Dimensional regularization of nonlinear sigma models on a finite time interval

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Abstract

We extend dimensional regularization to the case of compact spaces. Contrary to previous regularization schemes employed for nonlinear sigma models on a finite time interval ("quantum mechanical path integrals in curved space") dimensional regularization requires only a covariant finite two-loop counterterm. This counterterm is nonvanishing and given by $\frac{1}{8}\hbar^2 R$. © 2000 Elsevier Science B.V. All rights reserved.

The regularization of nonlinear sigma models with higher-dimensional target spaces but on a one-dimensional worldline (quantum mechanical path integrals in curved space) has a long and confusing history. Early on, it was noticed by many authors that one obtains extra finite noncovariant counterterms of order $\hbar^2$ in the actions for the path integral if one goes from the hamiltonian to the lagrangian approach. These results were obtained in various ways: by using the Schroedinger equation for the transition element [1], Weyl ordering of the hamiltonian [2,3], canonical point transformations in path integrals with time slicing [3] or by making a change of variables at the operatorial level from field variables to collective coordinates and nonzero modes [4]. Also in standard four-dimensional gauge field theories such order $\hbar^2$ counterterms were found to be present if one chooses the Coulomb gauge [5,6] because gauge theories become nonlinear sigma models in this gauge.

Having fixed the counterterms in the action for the path integral has no meaning by itself. One must also specify the regularization scheme. Nonlinear sigma models contain double derivative couplings so they are superficially divergent at the one- and two-loop levels by power counting. In [7] it was noted that one should take into account the factor $\prod \sqrt{\det g_{ij}} = \exp\{\frac{1}{2}\delta(0) \int d\tau \, \mathrm{tr}(\ln g_{ij})\}$ in the measure; exponentiating this factor by means of "Lee–Yang ghosts" [8, 9], one obtains instead $\int d\tau (b^i g_{ij} c^j + a^i g_{ij} a^j)$ and the divergences canceled in the sums of diagrams. Different counterterms correspond to different regularization schemes for these individually divergent Feynman graphs. In fact, one first chooses a regularization scheme and then determines the corresponding counterterms.

The last decade two schemes were studied in detail [8–13]: (i) mode regularization (MR) [8,9] according to which the quantum fluctuations $q(\tau)$ around a background solution $x_{\mathrm{cl}}(\tau)$ are expanded in a Fourier

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sine series cut-off at mode N and all calculations are performed before letting N tend to infinity, and (ii) time slicing (TS) [10] according to which only N variables $q(\tau_1), \ldots, q(\tau_N)$ appear in the action at equally spaced points τ_i. In the latter case exact propagators were developed for finite N and the limit $N \to \infty$ could already be implemented in the Feynman rules themselves.

Of course, different regularization schemes give results which differ by finite local counterterms. In mode regularization these counterterms were fixed by requiring that the transition element $\langle x| \exp(-\frac{\tau}{\hbar}H)|y\rangle$ can also be obtained from a path integral with an action which differs from the naive action and which is fixed by requiring that the transition element satisfies the Schroedinger equation with the hamiltonian H. In time slicing one also obtains a path integral representation for $\langle x| \exp(-\frac{\tau}{\hbar}H)|y\rangle$ by inserting complete sets of position and momentum eigenstates, but here all steps are deductive and there is no need to impose the Schroedinger equation. Since Feynman graphs are regulated differently it comes as no surprise that also the counterterms are different. One finds

$$V_{\mathrm{MR}} = \frac{\hbar^2}{8} R - \frac{\hbar^2}{24} g^{ij} g^{kl} g_{mn} \Gamma^m_{ik} \Gamma^n_{jl},$$

$$V_{\mathrm{TS}} = \frac{\hbar^2}{8} R + \frac{\hbar^2}{8} g^{ij} \Gamma^l_{ik} \Gamma^k_{jl}. \tag{1}$$

With these counterterms, both schemes give the same answer corresponding to an hamiltonian H proportional to the covariant laplacian. Thus we see that a covariant quantum hamiltonian in the transition amplitude requires in both cases these noncovariant counterterms in the path integral to obtain the same covariant answer for the transition element. Numerous two- and three-loop calculations have confirmed these schemes [12–14]. Yet, it might simplify the calculations if a regularization scheme were found that only needs covariant counterterms. One might think of using geodesic time slicing, but the positions of the intermediate points $q(\tau_1), \ldots, q(\tau_N)$ would depend on the path considered and complexities overwhelm efforts in this direction.

