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Research Article

On Noncompact Fractional Order Differential Inclusions with Generalized Boundary Condition and Impulses in a Banach Space

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We provide existence results for a fractional differential inclusion with nonlocal conditions and impulses in a reflexive Banach space. We apply a technique based on weak topology to avoid any kind of compactness assumption on the nonlinear term. As an example we consider a problem in population dynamic described by an integro-partial-differential inclusion.

1. Introduction

The main result of this paper is an existence result for fractional inclusions with impulses and nonlocal boundary conditions.

Fractional calculus deals with the study of fractional order integrals and derivatives, a generalization of ordinary integral and differential operators. There are some different definitions of fractional derivatives: Riemann-Liouville, Hadamard, and Caputo are examples of fractional derivatives. For a survey on the subject see, for example, [1–3]. They all are very useful at describing the anomalous phenomena, providing an excellent tool for the description of memory and hereditary properties of various materials and processes. Roughly speaking, noninteger derivatives give more flexibility helping to model real-life problems. For instance, fractional derivatives found interesting applications in fractional variational principles and fractional control theory as well as in fractional Lagrangian and Hamiltonian dynamics. In particular, the Caputo fractional derivative is especially suitable for physical applications. Unlike the Riemann-Liouville fractional derivative, the Caputo derivative of a constant is zero and it allows a physical interpretation of the initial conditions as well as of boundary conditions.

The theory of fractional differential equations and inclusions in abstract spaces is now an important area of investigation. Besides the above-mentioned monographs, which contain several existence results for fractional differential equations, we also quote the following recent papers concerning fractional differential inclusions with nonlocal conditions: [4–7]. On the other hand, there are various examples in physics, population dynamics, biotechnology, and economics of processes characterized by the fact that the model parameters are subject to short-term perturbations in time. This problem involves impulses. For instance, in the periodic treatment of some diseases, impulses may correspond to administration of a drug treatment; in environmental sciences, impulses may correspond to seasonal changes or harvesting; in economics impulses may correspond to abrupt changes of prices. Adequate apparatus to solve such processes and phenomena are impulsive differential equations and inclusions. The first ones have been extensively investigated in finite and infinite-dimensional Banach spaces; see, for instance, the monographs [8, 9]. On the contrary, systems governed by impulsive differential inclusions are a more recent argument of research. This subject was studied at first by, for example, Watson and Ahmed; see [10, 11]; moreover we refer the interested reader to some papers of the last decade [12–14] and to the very recent monograph [15].

For the above reasons it is natural to study fractional differential inclusions with impulses. Bonanno et al. in [16] proved existence results for impulsive fractional differential equations by a variational approach. Henderson and Ouahab in [17] proved a Filippov-type theorem for an impulsive fractional differential inclusion with initial conditions in $\mathbb{R}$ (see also [18–20]). In the survey [21] Agarwal et al. collect some recent existence results for fractional differential equations and inclusions with impulses and various boundary conditions in $\mathbb{R}$, applying the Banach contraction principle, the Schaefer fixed point theorem, and the Leray-Schauder alternative. Benchora and Seba extended these results to Banach spaces by means of measures of noncompactness in [22].

We consider the following fractional evolution inclusion in a reflexive Banach space E in the presence of impulse effects:

$$\begin{gathered} {}^{C}D^{\alpha}x(t) \in F(t, x(t)), \quad \text{for a.a. } t \in [a,b],\ t \neq t_1, \dots, t_N, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)), \quad k = 1, \dots, N, \\ 0 < \alpha < 1 \end{gathered} \tag{1}$$

associated with a nonlocal boundary condition

$$x(a) \in M(x). \tag{2}$$

Here ${}^{C}D^{\alpha}$, $0 < \alpha < 1$, means the Caputo fractional derivative of x; $F : [a,b] \times E \multimap E$ is a multivalued map (multimap for short); $M : \mathscr{C}([a,b];E) \multimap E$ is a multivalued operator (multioperator for short) and $\mathscr{C}([a,b];E)$ is the space of piecewise continuous functions; I_k are given functions, $k = 1, \dots, N$, $x(t^+) = \lim_{s \to t^+} x(s)$, and $a = t_0 < t_1 < \cdots < t_N < t_{N+1} = b$. See Section 3 for the detailed assumptions.

The boundary condition considered is fairly general and includes the initial valued problem, the periodic and antiperiodic problem, and more general two-point problems as well as several nonlocal conditions. For instance, the following two particular cases are covered by our general approach (see Section 4 for details):

(i) $M(x) = (1/(b-a)) \int_a^b p(t)x(t)dt$ with $p \in L^1([a,b],\mathbb{R})$;

(ii) $M(x) = \sum_{i=1}^{n} \alpha_i x(s_i) + x_0$, with $x_0 \in E$, $\alpha_i \neq 0$, $s_i \in [a,b]$, $i = 1, \dots, n$;

(iii) $M(x) \equiv B$, with B a prescribed set.

Since the solutions of an impulse equation are no longer continuous and the Caputo derivative strongly depends on the initial time, according to [21, 23], we define the solution $x : [a,b] \to E$ of (1) as

$$\begin{aligned} & x(a) \in M(x), \\ & x(t) = x(a) + \sum_{a<t_k<t} \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s)\, ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s)\, ds \\ & \quad + \sum_{a<t_k<t} I_k(x(t_k)), \quad \forall t \in [a,b], \end{aligned} \tag{3}$$

where $f \in L^p([a,b];E)$ with $p > 1/\alpha$ and $f(s) \in F(s, x(s))$ for a.e. $s \in [a,b]$.

Notice that in all the above cited works in order to solve an impulsive fractional differential problem a finite dimensional framework is considered, or some compactness assumptions are required for the nonlinear term.

Unlike all those results, by means of a technic based on weak topology and developed in [24, 25], we are able to prove the existence of at least a solution of problem (1) avoiding any kind of compactness hypotheses on the nonlinear term F.

