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Multi-parametric analysis of strongly inhomogeneous periodic waveguides with internal cutoff frequencies / Kaplunov, J.; Nobili, Andrea. - In: MATHEMATICAL METHODS IN THE APPLIED SCIENCES. - ISSN 0170-4214. - STAMPA. - 40:9(2017), pp. 3381-3392. [10.1002/mma.3900]

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# Multi-parametric analysis of strongly inhomogeneous periodic waveguides with internal cutoff frequencies

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In this paper, we consider periodic waveguides in the shape of a inhomogeneous string or beam partially supported by an uniform elastic Winkler foundation. A multi-parametric analysis is developed to take into account the presence of internal cutoff frequencies and strong contrast of the problem parameters. This leads to asymptotic conditions supporting non-typical quasi-static uniform or, possibly, linear micro-scale displacement variations over the high-frequency domain. Macro-scale governing equations are derived within the framework of the Floquet-Bloch theory as well as using a high-frequency-type homogenization procedure adjusted to a string with variable parameters. It is found that, for the string problem, the associated macro-scale equation is the same as that applying to a string resting on a Winkler foundation. Remarkably, for the beam problem, the macro-scale behavior is governed by the same equation as for a beam supported by a two-parameter Pasternak foundation. Copyright © 0000 John Wiley & Sons, Ltd.

**Keywords:** Periodic waveguide; cutoff frequency; homogenization; contrast; high-frequency

## 1. Introduction

Periodic structures with internal cutoff frequencies are of interest for numerous applications: as an example, we mention elastically supported periodic strings and beams [1, 2, 3], composite materials [4], phononic crystals [5, 6, 7] and vibration absorbers in fluid carrying pipes [8]. It is well renowned that a string supported by a Winkler foundation exhibits a cutoff frequency [9, §1.5.2]

$$\omega_0 = \sqrt{\frac{\beta}{\rho}},$$

where  $\beta$  is Winkler foundation modulus and  $\rho$  is the string linear mass density. As it is shown in [10], a two-phase piecewise periodic string supported by a uniform Winkler foundation cannot be treated by the conventional “low-frequency” homogenization method [11, 12]. For the latter, the sought for macro-scale homogenized equation is of the same form as the original equation governing the behavior of the periodic system. Besides, a quasi-static uniform variation of the displacement

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Contract/grant sponsor: Fondazione Cassa di Risparmio di Modena, Sime n.2013.0662; GNFM-INdAM, prot. U2015/000125

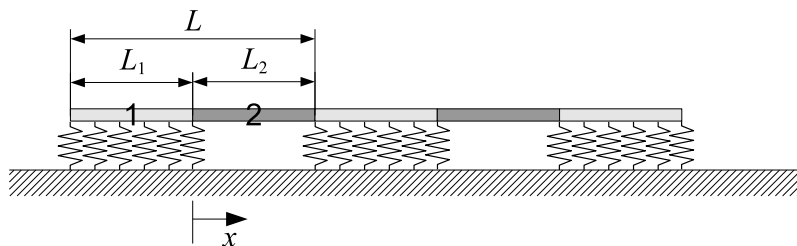


Figure 1. A piecewise homogeneous periodic waveguide with a internal cutoff

field is retrieved at the micro-scale (see also [13, 14]). On the other hand, the periodically supported string problem can be efficiently treated through a high-frequency asymptotic homogenization procedure, as established in [10, 15, 16, 17]. Within this approach and contrarily to the classical setup, the homogenized macro-scale equation takes, as a rule, a different form than the original equation. Moreover, sinusoidal variations at the micro-scale are found which correspond to the eigenforms of the unit cell. Dynamic homogenization has been the subject of a number of remarkable contributions among which we mention [18, 19, 20, 21, 22, 23, 24] and also [25, 26, 27], dealing with the important case of periodic waveguides with contrast properties.

In this paper, we show that quasi-static uniform (or, possibly, linear) micro-scale variation over the high-frequency range is still possible for strongly inhomogeneous periodic structures with internal cutoffs. As an example, a two-phase periodic waveguide in the shape of a string or a beam supported by an elastic Winkler foundation is studied. The foundation is assumed to be periodically discontinuous. Such feature is crucial for the subsequent analysis and, to the best of our knowledge, it appears in the literature only in the shape of point supports [28].

A dimensional analysis brings up the relevant dimensionless quantities (three for the string and four for the beam), expressed in terms of relative lengths, stiffnesses and densities. We develop a multi-parametric asymptotic approach assuming that two of the aforementioned parameters are small in each of the cases. The long-wave expansions of the obtained dispersion relations near the lowest cutoff frequencies are derived and the ODEs governing the macro-scale behavior deduced.

Remarkably, although for the string problem the macro-scale equation takes the same form as that for a string on a Winkler foundation, the beam macro-scale behavior is governed by the equation for a beam supported by a two-parameter Pasternak foundation (as opposed to a Winkler foundation, as it might be expected).

The associated quasi-static displacement fields are shown to be almost uniform at the micro-scale, as for a rigid body motion. Numerical testing of the asymptotic formulae for the lowest cutoff frequency and the displacement field exhibits excellent agreement.

The setup in which the string parameters vary along the unsupported region is also addressed using a two-scale approach. The derived macro-scale equation reduces to the asymptotic expression previously obtained for constant parameters.

