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From probabilities to Bell's inequalities: a pathway for secondary school quantum literacy

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Abstract. Quantum mechanics is a key part of modern science and technology, but its complexity makes it challenging to teach it in schools. We suggest a teaching method for secondary schools that focuses on Bell's inequalities and how they are tested experimentally. Using students' understanding of statistics and probability, we provide a step-by-step approach. This includes hands-on activities, like a simple card game, and discussions of CHSH experiments, to help students see the differences between classical and quantum probability. Early testing shows promising results, indicating that this method could be effective.

1. Introduction

The introduction of Bell's inequalities represents a pivotal moment in modern physics, bridging the gap between foundational philosophical questions and empirical scientific inquiry. Historically, the notions of reality and locality – whether physical properties exist independently of observation and whether distant events can influence each other instantaneously – were subjects of intense physical and philosophical debate [1]. In 1964, John Bell's groundbreaking work [2,3] transformed these abstract discussions into testable hypotheses by formulating a mathematical framework that quantitatively examines the correlations between measurements performed at spatially separated locations. His inequalities provided a criterion to distinguish between theories based on local hidden variables and the inherently probabilistic nature of quantum physics and, most importantly, could undergo experimental verification.

Landmark experiments, most notably those conducted by the 2022 Nobel winners, John Clauser, Alain Aspect and Anton Zeilinger [4], provided strong empirical evidence supporting the violation of Bell's inequalities (in the Clauser-Horne-Shimony-Holt (CHSH) version[5],**)** by the amount predicted by quantum theory, thus confirming that the observed nonlocality cannot be ascribed to the presence of hidden variables. Experiments tell us that Nature does not obey local realism (see below), and this departure is exactly described by quantum mechanics. Subsequent experiments performed for almost three decades refined the evidence obtaining loophole-free tests that leave little room for non-quantum alternatives. These experimental achievements have not only deepened our understanding of quantum mechanics but have also played a crucial role in the emergence of the second quantum revolution.

Nowadays, the practical implications of quantum mechanics extend far beyond fundamental physics, enabling quantum technologies, such as quantum sensing, simulations, cryptography and computing.

Introducing students to Bell's inequalities represents a valuable opportunity to explore one of the most significant scientific discoveries of the 20th century [6,7]. Understanding its conceptual and experimental importance helps deepen students' understanding of quantum mechanics, while illustrating the connection between theoretical physics, experimental validation, and technological advancement. It also encourages critical reflection on the meaning of statistical correlation.

We have developed an effective teaching approach for introducing Bell's inequalities and their experimental verification through a teaching- learning sequence. This approach builds on students' prior knowledge of statistics and probability, ensuring they have the mathematical tools needed to engage with the new concepts in quantum physics. The narrative begins by exploring the meaning of classical correlations, illustrating how they arise in everyday experiences and follow intuitive probabilistic rules, while also clarifying the distinction between correlation and causation. For classical systems, which adhere to local realism, we demonstrate, both analytically and through hands-on, game-based activities, that correlations are constrained by the limits set by Bell's inequalities. In particular, we introduce a simplified card game inspired by the CHSH version of Bell's inequalities, which effectively illustrates the classical boundaries. The sequence concludes with an activity where students simulate the key phases of the Nobel Prize-winning experiments. Preliminary post-assessment results suggest that this approach is effective in helping students grasp the significance of quantum correlations.

2. Entanglement, EPR and Bell's inequalities

Entanglement describes nonclassical correlations in quantum systems, challenging classical notions of locality and realism. It plays a crucial role in both the theoretical foundations and practical applications of quantum physics. For entanglement to occur, two systems must interact physically, resulting in a combined state described by a single wave function. This means the two subsystems cannot be described independently, i.e., their properties are intrinsically linked.

A natural way to introduce entanglement is through the ground state of a helium (He) atom, where two electrons occupy the 1s orbital with antiparallel spins (due to the Pauli exclusion principle). This example builds on familiar concepts: atomic orbitals as quantum states and spin properties demonstrated by the Stern-Gerlach (SG) experiment. The key observation is that the spins of the two electrons are perfectly anti-correlated in *any* direction, no matter how the SG apparatus is oriented. Thus, one cannot describe the spin of each electron separately; only the combined two-spin state makes physical sense. Mathematically, this is an *entangled state* (specifically, the singlet state), which cannot be expressed as a product of individual electron states [8]. Crucially, this holds true regardless of the chosen spin basis. i.e., the SG measurement axis.

