EXTREMAL CURVES IN NILPOTENT LIE GROUPS

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Abstract. We classify extremal curves in free nilpotent Lie groups. The classification is obtained via an explicit integration of the adjoint equation in Pontryagin Maximum Principle. It turns out that abnormal extremals are precisely the horizontal curves contained in algebraic varieties of a specific type. We also extend the results to the nonfree case.

1. Introduction

Let M be a differentiable manifold and $\mathcal{D} \subset TM$ a bracket generating distribution. A Lipschitz curve $\gamma : [0,1] \to M$ is *horizontal* if $\dot\gamma(t) \in \mathcal{D}(\gamma(t))$ for a.e. $t \in [0,1]$. Fixing a quadratic form on $\mathcal{D}$, one can define the length of horizontal curves. A distance on M can be defined by minimizing the length of horizontal curves connecting pair of points. This is known as sub-Riemannian or Carnot-Carathéodory distance. If the resulting metric space is proper, length minimizing curves between any given pair of points do exist.

The main open problem in the field is the regularity of length minimizing curves, see [9, Problem 10.1]. Length minimizing curves may in fact be *abnormal extremals* in the sense of Geometric Control Theory: while normal extremals are always smooth, abnormal ones are apriori only Lipschitz continuous. We refer the reader to Section 2 for precise definitions. Let us only recall here that a horizontal curve is an abnormal extremal if the end-point mapping is singular at this curve, i.e., its differential is not surjective. The notion of abnormal extremal is also related to that of rigid curve, see [3, 14]. Abnormal extremals depend only on the structure $(M, \mathcal{D})$ and not on the fixed quadratic form on $\mathcal{D}$; it is well known that they can be length minimizing, see [10, 12]. On the other hand, in some structures the presence of singularities prevent an abnormal extremal to be length minimizing [8, 11]. To our best knowledge, no example of minimizing nonsmooth horizontal curve is presently known.

The set of abnormal extremals of a structure $(M, \mathcal{D})$ is an interesting object that is not yet well-understood. See, however, the deep second order analysis [2]. In this paper, we describe this set when M is a (connected, simply connected) free nilpotent

Date: July 17, 2012.

2010 *Mathematics Subject Classification.* 53C17, 49K30.

Key words and phrases. Regularity of geodesics, Abnormal curves, Extremal curves, Free nilpotent groups, Carnot groups, SubRiemannian geometry, Algebraic variety.

This work was partially supported by the Fondazione CaRiPaRo Project "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems", Padova.

Lie group and $\mathcal{D}$ is the left-invariant subbundle spanned by a system of generators of its Lie algebra. We obtain a description that is constructive and of a purely algebraic type. It is constructive in the sense that it permits to construct abnormal extremals of any (feasible) desired type: Goh extremals, extremals with given corank, extremals of minimal order in the sense of [4], etc. The classification is purely algebraic in the sense that it only depends on the structure constants of the algebra of the group. These results also extend to other nilpotent groups. In particular, we consider connected, simply connected, nilpotent and stratified Lie groups (Carnot groups): this is of special interest because, by Mitchell's theorem, Carnot groups are the infinitesimal models of equiregular sub-Riemannian structures.

Let us give some flavour of the results contained in the paper. We start by fixing a basis $X_1, \dots, X_n$ of an n-dimensional free nilpotent Lie algebra $\mathfrak{g}$ generated by r elements $X_1, \dots, X_r$; the choice of this basis has to be done according to a very precise algorithm due to M. Hall Jr., see [7]. A detailed description of this algorithm is contained in Section 3. This basis determines a collection of *generalized structure constants*, see (4.42). Using these constants, for any $i = 1, \dots, n$ and for any multi-index $\alpha \in \mathcal{I} := \mathbb{N}^n = \{0, 1, 2, \dots\}^n$, we define certain linear mappings $\phi_{i\alpha} : \mathbb{R}^n \to \mathbb{R}$, see (4.43). When $|\alpha| := \alpha_1 + \cdots + \alpha_n$ is large we have $\phi_{i\alpha} = 0$ because of the nilpotency. For each $i = 1, \dots, n$ and $v \in \mathbb{R}^n$, we introduce the polynomials $P_i^v : \mathbb{R}^n \to \mathbb{R}$

$$P_i^v(x) = \sum_{\alpha \in \mathcal{I}} \phi_{i\alpha}(v) x^\alpha, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where we let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. These polynomials enjoy some remarkable properties that are discussed in Proposition 4.5.

A result due to M. Grayson and R. Grossman [6] ensures that $\mathfrak{g}$ is isomorphic to a certain algebra of vector fields in $\mathbb{R}^n$; it turns out that G can be identified with $\mathbb{R}^n$ via exponential coordinates of the second type associated with the vector fields corresponding to $X_1, \dots, X_n$, see Proposition 3.5. For any $v \in \mathbb{R}^n$, we call the set

$$Z_v = \{x \in \mathbb{R}^n : P_1^v(x) = \ldots = P_r^v(x) = 0\}$$

an *abnormal variety* of G of *corank* 1; when $v \neq 0$ we have $Z_v \neq \mathbb{R}^n$, see Proposition 4.3.

One of the results proved in the paper is the following theorem; see Definition 2.3 for the notion of corank.

Theorem 1.1. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group and let $\gamma : [0, 1] \to G$ be a horizontal curve with $\gamma(0) = 0$. The following statements are equivalent:*

A) *The curve γ is an abnormal extremal of corank $m \geq 1$.*
B) *There exist m linearly independent vectors $v_1, \dots, v_m \in \mathbb{R}^n$ such that $\gamma(t) \in Z_{v_1} \cap \ldots \cap Z_{v_m}$ for all $t \in [0, 1]$.*

A stronger version of Theorem 1.1 holds when γ is a *strictly abnormal* minimizer. In this case, the Goh condition (see Theorem 2.4) implies that for all $t \in [0, 1]$ we have

$$\gamma(t) \in \{x \in \mathbb{R}^n : P_1^v(x) = \ldots = P_s^v(x) = 0\}, \tag{1.2}$$

where $s = \dim(\mathfrak{g}_1 \oplus [\mathfrak{g}_1, \mathfrak{g}_1])$ and $\mathfrak{g}_1$ is the first layer of $\mathfrak{g}$, i.e., the subspace spanned by the generators of $\mathfrak{g}$. We call the algebraic sets as in (1.2) *Goh varieties* of G.

The proof of Theorem 1.1 relies upon Theorem 4.6, where, for any extremal (normal or abnormal), we explicitly compute the dual curve provided by Pontryagin Maximum Principle. We started from the usual version of the differential equation for the dual curve in Optimal Control Theory, see Theorem 2.1, formula (2.6). The coordinates $\lambda = (\lambda_1, \ldots, \lambda_n)$ of the dual curve in the basis of 1-forms dual to the fixed frame of vector fields, see (2.15), satisfy the system of equations (2.17). After studying some Lie groups of step 4, 5 and 6, we could guess the general formula for λ. This led us to the polynomials in (1.1): indeed, in Theorem 4.6, we prove that

$$\lambda_i(t) = P_i^v(\gamma(t)), \quad \text{for all } t \in [0, 1] \text{ and } i = 1, \ldots, n, \tag{1.3}$$

where $v := \lambda(0) \neq 0$ is the initial condition. To check formulae (1.3), we had to understand how the Jacobi identity enters the process of integration of the adjoint equation. The key technical tool is a combinatorial identity for iterated commutators in free nilpotent Lie algebras that is proved by induction in Lemma 3.2. We were not able to find it in the Lie algebra literature.

Formulae (1.3) provide an explicit integration of the second equation in the Hamiltonian system (2.9), see Remark 4.7. These formulae hold for both normal and abnormal extremals and, from the technical viewpoint, are the main result of the paper.

In Section 5, we extend our results from the free case to the case of nonfree Carnot groups. The results here are less explicit because they involve a "lifting" procedure from a nonfree group to a free one, see Theorem 5.4. However, the results are precise enough to deduce in a purely algebraic way some interesting facts on Carnot-Carathéodory geodesics, such as the C^∞ smoothness of length minimizing curves in Carnot groups of step 3 (see [13]), and Golè-Karidi's example [5] of a strictly abnormal extremal in a Carnot group. These and other examples are briefly discussed in Section 6.

The results of this paper raise several questions. As noticed, abnormal extremals are precisely horizontal curves lying inside abnormal varieties: consequently, the interplay between the horizontal distribution and the tangent space to the variety determines the structure of all possible singularities of abnormal extremals. The precise knowledge of these singularities could permit a fine analysis of the length minimality properties of abnormal extremals. It would be interesting to use the techniques developed in [8] and [11] to exclude corners and other kinds of singularities for length minimizers. See, however, the nonsmooth extremal of Section 6.4.

Also, it would be interesting to understand whether our results can be extended to wider classes of Lie groups or to other families of sub-Riemannian structures. For instance, one could wonder whether abnormal extremals of analytic sub-Riemannian manifolds (namely, analytic manifolds with analytic horizontal distribution) are contained in an analytic variety and whether such varieties can be characterized in some way.

As noted above, the family of abnormal extremals is a geometric object associated with a manifold M along with a bracket generating distribution $\mathcal{D} \subset TM$. In the same spirit, an abnormal variety is an intrinsic algebraic subset of a (free) Lie group. The study of abnormal varieties could be of interest in real algebraic geometry. Natural questions concern generic dimension, smoothness, structure of singularities and topology of abnormal varieties. In Section 6.4, we list the quadrics defining Goh extremals in the free nilpotent Lie group of rank 3 and step 4.

2. Extremal curves in sub-Riemannian geometry

In this section, we recall some basic facts concerning extremal curves in sub-Riemannian geometry. Let $X_1, \ldots, X_r$, $r \geq 2$, be linearly independent smooth vector fields in $\mathbb{R}^n$, $n \geq 3$, and let $\mathcal{D}$ denote the distribution of r-planes $\mathcal{D}(x) = \mathrm{span}\{X_1(x), \ldots, X_r(x)\}$ with $x \in \mathbb{R}^n$. $\mathcal{D}$ is called *horizontal distribution* and its sections are called horizontal vector fields. With respect to the standard basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ of $\mathbb{R}^n$ we have for any $j = 1, \ldots, r$

$$X_j = \sum_{i=1}^{n} X_{ji} \frac{\partial}{\partial x_i}, \tag{2.4}$$

where $X_{ji} : \mathbb{R}^n \to \mathbb{R}$ are smooth functions.

A Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^n$ is $\mathcal{D}$-*horizontal*, or simply *horizontal*, if there exists a vector of functions $h = (h_1, \ldots, h_r) \in L^\infty([0,1]; \mathbb{R}^r)$ such that

$$\dot\gamma = \sum_{j=1}^{r} h_j X_j(\gamma), \quad \text{a.e. on } [0,1].$$

The functions h are called controls of γ. Let g_x be the quadratic form on $\mathcal{D}(x)$ making $X_1, \ldots, X_r$ orthonormal. The horizontal length of a horizontal curve γ is then

$$L(\gamma) = \Big(\int_0^1 g_{\gamma(t)}(\dot\gamma(t)) dt \Big)^{1/2} = \Big(\int_0^1 |h(t)|^2 dt \Big)^{1/2}.$$

Here, we are adopting the L^2 definition for the length. For any couple of points $x, y \in \mathbb{R}^n$, we can define the distance

$$d(x,y) = \inf \Big\{ L(\gamma) : \gamma \text{ is horizontal, } \gamma(0) = x \text{ and } \gamma(1) = y \Big\}. \tag{2.5}$$

If the above set is nonempty for any $x, y \in \mathbb{R}^n$, then d is a distance on $\mathbb{R}^n$, usually called Carnot-Carathéodory distance. This holds when the vector fields $X_1, \ldots, X_r$ satisfy the Hörmander bracket-generating condition: at any point of $\mathbb{R}^n$, $X_1, \ldots, X_r$ along with their commutators of sufficiently large length span a vector space of full dimension n.

If the resulting metric space $(\mathbb{R}^n, d)$ is complete, the infimum in (2.5) is attained. We call a curve γ providing the minimum a *(length) minimizer*. If h is the vector of the controls of a minimizer γ, we call the pair (γ, h) an *optimal pair*. The minimizer, which in general is not unique, is found within the class of Lipschitz curves, which are differentiable almost everywhere. Pontryagin Maximum Principle provides necessary conditions for a horizontal curve to be a minimizer.