## 2. Periodically supported string

Let us consider a periodic waveguide constituted by a string in piecewise uniform tension periodically supported on a homogeneous Winkler elastic foundation (Fig.1). The governing equation for the transverse displacement  $w(x, t)$  is, for the supported regions  $S_n = \{x \in (-L_1 + nL, nL)\}$ ,  $n \in \mathbb{Z}$ , each with length  $L_1$ ,

$$-T_1 \partial_{xx}^2 w + \rho_1 \partial_{tt}^2 w + \beta w = 0, \quad (1)$$

and, for the free string regions  $U_n = \{x \in (nL, L_2 + nL)\}$ , each with length  $L_2$ ,

$$-T_2 \partial_{xx}^2 w + \rho_2 \partial_{tt}^2 w = 0, \quad (2)$$

where  $\rho_i$  and  $T_i$  are the constant linear mass density and tension of the string over the relevant regions  $i = 1, 2$ , respectively,  $\beta$  the Winkler elastic modulus (whose dimension is force over length squared), and  $n \in \mathbb{Z}$  (see [9] for more details). These

equations have periodic coefficients with period  $L = L_1 + L_2$  and they can be treated by means of the Floquet theory, e.g.[29, 30]. Accordingly, we may restrict attention to the single cell  $S_0 \cup U_0 = (-L_1, L_2)$ . Let us rewrite the equations (1,2) in dimensionless form

$$-\kappa_1^2 \partial_{x_1 x_1}^2 w + \partial_{\tau \tau}^2 w + w = 0, \quad x_1 \in (-1, 0), \quad (3)$$

and

$$-\frac{\kappa_2^2}{\alpha^2} \partial_{x_2 x_2}^2 w + \eta \partial_{\tau \tau}^2 w = 0, \quad x_2 \in (0, 1), \quad (4)$$

having introduced the dimensionless positive ratios

$$\kappa_i = \sqrt{\frac{T_i}{\beta L_1^2}}, \quad \eta = \frac{\rho_2}{\rho_1}, \quad \alpha = \frac{L_2}{L_1}, \quad (5)$$

together with  $x_i = x/L_i$ , the dimensionless axial co-ordinates, and  $\tau = t/\sqrt{\rho_1/\beta}$ , the dimensionless time. We look for the harmonic motion of the system, i.e.  $w(x_i, \tau) = u_i(x_i) \exp(i\Omega\tau)$ ,  $i = 1, 2$  and  $i^2 = 1$ , whence Eq.(3) and (4) become the linear ODEs with constant coefficients

$$\kappa_1^2 \frac{d^2 u_1}{dx_1^2} - (1 - \Omega^2) u_1 = 0, \quad x_1 \in (-1, 0), \quad (6a)$$

$$\kappa_1^2 \frac{d^2 u_2}{dx_2^2} + \chi_s^2 \Omega^2 u_2 = 0, \quad x_2 \in (0, 1), \quad (6b)$$

having let the shorthand notation  $u_1$  for  $u(x_1)$  and  $u_2$  for  $u(x_2)$ . Besides, it is let

$$\kappa = \frac{\kappa_2}{\kappa_1} = \sqrt{\frac{T_2}{T_1}} \quad \text{and} \quad \chi_s = \frac{\alpha \sqrt{\eta}}{\kappa}.$$

Eq.(6a) clearly shows that the supported string region possesses the internal cutoff frequency  $\Omega = 1$ . The conditions expressing continuity of displacement and tension at the supported/unsupported interface  $x_1 = x_2 = 0$  are

$$u_1(0) = u_2(0), \quad \frac{du_1}{dx_1}(0) = \epsilon_s \frac{du_2}{dx_2}(0), \quad (7)$$

where

$$\epsilon_s = \frac{\kappa^2}{\alpha}. \quad (8)$$

The Floquet-Bloch conditions read

$$u_1(-1) = u_2(1) \exp(iq), \quad \frac{du_1}{dx_1}(-1) = \epsilon_s \frac{du_2}{dx_2}(1) \exp(iq). \quad (9)$$

The general solution of Eq.(6a) is

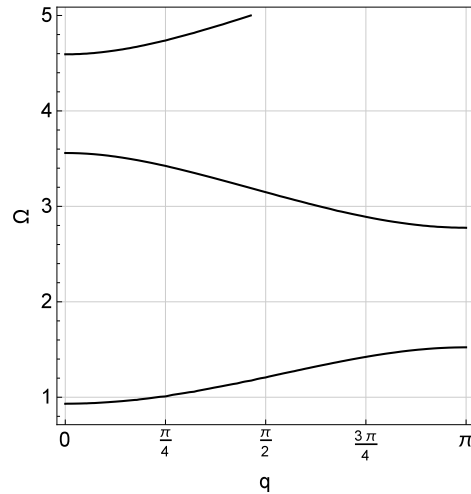
$$u_1 = A_1 \sin\left(\frac{\sqrt{\Omega^2 - 1}}{\kappa_1} x_1\right) + B_1 \cos\left(\frac{\sqrt{\Omega^2 - 1}}{\kappa_1} x_1\right), \quad (10)$$

where  $A_1$  and  $B_1$  are real constants provided  $\Omega > 1$ . The general solution of Eq.(6b) is

$$u_2 = A_2 \sin\left(\chi_s \frac{\Omega}{\kappa_1} x_2\right) + B_2 \cos\left(\chi_s \frac{\Omega}{\kappa_1} x_2\right). \quad (11)$$

The dispersion relation reads

$$\cos q - \cos\left(\frac{\sqrt{\Omega^2 - 1}}{\kappa_1}\right) \cos\left(\frac{\chi_s \Omega}{\kappa_1}\right) + \frac{(1 + \chi_s^2 \epsilon_s^2) \Omega^2 - 1}{2\chi_s \epsilon_s \Omega \sqrt{\Omega^2 - 1}} \sin\left(\frac{\sqrt{\Omega^2 - 1}}{\kappa_1}\right) \sin\left(\frac{\chi_s \Omega}{\kappa_1}\right) = 0 \quad (12)$$