Finally, our arguments are motivated by an application to a parabolic differential equation with the nonlinearity depending on an integral term. Precisely, in Section 5 we find a solution $z \in \mathscr{C}([a,b]; L^2(\Omega,\mathbb{R}))$ for the following problem in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{aligned} & D_t^{\alpha} z(t,x) \in \left[f_1\left(t, x, \int_{\Omega} k(x,\xi) z(t,\xi)\, d\xi\right), \right. \\ & \qquad \left. f_2\left(t, x, \int_{\Omega} k(x,\xi) z(t,\xi)\, d\xi\right) \right], \\ & \qquad t \in [0,T], \quad x \in \Omega, \\ & z(t_k^+, x) = z(t_k, x) + c_k, \quad k = 1, \dots, N,\ x \in \Omega, \\ & z(0,x) = \sum_{i=1}^{n} \alpha_i z(s_i, x), \quad x \in \Omega. \end{aligned} \tag{4}$$

This kind of models arises in the population dynamics; here the function $z(t,x)$ represents the density of the population at the point x and time t. For in this field memory effects are important, hence it is more realistic to use fractional order derivatives, which express the fact that the next state of the system depends not only upon its current state but also upon all of its historical states (see, e.g., [26–28]). Moreover, the above type of nonlinear functions arises also in mathematical problems dealing with heat flow in materials with memory and in viscoelastic problems, where the integral term represents the viscosity part; see, for example, [29].

2. Preliminaries

Let $(E, \|\cdot\|)$ be a reflexive Banach space and E_w denote the space E endowed with the weak topology. We denote by B the closed unit ball in E and for a set $A \subset E$, the symbol $\overline{A}^{w}$

means the weak closure of A. In the whole paper we denote by $\|\cdot\|_p$ and $\|\cdot\|_0$ the $L^p([a,b];\mathbb{R})$-norm, $1 < 1/\alpha < p \le \infty$, and the $C([a,b];E)$-norm, respectively; we consider the norm of a set $A \subset E$ defined as

$$\|A\| := \sup\{\|x\| : x \in A\}, \tag{5}$$

and by ν we denote the Lebesgue measure on $[a,b]$. Let $\mathscr{C}([a,b];E)$ be the space of all piecewise continuous functions $x : [a,b] \to E$ with discontinuity points at $t = t_k$, $k = 1,\ldots,N$, such that all values $x(t_k^+) = \lim_{s\to t_k^+} x(s)$ and $x(t_k^-) = \lim_{s\to t_k^-} x(s)$ are finite and $x(t_k) = x(t_k^-)$ for all such points.

The space $\mathscr{C}([a,b];E)$ is a normed space with the $\|\cdot\|_0$-norm.

For a map $f : [a,b] \to E$, the definition of the Riemann-Liouville fractional derivative with $0 < \alpha < 1$ is the following:

$$\left[D^\alpha f\right](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} ds, \tag{6}$$

with Γ the Euler function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx. \tag{7}$$

The Caputo fractional derivative is defined through the Riemann-Liouville fractional derivative as

$$\left[{}^C D^\alpha f\right](t) = D^\alpha \left[f(\cdot) - f(a)\right](t). \tag{8}$$

Let $BV([a,b];E)$ be the space of functions with bounded variation. We recall (see [30, Theorem 4.3]) that a sequence $\{x_n\} \subset BV([a,b];E)$ weakly converges to an element $x \in BV([a,b];E)$ if and only if

(1) $\|x_n(t)\| \le N$, for each $n \in \mathbb{N}$ and for each $t \in [a,b]$, for some constant $N > 0$;

(2) $x_n(t) \rightharpoonup x(t)$ for every $t \in [a,b]$.

Thus, the above characterization of weakly convergent sequences holds also for the space $\mathscr{C}([a,b];E)$.

Finally, for the sake of completeness, we recall some results that we will need in the sequel.

Firstly we state the Glicksberg-Ky Fan fixed point Theorem [31, 32].

Theorem 1. *Let X be a Hausdorff locally convex topological vector space, K a compact convex subset of X, and $G : K \multimap K$ an upper semicontinuous multimap with closed, convex values. Then G has a fixed point $x_* \in K : x_* \in G(x_*)$.*

We mention also two results from the Eberlein-Smulian theory.

Theorem 2 (see [33, Theorem 1, page 219]). *Let Ω be a subset of a Banach space X. The following assertions are equivalent:*

(1) *Ω is relatively weakly compact;*

(2) *Ω is relatively weakly sequentially compact.*

Corollary 3 (see [33, page 219]). *Let Ω be a subset of a Banach space X. The following assertions are equivalent:*

(1) *Ω is weakly compact;*

(2) *Ω is weakly sequentially compact.*

We recall the Krein-Smulian Theorem.

Theorem 4 (see [34, page 434]). *The convex hull of a weakly compact set in a Banach space E is weakly compact.*

3. Problem Setting

We study problem (1) under the following assumptions.

We assume that the multivalued nonlinearity $F : [a,b] \times E \multimap E$ has closed bounded and convex values and

($F1$) the multifunction $F(\cdot, c) : [a,b] \multimap E$ has a measurable selection for every $c \in E$; that is, there exists a measurable function $f : [a,b] \to E$ such that $f(t) \in F(t,c)$ for a.e. $t \in [a,b]$;

($F2$) the multimap $F(t,\cdot) : E \multimap E$ is weakly sequentially closed for a.e. $t \in [a,b]$; that is, it has a weakly sequentially closed graph;

(M) $M : \mathscr{C}([a,b];E) \multimap E$ is a weakly sequentially closed multioperator, with convex, closed, and bounded values, mapping bounded sets into bounded sets such that

$$\limsup_{\|u\|_0 \to \infty} \frac{\|M(u)\|}{\|u\|_0} = l \quad \text{with } l < 1, \tag{9}$$

where the norm of $M(u)$ is defined in (5);

(I_k) the functions $I_k : E \to E$, $k = 1,\ldots,N$, are weakly continuous, mapping bounded sets into bounded sets such that

$$\limsup_{\|c\| \to \infty} \frac{\left\|I_k(c)\right\|}{\|c\|} = 0, \quad k = 1,\ldots,N. \tag{10}$$

In the remaining part of this section we always assume the following assumption of local integral boundedness on the multivalued map F.