The obvious choice for regularization scheme is, of course, dimensional regularization, but in the past we did not succeed is using this scheme due to the following problems:

(i) for all interesting applications one needs the action defined on a *finite* time interval. This requires a modification of the standard formulation of dimensional regularization such that it can be applied to a finite time interval. This is the main problem which we solve below. Once this problem is solved, the calculation of the transition element and anomalies follows relatively straightforwardly.

(ii) generalizing terms such as $\dot{\phi}\dot{\phi}\dot{\phi}\dot{\phi}$ in the Feynman graphs; one must decide how to write them in n dimensions (as $\partial_\mu\phi\partial_\mu\phi\partial_\nu\phi\partial_\nu\phi$ or $\partial_\mu\phi\partial_\nu\phi\partial_\mu\phi\partial_\nu\phi$ for example). This problem has a simple solution [15] which we use below: one starts with the action $\partial_\mu\phi\partial^\mu\phi$ in n dimensions and then all Lorentz indices μ, ν in all contractions are unambiguous.

Recently, as a test project, we considered nonlinear sigma models on a infinite time interval. We were inspired to return to our attempts to use dimensional regularization for nonlinear sigma models by recent Letters by Kleinert and Chervyakov [15] who studied a nonlinear sigma model with a one-dimensional target space, and considered the same problem as Gervais and Jevicki [3], but using ordinary dimensional regularization. They studied a free particle in a box of length d by replacing the confining box by a smooth convex potential $V(x) = \frac{1}{2}\frac{m^2}{g}\tan^2(\sqrt{g}x)$ which grows to infinity near the walls ($x = \pm\frac{d}{2}$). The field redefinition $x \to \varphi = \frac{1}{\sqrt{g}}\tan(\sqrt{g}x)$ was made to obtain a nonlinear sigma model with a mass term $\frac{1}{2}m^2\varphi^2$. Using dimensional regularization it was found that both models gave the same results "so that there is no need for an artificial potential term of order $\hbar^2$ called for by previous authors" [15]. Of course, this refers to possible noncovariant counterterms since a one-dimensional model cannot test counterterms proportional to R. However, the interpretation "... artificial potential term ..." of the results of [1–6,8–14] may be misleading. In general counterterms up to order $\hbar^2$ are needed in any given regularization scheme as they mirror in this context the ordering ambiguities present in the canonical approach to quantum mechanics. It would be wrong to omit them. In the regularization schemes discussed before, the order $\hbar^2$ counterterms, including the noncovariant ones, are present and are definitely correct. We shall find later on with our modified dimensional regularization scheme on a finite time interval that no noncovariant counterterms

are present, as in [15]. One should view this as a property of a particular regularization scheme, in which the coefficients of the possible noncovariant counterterms happen to vanish.

The fact that no noncovariant counterterms were needed for infinite time intervals in the one-dimensional model of Kleinert and Chervyakov and in the D-dimensional model of [16] suggested to us to study dimensional regularization applied to general nonlinear sigma models. For an infinite time interval we indeed recently found that one only needs a covariant counterterm $\frac{1}{8}\hbar^2 R$ [16], but for massless nonlinear sigma models one must add by hand a noncovariant mass term $\frac{1}{2}m^2 x^2$ in order to regulate infrared divergences, and the result depends on m. For the really interesting applications (to anomalies and correlation functions of quantum field theories) one needs a finite time interval. In this case there are no infrared divergences and covariance can be maintained. In this letter we shall extend the method of dimensional regularization used in [15] to a finite time interval and show that for nonlinear sigma models one needs only a covariant counterterm $V_{DR} = \frac{1}{8}\hbar^2 R$.

