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UNIVERSITY STUDENTS AT WORK WITH MATHEMATICAL MACHINES TO TRACE CONICS

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This paper aims to investigate the way past experience with some tools to draw conics becomes part of the experience of designing a new drawer. In particular, it centres on the thinking processes of a group of university students who have the following task: to design a hyperbola drawer. The analysis is carried out using the perspectives of transfer of learning and instrumental approach, and focuses on utilization schemes and the interplay between scientific and technological aspects.

INTRODUCTION

I love analogies a lot, considering them as my very reliable masters, experts of all the mysteries of nature; in geometry, one has to pay attention to them, especially when they enclose –even if with expressions that seem absurd– infinite cases intermediate between their extremes (and a centre), and thus put before our eyes, in full light, the true essence of an object. Analogy also helped me a lot to draw conic sections. From reading Propositions 51 and 52 [*concerning the metric properties involving the foci*] from Apollonius’s Third Book, one can easily see how to trace ellipses and hyperbolas: these tracings can be made with a thread. [...] I regretted that for long I wasn’t able to describe the parabola in the same way. At the end, the analogy revealed to me that to trace this curve is not much more difficult (and the geometric theory does confirm it). (Kepler, 1604, Italian version, pp. 3-5; English translation of the authors)

This brief excerpt from the Italian version of the text *Ad Vitellionem paralipomena* shows how much relevant analogy was for Kepler in geometrical thinking. Kepler’s problem was that of drawing conic sections by means of a thread, from Apollonius’s Propositions. His use of analogy in the case of the parabola is strikingly meaningful for us, due to attention deserved to analogy and analogical reasoning by the literature in Mathematics Education (English, 1997). However, we do not want to adopt a specific meaning for analogy over the many considered in the research. Instead, we will refer to it in a naïve manner, as Kepler. We are interested in the spontaneous ways in which elements of situations that have been faced before are recalled in a new situation. This is exactly what Kepler makes. When referring to analogy to think of the construction of a new machine for the parabola, he applies knowledge acquired about hyperbola and ellipse.

To study the spontaneous ways said above, we present here a case study about some university students that are asked to construct a drawer to trace hyperbolas, after they

have investigated the functioning of other drawers for conics, using them concretely. Such drawers are for us mathematical machines. A mathematical machine is defined as a tool that forces a point to follow a trajectory or to be transformed on the basis of a given law (Maschietto, 2005; Bartolini Bussi & Maschietto, 2008). In this paper, we will centre on how the past experience of the students becomes part of the new experience in which the machine is no longer the starting point but the end point of the task. In so doing, we consider the perspectives about transfer of learning and instrumental approach, and we look at phenomena of transfer in terms of schemes that depend on the type of task.

TRANSFER OF LEARNING AND UTILIZATION SCHEMES

Transfer of learning

The notion of *transfer of learning* has been recently studied in a new perspective that integrates phenomena of cognition, emotion and bodily experience (Nemirovsky, 2011). Drawing on past studies about transfer, Nemirovsky considers transfer of learning as relevant when “it is immersed in the context of common and experiential phenomena of learning”. He defines transfer in terms of experience:

I see transfer as part of the study of how one experience becomes part of another. People can all sense that experiences do become part of other experiences. It is also clear, I think, that such participation can be lived in numerous ways, some of which I suggest calling “transfer”. (p. 309)

From this point of view, transfer of learning has a dynamic meaning that overcomes any operational definition, depending on the direct and participative engagement of learners. However, since the realm of ways in which an experience becomes part of another is wide, a growing number of studies would furnish information about the features of transfer of learning that characterize it within such realm and about the different ways in which it occurs. The point here is not “to ascertain mechanisms of transfer but to elucidate those experiences that are amenable to being described as transfer of learning.” (*ibid.*, p. 334). In this perspective, transfer of learning can be interpreted as strictly related to the subjective feelings of the subjects, instead of being ascribed to something stipulated or secured *a priori*.

In our context, we see the idea of transfer of learning as possibly related to analogy *à la Kepler*. In fact, Kepler uses his previous experience with the ellipse and the hyperbola to solve the problem of finding a way to trace a parabola with a thread. We may think of him as if he were thinking of a machine with tightened thread to obtain the tracing, his use of analogy being reasoning by continuity and extension from past experience. He transfers knowledge acquired about the other conics in the new task, in order to describe the parabola. Regarding our university students that have to face a similar task (thinking of a new machine), we may then ask: How does their past experience with the other conics and drawers become part of the new experience?