**Figure 2.** Dispersion curves for a string periodically supported on an elastic foundation ( $\kappa = 1, \epsilon_s = 0.4, \chi_s = 0.6$ )

and it gives rise to the usual pass/block bands depicted in Fig.2. The focus of the paper is on quasi-static uniform (or, possibly, linear) displacement variations at leading order long-wave approximation ( $q \ll 1$ ). As it can be seen from (6) and (10,11), these are obtained whenever  $\chi_s$  is small and  $\Omega$  is close to the internal cutoff frequency, i.e.

$$\chi_s \ll 1 \quad \text{and} \quad |\Omega - 1| \ll 1. \quad (13)$$

In this case, from the dispersion relation (12) it follows that

$$\delta_s = \epsilon_s \chi_s^2 = \alpha \eta \ll 1. \quad (14)$$

Here and below it is assumed that  $\kappa_1$  is of order unity. Thus, Eqs.(13) and (14) together show that the fundamental assumption for quasi-static behavior at leading order for  $q \ll 1$  is that  $\chi_s$  and  $\delta_s$  be both small, which entails

$$\eta \ll \alpha^{-1} \quad \text{and} \quad \eta \ll \alpha^{-2},$$

while it only demands  $\epsilon_s$  to be not too large, namely  $\epsilon_s \ll \chi_s^{-2}$ .

The asymptotic behavior of the lowest cutoff frequency  $\Omega = \Omega_*$  in small  $\delta_s$  and  $\chi_s$  follows from the transcendental equation (12) taken at  $q = 0$  and it is given by

$$\Omega_*^2 = 1 - \left(1 + \frac{\chi_s^2}{12\kappa_1^2}\right) \delta_s + \left(1 - \frac{1}{12\kappa_1^2}\right) \delta_s^2 + \dots \quad (15)$$

It is remarked that the cutoff frequency  $\Omega_*$  is close to the internal cutoff frequency of the string in supported region, i.e.  $\Omega_* \approx 1$ .

The related eigenforms are

$$u_1(x_1) = 1 - \left(\frac{1}{12\kappa_1^2} - \epsilon_s^2\right) \chi_s^2 + \frac{\delta_s}{2} \left[\frac{x_1}{\kappa_1^2}(1+x_1) + 1 + \frac{1}{6\kappa_1^2}\right] + \dots, \quad (16)$$

$$u_2(x_2) = 1 - \left\{\epsilon_s^2 + \frac{1}{12\kappa_1^2} [1 - 6x_2(1-x_2)]\right\} \chi_s^2 + \frac{\delta_s}{2} \left[1 + \frac{1}{6\kappa_1^2}\right] + \dots, \quad (17)$$

which show that  $u_1(x_1)$  and  $u_2(x_2)$  undertake a rigid body motion at leading order. It should be remarked that the above eigenform formulae are valid provided that  $\epsilon_s$  is not a large parameter.

Fig.3 compares the eigenforms evaluated through a numerical procedure with their asymptotic expressions (16), for the parameter set  $\epsilon_s = 0.4, \chi_s = 0.6, \kappa_1 = 1$ . For this choice of the parameters,  $\delta_s = 0.144$  is small and the lowest cutoff frequency

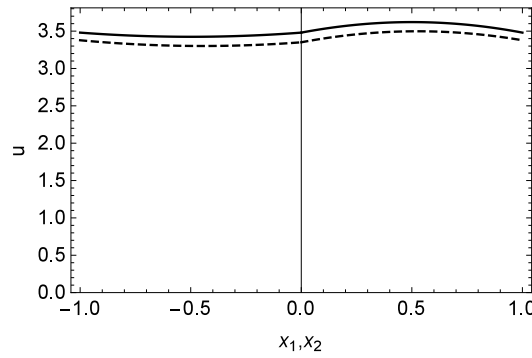


Figure 3. Eigensolutions near cutoff: numerical (solid) vs. asymptotic (dashed) for  $\epsilon_s = 0.4$ ,  $\chi_s = 0.6$  and  $\kappa_1 = 1$

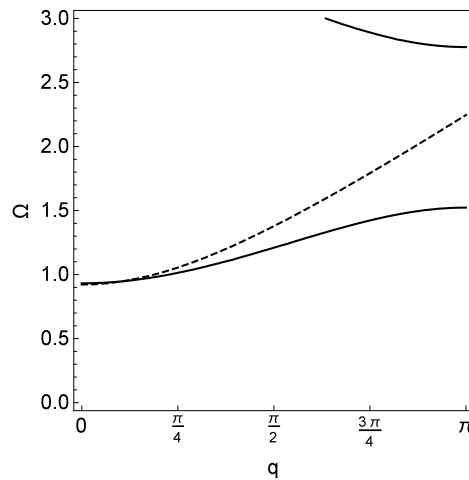


Figure 4. Asymptotic (dashed) vs. numerical (solid) dispersion curve for  $\epsilon_s = 0.4$ ,  $\chi_s = 0.6$ ,  $\kappa_1 = 1$

numerically occurs at  $\Omega^2 = 0.932716$ , as opposed to Eq.(15) which gives  $\Omega_*^2 = 0.856$ . Nonetheless, very good agreement is met for the eigenforms and an almost rigid body behavior found.