The model we consider is given by the following action

$$S[x^i] = \int_{-1}^{0} d\tau \, \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j. \tag{2}$$

Decomposing the paths $x^i(\tau)$ into a classical part $x_{cl}^i(\tau)$ satisfying suitable boundary conditions, and quantum fluctuations $q^i(\tau)$ which vanish at the boundary $(q^i(-1) = q^i(0) = 0)$ and decomposing the lagrangian into a free part $\frac{1}{2}g_{ij}(0)\dot{q}^i \dot{q}^j$ plus interactions, the propagator becomes formally

$$\langle 0|Tq^i(\tau)q^j(\sigma)|0\rangle = -g^{ij}(0)\Delta(\tau,\sigma),$$

$$\Delta(\tau,\sigma) = \sum_{n=1}^{\infty}\left[-\frac{2}{\pi^2 n^2}\sin(\pi n\tau)\sin(\pi n\sigma)\right]$$

$$= \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau). \tag{3}$$

In MR one truncates the sum to N modes and sends $N \to \infty$ at the end of the calculations, while in TS one uses $\frac{\partial}{\partial\sigma}\theta(\sigma-\tau) = \delta(\sigma-\tau)$ where $\delta(\sigma-\tau)$ acts like a Kronecker delta, implying for example that $\iint \delta(\sigma-\tau)\theta(\sigma-\tau)\theta(\sigma-\tau) = \frac{1}{4}$ (and not equal to $\frac{1}{3}$ as one might perhaps naively expect from replacing

the integrand with $\frac{1}{3}\frac{\partial}{\partial\sigma}\theta^3(\sigma-\tau)$. In general products of distributions are ambiguous, but going back to time slicing they are well defined). With these prescriptions one can unambiguously compute loop graphs.

To extend dimensional regularization to a compact time interval $-1 \leqslant \tau \leqslant 0$ we introduce D extra infinite dimensions $\mathbf{t} = (t^1, \ldots, t^D)$, and take the limit $D \to 0$ at the end, as in standard dimensional regularization [17]. We also require translational invariance in the extra dimensions. As action in the $D+1$ dimensions we take

$$S[x,a,b,c]$$

$$= \int d^{D+1}t\left[\frac{1}{2}g_{ij}(x)\left(\partial_\mu x^i \partial^\mu x^j + a^i a^j + b^i c^j\right)\right.$$

$$\left. + V_{DR}(x)\right], \tag{4}$$

where V_{DR} is the counterterm in dimensional regularization, $t^\mu = (\tau, \mathbf{t})$ with $\mu = 0, 1, \ldots, D$ and $d^{D+1}t = d\tau \, d^D\mathbf{t}$. The propagators for this action read

$$\Delta(t,s) = \int \frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n=1}^{\infty}\frac{-2}{(\pi n)^2 + \mathbf{k}^2}\sin(\pi n\tau)$$

$$\times \sin(\pi n\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}. \tag{5}$$

The coordinates $\mathbf{t}$ and $\mathbf{s}$ for the extra D dimensions run from $-\infty$ to ∞, and also the D continuous momenta $\mathbf{k}$ run from $-\infty$ to ∞. This propagator satisfies the Green equation [1]

$$\left(\partial_\tau^2 + \partial_{\mathbf{t}}^2\right)\Delta(t,s) = \delta^{D+1}(s,t) = \delta(\tau,\sigma)\delta^D(\mathbf{t}-\mathbf{s}), \tag{6}$$

where $\delta(\tau,\sigma) = \sum_{n=1}^{\infty}2\sin(\pi n\tau)\sin(\pi n\sigma)$ is the Dirac delta on the space of functions which vanish at $\tau, \sigma = -1, 0$.

In addition to the point particle coordinates $x^i(t)$ there are ghosts: one real commuting ghost $a^i(t)$ and two real anticommuting ghosts $b^i(t)$ and $c^i(t)$ [8,9]. They appear in the action in the combination $\partial_\mu x^i(t)\partial^\mu x^j(t) + a^i(t)a^j(t) + b^i(t)c^j(t)$, and have

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[1] An alternative ansatz is suggested by writing the sines in (5) as $(\exp i\pi n(\tau-\sigma) - \exp i\pi n(\tau+\sigma))$ and modifying it into $(\exp ik_\mu(t^\mu - s^\mu) - \exp ik_\mu(t^\mu + s^\mu))$, where $k_\mu = (\pi n, \mathbf{k})$. This last expression has been used for finite temperature physics (J. Zinn-Justin, private communication, unpublished), but is not suitable for our purposes as it does not satisfy the Green equation.