($F3$) For every $r > 0$ there exists a function $\mu_r \in L^p([a,b];\mathbb{R}_+)$ with $p > 1/\alpha$ such that for each $c \in E$, $\|c\| \le r$:

$$\|F(t,c)\| \le \mu_r(t) \quad \text{for a.e. } t \in [a,b]. \tag{11}$$

For our main result (see Theorem 12), instead of condition ($F3$), we need the stronger assumption below:

($F3'$) $\sup_{\|x\|\le n} \|F(t,x)\| \le \varphi_n(t)$, for a.a. $t \in [a,b]$, with $\varphi_n \in L^p([a,b];\mathbb{R})$, $p > 1/\alpha$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \left\{\int_a^b \left|\varphi_n(s)\right|^p ds\right\}^{1/p} = 0. \tag{12}$$

Definition 5. A continuous function $x : [a,b] \to E$ satisfying (2) is a *mild solution* to problem (1) if x may be represented in the following form:

$$x(t) = x(a) + \sum_{a<t_k<t} \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s)\,ds + \sum_{a<t_k<t} I_k(x(t_k)), \quad (13)$$

for any $t \in [a,b]$, where $f \in L^p([a,b];E)$ and $f(s) \in F(s, x(s))$ for a.a. $s \in [a,b]$.

3.1. Existence Result. Note that, with our hypotheses on F, given $q \in \mathscr{C}([a,b];E)$, the superposition multioperator $\mathscr{P}_F(q) : \mathscr{C}([a,b];E) \multimap L^p([a,b];E)$, with

$$\mathscr{P}_F(q) = \{f \in L^p([a,b];E) : f(t) \in F(t, q(t)) \text{ a.a. } t \in [a,b]\}, \quad (14)$$

is well defined as the following proposition shows.

Proposition 6. *For a multimap $F : [a,b] \times E \multimap E$ satisfying properties $(F1)$, $(F2)$, and $(F3)$, the set $\mathscr{P}_F(q) \neq \emptyset$ is nonempty for any $q \in \mathscr{C}([a,b];E)$.*

Proof. By $(F3)$ the multimap $F(t,\cdot)$ is locally weakly compact for a.e. $t \in [a,b]$; that is, for a.e. t and every $x \in E$, there is a neighbourhood V of x such that the restriction of $F(t,\cdot)$ to V is weakly compact. Hence by $(F2)$ and [35, Theorem 1.1.5.], we easily get that $F(t,\cdot) : E_w \multimap E_w$ is upper semicontinuous for a.e. $t \in [a,b]$. Thus, $F(t,\cdot) : E \multimap E_w$ is upper semicontinuous for a.e. $t \in [a,b]$. The thesis then follows reasoning as in the proof of [36, Proposition 2.2], recalling that a map $q \in \mathscr{C}([a,b];E)$ can be approximated by a sequence $\{q_n\}$ of step functions, such that

$$\sup_{t\in[a,b]} \|q_n(t) - q(t)\| \longrightarrow 0, \quad \text{for } n \longrightarrow \infty. \quad (15)$$

□

Let $S : L^p([a,b];E) \to C([a,b];E)$ be defined as

$$S(\phi)(t) = \sum_{a<t_k<t} \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \phi(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \phi(s)\,ds, \quad (16)$$

and let $\Psi : \mathscr{C}([a,b];E) \to \mathscr{C}([a,b];E)$ be given as

$$\Psi(q)(t) = \sum_{a<t_k<t} I_k(q(t_k)). \quad (17)$$

It is easy to verify that the fixed points of the multioperator $T : \mathscr{C}([a,b];E) \multimap \mathscr{C}([a,b];E)$ defined as

$$T(q) = M(q) + S\mathscr{P}_F(q) + \Psi(q) \quad (18)$$

are mild solutions of problem (1).

Lemma 7. *The operator S is linear and bounded.*

Proof. The linearity follows from the linearity of the integral operator. Moreover, for every $\tau_1, \tau_2 \in [a,b]$,

$$\left(\int_{\tau_1}^{\tau_2} \left((\tau_2 - s)^{\alpha-1}\right)^{p/(p-1)} ds\right)^{(p-1)/p} \le \left[\frac{p-1}{\alpha p - 1}\right]^{(p-1)/p} (b-a)^{\alpha - 1/p} =: C_p. \quad (19)$$

Thus, using Hölder inequality (see [37, 38]), we have

$$\begin{aligned} \|S(\phi)(t)\| &\le \sum_{a<t_k<t} \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|\phi(s)\|\,ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|\phi(s)\|\,ds \\ &\le \frac{1}{\Gamma(\alpha)} (N+1) C_p \|\phi\|_p = D_p \|\phi\|_p, \end{aligned} \quad (20)$$

with

$$D_p = \frac{N+1}{\Gamma(\alpha)} C_p. \quad (21)$$

□

Lemma 8. *The operator Ψ is weakly sequentially continuous.*

Proof. Let $\{q_n\} \subset \mathscr{C}([a,b];E)$ such that $q_n \rightharpoonup q$. Then by the weak continuity of the functions I_k we have that $I_k(q_n(t_k)) \rightharpoonup I_k(q(t_k))$ for any $k = 1, \ldots, N$; that is, $\Psi(q_n(t)) \rightharpoonup \Psi(q(t))$ for every t. Moreover, the weak continuity yields the existence of $R > 0$ such that $\|I_k(q_n(t_k))\| \le R$ for every $k = 1, \ldots, N$ and $n \in \mathbb{N}$. It follows that $\|\Psi(q_n)(t)\| \le \sum_{a<t_k<t} \|I_k(q_n(t_k))\| \le NR$. Thus we have the weak convergence of $\Psi(q_n)$ to $\Psi(q)$ in $\mathscr{C}([a,b];E)$. □

Proposition 9. *The multioperator T has a weakly sequentially closed graph.*

Proof. Let $\{q_m\} \subset \mathscr{C}([a,b];E)$ and $\{x_m\} \subset \mathscr{C}([a,b];E)$ satisfying $x_m \in T(q_m)$ for all m and $q_m \rightharpoonup q$, $x_m \rightharpoonup x$ in $\mathscr{C}([a,b];E)$; we will prove that $x \in T(q)$.

