained in bounded packets that travel unperturbed at the speed of light. This justified the introduction of photons as quantum entities which, although related to electrodynamics, do not solve any of its equations.

The considerations mentioned above do not imply that non-Maxwellian electromagnetic radiations do not exist in the real world. On the contrary, there are many more than theory can describe. They fill our daily lives. For example, the specially designed and tested antennas in [33, 34] produce signals that are not covered by classical theories. These devices generate very focused waves, approaching situations that can be defined as "perfect directivity". The wavefronts advance according to the rules of geometrical optics on almost parallel surfaces, where the transverse fields are concentrated in such a way as to have a rather compact shape. These tests reinforce the idea that the usual approach needs revision.

With these last considerations it seems that the series of negative news regarding electrodynamics is over. I would like to conclude this long section with some cases in which the Maxwell-Heaviside equations are at their best. Most of the spherical vector eigenfunctions, which play a crucial role in many problems of quantum mechanics, are related to closed lines of force both for the electric and magnetic components. They are therefore compatible with both the vector wave equations and the free-divergence conditions. An innovative application outside the classic schemes can be found in [35].

The set of Maxwell-Heaviside equations admits interesting solutions rotating about the axis of a cylinder. These can be used to simulate electromagnetic waves evolving inside toroid structures. Examples are discussed in detail in [36, 37, 38]. I report here the most simplified version in the system of cylindrical coordinates $(r, \phi, z)$. For an integer $k \geq 1$ and arbitrary constants $\omega > 0$, we have for any $z$:

$$\vec{E} = (E_1, E_2, E_3) = \left( \frac{kJ_k(\omega r)}{\omega r} \cos(c\omega t - k\phi),\ J'_k(\omega r) \sin(c\omega t - k\phi),\ 0 \right) \quad (30)$$

$$\vec{B} = (B_1, B_2, B_3) = \frac{1}{c}\Big(0,\ \ 0,\ -J_k(\omega r) \cos(c\omega t - k\phi)\Big) \quad (31)$$

where $J_k$ denotes the $k$-th Bessel function of the first kind, which satisfies:

$$J''_k(\xi) \ + \ \frac{J'_k(\xi)}{\xi} \ - \ \frac{k^2 J_k \xi)}{\xi^2} \ + \ J_k(\xi) \ = \ 0 \qquad\qquad \xi > 0 \quad (32)$$

Potentials are available:

$$\vec{A} = -\frac{1}{\omega}\Big(J_{k+1}(\omega r) \sin(c\omega t - k\phi),\ J_{k+1}(\omega r) \cos(c\omega t - k\phi),\ 0\Big) \quad (33)$$

$$\Phi = -\frac{1}{\omega}\ J_k(\omega r) \cos(c\omega t - k\phi) \quad (34)$$

which satisfy the Lorenz condition (20). In order to prove (6), remember the relations involving Bessel's functions: $J'_{k+1}(\xi) + (k+1)J_{k+1}(\xi)/\xi = J_k(\xi)$ and

$J_k'(\xi) - kJ_k(\xi)/\xi = -J_{k+1}(\xi)$. A far more general solution of this type is reported in [39], p. 257.

We may take for example $r$ between 0 and the first zero of $J_k$ in order to actually obtain a solution defined on a disk for any fixed $z$ (however, other boundary conditions may be taken into consideration). Qualitatively, the same type of solutions can be derived in the case of a torus (see figure 2), although, as far as I know, there are not exact representations in terms of known families of special functions. Numerical computations can be carried out both for the case o a typical ring and that of a Hill's type vortex.