Theorem 2.1. *Let (γ, h) be an optimal pair. Then there exist $\xi_0 \in \{0, 1\}$ and a Lipschitz curve $\xi : [0, 1] \to \mathbb{R}^n$ such that:*

i) $\xi_0 + |\xi| \neq 0$ *on* $[0, 1]$*;*
ii) $\xi_0 h_j + \langle \xi, X_j(\gamma)\rangle = 0$ *on* $[0, 1]$ *for all* $j = 1, \ldots, r$*;*
iii) *the coordinates* ξ_k*,* $k = 1, \ldots, n$*, of the curve* ξ *solve the system of differential equations*

$$\dot{\xi}_k = -\sum_{j=1}^{r} \sum_{i=1}^{n} \frac{\partial X_{ji}}{\partial x_k}(\gamma) h_j \xi_i, \quad \textit{a.e. on } [0, 1]. \tag{2.6}$$

We refer to [1, Chapter 12] for a proof of Theorem 2.1 in a more general framework. Equations (2.6) are called *adjoint equations*.

Definition 2.2. *We say that a horizontal curve $\gamma : [0, 1] \to \mathbb{R}^n$ is an* extremal *if there exist $\xi_0 \in \{0, 1\}$ and $\xi \in \mathrm{Lip}([0, 1]; \mathbb{R}^n)$ such that i), ii), and iii) in Theorem 2.1 hold.*

We say that γ is a normal extremal *if there exists such a pair (ξ_0, ξ) with $\xi_0 \neq 0$.*

We say that γ is an abnormal extremal *if there exists such a pair with $\xi_0 = 0$.*

We say that γ is a strictly abnormal extremal *if γ is an abnormal extremal but not a normal one.*

We call a curve ξ satisfying i), ii), and iii) in Theorem 2.1 a *dual curve* of γ. We identify the curve ξ with the curve of 1-forms in $\mathbb{R}^n$

$$\xi = \xi_1 dx_1 + \ldots + \xi_n dx_n.$$

The dual curve ξ is constructed in the following way. Let $\Phi : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow in $\mathbb{R}^n$ associated with the controls $h_1, \ldots, h_r \in L^2([0, 1])$ of γ. Namely, let $\Phi(t, x) = \gamma_x(t)$ where $\gamma_x : [0, 1] \to \mathbb{R}^n$ is the solution to the problem

$$\dot{\gamma}_x = \sum_{j=1}^{r} h_j X_j(\gamma_x) \quad \text{a.e. and } \gamma_x(0) = x \in \mathbb{R}^n.$$

We also let $\Phi_t(x) = \Phi(t, x)$. Then, dual curves $\xi : [0, 1] \to T^*\mathbb{R}^n$ are of the form

$$\xi(t) = (\Phi_t^{-1})^* \xi(0), \quad t \in [0, 1] \tag{2.7}$$

for suitable $\xi(0) \neq 0$. Above, $(\Phi_t^{-1})^* \xi(0)$ denotes the pull-back of 1-forms by the diffeomorphism $\Phi_t^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. We refer to [1] for this characterization of dual curves.

Normal extremals are smooth curves. In fact, by ii) the controls satisfy

$$h_j = -\langle \xi, X_j(\gamma) \rangle \quad \text{a.e. on } [0,1] \tag{2.8}$$

for any $j = 1, \dots, r$. This along with the adjoint equation (2.6) implies the C^∞-smoothness of γ. Moreover, the pair (γ, ξ) solves the system of Hamilton's equations

$$\dot\gamma = \frac{\partial H}{\partial \xi}(\gamma, \xi), \quad \dot\xi = -\frac{\partial H}{\partial x}(\gamma, \xi), \tag{2.9}$$

where H is the Hamiltonian function

$$H(x, \xi) = -\frac{1}{2} \sum_{j=1}^{r} \langle X_j(x), \xi \rangle^2.$$

For abnormal extremals we have, for any $j = 1, \dots, r$,

$$\langle \xi, X_j(\gamma) \rangle = 0 \quad \text{on } [0,1]. \tag{2.10}$$

Let us recall the definition of the end-point mapping with initial point $x_0 \in \mathbb{R}^n$. For any $h \in L^2([0,1]; \mathbb{R}^r)$, let γ^h be the solution of the problem

$$\dot\gamma^h = \sum_{j=1}^{r} h_j X_j(\gamma^h), \quad \gamma^h(0) = x_0.$$

The mapping $\mathcal{E} : L^2([0,1]; \mathbb{R}^r) \to \mathbb{R}^n$, $\mathcal{E}(h) = \gamma^h(1)$, is called the *end-point mapping* with initial point x_0. It is well known that a horizontal curve γ starting from x_0 with controls h is an abnormal extremal if and only if there exists $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$, such that

$$\langle d\mathcal{E}(h)v, \lambda \rangle = 0 \tag{2.11}$$

for all $v \in L^2([0,1]; \mathbb{R}^r)$. Here, $d\mathcal{E}(h)$ is the differential of $\mathcal{E}$ at the point h. Abnormal extremals are precisely the singular points of the end-point mapping. Let $\operatorname{Im} d\mathcal{E}(h) \subset \mathbb{R}^n$ denote the image of the differential $d\mathcal{E}(h) : L^2([0,1]; \mathbb{R}^r) \to \mathbb{R}^n$.

Definition 2.3. *The* corank *of an abnormal extremal* $\gamma : [0,1] \to \mathbb{R}^n$ *with controls* h *is the integer* $n - \dim\big(\operatorname{Im} d\mathcal{E}(h)\big) \geq 1$.

If γ has corank $m \geq 1$ then we have m linearly independent functions $\xi^1, \dots, \xi^m \in \operatorname{Lip}([0,1]; \mathbb{R}^n)$ each solving the system of adjoint equations (2.6).

The necessary condition (2.10) can be improved in the case of strictly abnormal minimizers.

Theorem 2.4. *Let* $\gamma : [0,1] \to \mathbb{R}^n$ *be a strictly abnormal length minimizer. Then any dual curve* $\xi \in \operatorname{Lip}([0,1], \mathbb{R}^n)$ *satisfies*

$$\langle \xi, [X_i, X_j](\gamma) \rangle = 0 \quad \text{on } [0,1] \tag{2.12}$$

for any $i, j = 1, \dots, r$.

Condition (2.12) is known as *Goh condition.* Theorem 2.4 can be deduced from second order open mapping theorems. We refer to [1, Chapter 20] for a systematic treatment of the subject.

Definition 2.5. *We say that a horizontal curve* $\gamma : [0,1] \to \mathbb{R}^n$ *is a* Goh extremal *if there exists a Lipschitz curve* $\xi : [0,1] \to \mathbb{R}^n$ *such that* $\xi \neq 0$, ξ *solves the adjoint equations* (2.6) *and* $\langle \xi, X_i(\gamma)\rangle = \langle \xi, [X_i, X_j](\gamma)\rangle = 0$ *on* $[0,1]$ *for all* $i, j = 1, \dots, r$.

Goal of this paper is to integrate the system of adjoint equations. We shall actually integrate an equivalent system of ordinary differential equations. To this aim, let us complete the system of vector fields $X_1, \dots, X_r$ to a frame of n linearly independent vector fields $X_1, \dots, X_n$. This is always possible locally. Then there are smooth functions c_{ij}^k such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \quad i, j = 1, \dots, n. \tag{2.13}$$

In Lie groups, if $X_1, \dots, X_n$ form a basis of left invariant vector fields, the functions c_{ij}^k are constants called *structure constants* of the group.

Let $\vartheta_1, \dots, \vartheta_n$ be the frame of 1-forms dual to the frame of vector fields $X_1, \dots, X_n$. In the standard basis $dx_1, \dots, dx_n$, we have for suitable functions $\vartheta_{ik} : \mathbb{R}^n \to \mathbb{R}$

$$\vartheta_i = \sum_{k=1}^{n} \vartheta_{ik} dx_k.$$

Then, for all $i, j = 1, \dots, n$, we have

$$\delta_{ij} = \vartheta_i(X_j) = \sum_{k=1}^{n} \vartheta_{ik} X_{jk}. \tag{2.14}$$

The coefficients X_{jk} are defined as in (2.4), for all $j = 1, \dots, n$. Here and hereafter, δ_{ij} is the Kronecker symbol.

Given an extremal curve γ with dual curve ξ, let $\lambda_1, \dots, \lambda_n \in \mathrm{Lip}([0,1])$ be the functions defined via the relation along γ

$$\xi_1 dx_1 + \dots + \xi_n dx_n = \lambda_1 \vartheta_1(\gamma) + \dots + \lambda_n \vartheta_n(\gamma). \tag{2.15}$$

We translate the adjoint equations (2.6) for ξ into a system of differential equations for the coordinates $\lambda_1, \dots, \lambda_n$ of ξ in the frame $\vartheta_1, \dots, \vartheta_n$. We can express γ in the standard coordinates of $\mathbb{R}^n$ as $\gamma = (\gamma_1, \dots, \gamma_n)$. In the case of Lie groups, the following theorem is proved in [5].

Theorem 2.6. *Assume that the vector fields* $X_1, \dots, X_r$ *satisfy*

$$X_{jh} = \delta_{jh} \qquad \text{for all } 1 \leq j, h \leq r. \tag{2.16}$$

Then the functions $\xi_1, \dots, \xi_n$ *solve equations* (2.6) *if and only if the functions* $\lambda_1, \dots, \lambda_n$ *satisfy the system of differential equations*

$$\dot{\lambda}_i = -\sum_{k=1}^{n} \sum_{j=1}^{r} c_{ij}^k(\gamma) \dot{\gamma}_j \lambda_k \quad \text{a.e. on } [0,1], \tag{2.17}$$

for any $i = 1, \dots, n$.

Proof. Let $h : [0,1] \to \mathbb{R}^r$ be the controls of γ:

$$\dot\gamma = h_1 X_1(\gamma) + \cdots + h_r X_r(\gamma) \quad \text{a.e. on } [0,1].$$

By (2.16) we have $\dot\gamma_j = h_j$ a.e. on $[0,1]$, $j = 1,\ldots,r$. On the one hand, by differentiating the identity

$$\xi_k = \sum_{i=1}^n \lambda_i \vartheta_{ik}(\gamma),$$

we obtain

$$\begin{aligned}
\dot\xi_k &= \sum_{i=1}^n \dot\lambda_i \vartheta_{ik} + \lambda_i \langle \nabla \vartheta_{ik}(\gamma), \dot\gamma\rangle \\
&= \sum_{i=1}^n \Big(\dot\lambda_i \vartheta_{ik} + \lambda_i \Big\langle \nabla \vartheta_{ik}(\gamma), \sum_{j=1}^r h_j X_j(\gamma) \Big\rangle \Big) \\
&= \sum_{i=1}^n \Big(\dot\lambda_i \vartheta_{ik} + \lambda_i \sum_{j=1}^r h_j \, X_j \vartheta_{ik}(\gamma) \Big).
\end{aligned}$$

On the other hand, equation (2.6) reads

$$\dot\xi_k = -\sum_{j=1}^r \sum_{i=1}^n h_j \frac{\partial X_{ji}}{\partial x_k}(\gamma) \sum_{h=1}^n \lambda_h \vartheta_{hi}(\gamma).$$

Taking the difference of the previous two identities, we get

$$0 = \sum_{i=1}^n \dot\lambda_i \vartheta_{ik} + \sum_{i=1}^n \sum_{j=1}^r \lambda_i h_j \, X_j \vartheta_{ik}(\gamma) + \sum_{j=1}^r \sum_{i=1}^n \sum_{h=1}^n h_j \frac{\partial X_{ji}}{\partial x_k}(\gamma) \lambda_h \vartheta_{hi}(\gamma),$$

for any $k = 1,\ldots,n$. It follows that for any $p = 1,\ldots,n$ we have

$$0 = \sum_{k=1}^n X_{pk} \Big(\sum_{i=1}^n \dot\lambda_i \vartheta_{ik} + \sum_{i=1}^n \sum_{j=1}^r \lambda_i h_j \, X_j \vartheta_{ik}(\gamma) + \sum_{j=1}^r \sum_{i,h=1}^n h_j \frac{\partial X_{ji}}{\partial x_k}(\gamma) \lambda_h \vartheta_{hi}(\gamma) \Big).$$

