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ON THE GENUS OF $\mathbb{S}^m \times \mathbb{S}^n$

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Abstract. By using a recursive algorithm, we construct edge-coloured graphs representing products of spheres and consequently we give upper bounds for the regular genus of $\mathbb{S}^m \times \mathbb{S}^n$, for each $m, n > 0$.

1. Introduction

Throughout this paper we shall work in the PL category. In the following the term "manifold" will denote a closed, connected one and "graph" a finite connected multigraph (i.e. without loops).

An $(n+1)$-*coloured graph (without boundary)* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a graph, regular of degree $n+1$, and $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\}$ a map such that $\gamma(e) \neq \gamma(f)$, for each pair e, f of adjacent edges of Γ. For each $B \subseteq \Delta_n$, the B-residues of (Γ, γ) are the connected components of the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$. For each $c \in \Delta_n$, we set $\hat{c} = \Delta_n \setminus \{c\}$ and we shall write Γ_{cd} instead of $\Gamma_{\{c,d\}}$.

An $(n+1)$-coloured graph is called *contracted* if and only if for every $c \in \Delta_n$, $\Gamma_{\hat{c}}$ is connected.

From now on we often drop the edge-colorations, writing Γ instead of (Γ, γ).

Let K be an n-dimensional pseudocomplex, the *disjoint star* $std(s, K)$ of a simplex s in K is the disjoint union of the n-simplexes containing s, with re-identification of the $(n-1)$-simplexes containing s and of all their faces; the *disjoint link* of s in K is the complex $lkd(s, K) = \{t \in std(s, K) | s \cap t = \emptyset\}$.

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A coloured n-complex is a homogeneous pseudocomplex K together with a "coloration" of its vertices by Δ_n, which is injective on every simplex.

Given an $(n+1)$-coloured graph Γ, we can construct a coloured n-complex $K(\Gamma)$ in the following way:

- take an n-simplex $s(v)$ for each $v \in V(\Gamma)$ and label its vertices by Δ_n;
- for each $c \in \Delta_n$ and each pair v, w of c-adjacent vertices in Γ, identify the $(n-1)$-faces of $s(v)$ and $s(w)$ opposite to the vertices labelled c, so that equally labelled vertices coincide.

The above construction can be easily reversed in order to associate an $(n+1)$-coloured graph $\Gamma(K)$ to each coloured n-complex K. Therefore these constructions give rise to a correspondence between $(n+1)$-coloured graphs and coloured n-complexes.

It is easy to see that $\Gamma(K(\Gamma)) = \Gamma$; conversely $K(\Gamma(K)) = K$ if and only if the disjoint star of every simplex in K is strongly connected. In this case $|K|$ is said to be *represented* by Γ.

A contracted $(n+1)$-coloured graph representing a manifold M is called a *crystallization* of M.

By results in [8] and [3], every n-manifold admits crystallizations.

The above definitions, together with a general survey on edge-coloured graphs, can be found in [4].

Given an $(n+1)$-coloured graph Γ, each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ of Δ_n defines a particular imbedding (called *regular*) of Γ into a closed surface F_ε, whose Euler characteristic is (see [5] and [6]):

$$(*) \qquad \chi(F_\varepsilon) = \sum_{i \in \mathbb{Z}_{n+1}} g_{\varepsilon_i \varepsilon_{i+1}}(\Gamma) + \frac{1}{2}(1-n)p(\Gamma)$$

where $g_{ij}(\Gamma)$ is the number of connected components of Γ_{ij} and $p(\Gamma)$ is the number of vertices of Γ.

F_ε is orientable or non-orientable according to Γ being bipartite or not.

The *regular genus* $\rho(\Gamma)$ of Γ is defined as:

$$\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) | \varepsilon \text{ is a cyclic permutation of } \Delta_n\}$$

where $\rho_\varepsilon(\Gamma)$ denotes the genus of F_ε.

Given an n-manifold M the *regular genus* of M is the minimum among the regular genera of the graphs representing M.

In the following we shall describe a construction, introduced in [7], which, starting from two coloured graphs representing two polyhedra, produces a coloured graph representing their product.

If we apply this construction to the product of spheres, we get several simplifications which, given $m, n > 0$, allow us to build, by inductive steps, a graph representing $\mathbb{S}^m \times \mathbb{S}^n$. Furthermore we obtain some relations among the numbers of coloured cycles in the resulting graphs, by which we can find a "minimal" permutation (i.e. a cyclic permutation defining a regular imbedding of minimal genus) and we can compute the genera of these graphs in a recursive way. We also give direct formulas in the particular cases of $m = 2, 3$.

2. Representing products by edge-coloured graphs

We briefly outline the construction introduced in [7], to obtain "products" of coloured graphs.

Let σ^m (resp. τ^n) be an m-dimensional (resp. n-dimensional) simplex, whose vertices are labelled by $\{v_0, \ldots, v_m\}$ (resp. by $\{w_0, \ldots, w_n\}$); then the set of the vertices of the product ball complex $\sigma^m \times \tau^n$ is $\{(v_r, w_s) | r \in \Delta_m, s \in \Delta_n\}$.