By the weak convergence of the sequence $\{q_m\}$ in $\mathscr{C}([a,b];E)$, it follows that there exists a constant $r > 0$ such that $\|q_m\|_0 < r$ for every $m \in \mathbb{N}$ and by the weak convergence $q_m(t) \rightharpoonup q(t)$ for every $t \in [a,b]$, it follows that $\|q(t)\| \le \liminf_{m\to\infty} \|q_m(t)\| \le r$ for all t (see [39, Proposition III.5]). The fact that $x_m \in T(q_m)$ means that there exist a sequence $\{f_m\}$, $f_m \in \mathscr{P}_F(q_m)$, and a sequence $w_m \in M(q_m)$ such that

$$x_m = w_m + Sf_m + \Psi(q_m). \quad (22)$$

We observe that, according to $(F3)$, $\|f_m(t)\| \le \eta_r(t)$ for a.a. t and every m; that is, $\{f_m\}$ is uniformly bounded and by the reflexivity of the space $L^p([a,b];E)$, we have the existence of a subsequence, denoted by the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^p([a,b];E)$.

By Lemma 7 the operator S is a weakly sequentially continuous operator; hence $Sf_m \rightharpoonup Sg$ in $C([a,b];E)$. Moreover, by the linearity and continuity of the operator L we have that $LSf_m \rightharpoonup LSg$. The operator M maps bounded sets in bounded sets and it is weakly sequentially closed; hence, up to subsequence, $w_m \rightharpoonup w$ in E, with $w \in M(q)$. Finally, by Lemma 8 the map Ψ is weakly sequentially continuous, yielding $\Psi(q_m) \rightharpoonup \Psi(q)$. In conclusion, we have

$$x_m \rightharpoonup w + Sg + \Psi(q) = x_0, \quad (23)$$

thus, implying, by the uniqueness of the weak limit in E, that $x_0 = x$.

To conclude we have only to prove that $g(t) \in F(t, q(t))$ for a.a. $t \in [a,b]$.

By Mazur's convexity theorem we have the existence of a sequence

$$\tilde{f}_m = \sum_{i=0}^{k_m} \lambda_{mi} f_{m+i}, \quad \lambda_{mi} \geq 0, \ \sum_{i=0}^{k_m} \lambda_{mi} = 1 \quad (24)$$

satisfying $\tilde{f}_m \to g$ in $L^p([a,b];E)$ and, up to subsequence, there is $N_0 \subset [a,b]$ with Lebesgue measure zero such that $\tilde{f}_m(t) \to g(t)$ for all $t \in [a,b] \setminus N_0$ (see [40, Chapter IV, Theorem 38]). With no loss of generality we can also assume that $F(t,\cdot) : E_w \multimap E_w$ is weakly sequentially closed and $\sup_{\|x\| \leq r} \|F(t,x)\| \leq \eta_r(t)$ for every $t \notin N_0$.

Fix $t_0 \notin N_0$ and assume, by contradiction, that $g(t_0) \notin F(t_0, q(t_0))$. By the reflexivity of the space E the restriction $F_{rB}(t_0,\cdot)$ of the multimap $F(t_0,\cdot)$ on the set rB is weakly compact. Hence, by Corollary 3, we have that $F_{rB}(t_0,\cdot)$ is a weakly closed multimap and by [35, Theorem 1.1.5] it is weakly upper semicontinuous. Since $\|q(t_0)\| \leq r$ and since $F_{rB}(t_0, q(t_0))$ is closed and convex, from the Hahn-Banach Theorem, there is a weakly open convex set $V \supset F_{rB}(t_0, q(t_0))$ satisfying $g(t_0) \notin \overline{V}$. Since $F_{rB}(t_0,\cdot)$ is weakly upper semicontinuous, we can also find a weak neighbourhood V_1 of $q(t_0)$ such that $F_{rB}(t_0, x) \subset V$ for all $x \in V_1$ with $\|x\| \leq r$. Notice that $\|q_m(t_0)\| \leq r$ for all m. The convergence $q_m(t_0) \rightharpoonup q(t_0)$ as $m \to \infty$ then implies the existence of $m_0 \in \mathbb{N}$ such that $q_m(t_0) \in V_1$ for all $m > m_0$. Therefore $f_m(t_0) \in F_{rB}(t_0, q_m(t_0)) \subset V$ for all $m > m_0$. The convexity of V implies that $\tilde{f}_m(t_0) \in V$ for all $m > m_0$ and, by the convergence, we arrive to the contradictory conclusion that $g(t_0) \in \overline{V}$. We obtain that $g(t) \in F(t, q(t))$ for a.a. $t \in [a,b]$. □

Proposition 10. *The multioperator T is weakly compact.*

Proof. We first prove that T is weakly relatively sequentially compact.

Indeed let $\{q_m\} \subset \mathscr{C}([a,b];E)$ be a bounded sequence and take $\{q_m\}$ and $\{x_m\} \subset \mathscr{C}([a,b];E)$ satisfying $x_m \in T(q_m)$ for all m. By the definition of the multioperator T, there exist a sequence $\{f_m\}$, $f_m \in \mathscr{P}_F(q_m)$, and a sequence $w_m \in M(q_m)$ such that

$$x_m = w_m + Sf_m + \Psi(q_m). \quad (25)$$

Reasoning as in Proposition 9, we have that there exist a subsequence, denoted by the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^p([a,b];E)$. Moreover, since the multioperator M and the operators I_k map bounded sets into bounded sets and $\{q_m\}$ is bounded, we obtain that, up to subsequence, $w_m \rightharpoonup \overline{w} \in E$ and $I_k(q_m(t_k)) \rightharpoonup \overline{x}_k \in E$ as $m \to \infty$, implying $\Psi(q_m)(t) \rightharpoonup \sum_{a<t_k<t} \overline{x}_k = \overline{y}(t) \in \mathscr{C}([a,b];E)$. According to the weak convergence, there exists $R > 0$ such that $\|\Psi(q_m)(t)\| \leq \sum_{a<t_k<t} \|I_k(q_m(t_k))\| \leq NR$. Thus we have the weak convergence of $\Psi(q_m)$ to $\overline{y}$ in $\mathscr{C}([a,b];E)$. Therefore

$$x_m \rightharpoonup \overline{w} + Sg + \overline{y} \quad (26)$$

in $\mathscr{C}([a,b];E)$. It follows that T is weakly relatively sequentially compact and hence weakly relatively compact by Theorem 2. □