Notice now that, by (2.14),

$$0 = \sum_{k=1}^n X_{pk}(X_j \vartheta_{ik}) + \sum_{k=1}^n (X_j X_{pk}) \vartheta_{ik}\,; \tag{2.18}$$

moreover,

$$\sum_{k=1}^n X_{pk} \frac{\partial X_{ji}}{\partial x_k}(\gamma) = (X_p X_{ji})(\gamma). \tag{2.19}$$

Using (2.14), (2.18), and (2.19), we obtain

$$\begin{aligned}
0 &= \dot\lambda_p - \sum_{i,k=1}^{n}\sum_{j=1}^{r} \lambda_i h_j\,(X_j X_{pk})(\gamma)\,\vartheta_{ik}(\gamma) + \sum_{j=1}^{r}\sum_{i,h=1}^{n} \lambda_h h_j (X_p X_{ji})(\gamma)\,\vartheta_{hi}(\gamma) \\
&= \dot\lambda_p - \sum_{i,k=1}^{n}\sum_{j=1}^{r} \lambda_i h_j\,(X_j X_{pk})(\gamma)\,\vartheta_{ik}(\gamma) + \sum_{j=1}^{r}\sum_{k,i=1}^{n} \lambda_i h_j (X_p X_{jk})(\gamma)\,\vartheta_{ik}(\gamma) \\
&= \dot\lambda_p + \sum_{i,k=1}^{n}\sum_{j=1}^{r} \lambda_i h_j \vartheta_{ik}(\gamma)\big(X_p X_{jk} - X_j X_{pk}\big)(\gamma).
\end{aligned}$$

From (2.13) we deduce that

$$(X_p X_{jk} - X_j X_{pk})(\gamma) = \sum_{\ell=1}^{n} c_{pj}^{\ell}(\gamma) X_{\ell k}(\gamma),$$

and hence, by (2.14), we have

$$0 = \dot\lambda_p + \sum_{i,k,\ell=1}^{n}\sum_{j=1}^{r} \lambda_i h_j c_{pj}^{\ell}(\gamma)\vartheta_{ik}(\gamma) X_{\ell k}(\gamma) = \dot\lambda_p + \sum_{i=1}^{n}\sum_{j=1}^{r} \lambda_i h_j c_{pj}^{i}(\gamma).$$

This is equation (2.17). The same computations show that (2.6) follows from (2.17). □

3. Hall basis theorem for free nilpotent Lie algebras

In this section, we develop the algebraic tools needed in Section 4 to integrate the system of differential equations (2.17) in free nilpotent Lie groups. In groups, the structure functions c_{ij}^k appearing in (2.13) and (2.17) are in fact constants, as soon as the vector fields $X_1, \dots, X_n$ are left invariant. The technical difficulty is to make transparent how the Jacobi identity enters the integration process. The algebraic preliminaries for this integration are fixed in Lemma 3.2.

Let $\mathfrak{g}$ be a free nilpotent real Lie algebra of dimension n, step $s \geq 2$ and rank $r \geq 2$. The algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s,$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$ for $i = 1, \dots, s-1$ and $\mathfrak{g}_i = \{0\}$ for $i > s$. The first layer $\mathfrak{g}_1$ has dimension r.

We recall the algorithm for the construction of a basis for $\mathfrak{g}$ due to M. Hall Jr. [7]. Let $X_1, \dots, X_r$ be a basis for $\mathfrak{g}_1$. To each X_i we assign the degree $d(i) = \deg(X_i) = 1$ for $i = 1, \dots, r$. By induction, we complete $X_1, \dots, X_r$, to a basis $X_1, \dots, X_n$ for $\mathfrak{g}$, and we assign to each X_j a degree $d(j) = \deg(X_j) \in \{1, \dots, s\}$. Let us assume that the elements $X_1, \dots . X_k$, with degree at most $d-1$, are already defined and that they are ordered with the property:

$$d(i) \leq d-1,\ d(j) \leq d-1, \text{ and } d(i) < d(j) \ \Rightarrow\ i < j.$$

By definition, the commutator $[X_i, X_j]$ is an element of the Hall basis of degree d if:

(3.20a) $\quad i > j,$

(3.20b) $\quad X_i$ and X_j are elements of the basis of degree $\leq d-1$,

(3.20c) $\quad d(i) + d(j) = d,$

(3.20d) $\quad$ if X_i was constructed as $[X_h, X_k]$, then $k \leq j$.

Hall proved that this algorithm produces a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. The basis is ordered by subindices in such a way that $d(i) < d(j)$ implies $i < j$.

Following the genesis of the basis, it is easy to see by induction on the degree that, for any $\ell = 1, \ldots, n$ there exists a unique string of indices $\ell_0, \ell_1, \ell_2, \ldots, \ell_h \in \{1, \ldots, n\}$, with $h \in \{0, \ldots, s-1\}$, such that

$$X_\ell = [\cdots[[[X_{\ell_0}, X_{\ell_1}], X_{\ell_2}], \ldots], X_{\ell_h}] \tag{3.21}$$

and

(3.22a) $\quad \ell_0 > \ell_1,$

(3.22b) $\quad d(\ell_0) = d(\ell_1) = 1,$

(3.22c) $\quad \ell_1 \leq \ell_2 \leq \ldots \leq \ell_h,$

(3.22d) $\quad$ if $X_k = [\cdots[[[X_{\ell_0}, X_{\ell_1}], X_{\ell_2}], \ldots], X_{\ell_{i-1}}]$, with $i = 2, \ldots, h$, then $\ell_i < k$.

We need some more notation. Let us first introduce the set of multi-indices $\mathcal{I} = \mathbb{N}^n$. For any $\alpha \in \mathcal{I}$, we let $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $u_\alpha = \max\{i : \alpha_i \neq 0\}$. Let us agree that $\alpha \leq \beta$ for $\alpha, \beta \in \mathcal{I}$ means $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, n$.

Fix a Hall basis $X_1, \ldots, X_n$. For $j_0, j_1, j_2, \ldots, j_k \in \{1, \ldots, n\}$, we let

$$[X_{j_0}, X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}] := [\cdots[[[X_{j_0}, X_{j_1}], X_{j_2}], X_{j_3}], \ldots, X_{j_k}]. \tag{3.23}$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{I}$, we define the iterated commutator

$$[\cdot, X_\alpha] := [\cdot, \underbrace{X_1, \ldots, X_1}_{\alpha_1 \text{ times}}, \underbrace{X_2, \ldots, X_2}_{\alpha_2 \text{ times}}, \ldots, \underbrace{X_n, \ldots, X_n}_{\alpha_n \text{ times}}]. \tag{3.24}$$

We agree that $[\cdot, X_{(0,\ldots,0)}] = \mathrm{Id}$.

For any $\ell \in \{1, \ldots, n\}$ there exist unique $\ell_0 \in \{2, \ldots, r\}$ and $h = h(\ell) \in \{1, \ldots, s-1\}$ such that X_ℓ is given by the representation (3.21) subject to (3.22a)–(3.22d). For each ℓ we define the multi-index $I(\ell) \in \mathcal{I}$ as

$$I(\ell)_j = \#\{s \geq 1 : \ell_s = j\}. \tag{3.25}$$

For example, if $X_\ell = [X_3, X_2]$, then $\ell_0 = 3$ and $I(\ell) = (0, 1, 0, \ldots, 0)$. Also, if $\ell \in \{1, \ldots, r\}$ then $\ell_0 = \ell$ and $I(\ell) = (0, \ldots, 0)$.

With this notation, we have the following properties, for any $\ell = 1, \ldots, n$:

$$X_\ell = [X_{\ell_0}, X_{I(\ell)}], \tag{3.26}$$

$$u_{I(\ell)} = \ell_{h(\ell)}. \tag{3.27}$$

Moreover, given $\ell, k \in \{1, \dots, n\}$, we have

$$[X_\ell, X_k] \text{ is a base vector} \iff u_{I(\ell)} \le k < \ell. \tag{3.28}$$

Definition 3.1. *We say that $X_\ell = [X_{\ell_0}, X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_h}]$ is a direct discendant of X_j if $X_j = [X_{\ell_0}, X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_k}]$, for some $k \in \{0, \dots, h\}$. In this case, we write $j \preceq \ell$.*

The relation $\preceq$ is a partial order on indices. Notice that $j \preceq \ell$ implies $I(j) \le I(\ell)$, but not viceversa.

The next combinatorial lemma plays a central role in the next section. For any $\beta \in \mathcal{I}$ and $q = 1, \dots, r$, let us define the set of indices

$$\mathcal{A}_{\beta,q} = \big\{\ell \in \{1, \dots, n\} : q \preceq \ell,\, I(\ell) \le \beta\big\}.$$

The statement $q \preceq \ell$ is equivalent to $\ell_0 = q$. We will also use the notation $\beta! := \beta_1!\,\beta_2! \cdots \beta_n!$, $|\beta| := \beta_1 + \cdots + \beta_n$, and $\mathrm{e}_k := (\delta_{1k}, \delta_{2k}, \dots, \delta_{nk}) = (0, \dots, 1, \dots, 0)$.

Lemma 3.2. *For all $\beta \in \mathcal{I}$, $i = 1, ..., n$, and $q = 1, ..., r$, we have*

$$[[X_i, X_q], X_\beta] = \sum_{\ell \in \mathcal{A}_{\beta,q}} c_{\ell\beta} [X_i, X_{\beta - I(\ell) + \mathrm{e}_\ell}], \tag{3.29}$$

where we let

$$c_{\ell\beta} = \frac{\beta!}{(\beta - I(\ell))!\,I(\ell)!}. \tag{3.30}$$

Proof. The case $\beta = 0$ is straightforward. The proof is by induction on $|\beta| \ge 1$.

Base of induction. Assume $|\beta| = 1$, i.e., $\beta = \mathrm{e}_k$, for some $k \in \{1, 2, \dots, n\}$. Hence, $\beta! = 1$. Let $\ell \in \mathcal{A}_{\beta,q}$. Since $I(\ell) \le \beta$, then either $I(\ell) = (0, \dots, 0)$ or $I(\ell) = \mathrm{e}_k$. In both cases we have $c_{\ell\beta} = 1$. Since $\ell_0 = q$, then X_ℓ is either X_q, in which case $\beta - I(\ell) + \mathrm{e}_\ell = \mathrm{e}_k + \mathrm{e}_q$, or $X_\ell = [X_q, X_k]$, in which case $\beta - I(\ell) + \mathrm{e}_\ell = \mathrm{e}_\ell$. Notice that the latter case is allowed only if $k < q$.

Therefore, equation (3.29) reduces to a trivial identity when $k \ge q$, whereas when $k < q$ it reduces to

$$[[X_i, X_q], X_k] = [[X_i, X_k], X_q] + [X_i, [X_q, X_k]],$$

which is true by the Jacobi identity.

Inductive step. Using induction, we prove the claim for any $\widetilde{\beta} \in \mathcal{I}$. We write $\widetilde{\beta} = \beta + \mathrm{e}_k$ for some $\beta \in \mathcal{I} \setminus \{0\}$ and $k = u_{\widetilde{\beta}} \in \{1, 2, \dots, n\}$. Hence we have $u_\beta \le k$ and $|\beta| < |\widetilde{\beta}|$. By induction, formula (3.29) holds for β and thus:

$$[[X_i, X_q], X_{\widetilde{\beta}}] = \sum_{\ell \in \mathcal{A}_{\beta,q}} c_{\ell\beta} [[X_i, X_{\beta - I(\ell) + \mathrm{e}_\ell}], X_k].$$

We split the sum into two sums according to whether $\ell \le k$ or $\ell > k$. In the case when $\ell \le k$, the coordinates of the multi-index $\beta - I(\ell) + e_\ell$ from the $(k+1)$-th onward are null and hence,

$$[[X_i, X_{\beta - I(\ell) + \mathrm{e}_\ell}], X_k] = [X_i, X_{\beta - I(\ell) + \mathrm{e}_\ell + \mathrm{e}_k}].$$

In the case $\ell > k$, also because $u_{I(\ell)} \leq k$, the commutator $[X_\ell, X_k]$ is an element of the Hall basis, see (3.28). We denote this element by $X_{\ell.k} = [X_\ell, X_k]$. Then by the Jacobi identity we obtain

$$\begin{aligned}[][[X_i, X_{\beta-I(\ell)+\mathrm{e}_\ell}], X_k] &= [[[X_i, X_{\beta-I(\ell)}], X_\ell], X_k] \\ &= [[[X_i, X_{\beta-I(\ell)}], X_k], X_\ell] + [[X_i, X_{\beta-I(\ell)}], [X_\ell, X_k]] \\ &= [X_i, X_{\beta-I(\ell)+\mathrm{e}_k+\mathrm{e}_\ell}] + [X_i, X_{\beta-I(\ell)+\mathrm{e}_{\ell.k}}].\end{aligned}$$

To conclude the proof, we need to show that the sum

$$\sum_{\ell\in\mathcal{A}_{\beta,q}} c_{\ell\beta}[X_i, X_{\beta-I(\ell)+\mathrm{e}_\ell+\mathrm{e}_k}] + \sum_{\ell\in\mathcal{A}_{\beta,q},\ell>k} c_{\ell\beta}[X_i, X_{\beta-I(\ell)+\mathrm{e}_{\ell.k}}] \tag{3.31}$$

is equal to $\sum_{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}} c_{\ell,\beta+\mathrm{e}_k}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}]$, i.e., to

$$\sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k\leq\beta_k}} c_{\ell,\beta+\mathrm{e}_k}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}] + \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=\beta_k+1}} c_{\ell,\beta+\mathrm{e}_k}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}]\,. \tag{3.32}$$

For fixed β, we introduce the notation

$$\Phi(\ell) = c_{\ell\beta}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}],$$

that is well defined if $I(\ell)_k \leq \beta_k$.