Let $\mathbf{A}(\sigma^m, \tau^n)$ $(m, n > 0)$ be the matrix with $(m+n+1)$ columns, whose $\binom{m+n}{n}$ rows are sequences of elements of

$$\{v_0, \ldots, v_m\} \times \{w_0, \ldots, w_n\}$$

of the following type:

$$(v_m, w_n) = (v_{r_m}, w_{s_n}), \ldots, (v_{r_0}, w_{s_0}) = (v_0, w_0)$$

$$0 \leq r_0 \leq r_1 \leq \cdots \leq r_m = m, 0 \leq s_0 \leq s_1 \leq \cdots \leq s_n = n.$$

These elements can be thought as "words" of length $(m+n+1)$ in the alphabet $\{v_0, \ldots, v_m\} \times \{w_0, \ldots, w_n\}$, lexicographically ordered, where each "letter" is obtained by decreasing by one, at each step, the index of one and only one of the two components v_r and w_s.

The sequences represent the $(m+n+1)$ vertices of $\{v_0, \ldots, v_m\} \times \{w_0, \ldots, w_n\}$ which span the maximal simplexes of a simplicial triangulation $\sigma^m \boxtimes \tau^n$ of $\sigma^m \times \tau^n$ (see [2], [9]).

The matrix $\mathbf{A}(\sigma^m, \tau^n)$ can be constructed according to the following scheme:

$$\mathbf{A}(\sigma^m, \tau^n) = \begin{pmatrix} (v_m, w_n) & \\ \cdot & \mathbf{B} = \mathbf{A}(\sigma^{m-1}, \tau^n) \\ \cdot & \\ \cdot & \mathbf{C} = \mathbf{A}(\sigma^m, \tau^{n-1}) \\ (v_m, w_n) & \end{pmatrix}$$

where $\mathbf{A}(\sigma^{m-1}, \tau^n)$ (resp. $\mathbf{A}(\sigma^m, \tau^{n-1})$) represents the simplicial complex $\sigma^{m-1} \boxtimes \tau^n$ (resp. $\sigma^m \boxtimes \tau^{n-1}$), obtained by deleting the vertex v_m (resp. w_n) from σ^m (resp. τ^n) and has $\binom{m+n-1}{n}$ (resp. $\binom{m+n-1}{m}$) rows.

Let Γ' (resp. Γ'') be an $(m+1)$-coloured (resp. $(n+1)$-coloured) graph, an $(m+n+1)$-coloured graph $\Gamma' \boxtimes \Gamma''$ representing $|K(\Gamma') \times K(\Gamma'')|$ can be obtained in the following way:

- for each pair (α^i, β_j) of vertices of $V(\Gamma') \times V(\Gamma'')$, consider the $\binom{m+n}{n}$ vertices $\delta^i_j(k)$ which are in one-to-one correspondence with the rows of the matrix $\mathbf{A}(\sigma^m_i, \tau^n_j)$, where σ^m_i (resp. τ^n_j) is the m-simplex (resp. n-simplex) of $K(\Gamma')$ (resp. of $K(\Gamma'')$) represented by α^i (resp. by β_j);
- set $V(\Gamma' \boxtimes \Gamma'') = \{\delta^i_j(k) | i = 1, \dots, \#V(\Gamma'), j = 1, \dots, \#V(\Gamma''), k = 1, \dots, \binom{m+n}{n}\}$;
- for each vertex $\delta^i_j(k) \in V(\Gamma' \boxtimes \Gamma'')$ $(i = 1, \dots, \#V(\Gamma'), j = 1, \dots, \#V(\Gamma''), k = 1, \dots, \binom{m+n}{n})$, let us denote by ω_k its corresponding row of $\mathbf{A}(\sigma^m_i, \tau^n_j)$; then:

a) for each $d \in \Delta_{m+n}$, delete from ω_k the unique element (v_r, w_s) such that $r + s = d$, yielding a sequence $\omega_k(\hat{d})$. If there exists another row ω_h of $\mathbf{A}(\sigma^m_i, \tau^n_j)$ such that $\omega_k(\hat{d}) = \omega_h(\hat{d})$, then the way the matrix is constructed guarantees that it is unique; in this case join $\delta^i_j(k)$ and $\delta^i_j(h)$ by a d-coloured edge;

b) if v_r (resp. w_s) appears exactly once in a pair $(v_r, w_{s'})$ (resp. $(v_{r'}, w_s)$) of ω_k for some $r \in \Delta_m$ (resp. for some $s \in \Delta_n$), let α^t (resp. β_t) be the vertex of Γ' (resp. of Γ'') r-adjacent with α^i (resp. s-adjacent with β_j). Join $\delta^i_j(k)$ and $\delta^t_j(k)$ (resp. $\delta^i_j(k)$ and $\delta^i_t(k)$) by a d-coloured edge, with $d = r + s'$ (resp. $d = r' + s$).

In the particular case of products of spheres, we can simplify the above procedure by using the standard $(p+1)$-coloured graph $\Gamma^{(p)}$ representing $\mathbb{S}^p$ and having two vertices joined by $p+1$ edges.

Starting from $\Gamma^{(m)}$ and $\Gamma^{(n)}$, we construct $\Gamma^{(m)} \boxtimes \Gamma^{(n)}$ as follows:

- $\#V(\Gamma^{(m)} \boxtimes \Gamma^{(n)}) = 4\binom{m+n}{n}$;

- if $\omega_k(\hat{d}) = \omega_h(\hat{d})$, join $\delta_j^i(k)$ and $\delta_j^i(h)$ $(i,j = 1,2)$ by a d-coloured edge;
- if v_r (resp. w_s) appears exactly once in a pair $(v_r, w_{s'})$ (resp. $(v_{r'}, w_s)$) of ω_k, join $\delta_1^1(k)$ with $\delta_1^2(k)$ (resp. with $\delta_2^1(k)$) and $\delta_2^2(k)$ with $\delta_2^1(k)$ (resp. with $\delta_1^2(k)$) by a d-coloured edge, with $d = r + s'$ (resp. $d = r' + s$).