The leading order long-wave approximation ( $q \ll 1$ ) of the dispersion relation (12) can be written as

$$\Omega^2 - \Omega_*^2 = \kappa_1^2 q^2 \epsilon \tag{18}$$

and its accuracy is shown in Fig.4 for the usual parameter set. The corresponding macro-model ODE is a continuously supported string equation

$$\partial_{xx}^2 w - \frac{1}{c_*^2} \partial_{tt}^2 w - \beta_* w = 0 \tag{19}$$

where

$$c_*^2 = \frac{\epsilon \kappa_1^2 \beta (L_1 + L_2)^2}{\rho_1}, \quad \beta_* = \frac{\Omega_*^2}{\epsilon \kappa_1^2 (L_1 + L_2)^2}$$

given that  $q^2 = -(L_1 + L_2)^2 \partial_{xx}^2$ .

### 3. Two-scale procedure for a periodically supported string

Let us now consider the case when the tension along the unsupported region is given by a periodic function with period  $L$ , i.e.  $T_2(x) = T_2(x + L)$ . It is observed that this condition implies either horizontal motion or a restraining device to prevent it and, for simplicity, we assume the latter. Conversely, it is further assumed that the string tension  $T_1$  is still constant along the supported region and, therefore, a cutoff frequency can still be clearly defined. Then, governing equation for the

transverse displacement  $w(x, t)$  in the supported interval is again Eq.(1), while the governing equation for the free string becomes

$$-\partial_x (T_2 \partial_x w) + \rho_2 \partial_{tt} w = 0, \quad x \in U_{n \in \mathbb{Z}}. \quad (20)$$

The set of dimensionless governing Eqs.(6) is replaced by the following pair of ODEs, the second of which is an equation with variable coefficients,

$$\kappa_1^2 \frac{d^2 u_1}{dx_1^2} - (1 - \Omega^2) u_1 = 0, \quad x_1 \in (-1, 0), \quad (21a)$$

$$\kappa_1^2 \left[ \frac{d^2 u_2}{dx_2^2} + 2 \frac{d \ln \kappa}{dx_2} \frac{du_2}{dx_2} \right] + \chi_s^2 \Omega^2 u_2 = 0, \quad x_2 \in (0, 1), \quad (21b)$$

where it is understood that  $\kappa = \kappa(x_2)$  and  $\chi_s = \chi_s(x_2)$ ; in addition,  $\frac{d \ln \kappa}{dx_2}$  is the logarithmic derivative of  $\kappa(x_2)$ . The same set of equations governs the behavior of a string whose linear mass density is constant along the supported region and periodically variable along the unsupported one, i.e.  $\rho_1 = \text{const}$  and  $\rho_2 = \rho_2(x_2)$  with  $\rho_2(x_2) = \rho_2(x_2 + L)$ . The conditions expressing continuity at  $x_1 = x_2 = 0$  are

$$u_1(0) = u_2(0), \quad \frac{du_1}{dx_1}(0) = \epsilon_s \frac{du_2}{dx_2}(0), \quad (22)$$

where

$$\epsilon_s = \frac{\kappa^2(0)}{\alpha}. \quad (23)$$

Let, for the sake of definiteness, assume the following asymptotic relation between the small parameters of the previous section

$$\delta_s \sim \chi_s^4.$$

In this case,  $\epsilon_s \sim \chi_s^2$  is also a small parameter and we set

$$\chi_s = \sqrt{\epsilon_s} \chi_0,$$

where it is recalled that  $\chi_0 = \chi_0(x_2)$ . Then, Eq.(21b) becomes

$$\kappa_1^2 \left( \frac{d^2 u_2}{dx_2^2} + 2 \frac{d \ln \kappa}{dx_2} \frac{du_2}{dx_2} \right) + \epsilon_s \chi_0^2 \Omega^2 u_2 = 0. \quad (24)$$

In this section, we adopt a two-scale approach [31] setting

$$u_i = u_i(\xi_i, X), \quad i = 1, 2, \quad (25)$$

being  $X = \epsilon_s x_2$  the slow variable,  $\xi_i = x_i$ ,  $i = 1, 2$ . Then, with the usual understanding for the integer power of a linear operator,

$$\kappa_1^2 \left( \partial_{\xi_1} + \frac{\epsilon_s}{\alpha} \partial_X \right)^2 u_1 - (1 - \Omega^2) u_1 = 0, \quad \xi_1 \in (-1, 0), \quad (26a)$$

$$\kappa_1^2 [(\partial_{\xi_2} + \epsilon_s \partial_X)^2 + 2(\partial_{\xi_2} \ln \kappa)(\partial_{\xi_2} + \epsilon_s \partial_X)] u_2 + \epsilon_s \chi_0^2 \Omega^2 u_2 = 0, \quad \xi_2 \in (0, 1). \quad (26b)$$

The conditions expressing continuity at  $\xi = 0$  are

$$u_1(0, X) = u_2(0, X), \quad (27a)$$

$$\left( \partial_{\xi_1} + \frac{\epsilon_s}{\alpha} \partial_X \right) u_1(0, X) = \epsilon_s (\partial_{\xi_2} + \epsilon_s \partial_X) u_2(0, X), \quad (27b)$$

while periodicity yields

$$u_1(-1, X) = u_2(1, X), \tag{28a}$$

$$\left(\partial_{\xi_1} + \frac{\epsilon_s}{\alpha} \partial_X\right) u_1(-1, X) = \epsilon_s (\partial_{\xi_2} + \epsilon_s \partial_X) u_2(1, X). \tag{28b}$$