Let us rewrite (3.31). In the second summation in (3.31), we perform the change of indices $\widetilde{\ell} = \ell.k$. Then we have $I(\widetilde{\ell}) = I(\ell) + \mathrm{e}_k$ and the summation becomes

$$\begin{aligned}&\sum_{\substack{\widetilde{\ell}\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\widetilde{\ell})_k\geq 1}} \frac{\beta!}{(\beta+\mathrm{e}_k-I(\widetilde{\ell}))!(I(\widetilde{\ell})-\mathrm{e}_k)!}[X_i, X_{\beta+\mathrm{e}_k-I(\widetilde{\ell})+\mathrm{e}_{\widetilde{\ell}}}] \\ &= \sum_{\substack{\widetilde{\ell}\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 1\leq I(\widetilde{\ell})_k\leq\beta_k}} \frac{I(\widetilde{\ell})_k}{\beta_k+1-I(\widetilde{\ell})_k}\Phi(\widetilde{\ell}) + \sum_{\substack{\widetilde{\ell}\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\widetilde{\ell})_k=\beta_k+1}} \frac{\beta!}{(\beta+\mathrm{e}_k-I(\widetilde{\ell}))!(I(\widetilde{\ell})-\mathrm{e}_k)!}[X_i, X_{\beta+\mathrm{e}_k-I(\widetilde{\ell})+\mathrm{e}_{\widetilde{\ell}}}].\end{aligned}$$

Therefore, the sum in (3.31) is

$$\begin{aligned}(3.31) = &\sum_{\ell\in\mathcal{A}_{\beta,q}} \Phi(\ell) + \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 1\leq I(\ell)_k\leq\beta_k}} \frac{I(\ell)_k}{\beta_k+1-I(\ell)_k}\Phi(\ell) \\ &+ \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=\beta_k+1}} \frac{\beta!}{(\beta+\mathrm{e}_k-I(\ell))!(I(\ell)-\mathrm{e}_k)!}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}].\end{aligned} \tag{3.33}$$

We split the first summation in (3.33) according to whether $I(\ell)_k = 0$ or not. Notice that $\ell \in \mathcal{A}_{\beta,q}$ is equivalent to $\ell \in \mathcal{A}_{\beta+\mathrm{e}_k,q}$ and $I(\ell)_k \le \beta_k$. Then we have

$$\begin{aligned}\sum_{\ell\in\mathcal{A}_{\beta,q}} \Phi(\ell) &= \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=0}} \Phi(\ell) + \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 1\le I(\ell)_k\le\beta_k}} \Phi(\ell)\\ &= \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=0}} \frac{\beta_k+1}{\beta_k+1-I(\ell)_k}\Phi(\ell) + \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 1\le I(\ell)_k\le\beta_k}} \Phi(\ell)\end{aligned}$$

and, by (3.33), we obtain

$$\begin{aligned}(3.31) = &\sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=0}} \frac{\beta_k+1}{\beta_k+1-I(\ell)_k}\Phi(\ell) + \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 1\le I(\ell)_k\le\beta_k}} \Phi(\ell) + \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 1\le I(\ell)_k\le\beta_k}} \frac{I(\ell)_k}{\beta_k+1-I(\ell)_k}\Phi(\ell)\\ &+ \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=\beta_k+1}} \frac{\beta!}{(\beta+\mathrm{e}_k-I(\ell))!(I(\ell)-\mathrm{e}_k)!}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}]\,.\end{aligned}$$

Hence, the sum in (3.31) is

$$\begin{aligned}(3.31) = &\sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ 0\le I(\ell)_k\le\beta_k}} \frac{\beta_k+1}{\beta_k+1-I(\ell)_k}\Phi(\ell)\\ &+ \sum_{\substack{\ell\in\mathcal{A}_{\beta+\mathrm{e}_k,q}\\ I(\ell)_k=\beta_k+1}} \frac{\beta!}{(\beta+\mathrm{e}_k-I(\ell))!(I(\ell)-\mathrm{e}_k)!}[X_i, X_{\beta+\mathrm{e}_k-I(\ell)+\mathrm{e}_\ell}]\end{aligned}$$

and it can be easily checked that the right hand side of the previous formula equals (3.32). This concludes the proof. □

Let $\mathfrak{f}_{s,r}$ denote the free nilpotent Lie algebra over $\mathbb{R}$ of step s and rank r. Let $n = \dim(\mathfrak{f}_{r,s})$ be the dimension of the algebra. Let us fix a Hall basis for $\mathfrak{f}_{s,r}$. Then for any $\ell \in \{1,\dots,n\}$ we have a multi-index $I(\ell) \in \mathcal{I}$, see (3.25), and moreover there is a partial order $\preceq$ on indices, see Definition 3.1. Finally, recall the notation $x^\alpha = x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}$ for $x\in\mathbb{R}^n$ and $\alpha\in\mathcal{I}$.

The following theorem is proved by M. Grayson and R. Grossman, see [6, Theorem 2.1].

Theorem 3.3. *The vector fields $X_1,\dots,X_r$ in $\mathbb{R}^n$*

$$X_i(x) = \sum_{\ell:i\preceq\ell} \frac{(-1)^{|I(\ell)|}}{I(\ell)!} x^{I(\ell)}\frac{\partial}{\partial x_\ell}, \quad x\in\mathbb{R}^n, \tag{3.34}$$

with $i = 1,\dots,r$, generate a Lie algebra isomorphic to $\mathfrak{f}_{s,r}$.

The index $\ell = i$ is included in the sum in (3.34) and gives the summand $\partial/\partial x_i$. In [6], the authors use a slightly different notation. Moreover, there is no $\ell \ne 1$ such that $1 \preceq \ell$ and thus $X_1 = \partial/\partial x_1$. Finally, notice that $X_1, \dots, X_r$ satisfy assumption (2.16).

By Hall's construction, the generators $X_1, \dots, X_r$ can be completed to a basis $X_1, \dots, X_n$ of the Lie algebra. We call $X_1, \dots, X_n$ a *Hall-Grayson-Grossman basis* of vector fields in $\mathbb{R}^n$.

In [6], the authors also describe the elements X_j with $j > r$. For $\alpha \in \mathcal{I}$ with $\alpha \neq 0$, define the *minimal order* of the monomial x^α as

$$m(x^\alpha) = \min \big\{ j \in \{1, \dots, n\} : \alpha_j > 0 \big\}.$$

For a polynomial $P(x) = \sum_{h=1}^N c_h x^{\alpha_h}$ with $c_h \neq 0$ and $\alpha_h \in \mathcal{I}$, $\alpha_h \neq 0$, we define the minimal order $m(P) = \max_{h=1,\dots,N} m(x^{\alpha_h})$.

For $i, \ell \in \{1, \dots, n\}$ with $i \preceq \ell$, let us define the monomial

$$P_{i\ell}(x) = \frac{(-1)^{|I(\ell)|-|I(i)|}}{(I(\ell) - I(i))!} x^{I(\ell)-I(i)}, \quad x \in \mathbb{R}^n. \tag{3.35}$$

Notice that (3.34) can be rewritten as

$$X_i(x) = \sum_{\ell : i \preceq \ell} P_{i\ell}(x) \frac{\partial}{\partial x_\ell}, \qquad i = 1, \dots, r.$$

Lemma 3.4. ([6, Lemma 2.3]) *If X_i is an element of the Hall basis constructed as $X_i = [X_j, X_k]$, then*

$$X_i(x) = \sum_{\ell : i \preceq \ell} P_{i\ell}(x) \frac{\partial}{\partial x_\ell} + \sum_{\ell=1}^n Q_{i\ell}(x) \frac{\partial}{\partial x_\ell}, \quad x \in \mathbb{R}^n, \tag{3.36}$$

where $Q_{i\ell}$ are polynomials with no constant terms satisfying $m(Q_{i\ell}) < k$ and $P_{i\ell}$ are monomials of the form (3.35) *satisfying $k \leq m(P_{i\ell}) < i$ for all $\ell \neq i$.*

The exponential mapping of the second type related to an ordered system of vector fields $X_1, \dots, X_n$ is the mapping $\Psi : \mathbb{R}^n \to \mathbb{R}^n$, when globally defined,

$$\Psi(x) = \mathrm{e}^{x_1 X_1} \circ \dots \circ \mathrm{e}^{x_n X_n}(0), \quad x \in \mathbb{R}^n, \tag{3.37}$$

where e^{tX} denotes the flow of the vector field X with parameter $t \in \mathbb{R}$. When G is a Lie group with group law $\cdot$ and $X_1, \dots, X_n \in \mathfrak{g} = \mathrm{Lie}(G)$, we can equivalently define the mapping $\Psi : \mathbb{R}^n \to G$ as

$$\Psi(x) = \exp(x_n X_n) \cdot \ldots \cdot \exp(x_1 X_1), \quad x \in \mathbb{R}^n, \tag{3.38}$$

where $\exp : \mathfrak{g} \to G$ is the exponential mapping.

We prove that a Hall-Grayson-Grossman basis of vector fields in $\mathbb{R}^n$ induces exponential coordinates of the second type.

Proposition 3.5. *Let $X_1, \dots, X_n$ be a Hall-Grayson-Grossman basis of vector fields in $\mathbb{R}^n$. Then we have $x = \Psi(x)$ for all $x \in \mathbb{R}^n$.*

Proof. Let $\mathrm{e}_1, \dots, \mathrm{e}_n$ be the standard basis of $\mathbb{R}^n$. By the structure (3.36) of X_n, we deduce that $\mathrm{e}^{x_n X_n}(0) = x_n \mathrm{e}_n$. Assume by induction that for some $1 \le i < n$ we have

$$\Psi(x) = \mathrm{e}^{x_1 X_1} \circ \dots \circ \mathrm{e}^{x_i X_i}\Big(\sum_{h=i+1}^{n} x_h \mathrm{e}_h \Big). \tag{3.39}$$

Assume first that $i > r$. Then we have $X_i = [X_j, X_k]$ for some $k < j < i$, and X_i is of the form (3.36). Since $m(Q_{i\ell}) < k < i$ for all $\ell = 1, \dots, n$, the polynomial $Q_{i\ell}$ vanishes along the flow of X_i starting from $\sum_{h=i+1}^{n} x_h \mathrm{e}_h$. Because $m(P_{i\ell}) < i$ for $\ell \neq i$ with $i \preceq \ell$, the monomial $P_{i\ell}$ also vanishes along the same flow. When $\ell = i$, we have $P_{i\ell} = 1$. It follows that

$$\mathrm{e}^{x_i X_i}\Big(\sum_{h=i+1}^{n} x_h \mathrm{e}_h \Big) = \sum_{h=i}^{n} x_h \mathrm{e}_h. \tag{3.40}$$

This proves the inductive step when $i > r$.