The so-called Einstein-Podolski-Rosen (EPR) paradox can then be introduced using Bohm's formulation: two electrons are sent to distant locations, where experimenters Alice and Bob measure their spins along arbitrary axes. Due to entanglement, if Alice measures the spin of her electron along one axis, Bob's measurement along the *same* axis is immediately determined, regardless of distance or Alice's choice of axis. In analyzing these results, EPR assumed two principles: realism and locality. Realism means physical properties (observables) have definite values independent of measurement. As EPR argued: if the outcome of a measurement can be predicted with certainty without disturbing the system, that outcome corresponds to a real property. Or, as famously paraphrased: *"The moon exists even when no one is looking."* Locality means no influence can propagate faster than light. Thus, Bob's measurement outcomes cannot depend on Alice's choices, if they are spacelike-separated.

In Bohm's experiment, Alice's measurement allows Bob to predict his electron's spin along the same axis *without any interaction*. By realism, this spin component must correspond to a real property. But if Bob then measures a *different* spin component, quantum mechanics forbids simultaneous definite values for incompatible observables. This contradiction is the EPR paradox, which leaves three possible resolutions. Realism fails; properties do not exist prior to measurement. Locality fails; measurements on one subsystem instantaneously affect the other. Quantum mechanics is incomplete: there are hidden variables which determine outcomes.

EPR favored the third option, but Bell's inequalities (see Appendix A for a proof of CHSH version) provided a testable criterion to settle the debate. Indeed, experiments have since confirmed violations of Bell's inequalities, ruling out local realism. Nature does not obey a local-realist theory.

3. Classical and quantum correlations

One of the main sources of confusion and misunderstanding regarding classical correlations is the way entanglement is often presented as a case of perfect correlations. While it is true that entangled particles exhibit strong correlations, perfect correlations are not unique to quantum mechanics as they can also occur in classical systems. Many examples can be put forward to show perfect classical correlations, such as a pair of gloves or socks, which nevertheless do not imply any quantum effects. The key distinction is that in classical systems, these correlations can always be explained by preexisting local properties, whereas in quantum mechanics, the observed correlations can surpass what classical models can predict. This is what happens when entangled states are measured: their outcomes can violate Bell's inequalities.

A second important conceptual distinction that often leads to confusion is the relationship between correlation and causation. Correlation simply refers to the statistical relationship between two or more variables, meaning that knowing the state of one provides information about the state of the other. However, correlation does not imply causation; just because two events appear to be related does not mean one causes the other. In classical physics, correlations often have an underlying causal explanation, such as a common cause or direct interaction between variables. In quantum mechanics, however, entanglement presents correlations that lack any classical causal explanation, as they cannot be attributed to hidden variables or direct communication between particles.

To introduce students to Bell's inequalities, we start with a hands-on demonstration using a simplified card game, which mimics the probabilistic predictions about the outcomes of measurements performed on the two parts of a bipartite system. By playing the game, students convince themselves that the amount of correlation is limited, according to the CHSH version of Bell's inequality, which itself can be mathematically proven through elementary calculations at the level of high-school algebra, as demonstrated in Appendix A.

Then, we introduce real CHSH experiments that demonstrate how quantum correlations violate Bell's inequalities, providing empirical evidence against local hidden variable theories. By analysing real or simulated experimental data, students see firsthand how quantum mechanics predicts correlations that exceed classical bounds.

Finally, our sequence includes interactive discussions on historical and modern experimental verifications of Bell's theorem, from pioneering experiments [9] to contemporary loophole-free tests [10]. These discussions emphasize the robustness of quantum mechanical predictions and the ongoing efforts to refine our understanding of quantum correlations. Students explore the implications of these findings for emerging technologies such as quantum computing and quantum cryptography, fostering an appreciation for the practical significance of distinguishing classical and quantum correlations.