It is clear that the structure of this "product" graph depends only on the structure of the matrix $\mathbf{A}(\sigma^m, \tau^n)$; moreover, the inductive construction of $\mathbf{A}(\sigma^m, \tau^n)$ allows us to describe a method to build $\Gamma^{(m)} \boxtimes \Gamma^{(n)}$, starting from $\Gamma^{(m-1)} \boxtimes \Gamma^{(n)}$ and $\Gamma^{(m)} \boxtimes \Gamma^{(n-1)}$, without further reference to $\mathbf{A}(\sigma^m, \tau^n)$. Construct an $(m+n+1)$-coloured graph $\Gamma^{(m,n)}$ as follows:

- $V(\Gamma^{(m,n)}) = V(\Gamma^{(m-1)} \boxtimes \Gamma^{(n)}) \cup V(\Gamma^{(m)} \boxtimes \Gamma^{(n-1)}) = \{\bar{\delta}_j^i(k) | i,j = 1,2 \ \ k = 1, \ldots, \binom{m+n-1}{n}\} \cup \{\bar{\bar{\delta}}_j^i(k) | i,j = 1,2 \ \ k = 1, \ldots, \binom{m+n-1}{m}\}$;
- for each $k = 1, \ldots, \binom{m+n-1}{n}$ (resp. $k = 1, \ldots, \binom{m+n-1}{m}$) join $\bar{\delta}_1^1(k)$ with $\bar{\delta}_1^2(k)$ (resp. $\bar{\bar{\delta}}_1^1(k)$ with $\bar{\bar{\delta}}_2^1(k)$) and $\bar{\delta}_2^2(k)$ with $\bar{\delta}_2^1(k)$ (resp. $\bar{\bar{\delta}}_2^2(k)$ with $\bar{\bar{\delta}}_1^2(k)$) by an $(m+n)$-coloured edge;
- for each $k = \binom{m+n-2}{n} + 1, \ldots, \binom{m+n-2}{n} + \binom{m+n-2}{n-1}$ join $\bar{\delta}_j^i(k)$ and $\bar{\bar{\delta}}_j^i(k - \binom{m+n-2}{n})$ $(i,j = 1,2)$ by an $(m+n-1)$-coloured edge; for the remaining vertices of $\Gamma^{(m,n)}$ re-establish the edges as they are in $\Gamma^{(m-1)} \boxtimes \Gamma^{(n)}$ and $\Gamma^{(m)} \boxtimes \Gamma^{(n-1)}$.

Proposition 1. $\Gamma^{(m,n)} = \Gamma^{(m)} \boxtimes \Gamma^{(n)}$.

Proof. Note that, for each $d \neq m+n-1$, if two rows of the submatrix $\mathbf{B}$ (resp. $\mathbf{C}$) of $\mathbf{A}$ corresponding to $\mathbf{A}(\sigma^{m-1}, \tau^n)$ (resp. $\mathbf{A}(\sigma^m, \tau^{n-1})$), say ω_k and ω_h, lead to equal sequences $\omega_k(\hat{d})$ and $\omega_h(\hat{d})$ in $\mathbf{B}$ (resp. in $\mathbf{C}$) they also lead to equal sequences in $\mathbf{A}$; furthermore if v_r or $w_r \neq w_n$ (resp. w_s or $v_s \neq v_m$) appears once in a row of $\mathbf{B}$ (resp. of $\mathbf{C}$), then it appears once in the same row of $\mathbf{A}$. Thus all d-coloured edges ($d \neq m+n-1$) of $\Gamma^{(m-1,n)}$ and $\Gamma^{(m,n-1)}$ remain unchanged in $\Gamma^{(m,n)}$;

Furthermore, following the more detailed scheme below for the matrix $\mathbf{A}(\sigma^m, \tau^n)$, it is easy to see that:

$$\mathbf{A} = \mathbf{A}(\sigma^m, \tau^n) = \begin{pmatrix} (v_m, w_n) & (v_{m-1}, w_n) & \mathbf{B}'' = \mathbf{A}(\sigma^{m-2}, \tau^n) \\ \cdot & (v_{m-1}, w_n) & \mathbf{B}' \ = \mathbf{A}(\sigma^{m-1}, \tau^{n-1}) \\ \cdot & (v_m, w_{n-1}) & \mathbf{C}' \ = \mathbf{A}(\sigma^{m-1}, \tau^{n-1}) \\ (v_m, w_n) & (v_m, w_{n-1}) & \mathbf{C}'' \ = \mathbf{A}(\sigma^m, \tau^{n-2}) \end{pmatrix}$$

a) w_n (resp. v_m) appears once in all rows of the submatrix $\mathbf{B}'$ (resp. $\mathbf{C}'$) corresponding to $\mathbf{A}(\sigma^{m-1}, \tau^{n-1})$, but twice in all the corresponding rows of $\mathbf{A}$, i.e. all the $(m+n-1)$-coloured edges of

$\Gamma^{(m-1,n)}$ and $\Gamma^{(m,n-1)}$ joining the vertices corresponding to $\mathbf{B}'$ and $\mathbf{C}'$ disappear in $\mathbf{A}$;

b) each row of $\mathbf{B}'$, with the element (v_{m-1}, w_n) deleted, is equal to a row of $\mathbf{C}'$, with the element (v_m, w_{n-1}) deleted, therefore the corresponding vertices are joined by $(m+n-1)$-coloured edges;

c) v_m (resp. w_n) appears once in the first $\binom{m+n-1}{n}$ (resp. in the last $\binom{m+n-1}{m}$) rows of $\mathbf{A}$, therefore the corresponding vertices are joined by $(m+n)$-coloured edges.