Assume now $i \in \{1, \dots, r\}$. We have $X_1 = \partial/\partial x_1$ and for $i = 2, \dots, r$ the condition $i \preceq \ell$, $\ell \neq i$, implies that $m(P_{i\ell}) < i$. The same argument as above proves (3.40) also when $i = 1, \dots, r$. □

4. Integration of the adjoint equations in free nilpotent groups

Let G be a free nilpotent Lie group of dimension n and rank r. The Lie algebra $\mathfrak{g}$ of G is isomorphic to a Lie algebra of vector fields in $\mathbb{R}^n$ that are left invariant with respect to some product structure. Let $X_1, ..., X_n$ be a Hall-Grayson-Grossman basis of this Lie algebra. Recall that the generators $X_1, \dots, X_r$ have the form (3.34). In this section, we integrate the system of differential equations (2.17) in $\mathbb{R}^n$ with the structure constants $c_{ij}^k \in \mathbb{R}$ determined by the basis $X_1, ..., X_n$:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \quad i, j = 1, \dots, n. \tag{4.41}$$

Let us also introduce the *generalized structure constants* $c_{i\alpha}^k \in \mathbb{R}$ for any multi-index $\alpha \in \mathcal{I} = \mathbb{N}^n$ and $i, k \in \{1, \dots, n\}$. These constants are defined via the relation

$$[X_i, X_\alpha] = \sum_{k=1}^{n} c_{i\alpha}^k X_k. \tag{4.42}$$

Recall the definition (3.24) for the iterated commutator $[X_i, X_\alpha]$.

For any $i = 1, \dots, n$ and $\alpha \in \mathcal{I}$, define the linear mapping $\phi_{i\alpha} : \mathbb{R}^n \to \mathbb{R}$

$$\phi_{i\alpha}(v) = \frac{(-1)^{|\alpha|}}{\alpha!} \sum_{k=1}^{n} c_{i\alpha}^k v_k, \quad v = (v_1, \dots, v_n) \in \mathbb{R}^n. \tag{4.43}$$

Notice that $\phi_{i0}(v) = v_i$. The following polynomials are central objects in the integration formulae for the adjoint equations.

Definition 4.1. *For each $i \in \{1, \dots, n\}$ and $v \in \mathbb{R}^n$, we call the polynomial P_i^v : $\mathbb{R}^n \to \mathbb{R}$*

$$P_i^v(x) = \sum_{\alpha \in \mathcal{I}} \phi_{i\alpha}(v) x^\alpha, \quad x \in \mathbb{R}^n, \tag{4.44}$$

extremal polynomial *of the free nilpotent group G with respect to the basis $X_1, \dots, X_n$ of $\mathfrak{g} = \mathrm{Lie}(G)$.*

Remark 4.2. Notice that the generalized structure constants $c_{i\alpha}^k$ in (4.42) satisfy $c_{i\alpha}^k = 0$ if $d(i) + |\alpha| > s$, where s is the step of the Lie algebra. Then the polynomial $P_i^v(x)$ has homogeneous degree at most $s - d(i)$. Recall that, by definition, the homogeneous degree of a monomial $x^\alpha, \alpha \in \mathcal{I}$, is $d(\alpha) := \sum_{j=1}^n \alpha_j \, d(j)$ and the homogeneous degree of a polynomial $\sum_{i=1}^N c_i x^{\alpha_i}$, with $c_i \neq 0$ and $\alpha_i \in \mathcal{I}$, is $\max_{i=1,\dots,N} d(\alpha_i)$.

Proposition 4.3. *Extremal polynomials have the following properties.*
(i) *If $\ell, i = 1, \dots, n$ are such that $i \preceq \ell$ and $v_\ell \neq 0$, then $P_i^v \neq 0$.*
(ii) *If $v \in \mathbb{R}^n$ is such that $P_i^v = 0$ for all $i = 1, \dots, r$, then $v = 0$.*
$P_i^v = 0$ for all
(iii) *For all $i = 1, \dots, n$ and $v \in \mathbb{R}^n$ we have $P_i^v(0) = v_i$.*

Proof. Let us prove (i). As $i \preceq \ell$, we have $X_\ell = [X_i, X_\alpha]$ for $\alpha = I(\ell) \in \mathcal{I}$. It follows that $c_{i\alpha}^k = \delta_{\ell k}$, and thus, as $v_\ell \neq 0$,

$$P_i^v(x) = \frac{(-1)^{|\alpha|}}{\alpha!} v_\ell x^\alpha + \sum_{\beta \in \mathcal{I}, \, \beta \neq \alpha} \phi_{i\beta}(v) x^\beta \neq 0.$$

Statement (ii) is an easy consequence of (i), while the proof of (iii) is elementary and relies upon the identity $c_{i0}^k = \delta_{ik}$. ☐

Remark 4.4. We point out the following implication: if $r_2 := \dim \mathfrak{g}_2$ and $v \in \mathbb{R}^n$ is such that $P_i^v = 0$ for all $i = r+1, \dots, r + r_2$, then $v_{r+1} = v_{r+2} = \cdots = v_n = 0$.

Indeed, arguing by contradiction we assume that there exists j with $d(j) \geq 2$ such that $v_j \neq 0$. By Proposition 4.3 (iii) we have

$$v_{r+1} = v_{r+2} = \cdots = v_{r+r_2} = 0,$$

hence $d(j) \geq 3$ and we can write

$$X_j = [\cdots [X_{j_0}, X_{j_1}], \dots], X_{j_h}]$$

for a suitable $h \geq 2$. The element $[X_{j_0}, X_{j_1}]$ belongs to the Hall basis and, in particular, $[X_{j_0}, X_{j_1}] = X_k$ for a suitable k of degree 2. Since $k \preceq j$, Proposition 4.3 (i) gives $P_k^v \neq 0$, a contradiction.

The following lemma shows that the linear subspace generated by extremal polynomials is closed under derivatives along left invariant vector fields, in a way related to the structure of $\mathfrak{g}$.

Proposition 4.5. *For any $v \in \mathbb{R}^n$ and $i, j \in \{1, \dots, n\}$ there holds*

$$X_i P_j^v = \sum_{k=1}^n c_{ij}^k P_k^v. \tag{4.45}$$

In particular, $X_i P_i^v = 0$ for any $i = 1, \dots, n$ and $v \in \mathbb{R}^n$.

Proof. Let $v \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$ be fixed. We argue by induction on the degree of i. If $d(i) = 1$, i.e., $i = 1, \dots, r$, we have by (3.34)

$$\begin{aligned} X_i P_j^v &= \Bigg(\sum_{i \preceq \ell} \frac{(-1)^{|I(\ell)|}}{I(\ell)!} x^{I(\ell)} \frac{\partial}{\partial x_\ell} \Bigg) \sum_{\alpha \in \mathcal{I}} \phi_{j\alpha}(v) x^\alpha \\ &= \sum_{k=1}^n v_k \sum_{\alpha \in \mathcal{I}} \sum_{i \preceq \ell} c_{j\alpha}^k \alpha_\ell \frac{(-1)^{|\alpha| + |I(\ell)|}}{\alpha! I(\ell)!} x^{\alpha - \mathrm{e}_\ell + I(\ell)}. \end{aligned}$$

We now use the following polynomial identity

$$\sum_{\alpha \in \mathcal{I}} \sum_{i \preceq \ell} c_{j\alpha}^k \alpha_\ell \frac{(-1)^{|\alpha| + |I(\ell)|}}{\alpha! I(\ell)!} x^{\alpha - \mathrm{e}_\ell + I(\ell)} = - \sum_{h=1}^n \sum_{\beta \in \mathcal{I}} c_{ji}^h c_{h\beta}^k \frac{(-1)^{|\beta|}}{\beta!} x^\beta. \tag{4.46}$$

This identity will be proven later, in a crucial step of the proof of Theorem 4.6, see equation (4.53). From (4.46) we obtain

$$X_i P_j^v = - \sum_{k=1}^n v_k \sum_{h=1}^n \sum_{\beta \in \mathcal{I}} c_{ji}^h c_{h\beta}^k \frac{(-1)^{|\beta|}}{\beta!} x^\beta = \sum_{h=1}^n c_{ij}^h P_h^v.$$

This ends the proof of the induction base.

If $d(i) \geq 2$, we have $X_i = [X_p, X_u]$ for some p, u with $d(p), d(u) < d(q)$. By inductive assumption, we have

$$\begin{aligned} X_i P_j^v &= X_p X_u P_j^v - X_u X_p P_j^v \\ &= X_p \Big(\sum_{k=1}^n c_{uj}^k P_k^v \Big) - X_u \Big(\sum_{k=1}^n c_{pj}^k P_k^v \Big) \\ &= \sum_{h,k=1}^n (c_{uj}^k c_{pk}^h P_h^v - c_{pj}^k c_{uk}^h P_h^v) \\ &= - \sum_{h,k=1}^n (c_{ju}^k c_{pk}^h + c_{pj}^k c_{uk}^h) P_h^v. \end{aligned}$$

The Jacobi identity $[[X_j, X_u], X_p] + [[X_u, X_p], X_j] + [[X_p, X_j], X_u] = 0$ yields

$$\sum_{k=1}^n c_{ju}^k c_{kp}^h + c_{up}^k c_{kj}^h + c_{pj}^k c_{ku}^h = 0.$$

Using this identity and $c_{up}^k c_{jk}^h = c_{pu}^k c_{kj}^h = \delta_{ik} c_{kj}^h$, we obtain

$$X_i P_j^v = \sum_{h,k=1}^n c_{up}^k c_{jk}^h P_h^v = \sum_{h=1}^n c_{ij}^h P_h^v.$$

This completes the proof. □

We identify a free nilpotent Lie group G with $\mathbb{R}^n$ via exponential coordinates of the second type related to a Hall-Grayson-Grossman basis $X_1, \dots, X_n$, as explained in Proposition 3.5.

Theorem 4.6. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group, let $\gamma : [0,1] \to G$ be a horizontal curve such that $\gamma(0) = 0$, and let $\lambda : [0,1] \to \mathbb{R}^n$ be a Lipschitz curve. The following statements are equivalent:*

A) *The curve λ solves the system of equations* (2.17).
B) *There exists $v \in \mathbb{R}^n$ such that, for all $i = 1, \dots, n$, we have*

$$\lambda_i(t) = P_i^v(\gamma(t)), \quad t \in [0,1], \tag{4.47}$$

and in fact $v = \lambda(0)$.

Proof. We show that the curve λ defined by (4.47) solves the system (2.17) with initial condition $\lambda(0) = v$. By the uniqueness of the solution, this will prove the equivalence of the statements A) and B). Notice that the curve λ defined via (4.47) satisfies $\lambda(0) = v$ by Proposition 4.3 part (iii).

Recall that γ is Lipschitz-continuous and is therefore differentiable almost everywhere. We preliminarily compute the derivative of $t \mapsto \gamma(t)^\alpha$ for any multi-index $\alpha \in \mathcal{I}$, at any differentiability point. We have

$$\frac{d}{dt}\gamma^\alpha = \sum_{\ell=1}^n \alpha_\ell \gamma_\ell^{\alpha_\ell - 1}\dot\gamma_\ell \prod_{p\neq\ell}\gamma_p^{\alpha_p} = \sum_{\ell=1}^n \alpha_\ell \gamma^{\alpha - \mathrm{e}_\ell}\dot\gamma_\ell. \tag{4.48}$$

Since γ is horizontal we have

$$\dot\gamma = \sum_{q=1}^r \dot\gamma_q X_q(\gamma) \qquad \text{a.e. on } [0,1], \tag{4.49}$$

where $X_1, \dots, X_n$ is the fixed Hall basis and $r \geq 2$ is the rank of the group.

