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Nonlinear vibrations of functionally graded shells subjected to harmonic external load

Matteo Strozzi*[1], Francesco Pellicano[1]

Abstract
The nonlinear vibrations of functionally graded (FGM) circular cylindrical shells are analysed. The Sanders-Koiter theory is applied in order to model the nonlinear dynamics of the system. The shell deformation is described in terms of longitudinal, circumferential and radial displacement fields. Simply supported boundary conditions are considered. The displacement fields are expanded by means of a double mixed series based on Chebyshev polynomials for the longitudinal variable and harmonic functions for the circumferential variable. Both driven and companion modes are considered. Numerical analyses are carried out in order to characterize the nonlinear response when the shell is subjected to a harmonic external load. A convergence analysis is carried out to obtain the correct number of axisymmetric and asymmetric modes describing the actual nonlinear behaviour. The influence of the material distribution on the nonlinear response is analysed considering different configurations and volume fractions of the constituent materials. The effect of the companion mode participation on the nonlinear response of the shell is analysed.

Keywords
Nonlinear vibrations, functionally graded materials, circular cylindrical shells

[1] Department of Engineering "Enzo Ferrari", University of Modena and Reggio Emilia, Modena, Italy
* **Corresponding author:** matteo.strozzi@unimore.it

Introduction

Functionally graded materials (FGMs) are composite materials obtained by combining two or more different constituent materials, which are distributed along the thickness in accordance with a volume fraction law.

The idea of FGMs was first introduced in 1984/87 by a group of Japanese material scientists [1]. They studied many different physical aspects such as temperature and thermal stress distributions, static and dynamic responses.

Loy et al. [2] analysed the vibrations of the cylindrical shells made of FGM, considering simply supported boundary conditions. They found that the natural frequencies are affected by the constituent volume fractions and configurations of the constituent materials.

Pradhan et al. [3] studied the vibration characteristics of FGM circular cylindrical shells made of stainless steel and zirconia, under different boundary conditions. They found that the natural frequencies depend on the material distributions and boundary conditions.

Amabili [4] analysed the nonlinear vibrations and stability of isotropic and FGM shells. He carried out a comparison of thin shells theories for large-amplitude vibrations of circular cylindrical shells and analysed the effect of the companion mode participation on the nonlinear response.

Pellicano [5] studied the dynamic instability of a cylindrical shell carrying a top mass under base excitation. He investigated the shell response with a resonant harmonic forcing applied taking into account geometric nonlinearities, electrodynamic shaker equations and shell-shaker interaction.

The method of solution used in the present work was developed by Strozzi et al. in Ref. [6].

In this paper, the nonlinear vibrations of FGM cylindrical shells are analysed. The Sanders-Koiter theory is applied to model the nonlinear dynamics of the system. Simply supported boundary conditions are studied. Both driven and companion modes are considered allowing for the travelling-wave response in the circumferential direction. The model is validated in the linear field by means of data present in the literature. Numerical analyses are carried out in order to characterize the nonlinear response when the shell is subjected to a harmonic external load. A convergence analysis is carried out by considering different axisymmetric and asymmetric modes. The present study is focused on determining the nonlinear character of the shell dynamics as the material distribution varies.

1. Functionally graded materials

A general material property P_{fgm} of an FGM depends on the material properties and the volume fractions of the constituent materials, and it is expressed in the form

$$P_{fgm}(T,z)=\sum_{i=1}^{k}\tilde{P}_i(T)V_{fi}(z) \tag{1}$$

where $\tilde{P}_i$ and V_{fi} are the material property and the volume fraction of the constituent material *i*.

For an FGM thin cylindrical shell made of two different constituent materials, the Young's modulus *E*, the Poisson's ratio *ν* and the mass density *ρ* are expressed as

$$E_{fgm}(T,z)=\left(E_2(T)-E_1(T)\right)\left(\frac{z+h/2}{h}\right)^p+E_1(T) \tag{2}$$

$$\nu_{fgm}(T,z)=\left(\nu_2(T)-\nu_1(T)\right)\left(\frac{z+h/2}{h}\right)^p+\nu_1(T) \tag{3}$$

$$\rho_{fgm}(T,z)=\left(\rho_2(T)-\rho_1(T)\right)\left(\frac{z+h/2}{h}\right)^p+\rho_1(T) \tag{4}$$

where the power-law exponent *p* is a positive real number, ($0 \le p \le \infty$), and *z* describes the radial distance measured from the middle surface of the shell, ($-h/2 \le z \le h/2$).