propagators

$$\langle 0|Ta^i(t)a^j(s)|0\rangle = g^{ij}(0)\Delta_{\mathrm{gh}}(t,s),$$

$$\langle 0|Tb^i(t)c^j(s)|0\rangle = -2g^{ij}(0)\Delta_{\mathrm{gh}}(t,s),$$

$$\Delta_{\mathrm{gh}}(t,s) = \int \frac{d^D\mathbf{k}}{(2\pi)^D} \sum_{n=1}^{\infty} 2\sin(\pi n\tau)$$
$$\times \sin(\pi n\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}$$
$$= \delta^{D+1}(t,s) = \delta(\tau,\sigma)\delta^D(\mathbf{t}-\mathbf{s}). \qquad (7)$$

These ghosts arise after one integrates over the momenta in the path integral, and contribute to higher loops in exactly the same way as ghosts in gauge theories. Although their propagators are formally equal to delta functions which vanish in standard dimensional regularization, they do contribute in our case because there are no infrared divergences, so that the usual cancellation between infrared and ultraviolet divergences in $\int d^{D+1}k = 0$ does not take place.

We can now calculate loop graphs treating the D-dimensional momenta as in ordinary dimensional regularization, and performing the sums over n as in finite temperature physics. We compute all two-loops graphs which contribute to the vacuum energy. For this case we have $x_{\mathrm{cl}}^i(\tau) = 0$. We shall give details of the calculations in an example below, but first summarize our result in the Table 1, where we give the results for each of the diagrams which contribute to the two-loop vacuum energy. In the last column we quote the tensor structure of the graphs with the shorthand notation $\partial^2 g \equiv g^{ij}g^{kl}\partial_k\partial_l g_{ij}$, $\partial^j g_j \equiv g^{ik}g^{jl}\partial_k\partial_l g_{ij}$, $\partial_k g \equiv g^{ij}\partial_k g_{ij}$ and $g_k \equiv g^{ij}\partial_i g_{jk}$. We record the results for time slicing, mode regularization and our version of dimensional regularization, respectively.[2]

It is clear that there are only differences for B_3 and B_4. The computations in DR are done by using partial integration to bring all integrals in a form that can unambiguously be computed at $D \to 0$. The various manipulation are justified in dimensional regularization. In particular, partial integration is always allowed in the extra D dimension because of momentum con-

servation while it can be done in the finite time interval whenever there is an explicit function vanishing at the boundary (e.g., the propagator of the coordinates without derivatives). Let us use the notation $\frac{\partial}{\partial t^\mu}\Delta(t,s) = {}_\mu\Delta(t,s)$ and $\frac{\partial}{\partial s^\mu}\Delta(t,s) = \Delta_\mu(t,s)$ so that Eq. (6) yields ${}_{\mu\mu}\Delta(t,s) = \Delta_{\mathrm{gh}}(t,s) = \delta^{D+1}(t,s)$. The rule for contracting which indices with which indices follows from the action in (4). We find then for B_4 in dimensional regularization

$$B_4(\mathrm{DR}) = \int_{-1}^{0}d\tau \int_{-1}^{0}d\sigma\, {}^{\bullet}\Delta(\Delta^{\bullet})\,{}^{\bullet}_{\bullet}\Delta$$

$$\longrightarrow \int d^{D+1}t \int d^{D+1}s ({}_\mu\Delta)(\Delta_\nu)({}_\mu\Delta_\nu)$$

$$= \int d^{D+1}t \int d^{D+1}s ({}_\mu\Delta)_\mu\left(\tfrac{1}{2}(\Delta_\nu)^2\right)$$

$$= -\frac{1}{2}\int d^{D+1}t \int d^{D+1}s ({}_{\mu\mu}\Delta)(\Delta_\nu)^2$$

$$= -\frac{1}{2}\int d^{D+1}t \int d^{D+1}s \delta^{D+1}(t,s)(\Delta_\nu)^2$$

$$= -\frac{1}{2}\int d^{D+1}t (\Delta_\nu)^2\big|_t$$

$$\longrightarrow -\frac{1}{2}\int_{-1}^{0}d\tau\, {}^{\bullet}\Delta^2\big|_\tau = -\frac{1}{24}, \qquad (8)$$