For any $\ell = 1, \dots, n$ we can compute the derivative $\dot\gamma_\ell$ starting from (4.49) and from the formula (3.34) for the generators $X_1, \dots, X_r$ of the Lie algebra. We have to consider the index $\ell_0 = 1, \dots, r$ and the multi-index $I(\ell) \in \mathcal{I}$ such that $X_\ell = [X_{\ell_0}, X_{I(\ell)}]$, see (3.26), and then look at the ℓ-th coordinate of the vector field X_{ℓ_0}. We then find

$$\dot\gamma_\ell = \dot\gamma_{\ell_0}\frac{(-1)^{|I(\ell)|}}{I(\ell)!}\gamma^{I(\ell)} \qquad \text{a.e. on } [0,1]. \tag{4.50}$$

By (4.47), (4.48), (4.50) and the definition of P_i^v, we obtain

$$\dot\lambda_i = \sum_{\alpha\in\mathcal{I}}\sum_{j=1}^n\sum_{\ell=1}^n c_{i\alpha}^j v_j \alpha_\ell \dot\gamma_{\ell_0}\frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!}\gamma^{I(\ell)+\alpha-\mathrm{e}_\ell},$$

which is equivalent to

$$\dot\lambda_i = \sum_{q=1}^{r}\dot\gamma_q \sum_{j=1}^{n} v_j \sum_{\alpha\in\mathcal{I}}\sum_{\ell:\ell_0=q} c^j_{i\alpha}\alpha_\ell \frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!}\gamma^{\alpha-\mathrm{e}_\ell+I(\ell)}. \tag{4.51}$$

On the other hand, the right hand side of (2.17) is

$$-\sum_{q=1}^{r}\dot\gamma_q\sum_{k=1}^{n} c^k_{iq}\lambda_k = -\sum_{q=1}^{r}\dot\gamma_q\sum_{j=1}^{n} v_j\sum_{k=1}^{n}\sum_{\beta\in\mathcal{I}} c^k_{iq}c^j_{k\beta}\frac{(-1)^{|\beta|}}{\beta!}\gamma^\beta. \tag{4.52}$$

Comparing the lines (4.51) and (4.52), we see that it is sufficient to show that for any fixed $i, j = 1,\dots,n$ and $q = 1,\dots,r$ the following identity of polynomials is verified

$$\sum_{\alpha\in\mathcal{I}}\sum_{\ell:\ell_0=q} c^j_{i\alpha}\alpha_\ell \frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!}x^{\alpha-\mathrm{e}_\ell+I(\ell)} = -\sum_{k=1}^{n}\sum_{\beta\in\mathcal{I}} c^k_{iq}c^j_{k\beta}\frac{(-1)^{|\beta|}}{\beta!}x^\beta. \tag{4.53}$$

On the other hand, this identity is equivalent to the following combinatorial identity, for any fixed $\beta\in\mathcal{I}$:

$$\sum_{\substack{\ell:\ell_0=q\\ \alpha:\alpha-\mathrm{e}_\ell+I(\ell)=\beta}} c^j_{i\alpha}\alpha_\ell \frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!} = -\sum_{k=1}^{n} c^k_{iq}c^j_{k\beta}\frac{(-1)^{|\beta|}}{\beta!}.$$

Since $X_1,\dots,X_n$ is a basis, this is in turn equivalent to

$$\begin{aligned}\sum_{j=1}^{n}\Big(\sum_{\substack{\ell:\ell_0=q\\ \alpha:\alpha-\mathrm{e}_\ell+I(\ell)=\beta}} c^j_{i\alpha}\alpha_\ell \frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!}\Big)X_j &= -\sum_{k=1}^{n} c^k_{iq}\frac{(-1)^{|\beta|}}{\beta!}\sum_{j=1}^{n} c^j_{k\beta}X_j\\ &= -\sum_{k=1}^{n}\frac{(-1)^{|\beta|}}{\beta!}c^k_{iq}[X_k, X_\beta]\\ &= -\frac{(-1)^{|\beta|}}{\beta!}[[X_i, X_q], X_\beta].\end{aligned}$$

We may rearrange the last identity in the following way:

$$\sum_{\substack{\ell:\ell_0=q\\ \alpha:\alpha-\mathrm{e}_\ell+I(\ell)=\beta}} \alpha_\ell \frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!}[X_i, X_\alpha] = -\frac{(-1)^{|\beta|}}{\beta!}[[X_i, X_q], X_\beta]. \tag{4.54}$$

Notice that the condition $\alpha - \mathrm{e}_\ell + I(\ell) = \beta$ implies $I(\ell)\le\beta$, because $I(\ell)_\ell = 0$. In particular, the left hand side of (4.54) is

$$\sum_{\substack{\ell:\ell_0=q\\ \alpha:\alpha-\mathrm{e}_\ell+I(\ell)=\beta}} \alpha_\ell \frac{(-1)^{|\alpha|+|I(\ell)|}}{\alpha! I(\ell)!}[X_i, X_\alpha] = -\sum_{\substack{\ell:\ell_0=q\\ I(\ell)\le\beta}} \frac{(-1)^{|\beta|}}{(\beta-I(\ell))! I(\ell)!}[X_i, X_{\beta-I(\ell)+\mathrm{e}_\ell}].$$

We conclude that (4.54) is equivalent to the identity

$$\sum_{\substack{\ell:\ell_0=q\\ I(\ell)\le\beta}} \frac{1}{(\beta-I(\ell))! I(\ell)!}[X_i, X_{\beta-I(\ell)+\mathrm{e}_\ell}] = \frac{1}{\beta!}[[X_i, X_q], X_\beta],$$

which is precisely identity (3.29) in Lemma 3.2. This concludes the proof of the theorem. □

Remark 4.7. Theorem 4.6 provides an explicit integration of the second equation in the Hamiltonian system (2.9). The first Hamilton's equation in (2.9), namely the equation $\dot\gamma = H_\xi(\gamma,\xi)$, reads

$$\dot\gamma = -\sum_{j=1}^{r} \langle \xi, X_j(\gamma)\rangle X_j(\gamma).$$

By (2.15) and (4.47), we have

$$\langle \xi, X_j(\gamma)\rangle = \lambda_j(\gamma(t)) = P_j^v(\gamma(t))$$

with $v = \lambda(0)$. By the formula (3.34) for X_j, we obtain the following system of ordinary differential equations for normal extremals $\gamma : [0,1] \to G = \mathbb{R}^n$

$$\dot\gamma = -\sum_{j=1}^{r} \sum_{j \preceq \ell} \frac{(-1)^{|I(\ell)|}}{I(\ell)!} \gamma^{I(\ell)} P_j^v(\gamma) \frac{\partial}{\partial x_\ell}.$$

In order to characterize abnormal curves, we introduce the following algebraic varieties.

Definition 4.8. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group of rank r. For any $v \in \mathbb{R}^n$, $v \neq 0$, we call the set*

$$Z_v = \{x \in \mathbb{R}^n : P_1^v(x) = \ldots = P_r^v(x) = 0\}$$

an abnormal variety *of* G of corank 1.

For linearly independent vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, $m \geq 2$, we call the set $Z_{v_1} \cap \ldots \cap Z_{v_m}$ an abnormal variety *of* G of corank m.

Abnormal varieties depend on the system of coordinates of the second type induced on G by a Hall-Grayson-Grossman basis $X_1, \ldots, X_n$ for the Lie algebra of G.

Theorem 4.9. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group and let $\gamma : [0,1] \to G$ be a horizontal curve with $\gamma(0) = 0$. The following statements are equivalent:*

A) *The curve γ is an abnormal extremal of corank $m \geq 1$.*
B) *There exists m linearly independent vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ such that $\gamma(t) \in Z_{v_1} \cap \ldots \cap Z_{v_m}$ for all $t \in [0,1]$.*

Proof. Recall that the property of having corank m for γ is equivalent to the existence of m linearly independent solutions to the system (2.17).

Let γ be an abnormal extremal and let $\lambda : [0,1] \to \mathbb{R}^n$ be a Lipschitz curve solving the system (2.17) and such that $\lambda \neq 0$ pointwise. By Theorem 4.6, there is $v \in \mathbb{R}^n$, $v \neq 0$, such that $\lambda_i = P_i^v(\gamma)$ for all $i = 1, \ldots, r$. For abnormal extremals we have $\lambda_1 = \ldots = \lambda_r = 0$ on $[0,1]$, i.e., $\gamma(t) \in Z_v$ for all $t \in [0,1]$. This shows that B) follows from A).

On the other hand, if B) holds then the curve λ defined in (4.47) for any $v = v_1, \dots, v_m$ satisfies the system (2.17) by Theorem 4.6, and moreover $\lambda_1 = \dots = \lambda_r = 0$. From $v \neq 0$ it follows that $\lambda(0) = v \neq 0$, and from the uniqueness of the solution to (2.17) with initial condition it follows that $\lambda \neq 0$ pointwise on $[0,1]$. If $v_1, \dots, v_m$ are linearly independent, then the corresponding curves λ are also linearly independent. □

Remark 4.10. Notice that when $v \neq 0$ the zero set Z_v is nontrivial, i.e., $Z_v \neq \mathbb{R}^n$, because by Proposition 4.3 (ii) there is at least one index $i = 1, \dots, r$ such that $P_i^v \neq 0$ is not the zero polynomial.

The following corollary easily follows from Definition 2.5.

Corollary 4.11. *Let $G = \mathbb{R}^n$ be a free nilpotent Lie group with stratified algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$ and let $r_1 = \dim(\mathfrak{g}_1)$ and $r_2 = \dim(\mathfrak{g}_2)$. Let $\gamma : [0,1] \to G$ be a horizontal curve such that $\gamma(0) = 0$. The following statements are equivalent:*

A) *The curve γ is a Goh extremal.*

B) *There exists $v \in \mathbb{R}^n$, $v \neq 0$, such that for all $i = 1, \dots, r_1 + r_2$ and for all $t \in [0,1]$ there holds*

$$P_i^v(\gamma(t)) = 0. \tag{4.55}$$

Remark 4.12. Given $v \in \mathbb{R}^n$ with $v \neq 0$, we call the set

$$G_v = \{x \in \mathbb{R}^n : P_i^v(x) = 0 \text{ for all } i = 1, \dots, r_1 + r_2\}$$

a *Goh variety* of the free nilpotent group G. By Proposition 4.3 (iii) we have $v_i = 0$ for all $i = 1, \dots, r_1 + r_2$. Therefore, by Proposition 4.3 (ii) and Remark 4.4, there are at least one index $i \in \{1, \dots, r_1\}$ and one index $j \in \{r_1 + 1, \dots, r_1 + r_2\}$ such that $P_i^v \neq 0$ and $P_j^v \neq 0$.

The polynomials P_i^v with $i = r_1 + 1, \dots, r_1 + r_2$ completely describe the horizontal sections of G_v. Namely, let $\gamma : [0,1] \to \mathbb{R}^n$ be a *horizontal* curve such that $\gamma(0) = 0$ and

$$\gamma(t) \in \{x \in \mathbb{R}^n : P_i^v(x) = 0 \text{ for all } i = r_1 + 1, \dots, r_1 + r_2\}$$

for all $t \in [0,1]$, where $v \in \mathbb{R}^n$ is a vector such that $v_i = 0$, for all $i = 1, \dots, r_1$. Then, by Proposition 4.5 for any index $i = 1, \dots, r_1$ and for almost every $t \in [0,1]$ we have

$$\frac{d}{dt} P_i^v(\gamma(t)) = \sum_{k=1}^{r_1} \dot\gamma_k(t) X_k P_i^v(\gamma(t)) = \sum_{k=1}^{r_1} \dot\gamma_k(t) \sum_{\ell=1}^{n} c_{ki}^\ell P_\ell^v(\gamma(t)) = 0,$$

because the only indices ℓ involved in the previous sum are those with degree 2. We deduce that $P_i^v(\gamma(t)) = P_i^v(\gamma(0)) = P_i^v(0) = v_i = 0$.

5. Extremal curves in stratified groups

In this section, we give a partial classification of extremal curves in general stratified Lie groups. The description is not optimal because it involves a "lifting" procedure. A more complete result, e.g., a purely algebraic characterization of abnormal extremals, seems to require an explicit extension of Grayson-Grossman results to nonfree nilpotent groups.

Let $Y_1, \dots, Y_r$, $r \geq 2$, be smooth vector fields in $\mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$, with $m \leq n$, be a smooth mapping. Let us consider the following system of vector fields in $\mathbb{R}^m$:

$$X_1 = \pi_*(Y_1), \dots, X_r = \pi_*(Y_r), \tag{5.56}$$

where π_* is the differential of π. Here and hereafter we assume that $X_1, \dots, X_r$ are linearly independent at each point.

Let $\gamma : [0,1] \to \mathbb{R}^m$ be a horizontal curve for $X_1, \dots, X_r$ with $\gamma(0) = 0 \in \mathbb{R}^m$. Then there are controls $h_1, \dots, h_r \in L^2([0,1])$ such that

$$\dot\gamma(t) = \sum_{j=1}^r h_j(t) X_j(\gamma(t)), \quad \text{for a.e. } t \in [0,1]. \tag{5.57}$$

We call the curve $\kappa : [0,1] \to \mathbb{R}^n$ such that $\kappa(0) = 0$ and $\dot\kappa(t) = \sum_{j=1}^r h_j(t) Y_j(\kappa(t))$ for a.e. $t \in [0,1]$ the *lift of* γ to $\mathbb{R}^n$.