Utilization schemes

We adopt transfer of learning as a perspective to analyse how our students solve the design-like problem of thinking of a new machine, after they had concretely used other machines. The presence of artefacts (physical in past activities, potential in the new activity –being its goal) strongly influences the task assigned to the students. Concerning this influence, we see as interesting the notion of *utilization scheme* as it is studied by cognitive ergonomics research in the analysis of human action mediated by tools (Vérillon & Rabardel, 1995; Rabardel, 2002). The instrumental approach underlines that the use of an artefact to solve a specific task implies to activate certain utilization schemes. Rabardel (2002) defines such schemes as “stable and structured elements in the user’s activities and actions” (p.65). The approach pays attention to the distinction between artefact and instrument: the former is a material or symbolic object, constructed by human beings; the latter is a mixed entity made of the artefact and of associated utilization schemes. The schemes result from a personal construction of the subject or from the appropriation of social schemes already formed outside of him. They are related to accomplishing a specific task on the one hand and, on the other hand, to managing characteristics of the artefact that are strictly related to the given tasks.

A significant element for our context depends on the fact that the task of constructing a tool can be considered between technological and scientific activities, as Weisser (2005) highlights in the field of technology education. In particular, the machine that the students have to think of has the double status of artefact and instrument during the solution phase. Following Rabardel, the process of creating an instrument (that is, the instrumental genesis) has two components: instrumentation, subject-oriented and leading to the emergence and evolution of utilization schemes; instrumentalization, object-oriented and concerning the emergence and evolution of the instrument’s artefact component. Speaking of utilization schemes, Rabardel highlights that they are “the object of more or less formalized transmissions and transfers” (*ibid.*, p.84).

With respect to the question of how students’ past experience with the other conics and drawers becomes part of the new experience, we see as fundamental the role of utilization schemes. So, we may ask: Do the students transfer utilization schemes previously formed? How do acquired schemes shape new schemes for the new machine in a new kind of task (to use vs. to construct: the drawer is no longer the starting point but the end point of the activity)?

THE ACTIVITY

The activity is part of a university course on Elementary Mathematics from an Advanced Standpoint. The course can be attended at the second year of the Master’s Degree in Mathematics; its specific topic considers conic sections and their properties, since Greek Mathematics.

Regarding work methodology, the construct of *mathematics laboratory* is the basis of the course's activities. The mathematics laboratory is meant as a structured set of activities aimed to the construction of meanings for mathematical objects (Anichini *et al.*, 2004). It is defined as a space of interaction and collaboration, in which the tasks are addressed and solved using (physical and digital) tools.

Specifically, our students dealt with types of drawers for conics that use a tightened thread. These drawers base on the definition of conic sections as loci of points. Their essential elements are: a wooden flat surface; one pin/two pins for the focus/foci; a thread to materialize distance between each focus and any point it is stretched from. A pen that moves while stretching the thread draws a curve, its point belonging to the curve. For the ellipse, the drawer satisfies the gardener's method. For the parabola, see Figure 1: F is the focus, P is the generic point of the curve.

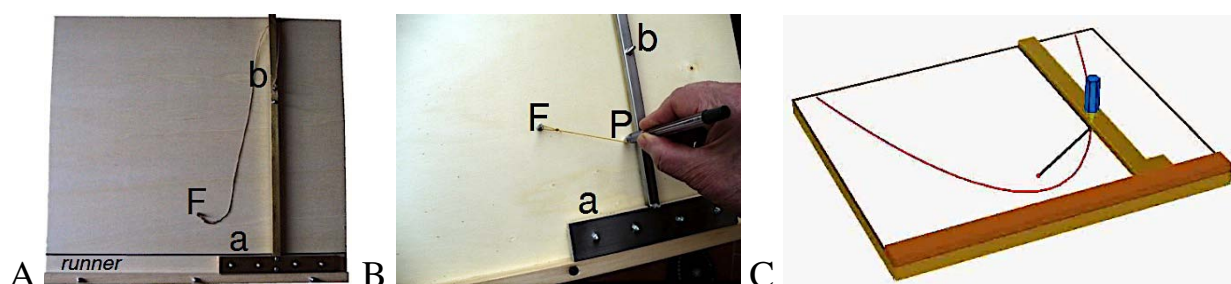


Figure 1. Parabola drawer with tightened thread

In five laboratory sessions the eight university students met five machines. They were divided into two groups, in which one of them had the role of observer. In the first three sessions the students have worked with: the Cavalieri's drawer for parabola, the parabola drawer and the ellipsograph both with tightened thread. The three machines were explored through three phases: to describe their physical structure, parts and spatial relationships; to centre on the product of the machine; to produce conjectures and proofs on that product. An individual report and a collective discussion lead by the teacher (one of the authors) concluded the activity. In this study, we focus on the fourth activity, whose task is completely different from the previous ones, asking the students to imagine how a machine with tightened thread for hyperbola is made. The students have the curve as starting point, but not the machine to trace that curve.