Proposition 11. *The multioperator T has convex and weakly compact values.*

Proof. Fix $q \in \mathscr{C}([a,b];E)$ since F and M are convex valued, then the set $T(q)$ is convex from the linearity of the integral. The weak compactness of $T(q)$ follows from Propositions 10 and 9. □

Theorem 12. *Under assumptions* $(F1)$, $(F2)$, $(F3')$, (M), *and* (I_k) *problem (1) has at least a mild solution.*

Proof. Fix $n \in \mathbb{N}$; consider Q_n the closed ball of radius n of $\mathscr{C}([a,b];E)$. We show that there exists $n \in \mathbb{N}$ such that the operator T maps the ball Q_n into itself.

According to (12), there exists a subsequence, still denoted by the sequence, such that

$$\lim_{n\to\infty} \frac{1}{n} \left\{ \int_a^b |\varphi_n(s)|^p \, ds \right\}^{1/p} = 0. \quad (27)$$

Assume to the contrary that there exist two sequences $\{q_n\}$ and $\{x_n\}$ such that $q_n \in Q_n$, $x_n \in T(q_n)$, and $x_n \notin Q_n$ for all $n \in \mathbb{N}$. By the definition of T, there exist a sequence $\{g_n\} \subset \mathscr{P}_F(q_n)$ and a sequence $w_n \in M(q_n)$ such that

$$x_n = w_n + Sg_n + \Psi(q_n). \quad (28)$$

From the assumption $x_n \notin Q_n$ we must have, for any n,

$$n < \|x_n\|_0 \leq \|w_n\| + D_p \|g_n\|_p + \|\Psi(q_n)\|_0, \quad (29)$$

where D_p is defined in (21). Moreover $q_n \in Q_n$ implies, by $(F3')$, that $\|g_n(t)\| \leq \varphi_n(t)$ for a.a. $t \in [a,b]$; hence $\|g_n\|_p \leq \|\varphi_n\|_p$. Consequently

$$n < \|M(q_n)\| + D_p \|\varphi_n\|_p + \|\Psi(q_n)\|_0. \quad (30)$$

Therefore

$$1 < \frac{\|M(q_n)\|}{n} + D_p \frac{\|\varphi_n\|_p}{n} + \frac{\|\Psi(q_n)\|_0}{n}. \quad (31)$$

Notice that if $\|q_n\|_0 \le H_1 < +\infty$ for any $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \frac{\|M(q_n)\|}{n} = 0, \tag{32}$$

because M maps bounded sets into bounded sets.

If $\lim_{n\to\infty}\|q_n\|_0 = +\infty$ by hypothesis we have

$$\lim_{n\to\infty} \frac{\|M(q_n)\|}{n} \le \limsup_{n\to\infty} \frac{\|M(q_n)\|}{\|q_n\|_0} \le \limsup_{\|u\|_0\to\infty} \frac{\|M(u)\|}{\|u\|_0} = l < 1. \tag{33}$$

In both cases

$$\lim_{n\to\infty} \frac{\|M(q_n)\|}{n} < 1. \tag{34}$$

Moreover fix $k \in \{1,\dots,N\}$; if $\|q_n(t_k)\| < H_2$ for any $n \in \mathbb{N}$ then since I_k maps bounded sets into bounded sets for any $k = 1,\dots,N$ it follows that

$$\lim_{n\to\infty} \frac{\|I_k(q_n(t_k))\|}{n} = 0. \tag{35}$$

If $\lim_{n\to\infty}\|q_n(t_k)\| = +\infty$ by hypothesis we have

$$\lim_{n\to\infty} \frac{\|I_k(q_n(t_k))\|}{n} \le \lim_{n\to\infty} \frac{\|I_k(q_n(t_k))\|}{\|q_n(t_k)\|} \le \limsup_{\|c\|\to\infty} \frac{\|I_k(c)\|}{\|c\|} = 0. \tag{36}$$

In conclusion

$$\lim_{n\to\infty} \frac{\|\Psi(q_n)\|_0}{n} = \lim_{n\to\infty} \sup_{t\in[a,b]} \frac{\left\|\sum_{a<t_k<t} I_k(q(t_k))\right\|}{n} \le \sum_{a<t_k<b} \lim_{n\to\infty} \frac{\|I_k(q(t_k))\|}{n} = 0. \tag{37}$$

Moreover by (27),

$$\lim_{n\to\infty} \frac{\|\varphi_n\|_p}{n} = 0. \tag{38}$$

Hence

$$1 \le \limsup_{n\to\infty} \left[\frac{\|M(q_n)\|}{n} + D_p \frac{\|\varphi_n\|_p}{n} + \frac{\|\Psi(q_n)\|_0}{n} \right] < 1, \tag{39}$$

giving the contradiction.