□

Using the above construction and starting from the $(r+2)$-coloured graphs $\Gamma^{(1,r)}$ and $\Gamma^{(r,1)}$ $(r \geq 1)$, it is possible to build by successive steps, the $(m+n+1)$-coloured graph $\Gamma^{(m,n)}$, for each $m, n > 0$.

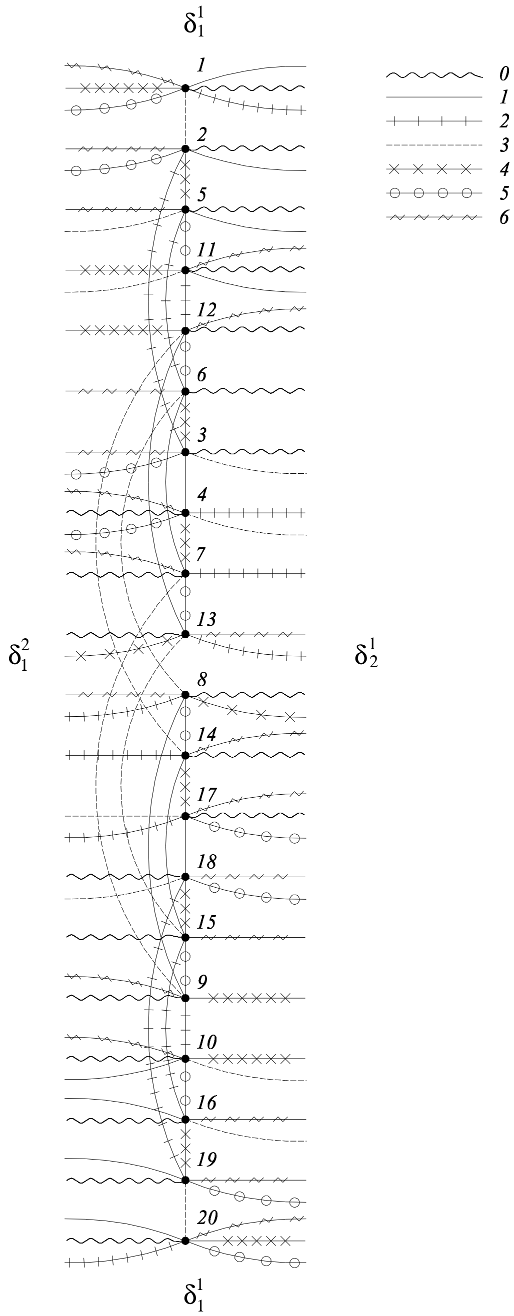
Remark 1. Note that all $\Gamma^{(m,n)}$ have a double simmetry. In fact, for each $k = 1, \ldots, \binom{m+n}{n}$, each edge between the vertices $\bar{\delta}_1^1(k)$ and $\bar{\delta}_1^2(k)$ (resp. $\bar{\bar{\delta}}_1^1(k)$ and $\bar{\bar{\delta}}_2^1(k)$) has a corresponding edge, with the same colour, between $\bar{\delta}_2^2(k)$ and $\bar{\delta}_2^1(k)$ (resp. $\bar{\bar{\delta}}_2^2(k)$ and $\bar{\bar{\delta}}_1^2(k)$).

An easily implemented program allows us to build $\Gamma^{(m,n)}$ for each $m, n > 0$.

As an example, figure 1 shows $\Gamma^{(3,3)}$. Since its number of vertices is too big $(= 80)$ to fit the picture, we only drew part of the graph, which, because of the simmetries, is sufficient to represent the whole of it.

3. The genus of $\Gamma^{(m,n)}$

Let us denote by g_{cd}, where $c, d \in \Delta_{m+n}$ (resp. $\bar{g}_{cd}$ where $c, d \in \Delta_{m+n-1}$) (resp. $\bar{\bar{g}}_{cd}$ where $c, d \in \Delta_{m+n-1}$) the number of connected components of $\Gamma_{cd}^{(m,n)}$ (resp. $\Gamma_{cd}^{(m-1,n)}$) (resp. $\Gamma_{cd}^{(m,n-1)}$). Moreover, let $\alpha_{m,n}^c$ (resp. $\beta_{m,n}^c$) $(c \in \Delta_{m+n-2})$ denote the number of $\{c, m+n\}$-residues of length two of $\Gamma^{(m,n)}$, whose vertices correspond to rows of the submatrix $\mathbf{A}(\sigma^m, \tau^{n-1})$ (resp. $\mathbf{A}(\sigma^{m-1}, \tau^n)$) of $\mathbf{A}(\sigma^m, \tau^n)$ (see the scheme above).



δ_1^1
1
2
5
11
12
6
3
4
7
13
δ_1^2
δ_2^1
8
14
17
18
15
9
10
16
19
20
δ_1^1
0
1
2
3
4
5
6

Figure 1.