Let us take the regular expansions

$$u_1(\xi_1, X) = p_0(\xi_1, X) + \epsilon_s p_1(\xi_1, X) + \epsilon_s^2 p_2(\xi_1, X) + \dots, \tag{29a}$$

$$u_2(\xi_2, X) = q_0(\xi_2, X) + \epsilon_s q_1(\xi_2, X) + \epsilon_s^2 q_2(\xi_2, X) + \dots, \tag{29b}$$

and, likewise, for the frequency (e.g.[10])

$$\Omega^2 = 1 + \epsilon_s \Omega_1^2 + \epsilon_s^2 \Omega_2^2 + \epsilon_s^3 \Omega_3^2 + \dots \tag{30}$$

Here and below we understand  $\xi = \xi_1$  ( $\xi = \xi_2$ ) when the (un)-supported string is dealt with. Then, we get the usual succession of linear problems in the expansion terms. Indeed, at order zero in  $\epsilon$ , we get

$$\begin{aligned} \kappa_1^2 \partial_{\xi\xi}^2 p_0 &= 0, \\ \kappa_1^2 [2(\ln \kappa)' \partial_{\xi} + \partial_{\xi\xi}^2] q_0 &= 0, \end{aligned}$$

where prime is short for the total derivative  $d/d\xi_2$ . The first equation admits the linear polynomial solution

$$p_0 = a_0(X) + b_0(X)\xi, \tag{31}$$

while the second equation is linear and first-order in  $\partial_{\xi} q_0$ , whence its solution is

$$q_0 = c_0(X) + d_0(X)\phi_1(\xi) \tag{32}$$

having let

$$\phi_1(\xi) = \int_0^{\xi} \kappa^{-2}(\sigma) d\sigma.$$

Plugging the solutions (31,32) into the expansions (29), we get from the conditions (27,28) at leading order

$$c_0(X) = a_0(X), \quad d_0(X) = b_0(X) \equiv 0,$$

whereupon, the zero order solution is just a rigid body motion

$$p_0 = q_0 = a_0(X).$$

At first order in  $\epsilon$ , we get

$$\begin{aligned} \kappa_1^2 \partial_{\xi\xi}^2 p_1 + \Omega_1^2 a_0 &= 0, \\ \kappa_1^2 [2(\ln \kappa)' \partial_{\xi} + \partial_{\xi\xi}^2] q_1 + 2\kappa_1^2 (\ln \kappa)' \frac{da_0}{dX} + \chi_0^2 a_0 &= 0 \end{aligned}$$

whose general solution is

$$\begin{aligned} p_1 &= -\frac{\Omega_1^2 a_0}{\kappa_1^2} \frac{\xi^2}{2} + a_1 \xi + b_1, \\ q_1 &= c_1 + d_1 \phi_1 - \frac{a_0}{\kappa_1^2} \phi_2 - \frac{da_0}{dX} \xi, \end{aligned}$$

where  $a_1, b_1, c_1$  and  $d_1$  are (yet) undetermined functions of  $X$  and

$$\phi_2(\xi) = \int_0^\xi \frac{\int_0^\sigma \chi_0^2(\gamma) \kappa^2(\gamma) d\gamma}{\kappa^2(\sigma)} d\sigma.$$

Solvability of the first order problem [31] yields  $\Omega_1 = 0$  and, after tedious calculations, we find

$$p_1 = -\frac{(1+\xi)}{\alpha} \frac{da_0}{dX},$$

$$q_1 = \frac{(\alpha+1)\phi_1(\xi) - \phi_1(1)(\alpha\xi+1)}{\alpha\phi_1(1)} \frac{da_0}{dX} + \frac{\phi_2(1)\phi_1(\xi) - \phi_1(1)\phi_2(\xi)}{\kappa_1^2\phi_1(1)} a_0.$$

At second order in  $\epsilon$ , we get

$$\kappa_1^2 \partial_{\xi\xi}^2 p_2 - \frac{\kappa_1^2}{\alpha^2} \frac{d^2 a_0}{dX^2} + \Omega_2^2 a_0 = 0,$$

$$\kappa_1^2 \left[ (2(\ln \kappa)' \partial_\xi + \partial_{\xi\xi}^2) q_2 + 2((\ln \kappa)' \partial_X + \partial_{\xi X}^2) q_1 + \frac{d^2 a_0}{dX^2} \right] + \chi_0^2 q_1 = 0,$$

whose general solution is

$$p_2 = \left( \frac{1}{\alpha^2} \frac{d^2 a_0}{dX^2} - \frac{\Omega_2^2}{\kappa_1^2} a_0 \right) \frac{\xi^2}{2} + a_2 \xi + b_2,$$

$$q_2 = c_2 + \int_0^\xi y(\sigma, X) d\sigma,$$

being

$$y(\sigma, X) = \frac{d_2}{\kappa^2} + \frac{a_0}{\kappa_1^4} y_1 + \frac{1}{\kappa_1^2} \frac{da_0}{dX} y_2 + \frac{d^2 a_0}{dX^2} y_3$$

and  $a_2, b_2, c_2, d_2$  are yet unknown functions of the slow variable. In the case of constant coefficients (i.e. for constant string tension,  $\kappa = const$ , and constant linear mass density,  $\chi_0 = const$ )

$$y_1 = \frac{\sigma^2(2\sigma-3)}{12} \chi_0^4,$$

$$y_2 = \chi_0^2 \sigma \left( \frac{2-\sigma}{2\alpha} + \sigma - 1 \right),$$

$$y_3 = \frac{1-2\sigma}{\alpha} - \sigma.$$

Solvability of the second order problem yields

$$\Omega_2^2 = -\phi_2'(1),$$

while conditions (27,28) give  $b_2 = c_2$ ,

$$a_2 = \frac{1}{\alpha^2} \frac{d^2 a_0}{dX^2} + \frac{1+\alpha}{\alpha\kappa(0)^2\phi_1(1)} \frac{da_0}{dX} + \frac{\phi_2(1)}{\kappa(0)^2\kappa_1^2\phi_1(1)} a_0.$$