In the sequel, we denote by $T^*\mathbb{R}^n$ and $T^*\mathbb{R}^m$ the cotangent spaces to $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and we denote by $\pi^* : T^*\mathbb{R}^m \to T^*\mathbb{R}^n$ the pull-back mapping induced by π. For the sake of completeness, we provide the proof of the following easy fact.

Proposition 5.1. *Let $\gamma : [0,1] \to \mathbb{R}^m$ be a horizontal curve for $X_1, \dots, X_r$ with $\gamma(0) = 0$ and let $\kappa : [0,1] \to \mathbb{R}^m$ be the lift of γ to $\mathbb{R}^n$. If $\xi : [0,1] \to T^*\mathbb{R}^m$ is a dual curve for γ then $\pi^*\xi : [0,1] \to T^*\mathbb{R}^n$ is a dual curve for κ.*

Proof. Let $h_1, \dots, h_r \in L^2([0,1])$ be the controls of γ as in (5.57) and let $\Phi : [0,1] \times \mathbb{R}^m \to \mathbb{R}^m$ be the flow in $\mathbb{R}^m$ associated with these controls. Namely, let $\Phi(t,x) = \gamma_x(t)$ where $\gamma_x : [0,1] \to \mathbb{R}^m$ is the solution to the problem

$$\dot\gamma_x = \sum_{j=1}^r h_j X_j(\gamma_x) \quad \text{a.e. and } \gamma_x(0) = x \in \mathbb{R}^m.$$

We also let $\Phi_t(x) = \Phi(t,x)$. Finally, let us denote by $\Psi : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ the flow in $\mathbb{R}^n$ associated with the same controls.

By the characterization (2.7) of dual curves, we have

$$\xi(t) = (\Phi_t^{-1})^* \xi(0), \tag{5.58}$$

for some $\xi(0) \in T_0^*\mathbb{R}^m$, $\xi(0) \neq 0$. Above, $(\Phi_t^{-1})^*\xi(0)$ is the pull-back via the diffeomorphism $\Phi_t^{-1} : \mathbb{R}^m \to \mathbb{R}^m$.

Let $\eta = \pi^*\xi$ be the pull-back to $\mathbb{R}^n$ of ξ by π. We claim that for any $t \in [0,1]$ we have

$$\eta(t) = (\Psi_t^{-1})^* \eta(0), \tag{5.59}$$

and thus $\eta : [0,1] \to T^*\mathbb{R}^n$ is a dual curve for the lift κ to $\mathbb{R}^n$ of γ.

To prove this claim, we preliminarily show that π commutes with the flows Φ and Ψ. Namely, for any $y \in \mathbb{R}^n$ and $t \in [0,1]$ we have:

$$\pi(\Psi_t(y)) = \Phi_t(\pi(y)). \tag{5.60}$$

This identity holds when $t = 0$ because $\pi(\Psi_0(y)) = \pi(y) = \Phi_0(\pi(y))$. Let us compute the derivatives in t of the left and of the right hand sides in (5.60). We have:

$$\begin{aligned}\frac{d}{dt}\pi(\Psi_t(y)) = \pi_*\Big(\frac{d}{dt}\Psi_t(y)\Big) \ &= \ \pi_*\Big(\sum_{j=1}^r h_j(t)Y_j(\Psi_t(y))\Big) \\ = \sum_{j=1}^r h_j(t)\pi_*(Y_j(\Psi_t(y))) \ &= \ \sum_{j=1}^r h_j(t)X_j(\pi(\Psi_t(y))).\end{aligned}$$

In the last line, we used (5.56). On the other hand, we have

$$\frac{d}{dt}\Phi_t(\pi(y)) = \sum_{j=1}^r h_j(t)X_j(\Phi_t(\pi(y))).$$

It follows that the curves $t \mapsto \pi(\Psi_t(y))$ and $t \mapsto \Phi_t(\pi(y))$ solve the same differential equation with the same initial condition. Now, identity (5.60) follows from the uniqueness of the solution. The same argument proves that

$$\pi \circ \Psi_t^{-1} = \Phi_t^{-1} \circ \pi, \quad t \in [0,1]. \tag{5.61}$$

Now we prove our main claim (5.59). By (2.7), by the composition rule for the pull-back, and by (5.61), we have

$$\begin{aligned}\eta(t) &= \pi^*(\xi(t)) = \pi^*((\Phi_t^{-1})^*\xi(0)) \\ &= (\Phi_t^{-1} \circ \pi)^*\xi(0) = (\pi \circ \Psi_t^{-1})^*\xi(0) \\ &= (\Psi_t^{-1})^*\pi^*(\xi(0)) = (\Psi_t^{-1})^*\eta(0).\end{aligned}$$

This ends the proof. □

Remark 5.2. The statement of Proposition 5.1 can be improved in the following sense: if γ is an abnormal (respectively, Goh) extremal with dual curve ξ, then κ is an abnormal (resp., Goh) extremal with dual curve $\pi^*\xi$.

Remark 5.3. Let g be a quadratic form on the distribution $\mathcal{D}$ on $\mathbb{R}^m$ spanned by $X_1, \dots, X_r$, and let $h = \pi^* g$ be the pull-back of g to the distribution $\mathcal{E}$ on $\mathbb{R}^n$ spanned by $Y_1, \dots, Y_r$. Then, π preserves the length of horizontal curves. Hence, if γ is a length minimizing curve in $(\mathbb{R}^m, \mathcal{D}, g)$, then the lift κ of γ to $\mathbb{R}^n$ is a length minimizing curve in $(\mathbb{R}^n, \mathcal{E}, h)$.

Now we pass to stratified Lie groups. Let G be a stratified m-dimensional Lie group of step $s \geq 2$ and rank $r \geq 2$. Its Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s,$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$ for $i = 1, \dots, s-1$ and $\mathfrak{g}_i = \{0\}$ for $i > s$. The first layer $\mathfrak{g}_1$ has dimension r.

Let F be the free nilpotent Lie group of step $s \geq 2$ and rank $r \geq 2$. Let $\mathfrak{f}$ be its Lie algebra and let $n = \dim(\mathfrak{f})$. Let $Y_1, \dots, Y_r$ be generators of $\mathfrak{f}$. There is a surjective Lie group homomorphism $\pi : F \to G$, with differential mapping $\pi_* : \mathfrak{f} \to \mathfrak{g}$, such that $\pi_* Y_i \in \mathfrak{g}_1$, $i = 1, \dots, r$. We complete $Y_1, \dots, Y_r$ to a Hall basis $Y_1, \dots, Y_n$ of $\mathfrak{f}$. By exponential coordinates of the second type, we can identify F with $\mathbb{R}^n$ and assume that $Y_1, \dots, Y_n$ is a Hall-Grayson-Grossman basis of left invariant vector fields on $\mathbb{R}^n = F$.

Let $S \subset \{1, 2, \dots, n\}$ be a set such that the vector fields $\pi_* Y_s$ with $s \in S$ form a basis for $\mathfrak{g}$. We have $m = \#S$ and $S = \{s_1 < s_2 < \dots < s_m\}$. We relabel the basis in the following way

$$X_i = \pi_* Y_{s_i}, \quad i = 1, \dots, m.$$

As $\mathfrak{g}$ has rank r, then the vector fields $X_i = \pi_* Y_i \in \mathfrak{g}_1$, $i = 1, \dots, r$, are generators of $\mathfrak{g}$. There are constants $\zeta_{ij} \in \mathbb{R}$ such that

$$\pi_* Y_i = \sum_{j=1}^{m} \zeta_{ij} X_j. \tag{5.62}$$

In the following theorem, $G = \mathbb{R}^m$ is a stratified Lie group with rank r and step s, and $F = \mathbb{R}^n$ is the free nilpotent Lie group with rank r and step s. The lift procedure from G to F is defined with respect to the systems of vector fields fixed above.

Theorem 5.4. *Let $\gamma : [0, 1] \to G$ be a horizontal curve with $\gamma(0) = 0$ and let $\kappa : [0, 1] \to F$ be the lift of γ to F. If γ is an extremal curve with dual curve $\lambda : [0, 1] \to \mathbb{R}^m$ then there is $v \in \mathbb{R}^n$ such that*

$$v_i = \sum_{j=1}^{m} \zeta_{ij} v_{s_j}, \tag{5.63}$$

and the coordinates of λ satisfy, for all $i = 1, \dots, m$,

$$\lambda_i(t) = P^v_{s_i}(\kappa(t)), \quad t \in [0, 1].$$

Proof. If γ is an extremal curve in G with dual curve λ, then κ is an extremal curve in F with dual curve $\mu = \pi^* \lambda$, by Proposition 5.1. By Theorem 4.6, there is $v \in \mathbb{R}^n$ such that the coordinates of μ in the dual basis of $Y_1, \dots, Y_n$ are

$$\mu_i(t) = P^v_i(\kappa(t)), \quad t \in [0, 1],$$

for any $i = 1, \dots, n$, and in fact we have $v = \mu(0) = \pi^* \lambda(0)$. By (5.62), we have for $i = 1, \dots, n$

$$\mu_i = \mu(Y_i) = \pi^* \lambda(Y_i) = \lambda(\pi_* Y_i) = \sum_{j=1}^{m} \zeta_{ij} \lambda(X_j) = \sum_{j=1}^{m} \zeta_{ij} \mu_{s_j}.$$

At $t = 0$, this identity implies the relation (5.63) for v. On the other hand, for any $i = 1, \dots, m$ we have

$$\lambda_i = \lambda(X_i) = \lambda(\pi_* Y_{s_i}) = \mu(Y_{s_i}) = \mu_{s_i} = P^v_{s_i}(\kappa).$$

□

6. Examples and applications

In this section, we discuss some examples on how theorems and formulae of Sections 4 and 5 can be applied.

6.1. **Regularity of geodesics in stratified groups of step 3.**

We give a short and alternative proof of a result proved in [13].

Theorem 6.1. *Let G be a stratified Lie group of step 3 with a smooth left invariant quadratic form g on the horizontal distribution $\mathcal{D}$. Any length minimizing curve in $(G, \mathcal{D}, g)$ is of class C^∞.*

Proof. By Proposition 5.1 and Remark 5.3, we can assume that G is free. Let n and r be the dimension and the rank of G, respectively. By contradiction, assume there is a length minimizing curve $\gamma : [0,1] \to G$ that is not of class C^∞. Then γ is a strictly abnormal extremal (normal extremals are of class C^∞) and thus a Goh extremal.

We can assume that $\gamma(0) = 0$. By Theorem 4.9, there are an index $i = 1, \dots, n$ with $d(i) = 2$ and $v \in \mathbb{R}^n$, $v \neq 0$, such that $P_i^v(\gamma) = 0$. Notice that the polynomial

$$P_i^v(x) = -\sum_{\substack{j:d(j)=3\\ \ell:d(\ell)=1}} c_{i\ell}^j v_j x_\ell \neq 0$$

has homogeneous degree 1. On differentiating in t the identity $P_i^v(\gamma) = 0$, we obtain

$$\sum_{\substack{j:d(j)=3\\ \ell:d(\ell)=1}} c_{i\ell}^j v_j \dot\gamma_\ell = 0. \tag{6.64}$$

Let $\mathfrak{f}$ be the (free) subalgebra of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \mathrm{Lie}(G)$ generated by the subspace $\mathfrak{f}_1 \subset \mathfrak{g}_1$

$$\mathfrak{f}_1 = \Big\{ \sum_{\ell:d(\ell)=1} x_\ell X_\ell \in \mathfrak{g}_1 : \sum_{\substack{j:d(j)=3\\ \ell:d(\ell)=1}} c_{i\ell}^j v_j x_\ell = 0 \Big\},$$

where $X_1, \dots, X_r \in \mathfrak{g}_1$ are Hall-Grayson-Grossman generators. As $P_i^v(x) \neq 0$, we have $\dim(\mathfrak{f}_1) = r - 1$. Let $F \subset G$ be the stratified Lie group with Lie algebra $\mathfrak{f}$. By (6.64), $\dot\gamma$ is in $\mathfrak{f}$ and hence the curve γ is in F. Moreover, γ is a length minimizer in F for the restricted quadratic form, and thus γ is a Goh extremal in F (because it is not smooth).