Lemma 2. *We have the following equalities*:

$$g_{cd} = \bar{g}_{cd} + \bar{\bar{g}}_{cd} \quad \text{for each } c, d \in \Delta_{m+n-2}$$

$$g_{c\ m+n} = \bar{g}_{c\ m+n-1} + \bar{\bar{g}}_{c\ m+n-1} \quad \text{for each } c \in \Delta_{m+n-2}$$

$$g_{m+n-1\ m+n} = \binom{m+n-2}{n-1} + 2\binom{m+n-2}{n} + 2\binom{m+n-2}{n-2}$$

$$g_{c\ m+n-1} = \bar{g}_{c\ m+n-1} + \bar{\bar{g}}_{c\ m+n-1} - \frac{1}{2}(\alpha^c_{m-1,n} + \beta^c_{m,n-1})$$

$$\text{for each } c \in \Delta_{m+n-3}$$

$$g_{m+n-2\ m+n-1} = \bar{g}_{m+n-2\ m+n-1} + \bar{\bar{g}}_{m+n-2\ m+n-1} - \binom{m+n-2}{n-1}.$$

Proof. By the construction of section 2 it is clear that all c-coloured edges ($c \in \Delta_{m+n-2}$) of $\Gamma^{(m,n)}$ are the same as in $\Gamma^{(m-1,n)}$ and $\Gamma^{(m,n-1)}$, while the $(m+n)$-coloured edges in $\Gamma^{(m,n)}$ take the places of the $(m+n-1)$-coloured edges of $\Gamma^{(m-1,n)}$ and $\Gamma^{(m,n-1)}$; therefore we obtain equalities 1) and 2).

To prove the third equality, recall the scheme for $\mathbf{A}(\sigma^m, \tau^n)$ in the proof of Proposition 1.

Note that, for each row ω_k of $\mathbf{B}''$ (resp. $\mathbf{C}''$), we have two $\{m+n-1, m+n\}$-residues, whose sets of vertices are $\{\bar{\delta}^1_1(k), \bar{\delta}^2_1(k)\}$ and $\{\bar{\delta}^2_2(k), \bar{\delta}^1_2(k)\}$ (resp. $\{\bar{\bar{\delta}}^1_1(k), \bar{\bar{\delta}}^1_2(k)\}$ and $\{\bar{\bar{\delta}}^2_2(k), \bar{\bar{\delta}}^2_1(k)\}$).

Furthermore, for each $k = \binom{m+n-2}{n} + 1, \ldots, \binom{m+n-2}{n} + \binom{m+n-2}{n-1}$, we have only one $\{m+n-1, m+n\}$-residue, whose set of vertices is $\{\bar{\delta}^1_1(k), \bar{\delta}^2_1(k), \bar{\delta}^2_2(k), \bar{\delta}^1_2(k), \bar{\bar{\delta}}^1_1(h), \bar{\bar{\delta}}^1_2(h), \bar{\bar{\delta}}^2_2(h), \bar{\bar{\delta}}^2_1(h)\}$, where $h = k - \binom{m+n-2}{n}$ (see figure 2). Equality 3) follows.

Let us now consider the $\{c, m+n-1\}$-residues of $\Gamma^{(m-1,n)}$ and $\Gamma^{(m,n-1)}$ ($c \in \Delta_{m+n-2}$); note that those having all vertices corresponding to rows of $\mathbf{B}''$ or $\mathbf{C}''$ don't change in $\Gamma^{(m,n)}$.

For $c \neq m+n-2$, we have the following situations:

(i) for every pair of length two $\{c, m+n-1\}$-residues of $\Gamma^{(m-1,n)}$ (resp. $\Gamma^{(m,n-1)}$) corresponding to a row ω_k of $\mathbf{B}'$ (resp. $\mathbf{C}'$), there exists exactly one $\{c, m+n-1\}$-residue of $\Gamma^{(m,n-1)}$ (resp. $\Gamma^{(m-1,n)}$) of length four, whose vertices correspond to the row ω_h of $\mathbf{C}'$, with $h = k - \binom{m+n-2}{n}$ (resp. of $\mathbf{B}'$ with $h = k + \binom{m+n-2}{n}$) and conversely;

(ii) for every pair of length four $\{c, m+n-1\}$-residues of $\Gamma^{(m-1,n)}$, whose sets of vertices are $\{\bar{\delta}^1_i(k), \bar{\delta}^1_i(h) | i = 1,2\}$ and $\{\bar{\delta}^2_i(k), \bar{\delta}^2_i(h) | i = 1,2\}$, corresponding to the rows ω_k and ω_h of $\mathbf{B}'$, there exists exactly two $\{c, m+n-1\}$-residues of $\Gamma^{(m,n-1)}$ of length four, whose sets of vertices are $\{\bar{\bar{\bar{\delta}}}_1^i(k'), \bar{\bar{\bar{\delta}}}_1^i(h') | i = 1,2\}$ and $\{\bar{\bar{\bar{\delta}}}_2^i(k'), \bar{\bar{\bar{\delta}}}_2^i(h') | i = 1,2\}$, corresponding to the rows $\omega_{k'}$ and $\omega_{h'}$ of C', with $k' = k - \binom{m+n-2}{n}$ and $h' = h - \binom{m+n-2}{n}$ and conversely.

These are the only $\{c, m+n-1\}$-residues which change in $\Gamma^{(m,n)}$. It is easy to see that in case (ii) the number of the residues doesn't change and in case (i) the three residues produce two of length four in $\Gamma^{(m,n)}$.