For constant coefficients,

$$d_2 = \kappa_0^2 \left( \frac{\alpha^2-1}{2\alpha^2} \frac{d^2 a_0}{dX^2} + \left( \frac{\alpha-2}{6\alpha\kappa_1^2} \chi_0^2 - \frac{\alpha+1}{\alpha} \right) \frac{da_0}{dX} + \frac{\chi_0^4}{24\kappa_1^4} a_0 \right).$$

and  $\Omega_2^2 = -\chi_0^2$ ,  $b_2 = c_2$  while

$$a_2 = \frac{1}{\alpha^2} \frac{d^2 a_0}{dX^2} + \left( \frac{1}{\alpha} + 1 \right) \frac{da_0}{dX} + \frac{\chi_0^2}{2\kappa_1^2} a_0.$$

Finally, at third order in  $\epsilon$ , it is found

$$p_3 = a_3 \xi^3 + b_3 \xi^2 + c_3 \xi + d_3$$

where, for constant coefficients,

$$a_3 = -\frac{1}{6\alpha^3} \frac{d^3 a_0}{dX^3} - \frac{\chi_0^2}{2\alpha\kappa_1^2} \frac{da_0}{dX},$$

$$a_2 = -\frac{1}{2\alpha^3} \frac{d^3 a_0}{dX^3} - \frac{1 + \alpha}{\alpha^2} \frac{d^2 a_0}{dX^2} - \frac{\chi_0^2}{\alpha\kappa_1^2} \frac{da_0}{dX} - \frac{\Omega_3^2}{2\kappa_1^2} a_0.$$

In this case, we get from the solvability of the third order problem

$$-\kappa_1^2 \frac{d^2 a_0}{dX^2} - \frac{\alpha^2}{(\alpha + 1)^2} \left( \Omega_3^2 + \frac{\chi_0^4}{12\kappa_1^2} \right) a_0 = 0, \quad (33)$$

which is identical to Eq.(19). Indeed, setting in Eq.(33)

$$\Omega_3^2 = \frac{\Omega^2 - 1 - \epsilon_s \Omega_1^2 - \epsilon_s^2 \Omega_2^2}{\epsilon^3} = \frac{\Omega^2 - 1 + \epsilon_s \chi_s^2}{\epsilon^3}$$

the dispersion relation (18) is retrieved, which is associated with Eq.(19). In the general case of a string with variable parameters, we also arrive at second order macro-scale governing equation with messy expressions for its coefficients. For example, in the case of constant tension,  $\kappa = const$ , and variable linear mass density,  $\chi_0 = \chi_0(x_2)$ , it is found

$$-\kappa_1^2 \frac{d^2 a_0}{dX^2} - \frac{\alpha^2}{(\alpha + 1)^2} \left( \Omega_3^2 + \frac{\bar{\Omega}^2}{\kappa_1^2} \right) a_0 = 0, \quad (34)$$

where

$$\bar{\Omega}^2 = 2 \int_0^1 \chi_0^2(\sigma) \int_0^\sigma \tau \chi_0^2(\tau) d\tau d\sigma - \left( \int_0^1 \sigma \chi_0^2(\sigma) d\sigma \right)^2.$$

## 4. Periodically supported beam

Let us consider bending of a piecewise homogeneous beam periodically supported by a Winkler elastic foundation. Similarly to the string case above (Fig.1), the governing equations for the transverse displacement  $w(x, t)$  are given by

$$(EI)_1 \partial_{xxxx} w + \rho_1 \partial_{tt}^2 w + \beta w = 0, \quad x \in S_n, \quad (35a)$$

$$(EI)_2 \partial_{xxxx} w + \rho_2 \partial_{tt}^2 w = 0, \quad x \in U_n, \quad (35b)$$

where  $\rho_i$  and  $(EI)_i$  are, respectively, the linear mass density and the flexural rigidity of the beam, which are constant in the relevant regions  $i = 1, 2$ , while  $\beta$  is the Winkler foundation modulus. These equations may be rewritten in dimensionless form for a single cell as follows

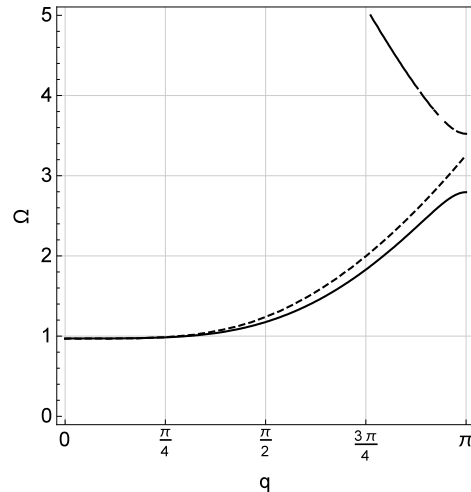
$$\gamma_1^4 \partial_{x_1 x_1 x_1 x_1} w + \partial_{\tau\tau}^2 w + w = 0, \quad x_1 \in (-1, 0), \quad (36a)$$

$$\frac{\gamma_2^4}{\alpha^4} \partial_{x_2 x_2 x_2 x_2} w + \eta \partial_{\tau\tau}^2 w = 0, \quad x_2 \in (0, 1), \quad (36b)$$

having introduced the dimensionless ratios

$$\gamma_i = \sqrt[4]{\frac{(EI)_i}{\beta L_i^4}}, \quad i = 1, 2, \quad (37)$$

together with the dimensionless axial co-ordinates  $x_i = x/L_i$ . Here, as before,  $\tau = t/\sqrt{\rho_1/\beta}$  is the dimensionless time.