4. CHSH card game

The objects of measurement in this CHSH game are playing cards, each characterized by two dichotomous properties: the color of their back (blue or red) and the color of their front (black for spades and clubs, red for hearts and diamonds). Since this is a classical system, both properties can be measured simultaneously for each card, with the outcomes assigned values of +1 to the red outcomes and -1 to the blue or black ones (see Fig. 1, where some of the possibilities are reported as examples).

ALICE		$A_i^b B_i^b$	$A_i^f B_i^b$	$A_i^b B_i^f$	$A_i^f B_i^f$	BOB		S_i
$A^b = +1$	$A^f = +1$	−1	−1	+1	+1	$B^b = -1$	$B^f = +1$	−2
$A^b = -1$	$A^f = -1$	−1	−1	+1	+1	$B^b = +1$	$B^f = -1$	−2
$A^b = -1$	$A^f = -1$	−1	+1	+1	−1	$B^b = +1$	$B^f = +1$	+2
$A^b = +1$	$A^f = +1$	+1	+1	−1	−1	$B^b = +1$	$B^f = -1$	+2
$A^b = -1$	$A^f = +1$	+1	−1	−1	+1	$B^b = -1$	$B^f = +1$	−2

Figure 1. Examples of possible results for card decks prepared for CHSH game (other combinations are also possible). The quantity $S_i = A_i^b B_i^b + A_i^f B_i^b + A_i^b B_i^f - A_i^f B_i^f$ (superscripts b and f stand for back and front, respectively) and in the present case $|\langle S \rangle| = |-0.4| < 2$.

In this setup, a dealer (Charlie) prepares two decks of N cards and distributes them to two players, Alice and Bob, ensuring that the decks contain matching pairs of cards in a predefined order. This guarantees that correlations – whether perfect, partial, or nonexistent – are maintained between the two decks. The players then proceed to measure their respective cards while preserving their original order and record the results in a table (see Fig. 1). The classical nature of the system allows all relevant properties to be measured simultaneously, making it possible to compute the value of the quantity $S_i = A_i^b B_i^b + A_i^f B_i^b + A_i^b B_i^f - A_i^f B_i^f$ (where A^b, A^f , B^b and B^f represent the possible measurements Alice and Bob can perform on each pair, superscripts b and f standing for back and front, respectively). The key observation here is that, since the system is entirely classical, it is always true that $S_i = \pm 2$, leading directly to the CHSH inequality bound $|\langle S \rangle| = \left|1/M \sum_{i=1}^{M} S_i\right| \leq 2$. This finding - which becomes self-evident upon compiling a few rows of the table - can be also demonstrated in a more formal – yet simple – way, as reported in Appendix A.