We can repeat the above reduction argument to conclude that γ is a Goh extremal in a free nilpotent group of rank 2. Now the equation

$$P_i^v(\gamma) = -\sum_{\substack{j:d(j)=3\\ \ell\in\{1,2\}}} c_{i\ell}^j v_j \gamma_\ell = 0,$$

where P_i^v is a nonzero polynomial, implies that γ is a line, and thus a smooth curve. This is a contradiction. □

6.2. **Regularity of generic length minimizing curves in stratified Lie groups of rank 2.** We prove that in stratified Lie groups of rank 2 length minimizing curves are "generically" smooth. This is a special case of a deeper series of results by W. Liu and H. J. Sussmann on *regular abnormal extremals*, see [12].

Let G be a stratified Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ of rank 2 and step $s \geq 3$. We have $r_1 = \dim(\mathfrak{g}_1) = 2$, $r_2 = \dim(\mathfrak{g}_2) = 1$, and $r_3 = \dim(\mathfrak{g}_3) \in \{1, 2\}$. We fix a Hall-Grayson-Grossman basis $X_1, \dots, X_n$, and we identify G with $\mathbb{R}^n$. For a dual curve $\lambda : [0, 1] \to \mathbb{R}^n$, we let $\lambda^{(3)} = \lambda_4$ when $r_3 = 1$, and $\lambda^{(3)} = (\lambda_4, \lambda_5)$ when $r_3 = 2$.

Proposition 6.2. *Let $\gamma : [0, 1] \to G = \mathbb{R}^n$ be a Goh extremal with dual curve $\lambda : [0, 1] \to \mathbb{R}^n$. Assume that $\lambda^{(3)}(t) \neq 0$ for all $t \in [0, 1]$. Then γ is an analytic curve.*

Proof. By Proposition 5.1 and Remark 5.2, we can assume that G is free; in particular, $\dim(\mathfrak{g}_3) = 2$. By Corollary 4.11, there exists a $v \in \mathbb{R}^n$, $v \neq 0$, such that for all $t \in [0, 1]$ we have

$$\gamma(t) \in \Sigma = \big\{x \in \mathbb{R}^n : P_3^v(x) = 0\big\}, \tag{6.65}$$

where $P_3^v(x) \neq 0$ is a nonzero polynomial of the form (4.44).

By Proposition 4.5, we have

$$X_1 P_3^v(x) = -P_4^v(x) \quad \text{and} \quad X_2 P_3^v(x) = -P_5^v(x).$$

By Theorem 4.6, $\lambda_4 = P_4^v(\gamma)$ and $\lambda_5 = P_5^v(\gamma)$, and the assumption $\lambda^{(3)} \neq 0$ on $[0, 1]$ implies that

$$(X_1 P_3^v(\gamma))^2 + (X_2 P_3^v(\gamma))^2 \neq 0.$$

In particular, $\nabla P_3^v \neq 0$ and thus Σ is an analytic hypersurface of $\mathbb{R}^n$ in a neighborhood of the support of γ. Moreover, the distribution $\mathcal{D}(x) = \operatorname{span}\big\{X_1(x), X_2(x)\big\}$ is transversal to $T_x\Sigma$:

$$\dim(T_x\Sigma \cap \mathcal{D}(x)) = 1, \quad \text{for } x \in \gamma([0, 1]).$$

Inasmuch as γ is horizontal and by (6.65), this implies that γ is analytic. □

Remark 6.3. By Theorem 5 in [12, p. 59], Goh extremals as in Proposition 6.2 are locally length minimizing.

6.3. **A strictly abnormal curve.** We review Golé-Karidi's example [5] of a strictly abnormal curve in a stratified group.

Let G be a (free) nilpotent Lie group of rank 2 and step $s \geq 4$, as in the previous example. We fix a Hall basis $X_1, \dots, X_n$ and we highlight the first commutators

$$X_3 = [X_2, X_1], \quad X_4 = [X_3, X_1], \quad X_5 = [X_3, X_2], \quad X_6 = [[X_3, X_1], X_1]. \tag{6.66}$$

We look for a Goh extremal $\gamma : [0, 1] \to G = \mathbb{R}^n$. To this aim, let us consider the polynomial

$$P_3^v(x) = \sum_{\alpha \in \mathcal{I}} \phi_{3\alpha}(v) x^\alpha = \sum_{\alpha \in \mathcal{I}} \frac{(-1)^{|\alpha|}}{\alpha!} \sum_{j=1}^n c_{3\alpha}^j v_j x^\alpha.$$

We look for a $v \in \mathbb{R}^n$, $v \neq 0$, and for a curve γ such that $P_3^v(\gamma) = 0$. The identities $P_1^v(\gamma) = P_2^v(\gamma) = 0$ will then follow from Proposition 4.5, by the argument of Remark 4.12, provided that we can choose $v_1 = v_2 = 0$.

The coefficient of v_5 in the polynomial $P_3^v(x)$ is

$$\sum_{\alpha \in \mathcal{I}} \frac{(-1)^{|\alpha|}}{\alpha!} c_{3\alpha}^5 x^\alpha = -x_2,$$

because the unique multi-index $\alpha \in \mathcal{I}$ such that $c_{3\alpha}^5 \neq 0$ is $\alpha = (0, 1, 0, \dots, 0)$ and for this α we have $c_{3\alpha}^5 = 1$, see (6.66).

The coefficient of v_6 in the polynomial $P_3^v(x)$ is

$$\sum_{\alpha \in \mathcal{I}} \frac{(-1)^{|\alpha|}}{\alpha!} c_{3\alpha}^6 x^\alpha = \frac{1}{2} x_1^2,$$

because the unique multi-index $\alpha \in \mathcal{I}$ such that $c_{3\alpha}^6 \neq 0$ is $\alpha = (2, 0, \dots, 0)$ and for this α we have $c_{3\alpha}^6 = 1$, see again (6.66).

With $v_5 = v_6 = 1$ and with $v_j = 0$ otherwise, the polynomial $P_3^v(x)$ is

$$P_3^v(x) = \frac{1}{2} x_1^2 - x_2.$$

Now we clearly see that the horizontal curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ such that $\gamma_1(t) = t$ and $\gamma_2(t) = \frac{1}{2} t^2$ is a Goh extremal, by Corollary 4.11 and Proposition 4.5.

We prove that γ is a strictly abnormal curve. If γ were a normal extremal with dual curve λ, then by (2.8) and (2.15) we would have

$$\dot{\gamma}_j = h_j = -\langle \xi, X_j \rangle = \lambda_j, \quad j = 1, 2.$$

This yields $\lambda_1 = 1$ and $\lambda_2 = t$. Moreover, the system of equations (2.17) is

$$\dot{\lambda}_i = -\sum_{k=1}^{n} \left\{ c_{i1}^k + c_{i2}^k t \right\} \lambda_k, \quad i = 1, \dots, n.$$

When $i = 1$ we obtain $t\lambda_3 = 0$, whereas for $i = 2$ we obtain $\lambda_3 = -1$. This is a contradiction.

Remark 6.4. As proved in [5], the curve γ above is locally length minimizing. Indeed, with the previous choice of v, we have by Proposition 4.5 that

$$\lambda_5(t) = P_5^v(\gamma(t)) = -(X_2 P_3^v)(\gamma(t)) = 1\,.$$

Therefore, γ satisfies the assumptions in Proposition 6.2 and is locally length minimizing by [12].

Let us give here the complete formula for the polynomial $P_3^v(x)$ when G is a free nilpotent Lie group of rank 2 and step $s = 6$. This group is diffeomorphic to $\mathbb{R}^{23}$.

The polynomial is

$$\begin{aligned}P_3^v(x) &= v_3 - v_4x_1 - v_5x_2 + v_6\frac{x_1^2}{2} + v_7x_1x_2 + v_8\frac{x_2^2}{2} - v_9\frac{x_1^3}{6} - v_{10}\frac{x_1^2x_2}{2} - v_{11}\frac{x_1x_2^2}{2} - v_{12}\frac{x_2^3}{6}\\ &\quad + v_{13}(x_4 + x_1x_3) + v_{14}(x_5 + x_2x_3)\\ &\quad + v_{15}\frac{x_1^4}{24} + v_{16}\frac{x_1^3x_2}{6} + v_{17}\frac{x_1^2x_2^2}{4} + v_{18}\frac{x_1x_3^2}{6} + v_{19}\frac{x_2^4}{24}\\ &\quad + v_{20}(x_6 - \frac{x_1^2x_3}{2}) + v_{21}(x_7 - x_1x_2x_3) + v_{22}(x_8 - \frac{x_2^2x_3}{2}) + v_{23}(x_2x_4 - x_1x_5).\end{aligned}$$

The homogeneous degree of the polynomial is at most 4. The variables $x_9, \dots, x_{23}$ do not appear.

6.4. **Rank 3 and step 4.** Let G be the free nilpotent Lie group of rank 3 and step 4. This group is diffeomorphic to $\mathbb{R}^{32}$. By Corollary 4.11 and Remark 4.12, Goh extremals of G starting from 0 are precisely the horizontal curves γ contained in the algebraic set

$$\Sigma = \big\{x \in \mathbb{R}^{32} : P_4^v(x) = P_5^v(x) = P_6^v(x) = 0\big\},$$

for some $v \in \mathbb{R}^{32}$ such that $v_1 = \ldots = v_6 = 0$. We list the the polynomials defining this algebraic set:

$$\begin{aligned}P_4^v(x) &= -x_1v_7 - x_2v_8 - x_3v_9 + x_5v_{30} + x_6v_{31}\\ &\qquad + \frac{x_1^2}{2}v_{15} + x_1x_2v_{16} + x_1x_3v_{17} + \frac{x_2^2}{2}v_{18} + x_2x_3v_{19} + \frac{x_3^2}{2}v_{20}\\ P_5^v(x) &= -x_1v_{10} - x_2v_{11} - x_3v_{12} - x_4v_{30} + x_6v_{32}\\ &\qquad + \frac{x_1^2}{2}v_{21} + x_1x_2v_{22} + x_1x_3v_{23} + \frac{x_2^2}{2}v_{24} + x_2x_3v_{25} + \frac{x_3^2}{2}v_{26}\\ P_6^v(x) &= x_1(v_9 - v_{11}) - x_2v_{13} - x_3v_{14} - x_4v_{31} - x_5v_{32} + x_1^2(-\frac{1}{2}v_{17} + \frac{1}{2}v_{22} + v_{30})\\ &\qquad + x_1x_2(-v_{19} + v_{24} + v_{31}) + x_1x_3(-v_{20} + v_{25}) + \frac{x_3^2}{2}v_{29}.\end{aligned}$$

The set Σ is an intersection of quadrics.

When $v_7 = 1$ $v_{18} = 2$ and $v_j = 0$ otherwise, we have $P_4^v(x) = x_2^2 - x_1$, $P_5^v(x) = P_6^v(x) = 0$. Let $\phi : [0,1] \to \mathbb{R}$ be *any* Lipschitz function with $\phi(0) = 0$. The horizontal curve $\gamma : [0,1] \to \mathbb{R}^{32}$ such that $\gamma(0) = 0$, $\gamma_1(t) = t^2$, $\gamma_2(t) = t$ and $\gamma_3(t) = \phi(t)$ is a Goh extremal with purely Lipschitz regularity.

An interesting question raised by the referee is whether the curve γ constructed above is length minimizing (for any choice of the metric in the horizontal bundle). In its generality, the question is open. We can only give the following partial answer in the negative, related to the regularity of ϕ.

Assume that there exists a point $t_0 \in (0,1)$ such that the following limits exist and are different

$$\lim_{t\to 0^+} \frac{\phi(t_0+t) - \phi(t_0)}{t} \neq \lim_{t\to 0^-} \frac{\phi(t_0+t) - \phi(t_0)}{t}. \tag{6.67}$$

Then we claim that γ is not length minimizing (for any choice of the metric in the horizontal bundle).

We sketch the argument. We can assume that $\gamma(t_0) = 0$. We perform a blow-up of the curve γ, as in [8, Section 2]. The curve obtained by the blow-up of γ consists of two half-lines emanating from 0 and forming a corner, by (6.67). This curve will be contained in a Carnot group of rank 2 and step at most 4 (see Remark 2.5 in [8]). The length minimality of γ would imply the length minimality of the limit curve. However, by Example 4.6 in [8], length minimizing curves in Carnot groups of rank 2 and step at most 4 are C^∞ smooth. Thus (6.67) prevents γ to be length minimizing.

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