where the symbol $|_\tau$ means that one should set $\sigma = \tau$. Similarly

$$B_3(\mathrm{DR}) = \int_{-1}^{0}d\tau \int_{-1}^{0}d\sigma\, \Delta\left({}^{\bullet}_{\bullet}\Delta^{\bullet 2} - \Delta_{\mathrm{gh}}^2\right)$$

$$\longrightarrow \int d^{D+1}t \int d^{D+1}s\, \Delta\big(({}_\mu\Delta_\nu)({}_\mu\Delta_\nu)$$
$$- ({}_{\mu\mu}\Delta)({}_{\nu\nu}\Delta)\big)$$

$$= \int d^{D+1}t \int d^{D+1}s \big(-({}_\mu\Delta)(\Delta_\nu)({}_\mu\Delta_\nu)$$
$$-\Delta(\Delta_\nu)({}_{\mu\mu}\Delta_\nu) + ({}_\mu\Delta)({}_\mu\Delta)({}_{\nu\nu}\Delta)$$
$$+\Delta({}_\mu\Delta)({}_{\mu\nu\nu}\Delta)\big)$$

$$= \int d^{D+1}t \int d^{D+1}s \big(-({}_\mu\Delta)(\Delta_\nu)({}_\mu\Delta_\nu)$$
$$+({}_\mu\Delta)({}_\mu\Delta)({}_{\nu\nu}\Delta)\big)$$

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[2] To check the statement in the caption of Table 1 one may use that $R = \partial^2 g - \partial^j g_j - \frac{3}{4}(\partial_k g_{ij})^2 + \frac{1}{2}(\partial_i g_{jk})\partial_j g_{ik} + \frac{1}{4}(\partial_j g)^2 - (\partial_j g)g^j + g_j^2$ and the $\Gamma\Gamma$ terms for TS are given by $-\frac{1}{8}\Gamma\Gamma = \frac{1}{32}(\partial_i g_{jk})^2 - \frac{1}{16}(\partial_i g_{jk})(\partial_j g_{ik})$ while for MR one has $\frac{1}{24}\Gamma\Gamma = \frac{1}{32}(\partial_i g_{jk})^2 - \frac{1}{48}(\partial_i g_{jk})(\partial_j g_{ik})$.

Table 1
2-loop results with time slicing (TS), mode regularization (MR) and dimensional regularization (DR). Dots denote derivatives while hatched lined denote ghosts. For each scheme the sum of all graphs and the counterterms is $-\frac{1}{12}R$

Integral	Results			Diagram	Tensor structure
	TS	MR	DR		
$A_1 \equiv \int \Delta\vert_\tau (\bullet\!\Delta^\bullet + \Delta_{\mathrm{gh}})\vert_\tau$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$		$-\frac{1}{4}\partial^2 g$
$A_2 \equiv \int (\bullet\!\Delta\vert_\tau)^2$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$		$-\frac{1}{2}\partial^j g_j$
$B_3 \equiv \int\int \Delta(\bullet\!\Delta^{\bullet\,2} - \Delta_{\mathrm{gh}}^2)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$		$-\frac{1}{4}(\partial_i g_{jk})^2$
$B_4 \equiv \int\int (\bullet\!\Delta^\bullet)\Delta^\bullet (\bullet\!\Delta)$	$-\frac{1}{6}$	$-\frac{1}{12}$	$-\frac{1}{24}$		$-\frac{1}{2}(\partial_i g_{jk})\partial_j g_{ik}$
$B_5 \equiv \int\int \bullet\!\Delta\vert_\tau (\bullet\!\Delta^\bullet)\Delta^\bullet\vert_\sigma$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$		$-\frac{1}{2}g_j^2$
$B_2 \equiv \int\int (\bullet\!\Delta^\bullet + \Delta_{\mathrm{gh}})\vert_\tau \Delta^\bullet (\bullet\!\Delta^\bullet\vert_\sigma)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$		$-\frac{1}{2}(\partial_j g)g^j$
$B_1 \equiv \int\int (\bullet\!\Delta^\bullet + \Delta_{\mathrm{gh}})\vert_\tau \Delta(\bullet\!\Delta^\bullet + \Delta_{\mathrm{gh}})\vert_\sigma$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$		$-\frac{1}{8}(\partial_j g)^2$