Finally, let us consider the case $c = m+n-2$. The only $\{m+n-2, m+n-1\}$-residues changing in $\Gamma^{(m,n)}$, are as follows:

(iii) for each $k = \binom{m+n-2}{n} + 1, \ldots, \binom{m+n-2}{n} + \binom{m+n-3}{n-1}$ (resp. $k = \binom{m+n-3}{n-1} + 1, \ldots, \binom{m+n-2}{n-1}$), there is exactly one $\{m+n-2, m+n-1\}$-residue of length eight in $\Gamma^{(m-1,n)}$ (resp. in $\Gamma^{(m,n-1)}$), whose set of vertices is $\{\bar{\delta}_j^i(k), \bar{\delta}_j^i(h) | i,j = 1,2\}, h = k - \binom{m+n-3}{n-1}$ (resp. $\{\bar{\bar{\bar{\delta}}}_j^i(k), \bar{\bar{\bar{\delta}}}_j^i(h) | i,j = 1,2\}, h = k + \binom{m+n-3}{n-2}$), to which corresponds a pair of length two $\{m+n-2, m+n-1\}$-residues of $\Gamma^{(m,n-1)}$ (resp. of $\Gamma^{(m-1,n)}$), whose sets of vertices are $\{\bar{\bar{\bar{\delta}}}_1^1(k'), \bar{\bar{\bar{\delta}}}_1^2(k')\}$ and $\{\bar{\bar{\bar{\delta}}}_2^2(k'), \bar{\bar{\bar{\delta}}}_2^1(k')\}$, with $k' = k - \binom{m+n-2}{n}$ (resp. $\{\bar{\delta}_1^1(k'), \bar{\delta}_2^1(k')\}$ and $\{\bar{\delta}_2^2(k'), \bar{\delta}_1^2(k')\}$, with $k' = k + \binom{m+n-2}{n}$).

Since, as can be directly seen, every three residues which correspond, yield two of length six in $\Gamma^{(m,n)}$, equality 5) easily follows. □

Let us now consider the graphs $\Gamma^{(1,n)}$ $(n = 1, 2, \ldots)$, which are shown in figure 3.

An easy calculation gives:

$$g_{01} = g_{02} = \cdots = g_{0n} = 2n - 1$$

$$g_{0\ n+1} = 2n$$

$$g_{1\ n+1} = g_{2\ n+1} = \cdots = g_{n\ n+1} = 2n - 1$$

$$g_{cd} = 2(n-1) \quad \text{for each } c = 1, \ldots, n-1 \text{ and for each } d = 1, \ldots, n.$$

The following result guarantees that similar relations hold among the number of residues g_{cd} of $\Gamma^{(m,n)}$ $(m, n > 0)$:

Proposition 2. *For each $m, n > 0$, there exist constants $r_{m,n}, s_{m,n}, t_{m,n}, u_{m,n}$ such that*

$$g_{0c} = g_{c\ m+n} = r_{m,n} \quad \textit{for each } c = 1, \ldots, m+n-1$$

$$g_{0\ m+n} = s_{m,n}$$

$$g_{c\ c+1} = t_{m,n} \quad \textit{for each } c = 1, \ldots, m+n-2$$

$$g_{cd} = u_{m,n} \quad \textit{for each } c, d = 1, \ldots, m+n-1 \textit{ and } d \neq c+1.$$

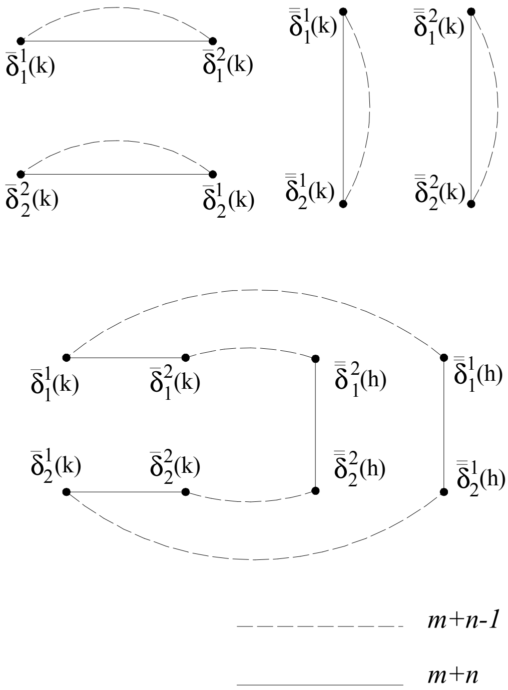


Figure 2.

Furthermore, if $m > 1$ *and* $n > 1$

$$r_{m,n} = \binom{m+n-2}{n-1} + 2\binom{m+n-2}{n} + 2\binom{m+n-2}{n-2}$$

and

$$\begin{aligned} r_{m,n} =& r_{m-1,n} + r_{m,n-1} \\ s_{m,n} =& s_{m-1,n} + s_{m,n-1} \\ t_{m,n} =& t_{m-1,n} + t_{m,n-1} \\ u_{m,n} =& u_{m-1,n} + u_{m,n-1} \quad (\text{if } m > 2 \quad \text{or } n > 2) \end{aligned}$$

with $t_{m,n} \leq u_{m,n} \leq r_{m,n} \leq s_{m,n}$, *for each* $\{m, n\} \neq \{1, 2\}$.