Fix, now $n \in \mathbb{N}$ such that $T(Q_n) \subseteq Q_n$. By Proposition 10 the set $V_n = \overline{T(Q_n)}^w$ is a weakly compact set. Let $W_n = \overline{\text{co}}(V_n)$, where $\overline{\text{co}}(V_n)$ denotes the closed convex hull of V_n. By Theorem 4 W_n is a weakly compact set. Moreover from the fact that $T(Q_n) \subset Q_n$ and that Q_n is a convex closed set we have that $W_n \subset Q_n$ and hence

$$T(W_n) = T(\overline{\text{co}}(T(Q_n))) \subseteq T(Q_n) \subseteq \overline{T(Q_n)}^w = V_n \subset W_n. \tag{40}$$

Therefore from Proposition 9 and from Corollary 3 we obtain that the restriction of the multimap T on W_n has a weakly closed graph; hence, it is weakly upper semicontinuous (see [35, Theorem 1.1.5]). The conclusion then follows from Theorem 1. □

Remark 13. In [41, Theorem 4.3], an existence result for mild solutions for a controllability problem associated with a semilinear differential inclusion is proved under the weaker growth condition:

$(F3'')$ there exists $\varphi \in L^1([a,b];\mathbb{R})$ such that, for every $x \in E$ and a.a. $t \in [a,b]$,

$$\|F(t,x)\| \le \varphi(t)(1 + \|x\|), \tag{41}$$

instead of $(F3')$. Following the proof's outline of the cited theorem and combing it with Propositions 9, 10, and 11, it is easy to obtain the existence of mild solutions for (1) with a Cauchy initial condition, under the same boundedness assumption $(F3'')$, with $\varphi \in L^p([a,b];\mathbb{R})$:

$$D_p \|\varphi\|_p < 1, \tag{42}$$

with D_p defined in (21), which allows also a linear growth on the nonlinear term.

Next theorem shows that, under condition $(F3'')$, if the impulse functions I_k are bounded, condition (42) can be dropped when we investigate the existence of a solution for the Cauchy problem.

Theorem 14. *Assume $(F1)$, $(F2)$, $(F3'')$, and (I_k). Moreover assume that the impulse functions I_k are bounded; then the problem*

$$\begin{gathered} {}^{C}D^{\alpha}x(t) \in F(t,x(t)), \quad \text{for a.a. } t \in [a,b],\ t \ne t_1,\dots,t_N, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)), \quad k = 1,\dots,N, \\ x(0) = x_0 \end{gathered} \tag{43}$$

has at least a mild solution.

Proof. Let D be a positive constant such that $\|I_k(c)\| \le D$ for all $c \in E$, $k = 1,\dots,N$ and denote $C = \|x_0\| + DN + (1/\Gamma(\alpha))\sum_{k=1}^{N}\int_{t_{k-1}}^{t_k}(\varphi(s)/(t_k-s)^{1-\alpha})ds + (1/\Gamma(\alpha))((p-1)/(\alpha p-1))^{(p-1)/p}\|\varphi\|_p(b-a)^{\alpha-1/p}$. According to the continuity in its third variable and the integrable boundedness of the function

$$h(t,s,L) = \begin{cases} \dfrac{e^{L(s-t)}\varphi(s)}{(t-s)^{1-\alpha}}, & t > s, \\ 0, & t \le s, \end{cases} \tag{44}$$

it is possible to find two positive constants L and R such that $((N+1)/\Gamma(\alpha))\max_{t\in[a,b]}\int_a^t e^{L(s-t)}(\varphi(s)/(t-s)^{1-\alpha})ds = \overline{\beta} < 1$ and $R \ge e^{-La}C(1-\overline{\beta})^{-1}$. Define

$$Q = \left\{ q \in \mathscr{C}([a,b],E) : \|q(t)\| \le Re^{Lt}\ \forall t \in [a,b] \right\}. \tag{45}$$

Trivially Q is bounded, convex, and closed and hence weakly compact. Let $q \in Q$ and consider

$$T(q)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} f(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s)\, ds + \sum_{a<t_k<t} I_k(x(t_k)), \tag{46}$$

with $f \in L^p([a,b], E)$ with $f(s) \in F(s, q(s))$ for a.a. $s \in [a,b]$. According to $(F3'')$ and Hölder's inequality, it follows that

$$\begin{aligned}
\|x(t)\| &\le \|x_0\| + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|f(s)\|\, ds \\
&\quad + \sum_{a<t_k<t} \|I_k(x(t_k))\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|f(s)\|\, ds \\
&\le \|x_0\| + DN + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \frac{\varphi(s)(1 + \|q(s)\|)}{(t_k - s)^{1-\alpha}} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(s)(1 + \|q(s)\|)}{(t-s)^{1-\alpha}} ds \\
&\le \|x_0\| + DN + \frac{1}{\Gamma(\alpha)} \sum_{a<t_k<t} \int_{t_{k-1}}^{t_k} \frac{\varphi(s)\left(1 + Re^{Ls}\right)}{(t_k - s)^{1-\alpha}} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(s)\left(1 + Re^{Ls}\right)}{(t-s)^{1-\alpha}} ds \\
&\le \|x_0\| + DN + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \frac{\varphi(s)}{(t_k - s)^{1-\alpha}} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(s)}{(t-s)^{1-\alpha}} ds \\
&\quad + \frac{R}{\Gamma(\alpha)} \sum_{k=1}^{N} \int_a^{t_k} \frac{\varphi(s)\, e^{Ls}}{(t_k - s)^{1-\alpha}} ds \\
&\quad + \frac{R}{\Gamma(\alpha)} \int_a^t \frac{\varphi(s)\, e^{Ls}}{(t-s)^{1-\alpha}} ds \\
&\le C + \frac{Re^{Lt}}{\Gamma(\alpha)} \\
&\quad \cdot \left[\sum_{k=1}^{N} \int_a^{t_k} \frac{e^{L(s-t)} \varphi(s)}{(t_k - s)^{1-\alpha}} ds + \int_a^t \frac{e^{L(s-t)} \varphi(s)}{(t-s)^{1-\alpha}} ds \right] \\
&\le C + \frac{Re^{Lt}}{\Gamma(\alpha)} (N+1) \max_{t \in [a,b]} \int_a^t e^{L(s-t)} \frac{\varphi(s)}{(t-s)^{1-\alpha}} ds \\
&= C + Re^{Lt} \overline{\beta} \\
&\le Re^{La} \left(1 - \overline{\beta}\right) + Re^{Lt} \overline{\beta} \\
&\le Re^{Lt} \left(1 - \overline{\beta}\right) + Re^{Lt} \overline{\beta} = Re^{Lt},
\end{aligned} \tag{47}$$

and hence $T(Q) \subseteq Q$. Since T is upper semicontinuous with convex and weakly compact values, as shown above, according to Theorem 1 we get the conclusion. □

4. Boundary Conditions

In this section we will examine in detail some examples of nonlocal boundary conditions shown in the introduction.