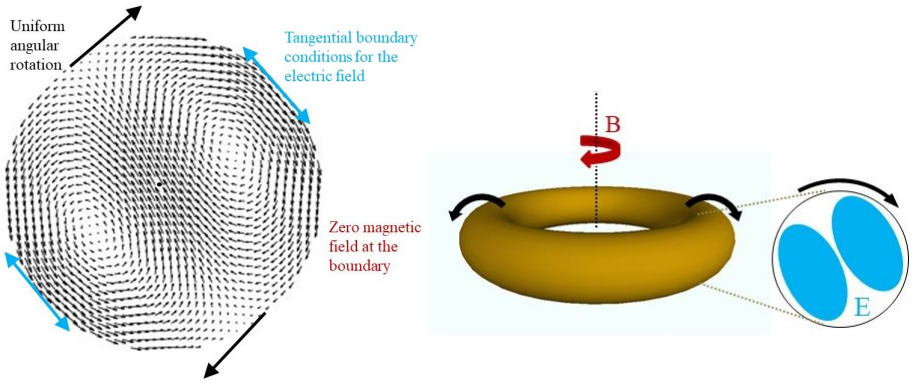


Figure 2: The Maxwell-Heaviside equations are particularly suitable for simulating electromagnetic phenomena defined and circulating within toroidal structures. Here we see an indicative simple example, where the fields evolve inside a ring that follows the classic movement of fluid dynamics vortexes. The electric field lies on the sections of the torus while the magnetic field forms closed lines around its main axis. In truth, the sections are not perfect circles, but their shape can be determined through numerical computations.

In the infinite cylinder solutions (30)-(31), by introducing the velocity field $\vec{v} = (v_1, v_2, v_3) = (0, c\omega r/k, 0)$ that indicates the direction in which the information moves, we get another example where both Statement 2 and Statement 4 are verified, i.e. the signal does not travel with constant intensity $c$ as far the velocity is concerned (it actually rotates at uniform angular velocity depending on $\omega$) and $\vec{E}$ displays a longitudinal component ($E_2$ parallel to $v_2$). This anomaly could be fixed with a generous use of general relativity techniques, although I am not aware of any specific paper on this matter that starts from the expressions of the fields plugged into the electromagnetic stress tensor and solves the corresponding Einstein's equations.

The moral of this last example is that, despite all predictions, except plane waves of infinite energy and some other scattered cases, the only situations in

which all equations (1)-(2) are solved lead to conclusions at odds with the facts that any physicist or engineer would be willing to believe.

# 3 Possible extensions and conclusions

The purpose of this article is to warn users about the errors that can occur in the field of electrodynamics equations. Some of the examples mentioned here are original, other have been considered in other works, although the contributions are scattered. Having rationally collected here the main sources of misunderstanding can further help the reader. This discussion also serves to put a limit on a blind subordination regarding the Maxwell-Heaviside model, generally considered totally reliable and untouchable.

The inconsistencies have generated a series of possible remedies, and many researchers have provided, depending on the circumstances, new versions of the equations. I place myself among these, citing my two books [30, 39]. It is not my intention to go into the details of these publications, which have however allowed me to suggest an interpretation of the structure of matter outside the usual quantum schemes [40]. My proposal is to weaken the hypothesis $\vec{\nabla} \cdot \vec{E} = 0$ also in vacuum, without dropping it but inserting it appropriately into the equations, while maintaining their physical meaning. In this way it is possible to include in the discussion spherical waves and photons (electromagnetic packets that travel at speed $c$ without dissipation and diffusion). It is also possible to include perfectly longitudinal waves and handle the case of isolated objects changing charge dynamically. A hypothesis is proposed that longitudinal electromagnetic waves are in an unstable state and, at the first perturbation or a transition to an asymmetric configuration, transform into transverse waves carrying the same message. The concept of anti-wave (the anti-matter analogue in the electromagnetic context) can be introduced, so to study the production of photons resulting from the annihilation of couples electron-positron. Similar conclusions, obtained through different approaches, have been discussed in other works [41, 42, 43].

Regarding other proposals, I will limit myself to mentioning the following contributions, therefore my list will be somewhat limited, given the large number of researchers who have spent their time reviewing the mythical equations. The main efforts are aimed at a re-elaboration of the Lagrangian of the system (see the pioneering paper [44]). The aim is to try to merge the classical and quantum vision by including solitary wave packets in the modified model (see, e.g. [45, 46]). The recent paper [47] also explores various possibilities in this direction.

In summary, this review paper does not show particular examples or possible new applications, but tries to inform the technological world about the possibility that the results of some electromagnetic problems can be influenced by mathematical passages that are not sufficiently justified and which may lead to conclusions not in line with reality. On several occasions it has been pointed out to me that the world of engineering does not seem to need reforms regarding

the theoretical aspects, since today more than ever electrodynamics has found fertile ground in countless home and industrial applications. I believe that the solutions proposed by the market, although not wrong, are instead correct consequences of models different from those hitherto considered inviolable.

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