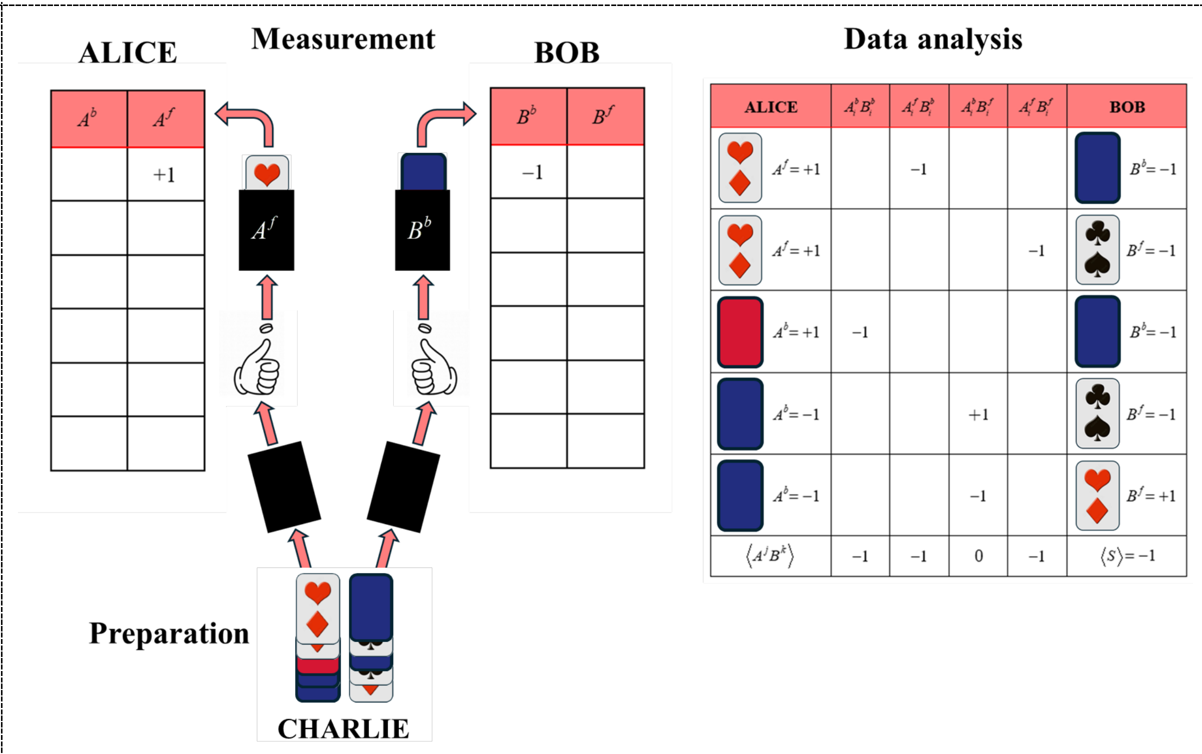

Figure 2. Second version of the CHSH game, reproducing the phases of the Nobel prize experiments. *Preparation:* Charlie prepares two decks of perfectly anti-correlated cards and sends them to Alice and Bob, respectively. *Measurement:* Each time they receive a card from Charlie, Alice and Bob choose randomly and independently the property to be measured (front colour or back colour). *Data analysis:* At the end of the experiment, Alice and Bob compile the table combining their results and calculate the average value $\langle S \rangle$.

With this game we can also naturally introduce the concepts of perfect correlation, $\langle \boldsymbol{S} \rangle = \mathbf{2}$, when each pair consists of identical cards, perfect anti-correlation, $\langle \boldsymbol{S} \rangle = -\mathbf{2}$, when each pair consists of opposite cards, and intermediate cases such as zero correlation, $\langle \boldsymbol{S} \rangle = \mathbf{0}$, when the card pairs are completely uncorrelated.

It is worth emphasizing some points to avoid misconceptions and to highlight the fundamental distinctions between classical and quantum cases:

- The presence of strong correlations between two systems does not necessarily imply entanglement. As shown by the CHSH game, correlations arise from predefined, deterministic properties of the cards, rather than from any fundamentally nonlocal mechanism. This helps dispel the misconception that strong correlations alone are uniquely quantum in nature.
- Regardless of how the decks are prepared, as long as the card properties are well-defined before measurement (realism) and no information which may affect the measurement result is exchanged between Alice and Bob during the game (locality), the Bell inequality remains satisfied. This reinforces the idea that violations of Bell's inequality require more than just correlation.

Unlike the classical case, where all properties of a card can be measured simultaneously, quantum mechanics includes fundamental limitations due to the non-commutativity of observables. Thus, in a true quantum CHSH experiment, it is not possible to evaluate S_i directly for each individual particle pair. Instead, the expectation value $\langle S \rangle$ must be obtained statistically by averaging over many experimental runs, where only one pair of measurements is performed per entangled pair. In actual experiments, $\langle S \rangle$ is computed by summing over the correlation functions of different measurement

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settings. This underscores the importance of collecting a sufficiently large dataset and avoiding biases in the choice of measurements.