The first example is an integral average condition:

(i)

$$M(x) = \frac{1}{b-a} \int_a^b p(t)\, x(t)\, dt, \quad \text{with } p \in L^1([0,T], \mathbb{R}). \tag{48}$$

It arises, for example, in age structure population models, where the boundary condition represents an average term taking into account the birth in the population, depending on the fertility rate p and on the total size of the population.

Assuming that $\|p\|_1/(b-a) < 1$, condition (M) is verified. Indeed, trivially M is a weakly continuous single valued operator; thus it is a weakly sequentially continuous multioperator. Moreover we have

$$\frac{\left\| (1/(b-a)) \int_a^b p(t)\, x(t)\, dt \right\|}{\|x\|_0} \le \frac{\|p\|_1}{b-a}. \tag{49}$$

Hence

$$\lim_{\|x\|_0 \to \infty} \frac{\left\| (1/(b-a)) \int_a^b p(t)\, x(t)\, dt \right\|}{\|x\|_0} \le \frac{\|p\|_1}{b-a} < 1. \tag{50}$$

The second example is a multipoint boundary value problem:

(ii)

$$\begin{gathered}
M(x) = \sum_{i=1}^{n} \alpha_i x(s_i) + x_0, \text{ with} \\
x_0 \in E, \quad \alpha_i \neq 0, \\
s_i \in [a,b], \quad i = 1, \ldots, n.
\end{gathered} \tag{51}$$

It has better application in physics than the classical initial problem, because it allows measurements at $t = s_i \in [a,b]$, $i = 1, \ldots, n$, rather than just at $t = a$. It can be applied, for example, to the description of the diffusion phenomenon of a small amount of gas in a transparent tube observed via the surface of the tube (see [42]).

Moreover if $\sum_{i=1}^{n} |\alpha_i| < 1$ condition (M) is satisfied. Indeed M is the translation of a linear and bounded single valued operator; hence it is a weakly sequentially closed multioperator. Furthermore

$$\frac{\left\|\sum_{i=1}^{n}\alpha_i x(s_i)+x_0\right\|}{\|x\|_0} \le \frac{\sum_{i=1}^{n}|\alpha_i|\,|x(s_i)|+\|x_0\|}{\|x\|_0} \le \frac{\|x\|_0\sum_{i=1}^{n}|\alpha_i|+\|x_0\|}{\|x\|_0} = \sum_{i=1}^{n}|\alpha_i| + \frac{\|x_0\|}{\|x\|_0}. \tag{52}$$

Hence

$$\lim_{\|x\|_0\to\infty}\frac{\left\|\sum_{i=1}^{n}\alpha_i x(s_i)+x_0\right\|}{\|x\|_0} \le \sum_{i=1}^{n}|\alpha_i| < 1. \tag{53}$$

In conclusion problem (1)-(2) with M given as in (i) or (ii) has a solution.

5. Applications

This application concerns the integrodifferential equation

$$\begin{gathered} D_t^{\alpha} z(t,x) \in \left[f_1\left(t,x,\int_{\Omega} k(x,\xi) z(t,\xi)\,d\xi\right), f_2\left(t,x,\int_{\Omega} k(x,\xi) z(t,\xi)\,d\xi\right)\right], \quad t\in[0,T],\quad x\in\Omega, \\ z(t_k^+,x) = z(t_k,x)+c_k,\quad k=1,\dots,N,\ x\in\Omega, \\ z(0,x)=\sum_{i=1}^{n}\alpha_i z(s_i,x),\quad x\in\Omega, \end{gathered} \tag{54}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a sufficiently regular boundary. This problem represents a model in population dynamic, $z(t,x)$ being the density of individuals at the point x and time t. The fractional order derivative takes into account memory effects; the multivalued nonlinearity represents the external influence on the process which is known up to some degree of uncertainty; the integral term describes the property that the state of the problem at a given point may include states in a suitable neighborhood.

We assume the following hypotheses:

(i) for all $r\in\mathbb{R}$, $i=1,2$, $f_i(\cdot,\cdot,r):[0,T]\times\Omega\to\mathbb{R}$ is measurable;

(ii) for a.a. $t\in[0,T]$ and $x\in\Omega$, $f_1(t,x,\cdot):\mathbb{R}\to\mathbb{R}$ is lower semicontinuous and $f_2(t,x,\cdot):\mathbb{R}\to\mathbb{R}$ is upper semicontinuous;

(iii) $f_1(t,x,r)\le f_2(t,x,r)$ in $[0,T]\times\Omega\times\mathbb{R}$;

(iv) there exist $\varphi\in L^p([0,T];\mathbb{R})$, with $p>1/\alpha$, and a nondecreasing function $\mu:[0,\infty)\to[0,\infty)$ such that, for a.a. $x\in\Omega$ and every $t\in[0,T]$, $r\in\mathbb{R}$ and $i=1,2$, we have $|f_i(t,x,r)|\le\varphi(t)\mu(|r|)$ with

$$\liminf_{r\to\infty}\frac{\mu(r)}{r}=0; \tag{55}$$

(v) $k:\Omega\times\Omega\to\mathbb{R}$ is measurable with $k(x,\cdot)\in L^2(\Omega;\mathbb{R})$ and $\|k(x,\cdot)\|_2\le 1$ for all $x\in\Omega$;

(vi) $\alpha_1,\dots,\alpha_n\ne 0$ with $\sum_{i=1}^{n}|\alpha_i|<1$.

Problem (54) can be represented in the form of the following abstract system in the Hilbert space $E=L^2(\Omega;\mathbb{R})$:

$$\begin{gathered} y'(t)\in F(t,y(t)), \\ y(t_k^+)=y(t_k)+I_k(y(t_k)), \\ y(0)=M(y), \end{gathered} \tag{56}$$

where $y:[0,T]\to E$ is defined as $y(t)=z(t,\cdot)$, $F:[0,T]\times E\multimap E$ is the multimap

$$F(t,y)(x)=\left[f_1\left(t,x,\int_{\Omega}k(x,\xi)y(\xi)\,d\xi\right), f_2\left(t,x,\int_{\Omega}k(x,\xi)y(\xi)\,d\xi\right)\right], \tag{57}$$

$I_k:E\to E$ reads as $I_k(y)\equiv c_k$, and $M:\mathscr{C}([0,T];E)\to E$ is the multimap defined as $M(y)(x)=\sum_{i=1}^{n}\alpha_i y(s_i)(x)$ for a.a. $x\in\Omega$.