The investigation of how they face this situation is the core of the paper. Data comes from the video-recording of one group, and from the students' written reports and the observer's notes. Our interests are on the way elements from the previous activities with drawers for parabola and ellipse are transferred in the new situation, and on how they originate new ways of writing, new ways of drawing, new ways of thinking.

ANALYSIS AND DISCUSSION

We present here some pieces of the work of one group (we label the group A and the students A1, A2, A3 and A4, the observer). Like for the other group, group B (Ferrara

& Maschietto, 2013), four phases can be captured, the first three depending on the means that the students use (paper and pencil; a wooden plan with two pins and one or two threads; a rod). For the sake of space constraints, we only focus on the phases 1 and 2 of group A's work.

In what follows, past experience is usually recalled by linguistic expressions of the kind: "let's think of how we did the other time", "last time". In addition, depending on the moment, technological aspects or scientific aspects can be at play, as well as gestures of usage can be produced (see Ferrara & Maschietto, *ibid.*). We will make explicit reference to utilization schemes and to these other aspects when necessary.

1) Work with paper and pencil on graphical representations

Group A begins its work with the metric definition of hyperbola accompanied by the algebraic expression and the standard graphical representation (with generic point P, foci, vertices, etc.). Immediately, the students recall their previous experience with the ellipsograph to detect first components of the new machine:

A1: Let's think of how we did the other time... we have two foci, two foci that were fixed [*pointing her fingers to the foci*]; maybe we could think that there are two pins

The students transfer the artefact component of the instrument ellipsograph: they look for those components of the machine that materialize elements of the definition of the curve (the pins for the foci, the tightened thread for the distance focus-point of the curve). The students think of the thread as a given part of the new machine, that is, as one of its artefact components. They also search for a link between the length of the thread and the constant k in the formula $|PF_1 - PF_2| = k$. However, the relationship (length of the thread, constant k) coming from the ellipsograph cannot be transferred as such in this case. In effect, for the ellipse, the length of the thread represents the sum of the two distances focus-point of the curve.

Resting on the formula $|PF_1 - PF_2| = k = 2a$, the students point out the connection between the constant difference and the distance between vertices (apparent on the graphical representation). This marks a new beginning: the established link (scientific aspect) affects the idea of the machine (technological aspect), because it entails the understanding that the components translated from the ellipsograph are not useful for the new machine. So, the parabola drawer is in turn recalled:

A3: You cannot do many things just using the two foci

A1: But for the parabola drawer, we also had the rod (b in Figure 1A and 1B)

The parabola drawer intervenes in thinking of the technological aspects of the new machine. A3 recalls through graphical representations utilization schemes for the rod, looking for their application in the present case, without success. Attention is drawn back to the thread as element that has to incorporate the condition about distances. Another utilization scheme associated to the parabola drawer is recovered: when the

thread is kept tightened, the equivalence of the distances point-focus and point-directrix is assured ($d(P, F)$ and $d(P, a)$ in Figure 1C). This exploration brings the group to conclude that the length of the thread does not count for the hyperbola and that constant k must be looked for in another way (scientific and technological aspects). Reference to the parabola drawer fails to help the students.

Recognizing the presence of material elements in the group discussions, the teacher now furnishes the students with a wooden plan with two pins and a thread. Group A begins to work with one thread, but then asks for a second thread.

2) Work with threads

The idea that guides the group action is to represent the parameter of the curve using the thread. For this reason, the students choose a certain segment of the thread (that corresponds to $2a$ in the formula), they tie one of its ends to a pin and they try to handle it (Figure 2A). A1 tests with the thread segment various configurations that all assure to satisfy the definition of hyperbola. In other terms, A1 begins a sort of process of instrumentation of the thread. The students also try to include in this kind of exploration a ruler that should play the role of the rod.

The observer A4 intervenes in the dialogue to suggest that the students think of the placement of the pencil to draw the curve. The task is brought back by the need for the tracing. The students must pass from a static configuration (test that a chosen point satisfies a certain relationship) to a dynamic one (a movement permits to trace the curve). A4's intervention supports the transfer of a specific utilization scheme of the previous drawers: the pencil guides the movement tightening the thread. As a result of this action, the pencil's point also corresponds to a point of the curve. So far, the students had not transferred this scheme that is instead crucial to draw the curve.