2. Sanders-Koiter nonlinear theory of cylindrical shells

The elastic strain energy U_s of a cylindrical shell (plane stress hypothesis $\sigma_z = 0$) is given by

$$U_s=\frac{1}{2}LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x+\sigma_\theta\varepsilon_\theta+\tau_{x\theta}\gamma_{x\theta})d\eta d\theta dz \tag{5}$$

The kinetic energy T_s of a cylindrical shell (rotary inertia neglected) is given by

$$T_s=\frac{1}{2}LR\int_0^1\int_0^{2\pi}\int_{-h/2}^{h/2}\rho(z)(\dot{u}^2+\dot{v}^2+\dot{w}^2)d\eta d\theta dz \tag{6}$$

The virtual work *W* done by the external forces (q_x, q_θ, q_z) distributed per unit area is written as

$$W=LR\int_0^1\int_0^{2\pi}(q_xu+q_\theta v+q_zw)d\eta d\theta \tag{7}$$

The Rayleigh's dissipation function (viscous damping coefficient *c*) is written as

$$F=\frac{1}{2}cLR\int_0^{2\pi}\int_0^1(\dot{u}^2+\dot{v}^2+\dot{w}^2)d\eta d\theta \tag{8}$$

3. Linear analysis

A modal vibration can be written in the form

$$u(\eta,\theta,t) = U(\eta,\theta)\, f(t) \qquad v(\eta,\theta,t) = V(\eta,\theta)\, f(t) \qquad w(\eta,\theta,t) = W(\eta,\theta)\, f(t) \tag{9}$$

where $U(\eta,\theta), V(\eta,\theta), W(\eta,\theta)$ describe the mode shape and $f(t)$ is the time law.

The mode shape is expanded by means of a double series in terms Chebyshev polynomials $T_m^*(\eta)$ in the axial direction and harmonic functions (cos $n\theta$, sin $n\theta$) in the circumferential direction

$$U(\eta,\theta) = \sum_{m=0}^{M_u}\sum_{n=0}^{N} \tilde{U}_{m,n} T_m^*(\eta)\cos n\theta \tag{10}$$

$$V(\eta,\theta) = \sum_{m=0}^{M_v}\sum_{n=0}^{N} \tilde{V}_{m,n} T_m^*(\eta)\sin n\theta \tag{11}$$

$$W(\eta,\theta) = \sum_{m=0}^{M_w}\sum_{n=0}^{N} \tilde{W}_{m,n} T_m^*(\eta)\cos n\theta \tag{12}$$

where $T_m^* = T_m\,(2\eta - 1)$, m is the polynomials degree and n denotes the number of nodal diameters.

4. Nonlinear analysis

The displacement fields $u(\eta,\theta,t), v(\eta,\theta,t), w(\eta,\theta,t)$ are expanded by using the linear mode shapes $U(\eta,\theta), V(\eta,\theta), W(\eta,\theta)$ and the conjugate mode shapes $U_c(\eta,\theta), V_c(\eta,\theta), W_c(\eta,\theta)$

$$u(\eta,\theta,t) = \sum_{j=1}^{N_u}\sum_{n=1}^{N}\left[U^{(j,n)}(\eta,\theta)\, f_{u,j,n}(t) + U_c^{(j,n)}(\eta,\theta)\, f_{u,j,n,c}(t)\right] \tag{13}$$

$$v(\eta,\theta,t) = \sum_{j=1}^{N_v}\sum_{n=1}^{N}\left[V^{(j,n)}(\eta,\theta)\, f_{v,j,n}(t) + V_c^{(j,n)}(\eta,\theta)\, f_{v,j,n,c}(t)\right] \tag{14}$$

$$w(\eta,\theta,t) = \sum_{j=1}^{N_w}\sum_{n=1}^{N}\left[W^{(j,n)}(\eta,\theta)\, f_{w,j,n}(t) + W_c^{(j,n)}(\eta,\theta)\, f_{w,j,n,c}(t)\right] \tag{15}$$