**Figure 5.** Dispersion curves (solid) and long-wave asymptotic approximation (dashed), Eq.(43), for  $\gamma_1 = 1$ ,  $\alpha = 0.9$ ,  $\epsilon_b = 0.4$  and  $\chi_b = 0.6$ . The first cutoff frequency is located at  $\Omega_* = 0.972354$ , which is very close to the internal cutoff, while the asymptotic expansion (42) provides  $\Omega_* = 0.970768$  (relative error  $-0.16\%$ ).

We look for the harmonic behavior of  $w$ , i.e.  $w(x_i, \tau) = u_i(x_i) \exp(i\Omega\tau)$ ,  $i = 1, 2$ , whence the Eqs.(36) become the pair of ODEs

$$\gamma_1^4 \frac{d^4 u_1}{dx_1^4} + (1 - \Omega^2)u_1 = 0, \quad x_1 \in (-1, 0), \quad (38a)$$

$$\gamma_1^4 \frac{d^4 u_2}{dx_2^4} - \chi_b^4 \Omega^2 u_2 = 0, \quad x_2 \in (0, 1), \quad (38b)$$

where the problem's parameters of interest are

$$\gamma = \frac{\gamma_2}{\gamma_1} = \sqrt[4]{\frac{(EI)_2}{(EI)_1}} \quad \text{and} \quad \chi_b = \frac{\alpha \sqrt[4]{\eta}}{\gamma}. \quad (39)$$

Similarly to Sec.2, Eq.(38a) indicates the existence of the internal cutoff frequency  $\Omega = 1$ .

The continuity conditions at  $x_1 = x_2 = 0$  are

$$u_1(0) = u_2(0), \quad \alpha \frac{du_1}{dx_1}(0) = \frac{du_2}{dx_2}(0), \quad \frac{d^2 u_1}{dx_1^2}(0) = \epsilon_b \frac{d^2 u_2}{dx_2^2}(0), \quad \alpha \frac{d^3 u_1}{dx_1^3}(0) = \epsilon_b \frac{d^3 u_2}{dx_2^3}(0), \quad (40)$$

where

$$\epsilon_b = \frac{\gamma^4}{\alpha^2}. \quad (41)$$

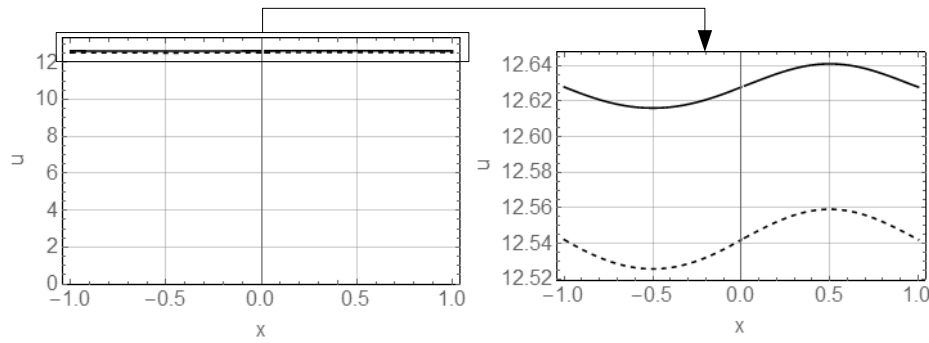
The periodicity conditions read

$$\begin{aligned} u_1(-1) &= u_2(1) \exp(iq), & \alpha \frac{du_1}{dx_1}(-1) &= \frac{du_2}{dx_2}(1) \exp(iq), \\ \frac{d^2 u_1}{dx_1^2}(-1) &= \epsilon_b \frac{d^2 u_2}{dx_2^2}(1) \exp(iq), & \alpha \frac{d^3 u_1}{dx_1^3}(-1) &= \epsilon_b \frac{d^3 u_2}{dx_2^3}(1) \exp(iq). \end{aligned}$$

The dispersion relation is presented in the Appendix in the form of a  $8 \times 8$  determinant. A plot is given in Fig.5 for the parameter set  $\gamma_1 = 1$ ,  $\alpha = 0.9$ ,  $\epsilon_b = 0.4$  and  $\chi_b = 0.6$ .

As in Sec.2, the lowest cutoff frequency ( $\Omega = \Omega_* \approx 1$ ,  $q = 0$ ) can be expanded from the dispersion relation for  $\delta_b = \epsilon_b \chi_b^4$  and  $\chi_b$  small, namely

$$\Omega_*^2 = 1 - \frac{\delta_b}{\alpha} \left( 1 + \frac{\chi_b^4}{720\gamma_1^4} \right) + \dots, \quad (42)$$



**Figure 6.** Numerical (solid) and asymptotic (dashed) eigenforms for  $\gamma_1 = 1$ ,  $\alpha = 0.9$ ,  $\epsilon_b = 0.4$  and  $\chi_b = 0.6$

provided that  $\gamma_1$  and  $\alpha$  are of order unity. We emphasize that the analogous formula for the string (15) does not involve the geometric parameter  $\alpha$  explicitly. The associated eigenforms are

$$u_1(x_1) = 1 + \frac{\alpha^{3/4}}{3\sqrt{2}\gamma_1} \delta_b^{1/4} + \dots$$

and

$$u_2(x_2) = 1 + \frac{x_2^2(x_2 - 1)^2}{24\gamma_1^4} \chi_b^4 + \frac{\alpha^{3/4}}{3\sqrt{2}\gamma_1} \delta_b^{1/4} + \dots,$$

which, at leading order, reduce to a rigid body motion, as in the case of a string. They are shown in Fig.6 together with their numerically evaluated counterparts.