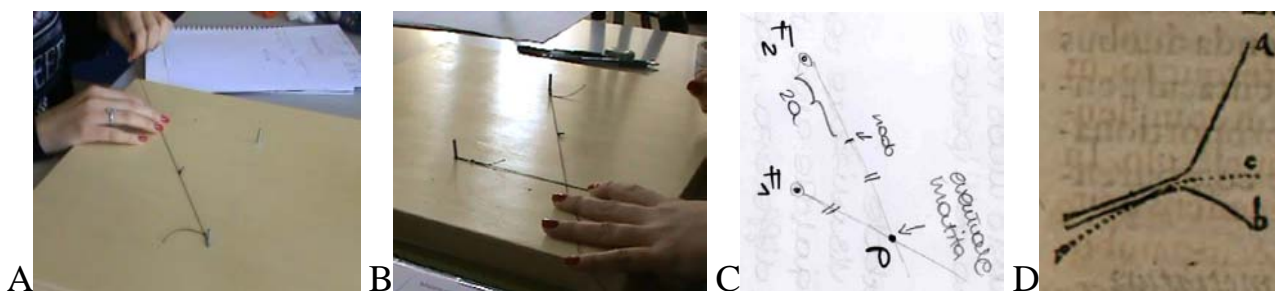


Figure 2

A new intervention of the teacher marks that these explorations do not consider the second focus, pushing the group to produce symmetric actions so to tie another thread to the second pin (Figure 2B). The students find configurations that seem to match sketches by Kepler (1604; e.g. Figure 2D).

The group tries to keep constant the difference $PF_2 - PF_1$ (Figure 2C) and to look for new gestures of usage in an instrumentation of the artefact with two threads. Indeed, moving the threads, the students want to preserve that difference when tracing the

curve. At the basis of this attempt, there is the detection of an isosceles triangle (scientific aspect, see Figure 2C). But the students abandon this way as soon as they are faced with a technological issue: “We are not successful in thinking of a tool that can replace my hands to move the two threads as desired” (from A1’s written report). In the case of the artefact with two threads, the construction of a new utilization scheme through the placement of the pencil is problematic for the students: they are only able to find discrete points but not the curve by continuous motion. The latter is another element of utilization schemes previously acquired that has to be transferred in the new situation, since it is a fundamental constraint for students’ action.

CONCLUSION

In this paper, we focused on how previous experience can become part of a new learning situation. In particular, we centred on the way a small group of university students recalls past experience with some mathematical machines for conic sections in order to face the task of constructing a new machine. This kind of task differs from the previous ones. Before, the students were asked to explore drawers for ellipse and parabola using them concretely (to understand how they are made, how they work, what they trace and why). In the new situation, they are required to think of and design a drawer for hyperbola. So, the machine is no longer the starting point of the activity, but the goal of it. Due to the presence of machines in the tasks, the notion of utilization schemes is interesting for us, especially in terms of their transfer and formation in the new situation.

Considering the perspectives of transfer of learning and of the instrumental approach, we have investigated if the students transfer utilization schemes previously formed, and especially how acquired schemes shape new schemes for the new machine in the new kind of task. Through the analysis of some work of group A, we have observed that the students’ process of constructing the instrument for hyperbola bounces between the technological side and the scientific side. The first side regards the fact of having a machine as goal of the activity and investigating its material components (that is, the artefact components). The second side refers to mathematical constraints that have to hold for tracing a given curve (an hyperbola) with that instrument. The relationship between the two sides is complex for the students for at least two reasons: on the one hand, the physical parts that constitute the machine have to materialize mathematical constraints; on the other hand, the curve has to be traced by a continuous motion with the instrument. Utilization schemes do just intervene in the search for such relationship.

Our analysis has shown that the students of group A try to construct a new artefact by transferring artefact components from the instruments to draw ellipse and parabola (e.g. the pins for the foci, the thread). The metric property that defines the hyperbola furnishes the mathematical constraint to be implemented in the machine. This implies that the students look for a condition on the length of the thread recalling utilization

schemes activated with the ellipsograph and the parabola drawer. The interventions of the observer and of the teacher help the students focus on other utilization scheme relevant for the machine: the motion of a pencil that keeps the thread tightened serves to trace the curve.

We believe that activities of this kind are relevant for mathematics learning because they encourage the students to make explicit theoretical principles under the machine. Following Koyré (1967), the construction of the new drawer corresponds to “creation of scientific thought or, better yet, the conscious realization of a theory” (p. 106).

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