To illustrate this latter point and the Nobel prize experiments, a second version of the game was played as a "living experiment", where participants take on the roles of Alice, Bob and Charlie, physically staging the measurement process of an actual Bell test, as depicted in Fig.2. In this version, perfectly anti-correlated card pairs are prepared by Charlie, and sent to Alice and Bob. By flipping a coin, both Alice and Bob randomly and autonomously choose – every time they receive their card – which property (front colour or back colour) they should measure. It is important here to emphasise that each of them flips their own coin, so that Alice's choice is totally uncorrelated from Bob's one. As clearly shown in the table reported in Fig.2, for each pair of cards (that is, for each row) only one entry can be filled-in, so that it is not actually possible to evaluate each S_i. Nevertheless, after a large enough number of measurements, the average products of outcomes $\langle \boldsymbol{A^f B^f} \rangle$, $\langle \boldsymbol{A^b B^f} \rangle$, $\langle \boldsymbol{A^f B^b} \rangle$, $\langle \boldsymbol{A^b B^b} \rangle$ can be evaluated, leading to a reliable estimate of $\langle \boldsymbol{S} \rangle$. As the cards we are using are obviously classical object, Bell's inequalities will not be violated and $\langle \boldsymbol{S} \rangle = \boldsymbol{-2}$, (perfect anti-correlation) in the statistical limit. Indeed, the scope of this latter version of the game is not to observe Bell's inequality violation (which will never occur for classical systems), nor to accumulate statistics, but rather to experience the subsequent steps of the experimental procedure, focusing on the fundamental role played by the random choice (represented by the flipping coins) of the measured quantity.

5. Implementation

The proposed educational approach has been successfully implemented in different learning contexts, including summer schools, online courses, teacher training programs, and public outreach initiatives. For instance, entanglement and Bell's inequalities were part of the topics of the 2023 edition of the exhibition *Dire l'indicibile* (*Speaking the unspeakable*) organized by the project Italian Quantum Weeks [11]. During that event, we engaged the public through interactive, guided tours, and the final survey results indicated strong interest and appreciation for both guides' expertise and hands-on activities.

A different activity was set up at the FIM Department in Modena. We organized a structured three-day program for 24 high school students (12th grade) having varying motivations for participation. The responses of the entrance questionnaire revealed limited prior knowledge of quantum mechanics, and almost no clue on entanglement. Post-assessment results demonstrated notable learning gains, particularly in understanding Bell inequalities and the CHSH experiment. As shown in Fig. 3, among 24 students, 50% correctly identified the role of realism and locality in Bell inequalities, and 83% recognized their violation in some quantum systems.

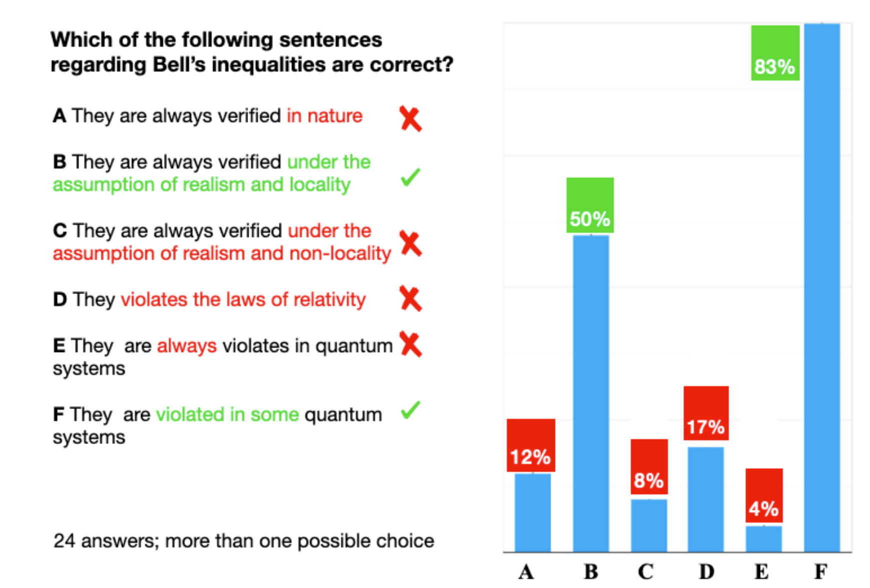

Figure 3. Results of the post-assessment test, regarding Bell's inequalities.