The Lagrange equations of motion for forced vibrations are expressed in the following form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad i \in [1, N_{\max}] \qquad (L = T_s - U_s) \tag{16}$$

The generalized forces Q_i are obtained by the differentiation of the Rayleigh’s dissipation function F (8) and of the virtual work done by the external forces W (7), in the form

$$Q_i = -\frac{\partial F}{\partial \dot{q}_i} + \frac{\partial W}{\partial q_i} \tag{17}$$

Using the Lagrange equations (16) a set of nonlinear ordinary differential equations is obtained; such system is then solved by using numerical methods.

5. Numerical results

The present study is carried out on an FGM made of stainless steel and nickel; its properties are graded in the thickness direction according to a volume fraction distribution, where p is the power-law exponent. The material properties, reported in Table 1, have been extracted from Ref. [2].

Table 1. Properties of stainless steel and nickel vs. coefficients of temperature (300 K)

	Stainless steel			**Nickel**		
	E	ν	ρ	E	ν	ρ
P_0	2.01×10^{11} Nm^{-2}	0.326	8166 kgm^{-3}	2.24×10^{11} Nm^{-2}	0.3100	8900 kgm^{-3}
P_{-1}	0 K	0 K	0 K	0 K	0 K	0 K
P_1	3.08×10^{-4} K^{-1}	$- 2.002 \times 10^{-4}$ K^{-1}	0 K^{-1}	$- 2.79 \times 10^{-4}$ K^{-1}	0 K^{-1}	0 K^{-1}
P_2	$- 6.53 \times 10^{-7}$ K^{-2}	3.797×10^{-7} K^{-2}	0 K^{-2}	$- 3.99 \times 10^{-9}$ K^{-2}	0 K^{-2}	0 K^{-2}
P_3	0 K^{-3}	0 K^{-3}	0 K^{-3}	0 K^{-3}	0 K^{-3}	0 K^{-3}
P	2.08×10^{11} Nm^{-2}	0.318	8166 kgm^{-3}	2.05×10^{11} Nm^{-2}	0.3100	8900 kgm^{-3}

In order to validate the present method, the natural frequencies of simply supported FGM shells are compared with those of Loy et al. [2], see Table 2. The comparison shows that the present method gives results quite close to Ref. [2], the differences being less than 1%.

Table 2. Comparisons for a simply supported FGM shell (h/R = 0.002, L/R = 20, p = 1)

Natural frequency (Hz)				**Difference** %
m	n	Present model	Ref. [2]	
1	3	4.1562	4.1569	0.02
1	2	4.4794	4.4800	0.01
1	4	7.0379	7.0384	0.01
1	5	11.241	11.241	0.00
1	1	13.211	13.211	0.00
1	6	16.455	16.455	0.00

The shell is excited by an external modally distributed radial force $q_z = f_{1,6} \sin \eta \cos 6\theta \cos \Omega t$; the amplitude of excitation is $f_{1,6} = 0.0012\, h^2 \rho\, \omega_{1,6}^2$ and the frequency of excitation is $\Omega \approx \omega_{1,6}$. The external forcing $f_{1,6}$ is normalized with respect to the mass, acceleration and thickness; the damping ratio is equal to $\xi_{1,6} = 0.0005$.

In Figure 1, a moderately thick and long shell is analysed (h/R = 0.025, L/R = 20, p = 1), the amplitude-frequency curves are obtained with the expansions of Table 3. The 6 dof model, with an insufficient number of axisymmetric modes, is clearly inaccurate; indeed, for this kind of shell the correct behaviour is softening. From the convergence analysis, one can claim that the 9 dof model gives satisfactory results with the minimal computational effort; therefore, in the following the 9 dof model of Table 3 will be used.