$$= -B_4 + \int d^{D+1}t \int d^{D+1}s \, (_\mu\Delta)^2 \delta^{D+1}(t,s)$$

$$= -B_4 + \int d^{D+1}t \, (_\mu\Delta)^2\vert_t = \frac{1}{8}. \tag{9}$$

We used the identity $(_{\mu\mu}\Delta_\nu) = (\Delta_{\nu\mu\mu})$ obvious from (5). Moreover to compute diagrams like A_1 it is useful to use an identity which can be quickly derived in $D+1$ dimensions: recalling that we denote with a subscript 0 the derivative along the origi-

nal compact time direction one has $((_\mu\Delta_\mu)(t,s) + (_{\mu\mu}\Delta)(t,s))\vert_{t=s} =_0 [(_0\Delta(t,s))\vert_{t=s}]$.

In the calculation of $B_4(\mathrm{DR})$ all steps are as in ordinary dimensional regularization, though one may question the following step

$$\int d^{D+1}t \int d^{D+1}s \, (_{\mu\mu}\Delta)(\Delta_\nu)^2$$

$$= \int d^{D+1}t \, (\Delta_\nu)^2\vert_t, \tag{10}$$

where we used that formally $_{\mu\mu}\Delta(t,s) = \delta^{D+1}(t,s)$. The symbol $\delta^{D+1}(t,s)$ is an analytically continued delta function, and it is not clear that one may treat that as a regular delta function which is defined for D integer and positive. However, we recall that the correct prescription of dimensional regularization is to carry out all integrals over spacetime at integer dimensions before analytically continuing the momenta to D dimensions. Using this we can show by explicit calculation that (10) is correct. The right-hand side of Eq. (10) reads

$$\frac{1}{4} \int d\sigma \, d^D\mathbf{s} \sum_{m_1 \neq 0} \int \frac{d^D\mathbf{q}_1}{(2\pi)^D} \frac{(1 - e^{2\pi i m_1 \sigma})}{(\pi m_1)^2 + \mathbf{q}_1^2}$$

$$\times \sum_{m_2 \neq 0} \int \frac{d^D\mathbf{q}_2}{(2\pi)^D} \frac{(1 - e^{2\pi i m_2 \sigma})}{(\pi m_2)^2 + \mathbf{q}_2^2}$$

$$\times (-\mathbf{q}_1 \cdot \mathbf{q}_2 - \pi^2 m_1 m_2)I, \qquad (11)$$

where I is unity. The integral over $\mathbf{s}$ gives the volume of the internal space which can be factored out, while the integrals over $\mathbf{q}_1$ and $\mathbf{q}_2$ are treated with ordinary dimensional regularization which makes the sums over m_1 and m_2 finite for sufficiently large negative D. Thus sums over modes of the finite time segment are made finite by dimensional regularization in the internal space. For the left-hand side of Eq. (10), one obtains a similar result after extracting the exponents containing $(\tau - \sigma)$ from each of the three propagators, and performing the integrals over $\mathbf{t}$ and τ, but now I is nontrivial

$$I = \frac{1}{2} \sum_{n \neq 0} (1 - e^{2\pi i n \sigma}) \left[\delta_{m_1 + m_2 + n, 0} \right.$$

$$\left. + \sum_l \delta_{m_1 + m_2 + n - (2l+1), 0} \frac{2e^{-(2l+1)\pi i \sigma}}{i\pi(2l+1)} \right]. \qquad (12)$$

We used that $\int_{-1}^0 d\tau \, e^{i\pi m \tau} = \delta_{m,0} + \sum_l \frac{2}{i\pi(2l+1)} \times \delta_{m-(2l+1),0}$ for any integer m. We can extend the sum over n to include $n = 0$. Performing the sum over n we obtain conditionally convergent series

$$I = \frac{1}{2}(1 - e^{-2\pi i(m_1 + m_2)\sigma})$$

$$+ \frac{1}{i\pi}(S(-\sigma) - e^{-2\pi i(m_1 + m_2)\sigma} S(\sigma)), \qquad (13)$$

where $S(\sigma) = \sum_l \frac{e^{(2l+1)i\pi\sigma}}{2l+1}$. The function $S(\sigma)$ is equal to $-i\pi/2$ for $-1 \leqslant \sigma \leqslant 0$, and $S(-\sigma) = -S(\sigma)$, hence I equals unity. This proves (10).