Additionally, 80% correctly identified key experimental phases, and 60% selected all correct responses or made only minor mistakes. However, conceptual challenges remained, particularly regarding quantum non-locality and the distinction between causality and correlation, highlighting areas for further pedagogical refinement. Despite the small sample size, students' feedback indicated strong satisfaction, with a positive correlation between assessment performance and engagement. These findings underscore the effectiveness of this instructional approach in fostering meaningful learning experiences in quantum physics for motivated high school students [8].

6. Discussion and conclusions

In conclusion, understanding the difference between classical and quantum correlations is a key part of modern teaching of physics that challenges students to rethink how they see reality. Our teaching approach introduces Bell's inequalities in a way that builds on students' knowledge of probability and statistics, helping them develop an intuitive grasp of quantum correlations. Through hands-on activities, interactive discussions, and real-world experiments, we connect classical and quantum physics, preparing students for deeper exploration of the quantum world.

Our findings suggest this method makes quantum mechanics more accessible and engaging, encouraging students to appreciate its profound implications. Preliminary results based on post-assessments show that this approach can foster valuable understanding of entanglement and nonlocality. By framing abstract quantum ideas in relatable ways, our method reduces learning barriers and supports meaningful understanding.

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Appendix A: CHSH Inequality

The CHSH (Clauser-Horne-Shimony-Holt) inequality sets a fundamental limit on correlations between measurements performed at distant location under the assumptions of *realism* (physical properties exist prior to measurement) and *locality* (no instantaneous influence at a distance). Below, we derive the inequality using basic algebra (the symbols and notations differ slightly from those used in the text to maintain greater generality). We have two experimenters, Alice and Bob, at distant locations. Each of them performs dichotomic measurements, denoted by A and B, each yielding, as possible outcomes ±1. The correlation function $C(A,B)$, is defined as the average product of outcomes over M trials:

$$C(A,B) = \langle AB \rangle = \frac{1}{M}\sum_{j=1}^{M} a_j\, b_j,$$

where $a_j, b_j \in \{+1,-1\}$. The correlation function satisfies $-1 \leq C(A,B) \leq 1$, with
$C = +1$ corresponding to perfect correlation (the two observables have the same value at every trial), with $C = -1$ to perfectly anticorrelated variables, and with $C = 0$ obtained for uncorrelated variables. Consider now a situation in which Alice chooses randomly between measurements A_1 or A_2, and Bob between B_1 or B_2. All measurements yield 1 or -1 as possible results. The CHSH quantity S combines their correlation functions in the following way:

$$S = \langle A_1B_1\rangle + \langle A_1B_2\rangle + \langle A_2B_1\rangle - \langle A_2B_2\rangle.$$

In each trial, we assume predefined outcomes (realism) and no influence between Alice and Bob's choices (locality). With these assumptions, both measured and unmeasured quantities have a defined values and we may write:

$$S_i = a_{1i}b_{1i} + a_{1i}b_{2i} + a_{2i}b_{1i} - a_{2i}b_{2i}.$$

Since b_{1i} , $b_{2i} \in \{+1, -1\}$, the sum and difference $(b_{1k} \pm b_{2k})$ can only be 0 and ± 2 (if $b_{1k} \neq b_{2k}$), or ± 2 and 0 (if $b_{1k} = b_{2k}$). We now factorize terms and rewrite S_i as

$$S_i = a_{1i}(b_{1i} + b_{2i}) + a_{2i}(b_{1i} - b_{2i}),$$

making clear that $|S_i| \leq 2$ for every trial. It is important to highlight that, due to the hypothesis of realism, this line of reasoning is valid even if we do not actually measure each quantity. Finally, we average over all trials and get

$$|S| = \left| \frac{1}{M} \sum_{i=1}^{M} S_i \right| \leq 2.$$

This is the *CHSH inequality*. Quantum mechanics instead predicts $S = 2\sqrt{2}$ for entangled states (e.g., photons with aligned polarizations), violating the inequality. Experimental violations of CHSH inequality thus confirms that Nature cannot be described by local realistic theories and that entanglement produces stronger correlations than classical physics allows. The simple derivation of the CHSH inequality is derived by assuming predefined outcomes (realism) and no faster-than-light influence (locality), bounding the sum of correlations using algebraic properties of the dichotomics observables. Its violation by quantum systems underscores the necessity of non-classical descriptions like entanglement.