Proof. If $c, d \neq m+n-1$ or $c, d \in \{m+n-2, m+n-1\}$, it follows easily by induction on m and n, and making use of equalities 1) - 3) and 5). An easy calculation shows that $r_{m,n} = r_{m-1,n} + r_{m,n-1}$. Furthermore, it is easy to see that, for each $c \neq m+n-2$, we have:

$$\alpha^c_{m-1,n} = \alpha^c_{m-2,n} + \alpha^c_{m-1,n-1} \text{ and } \beta^c_{m,n-1} = \beta^c_{m,n-2} + \beta^c_{m-1,n-1}.$$

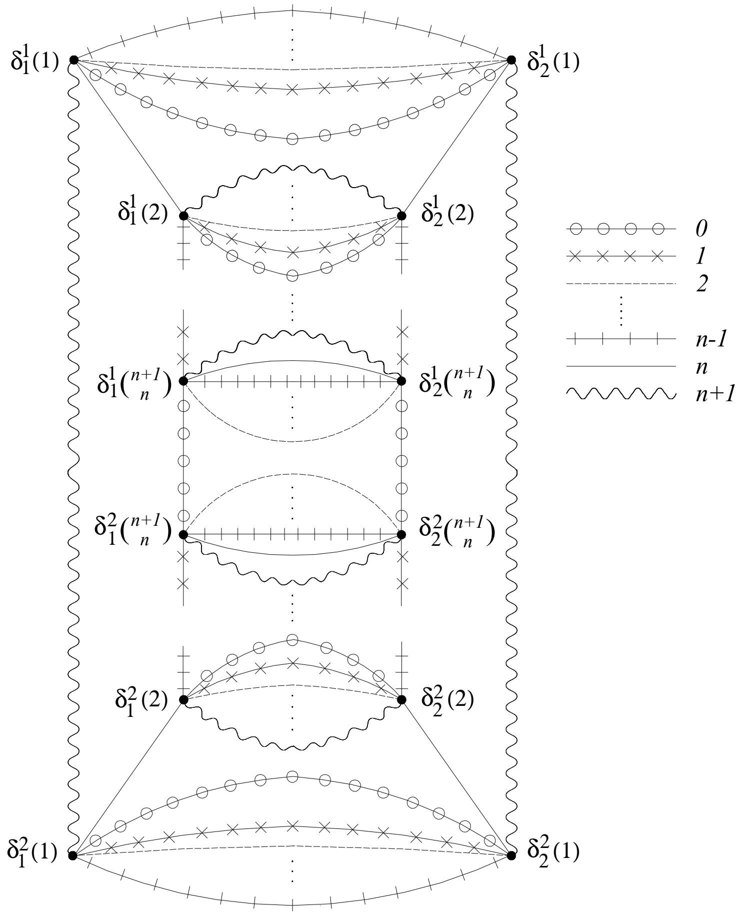


Figure 3.

Therefore, by applying induction to equality 4), we complete the proof. □

Let us consider now a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+n})$ of Δ_{m+n}. We can always suppose that $\varepsilon_{m+n} = m+n$. It is clear, by formula (*), that for an ε corresponding to a surface F_ε of minimal genus for $\Gamma^{(m,n)}$, the sum $\sum_{i \in \mathbb{Z}_{m+n}} g_{\varepsilon_i \varepsilon_{i+1}}$ must be maximal.

First note that, by Proposition 2, $g_{i\ i+1} \leq g_{ij}$ for each $i, j \neq 0, m+n$ and $i \neq m+n-1$.

Therefore it is sufficient to consider permutations which have all pairs $\varepsilon_i, \varepsilon_{i+1}$ (with $\varepsilon_i, \varepsilon_{i+1} \notin \{0, m+n\}$) made by non-consecutive numbers (i.e. $\varepsilon_{i+1} \neq \varepsilon_i + 1$ and conversely). There are essentially two types of such permutations:

$$\varepsilon^{(1)} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-1}, 0, \varepsilon_{k+1}, \dots, \varepsilon_{m+n-1}, m+n) \quad \text{if 0 is not "near" } (m+n)$$

$$\varepsilon^{(2)} = (0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m+n-2}, \varepsilon_{m+n-1}, m+n) \quad \text{if 0 is "near" } (m+n)$$

where all pairs $\varepsilon_i, \varepsilon_{i+1}$ are non-consecutive numbers.

If $m+n > 4$ we can always build such permutations in the following way:

- $\varepsilon^{(1)}$: if $(m+n)$ is even (resp. odd) put all even (resp. odd) numbers after 0 and all odd (resp. even) before 0;
- $\varepsilon^{(2)}$: if $(m+n)$ is even (resp. odd) put first all the odd (resp. even) numbers and then the even (resp. odd) ones, all in increasing order.