Let us show that Theorem 12 can be applied to the abstract formulation of the system (54). Notice first of all that Pettis measurability theorem (see [43, page 278]), the separability of $L^2([0,T];\mathbb{R})$, and conditions (i) and (ii) imply that the maps $t\mapsto f_i(t,\cdot,\int_{\Omega}k(\cdot,s)y(s)ds)$, $i=1,2$, are measurable selections of $F(\cdot,y)$ for every $y\in L^2(\Omega;\mathbb{R})$; hence condition $(F1)$ is satisfied. Moreover, according to (iv), we have, for a.a. $x\in\Omega$ and every $y\in L^2(\Omega;\mathbb{R})$,

$$\left|\int_{\Omega}k(x,\xi)y(\xi)\,d\xi\right| \le \int_{\Omega}|k(x,\xi)|\,|y(\xi)|\,d\xi \le \|k(x,\cdot)\|_2\,\|y\|_2 \le \|y\|_2, \tag{58}$$

and thus (v) implies, for a.a. $t\in[0,T]$ and every $y\in L^2(\Omega;\mathbb{R})$,

$$\left|f_i\left(t,x,\int_{\Omega}k(x,\xi)y(\xi)\,d\xi\right)\right| \le \varphi(t)\mu\left(\left|\int_{\Omega}k(x,\xi)y(\xi)\,d\xi\right|\right) \le \varphi(t)\mu(\|y\|_2) \tag{59}$$

for $i=1,2$. Hence the growth condition $(F3')$ is fulfilled with $\varphi_n(t)=\overline{\gamma}n+\varphi(t)\mu(n)\sqrt{\Omega}$.

We now prove that $F(t,\cdot)$ is weakly sequentially continuous for a.a. $t\in[0,T]$. To this aim consider the sequences $\{y_n\},\{\beta_n\}\subset L^2(\Omega;\mathbb{R})$ satisfying $y_n\rightharpoonup y$, $\beta_n\rightharpoonup\beta$ in $L^2(\Omega;\mathbb{R})$ and $\beta_n\in F(t,y_n)$ for all $n\in\mathbb{N}$. Since $\beta_n\rightharpoonup\beta$, applying Mazur's convexity lemma, we have the existence of a sequence

$$\widetilde{\beta}_n=\sum_{i=0}^{k_n}\delta_{n,i}\beta_{n+i},\quad \delta_{n,i}\ge 0,\ \sum_{i=0}^{k_n}\delta_{n,i}=1 \tag{60}$$

such that $\widetilde{\beta}_n \rightarrow \beta$ in $L^2(\Omega;\mathbb{R})$ and up to a subsequence denoted by the sequence $\widetilde{\beta}_n(x) \rightarrow \beta(x)$ for a.a. $x \in \Omega$. By definition we have, for a.a. $x \in \Omega$,

$$\sum_{i=0}^{k_n} \delta_{n,i} f_1\left(t, x, \int_\Omega k(x,\xi)\, y_{n+i}(\xi)\, d\xi\right) \leq \widetilde{\beta}_n(x) \leq \sum_{i=0}^{k_n} \delta_{n,i} f_2\left(t, x, \int_\Omega k(x,\xi)\, y_{n+i}(\xi)\, d\xi\right). \tag{61}$$

From (v) we get that $\int_\Omega k(x,\xi) y_n(\xi) d\xi \rightarrow \int_\Omega k(x,\xi) y(\xi) d\xi$ for every $x \in \Omega$. Passing to the limit as $n \rightarrow \infty$, according to (ii), we obtain that

$$f_1\left(t, x, \int_\Omega k(x,\xi)\, y(\xi)\, d\xi\right) \leq \beta(x) \leq f_2\left(t, x, \int_\Omega k(x,\xi)\, y(\xi)\, d\xi\right), \tag{62}$$

for a.a. $x \in \Omega$, that is, that $\beta \in F(t, y)$. We have showed that $F(t, \cdot)$ has weakly sequentially closed graph.

Trivially the constant functions I_k, $k = 1, \ldots, N$, are sequentially continuous with respect to the weak topology and map bounded sets into bounded sets. Moreover, according to Section 4, M is weakly sequentially closed, maps bounded sets into bounded sets, and satisfies condition (9); thus all the assumptions of Theorem 12 are satisfied and the existence of a solution of (54) is proved.

Remark 15. Reasoning as in the previous example and according to Theorem 14, it is possible to prove the existence of a solution for the following problem, arising from the same models:

$$\begin{aligned} D_t^\alpha z(t,x) \in \Big[& f_1\left(t, x, \int_\Omega k(x,\xi)\, z(t,\xi)\, d\xi\right), \\ & f_2\left(t, x, \int_\Omega k(x,\xi)\, z(t,\xi)\, d\xi\right)\Big] z(t,x), \\ & t \in [0,T], \quad x \in \Omega, \\ z(t_k^+, x) &= z(t_k, x) + c_k, \quad k = 1, \ldots, N, \ x \in \Omega, \\ z(0,x) &= z_0(x), \quad x \in \Omega, \end{aligned} \tag{63}$$

where f_1, f_2, and k satisfy conditions (i), (ii), (iii), and (v) of the previous example, and

(iv)′ there exist $\varphi \in L^p([0,T];\mathbb{R})$, with $p > 1/\alpha$, such that, for a.a. $x \in \Omega$ and every $t \in [0,T]$, $r \in \mathbb{R}$ and $i = 1, 2$, we have $|f_i(t,x,r)| \leq \varphi(t)$;

(vi) $z_0 \in L^2(\Omega, \mathbb{R})$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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