Table 3. Nonlinear convergence analysis. Modes selected for the expansions (13-15).

(j,n)	(1,6)	(1,12)	(1,18)	(3,6)	(3,12)	(3,18)	(1,0)	(3,0)	(5,0)	(7,0)
6 dof	u, v, w	v	–	–	–	–	u, w	–	–	–
9 dof	u, v, w	v	–	–	v	–	u, w	u, w	–	–
12 dof	u, v, w	v	–	u, v, w	v	–	u, w	u, w	–	–
15 dof	u, v, w	v	v	u, v, w	v	–	u, w	u, w	u, w	–
18 dof	u, v, w	v	v	u, v, w	v	v	u, w	u, w	u, w	u, w

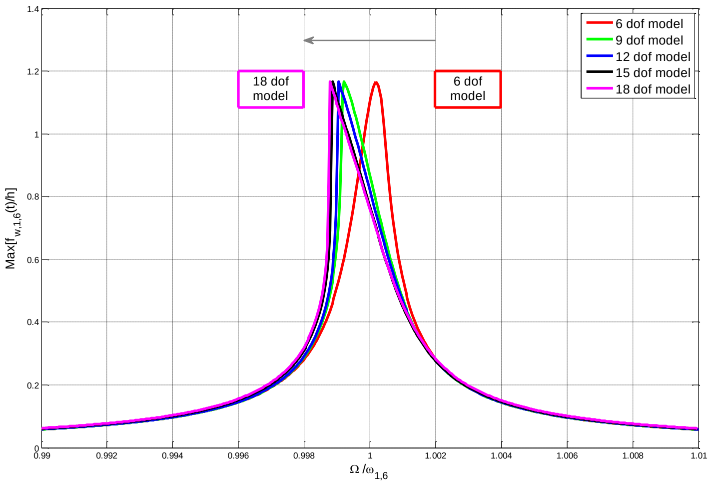


Figure 1. Convergence analysis, amplitude-frequency curves (h/R = 0.025, L/R = 20, p = 1). "–", 6 dof model; "–", 9 dof model; "–", 12 dof model; "–", 15 dof model; "–", 18 dof model.

The effect of the material distribution on the nonlinear response is analysed by considering two different FGM shells, Type I (nickel on the inner surface and stainless steel on the outer surface) and Type II (stainless steel on the inner surface and nickel on the outer surface).

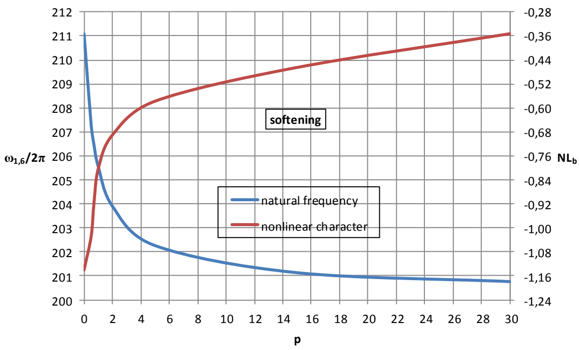
In Figures 2(a-b), the behaviour of the natural frequency $\omega_{1,6}$ and the nonlinear character vs. the exponent p (equations (2-4)) is shown for the FGM shell (h/R = 0.025, L/R = 20) having a softening nonlinear character identified by means of the following indicator

$$NL_b = \frac{\omega_{1,6\ nonlin} - \omega_{1,6\ lin}}{\omega_{1,6\ lin}} \times 1000 \tag{18}$$

where the nonlinear character is hardening when $NL_b > 0$, softening when $NL_b < 0$.

When the stiffer material is outside (Type I FGM, Figure 2(a)), an increment of the exponent p leads to a predominance of the material with a smaller Young's modulus (nickel) and this implies a decreasing of the natural frequencies, while an increase in the predominance of the weaker material produces a decrease of the nonlinearity of the system. In the case of FGM with stiffer material inside (Type II FGM, Figure 2(b)), the increment of the exponent p magnifies the presence of stainless steel, increasing the natural frequencies and the nonlinearity.