It may be useful to compare the calculations in DR with those using MR and TS. Consider the integral $\int_{-1}^0 \int_{-1}^0 (\Delta^{\bullet\bullet})(\Delta^\bullet)(\Delta)$. In DR we wrote the integrand as $(_\mu\Delta)(\Delta_\nu)(_\mu\Delta_\nu) = (_\mu\Delta)_\mu(\frac{1}{2}(\Delta_\nu)^2)$, and partially integrated the second μ derivative to obtain

$$\int d^{D+1}t \int d^{D+1}s \left(-\frac{1}{2}\right)(_{\mu\mu}\Delta)(\Delta_\nu)^2$$

$$= -\frac{1}{2} \int d^{D+1}t \int d^{D+1}s \, \delta^{D+1}(t,s)(\Delta_\nu)^2$$

$$= -\frac{1}{2} \int d^{D+1}t \, (\Delta_\nu)^2|_t = -\frac{1}{24}. \qquad (14)$$

In MR one can perform similar steps to arrive at $\int_{-1}^0 \int_{-1}^0 (-\frac{1}{2})(^{\bullet\bullet}\Delta)(\Delta^\bullet)^2$ but we do not set $^{\bullet\bullet}\Delta = \delta(\tau - \sigma)$ because in MR both $\delta(\tau - \sigma)$ and $\theta(\tau - \sigma)$ in $\Delta^\bullet$ are smeared so that one would need to work out integrals of products of such MR regulated distributions. Instead we use the symmetry of $^{\bullet\bullet}\Delta$ in τ and σ to replace $^{\bullet\bullet}\Delta$ by $\Delta^{\bullet\bullet}$ and obtain then

$$B_4(\text{MR}) = -\frac{1}{2} \int\limits_{-1}^0 d\tau \int\limits_{-1}^0 d\sigma \frac{1}{3} \partial_\sigma (\Delta^\bullet)^3 = -\frac{1}{12}, \qquad (15)$$

where we used that $\Delta^\bullet = \tau + \theta(\sigma - \tau)$. In DR this procedure is not possible when μ is different from ν. In the TS scheme one obtains directly without partial integration

$$B_4(\text{TS}) = \int\limits_{-1}^0 d\tau \int\limits_{-1}^0 d\sigma [\tau + \theta(\sigma - \tau)][\sigma + \theta(\tau - \sigma)]$$

$$\times [1 - \delta(\tau - s)] = -\frac{1}{6}, \qquad (16)$$

where we used that with TS one has

$$\int\limits_{-1}^0 \int\limits_{-1}^0 \theta(\tau - \sigma)\theta(\sigma - \tau)\delta(\tau - \sigma) = \frac{1}{4}$$

while

$$\int\limits_{-1}^0 \int\limits_{-1}^0 \delta(\tau - \sigma)\tau\theta(\tau - \sigma) = -\frac{1}{4}.$$

Summarizing: we have solved the problem of how to modify dimensional regularization such that it can be applied to finite time intervals. Our extension of dimensional regularization keeps translational invariance in the extra D dimensions. One can now regulate nonlinear sigma models on a finite time interval by using dimensional regularization. We were inspired by recent Letters [15] which applied standard dimensional regularization on an infinite time interval to a model with a mass term. Such a mass term is also needed in more general models on an infinite time interval [16] to regularize infrared divergences, but it breaks general covariance in target space. Our method applies both to massless and massive models and does not break general covariance. With this new method no noncovariant finite counterterms are needed. Other regularization schemes need such noncovariant counterterms, so from this perspective dimensional regularization is just another regularization scheme but with a simpler set of counterterms. Certainly all regularization schemes, each with its own counterterms, yield the same results. In this letter we tested this for the two-loop vacuum energy. Although the contributions to individual diagrams are different, we obtained the same results for the two-loop vacuum energy with this dimensional regularization and its covariant counterterms, $V_{\mathrm{DR}} = \frac{\hbar^2}{8} R$, as previously obtained with mode regularization and time slicing with their noncovariant counterterms. Presently, anomalies in higher dimensions are being calculated and we expect that our scheme will simplify such calculations. Also applications to scattering amplitudes may benefit from this scheme.

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