The long-wave approximation of the dispersion relation around the lowest cutoff frequency  $\Omega_*$  reads

$$\Omega^2 - \Omega_*^2 = -\frac{\delta_b}{6\alpha(1 + \alpha)} q^2 + \frac{\alpha\gamma_1^4 \epsilon_b}{(1 + \alpha)^2} q^4 + \dots \quad (43)$$

and its effectiveness is shown in Fig.5. Obviously, the  $O(q^4)$  term here needs be kept when  $q \gtrsim \chi^2$ . The macroscopic governing equation corresponding to (43) is given by the fourth-order PDE

$$EI_* \partial_{xxxx}^4 w - K_{2*} \partial_{xx}^2 w + K_{1*} w + \rho_1 \partial_{tt}^2 w = 0, \quad (44)$$

where

$$\begin{aligned} K_{1*} &= \beta \Omega_*^2, \\ K_{2*} &= -\frac{\delta_b \beta L_1^2 (L_1 + L_2)}{6L_2}, \\ EI_* &= \epsilon_b \alpha \beta \gamma_1^4 L_1 L_2 (L_1 + L_2)^2. \end{aligned}$$

It is worth mentioning that the derived equation is different from the original equation in that it corresponds to a beam supported by a two-parameter Pasternak foundation (see [32, 33, 34] and references therein) with coefficients  $K_{1*}$ ,  $K_{2*}$  instead of a Winkler foundation. Besides, the same equation governs the behavior of a beam pre-stressed by the longitudinal force  $K_{2*}$ , often termed a beam-column [35]. It is also worth noting that high-frequency homogenization of a similar system, in the absence of contrast, leads to a second-order PDE [15].

## 5. Conclusions

In this paper, high-frequency homogenization is adapted for periodic waveguides with internal cutoffs and contrast properties. The analysis is carried out in a Floquet-Bloch fashion for a piecewise uniform string or beam periodically supported by an

elastic Winkler foundation. The case of variable parameters is also considered, for the string problem, through a two-scale asymptotic procedure. In the special case of constant parameters, results match with those obtained by the Floquet-Bloch approach.

Unlike the usual outcome of high-frequency homogenization, micro-scale rigid body motion is retrieved at leading order and, unlike conventional low-frequency homogenization, such quasi-static behavior occurs near a non-zero cutoff frequency. This unusual phenomenon is due to a double cancellation. In fact, on the one hand, over the supported region, the inertia force balances the support reaction because of the proximity of global and internal cutoffs. This occurs when  $\delta_s, \delta_b \ll 1$ . On the other hand, over the unsupported region, the inertia force vanishes on account of strong contrast, namely at  $\chi_s, \chi_b \ll 1$ .

The asymptotic expansions of the dispersion relations in the vicinity of the cutoff frequencies are given and the associated macro-scale governing equations deduced. Although it might be expected that, for the string problem, the macro-scale equation corresponds to that for a string on a Winkler foundation, it is less obvious that, for the beam problem, the governing equation for a beam on a two-parameter Pasternak foundation arises. The latter can be also interpreted as an axially pre-stressed beam (i.e. beam-column) on a Winkler foundation. In this respect, we mention that axially pre-stressed thin structures recently found new interesting applications in the area of meta-materials [36].

## Acknowledgements

This work was partly supported by “Fondazione Cassa di Risparmio di Modena” within the framework of the Progetti di Ricerca finalizzata all’innovazione 2014, Sime n.2013.0662. JK gratefully acknowledges support from Modena University through the Long Visiting Professor programme 2014. AN gratefully acknowledges support from the National Group of Mathematical Physics (GNFM-INdAM) through the “Progetto Giovani Ricercatori 2015” scheme, prot. U2015/000125.

## Appendix

The dispersion relation for the beam problem is given by the determinant

$$\begin{vmatrix} 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ \alpha t & 0 & \alpha t & 0 & -r & 0 & -r & 0 \\ 0 & -t^2 & 0 & t^2 & 0 & \epsilon_b r^2 & 0 & -\epsilon_b r^2 \\ -\alpha t^3 & 0 & \alpha t^3 & 0 & \epsilon_b r^3 & 0 & -\epsilon_b r^3 & 0 \\ -\sin(t) & \cos(t) & -\sinh(t) & \cosh(t) & -\sin(r) & -\cos(r) & -\sinh(r) & -\cosh(r) \\ \alpha t \cos(t) & \alpha t \sin(t) & \alpha t \cosh(t) & \alpha(-t) \sinh(t) & -r \cos(r) & r \sin(r) & -r \cosh(r) & -r \sinh(r) \\ t^2 \sin(t) & -t^2 \cos(t) & -t^2 \sinh(t) & t^2 \cosh(t) & \epsilon_b r^2 \sin(r) & \epsilon_b r^2 \cos(r) & -\epsilon_b r^2 \sinh(r) & -\epsilon_b r^2 \cosh(r) \\ -\alpha t^3 \cos(t) & -\alpha t^3 \sin(t) & \alpha t^3 \cosh(t) & -\alpha t^3 \sinh(t) & \epsilon_b r^3 \cos(r) & -\epsilon_b r^3 \sin(r) & -\epsilon_b r^3 \cosh(r) & -\epsilon_b r^3 \sinh(r) \end{vmatrix} = 0,$$

having let the shorthand notation

$$t = \frac{\sqrt[4]{\omega^2 - 1}}{\gamma_1} \quad \text{and} \quad r = \frac{\chi \sqrt{\omega}}{\gamma_1}.$$

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