(a)



(b)

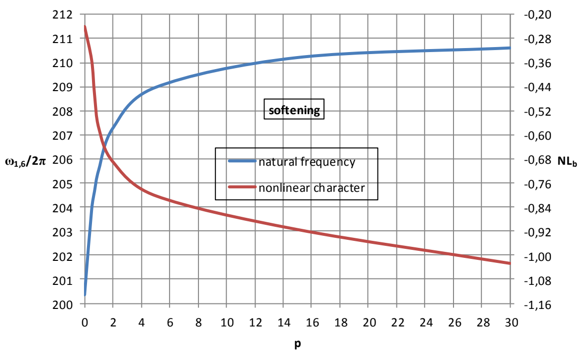


Figure 2. Natural frequency $\omega_{1,6}$ and nonlinear character NL_b vs. exponent parameter p for the cylindrical shell (h/R = 0.025, L/R = 20). (a) Type I FGM. (b) Type II FGM.

The effect of the companion mode participation on the nonlinear response is analysed. In Figures 3(a-b), the amplitude-frequency curve with the companion mode participation is presented (h/R = 0.025, L/R = 20, p = 1, mode (1,6)). The response $f_{w,1,6}(t)$ with companion mode participation, solid blue line of Figure 3(a), is very similar to the response without companion mode participation, dashed black line, see Figure 1. The companion mode, Figure 3(b), produces a variation in the small region close to the resonance ($0.9996 < \Omega/\omega_{1,6} < 0.9999$), where the companion mode is excited by means of a 1:1 internal resonance which induces an energy transfer between the two conjugate modes.

(a) (b)

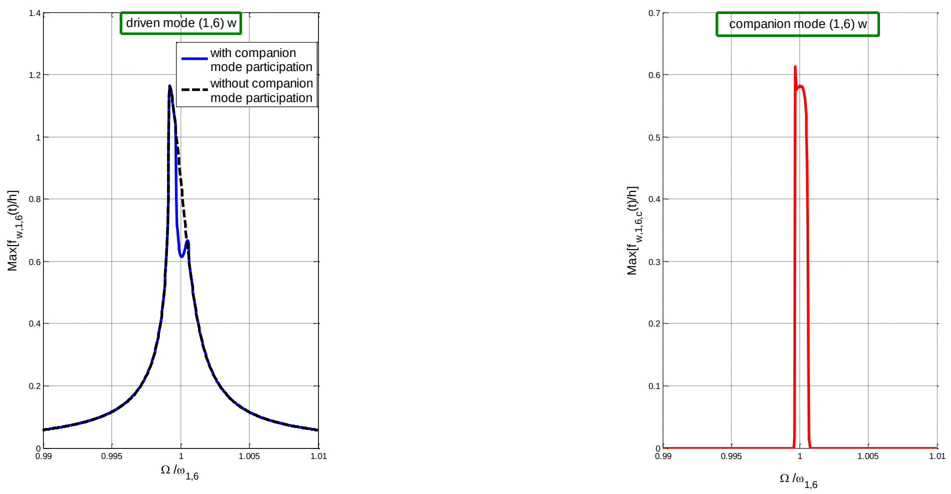


Figure 3. Amplitude-frequency curves of the FGM shell (h/R = 0.025, L/R = 20, p = 1). 14 dof model. (a) "– –", driven mode (1,6) *w* without companion mode participation; "–", driven mode (1,6) *w* with companion mode participation. (b) Companion mode (1,6) *w*.

6. Numerical results

The nonlinear vibrations of FGM cylindrical shells are analysed. The Sanders-Koiter theory is applied to model the nonlinear dynamics of the system in the case of finite amplitude of vibration.

A convergence analysis is carried out. The fundamental role of the axisymmetric and higher-order asymmetric modes is clarified in order to obtain the actual nonlinearity.

The effect of the material distribution is analysed. The relationships between the power-law exponent, the corresponding natural frequency and nonlinearity are studied.

The effect of the companion mode participation on the nonlinear response is analysed. Both driven and companion modes are considered. Nonlinear amplitude-frequency curves are obtained.

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