From now on we suppose $m+n > 4$. Let us compute $\sum_{i\in\mathbb{Z}_{m+n}} g_{\varepsilon_i\varepsilon_{i+1}}$ for $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$:

$$\begin{aligned} &\varepsilon^{(1)} : g_{\varepsilon_0\varepsilon_1} + \cdots + g_{\varepsilon_{k-2}\varepsilon_{k-1}} + g_{\varepsilon_{k-1}0} + g_{0\varepsilon_{k+1}} + \cdots \\ &\quad + g_{\varepsilon_{m+n-1}m+n} + g_{m+n\varepsilon_0} \\ &= (k-1)u_{m,n} + r_{m,n} + r_{m,n} + (m+n-k-2)u_{m,n} + r_{m,n} + r_{m,n} \\ &= 4r_{m,n} + (m+n-3)u_{m,n} \end{aligned}$$

$$\begin{aligned} &\varepsilon^{(2)} : g_{0\varepsilon_1} + g_{\varepsilon_1\varepsilon_2} + \cdots + g_{\varepsilon_{m+n-2}\varepsilon_{m+n-1}} + g_{\varepsilon_{m+n-1}m+n} + g_{m+n\ 0} \\ &= r_{m,n} + (m+n-2)u_{m,n} + r_{m,n} + s_{m,n} \\ &= 2r_{m,n} + s_{m,n} + (m+n-2)u_{m,n} \end{aligned}$$

It is easy to see, by using induction, that $2r_{m,n} = s_{m,n} + t_{m,n}$. Since $t_{m,n} \le u_{m,n}$ we have $2r_{m,n} \le s_{m,n} + u_{m,n}$.

Comparing the above inequalities with the formulas just found, we have:

$$\sum_{i\in\mathbb{Z}_{m+n}} g_{\varepsilon_i^{(1)}\varepsilon_{i+1}^{(1)}} \le \sum_{i\in\mathbb{Z}_{m+n}} g_{\varepsilon_i^{(2)}\varepsilon_{i+1}^{(2)}}$$

Hence, by applying formula (*) to $\Gamma^{(m,n)}$ and $\varepsilon^{(2)}$, we can state the following result for the genus of the "product" graphs:

PROPOSITION 3. *For each $m, n > 0$, $m+n > 4$, we have:*

$$\rho(\Gamma^{(m,n)}) = 1 - r_{m,n} - \frac{1}{2}s_{m,n} - \frac{1}{2}(m+n-2)u_{m,n} + (m+n-1)\binom{m+n}{n}.$$

Remark 2. If $m+n=4$, the only interesting case for the genus is for $m=n=2$ (since all $\Gamma^{(1,n)}$ have genus 1 (see [7])). We can't find a permutation of type $\varepsilon^{(2)}$ for Δ_4, since we always have at least two consecutive numbers, therefore we must compare the sum of the $g_{\varepsilon_i\varepsilon_{i+1}}$'s for the two permutations: (3,1,0,2,4) and (0,1,3,2,4). The calculation shows that both permutations are minimal and the genus of $\Gamma^{(2,2)}$ turns out to be 4. Actually this is the regular genus of $\mathbb{S}^2 \times \mathbb{S}^2$, as proved in [7].

Let us consider some particular cases:

Proposition 4. *For each* $n \geq 3$, $\rho(\Gamma^{(2,n)}) = n^2 - 1$.

Proof.

$$\begin{aligned} r_{2,n} =& 2n-1+r_{2,n-1} \\ s_{2,n} =& 2n+s_{2,n-1} \\ t_{2,n} =& 2(n-1)+t_{2,n-1} \\ u_{2,n} =& 2(n-1)+u_{2,n-1}. \end{aligned}$$

Moreover $t_{2,n} = s_{2,n-1}$ for each $n \geq 1$. In fact $t_{2,2} = s_{2,1} = 4$ (see figure 3) and supposing that $t_{2,n-1} = s_{2,n-2}$, it follows:

$$t_{2,n} = 2(n-1)+t_{2,n-1} = 2(n-1)+s_{2,n-2} = s_{1,n-1}+s_{2,n-2} = s_{2,n-1}$$

Similar calculations give: $r_{2,n} = n + s_{2,n-1}$ and $u_{2,n} = 2 + s_{2,n-1}$.
Furthermore:

$$\begin{aligned} s_{2,n} &= 2n+s_{2,n-1} = 2n+2(n-1)+s_{2,n-2} \\ &= \cdots = 2n+2(n-1)+2(n-2)+\cdots+4+4 \\ &= 2(n+(n-1)+(n-2)+\cdots+2+1)-2+4 = n(n+1)+2 \end{aligned}$$

Applying the equalities above, we have:

$$\begin{aligned} r_{2,n} &= n+n(n-1)+2 = n^2+2 \\ u_{2,n} &= 2+n(n-1)+2 = n^2-n+4 \end{aligned}$$

Suppose now $n > 2$ and compute the genus of $\Gamma^{(2,n)}$ using Proposition 3.

$$\begin{aligned} \rho(\Gamma^{(2,n)}) &= 1 - r_{2,n} - \frac{1}{2}s_{2,n} - \frac{1}{2}nu_{2,n} + (n+1)\binom{n+2}{n} \\ &= 1 - n^2 - 2 - \frac{1}{2}(n(n+1)+2) - \frac{1}{2}n(n^2-n+4) \\ &+ \ \frac{1}{2}(n+1)^2(n+2) = n^2-1. \end{aligned}$$

□

As a direct consequence of the formula above, we have

Corollary 4. *For each* $n \geq 3$, $\mathcal{G}(\mathbb{S}^2 \times \mathbb{S}^n) \leq n^2 - 1$.

Remark 3. If $n = 3$ the statement of Corollary 4 is actually an equality, as proved in [1, Corollary I].

Proposition 4 and Corollary 4, together with Remarks 2 and 3, suggest the following:

Conjecture. For each $n \geq 3$, $\mathcal{G}(\mathbb{S}^2 \times \mathbb{S}^n) = n^2 - 1$.

Proposition 5. *For each* $n \geq 1$, $\rho(\Gamma^{(3,n)}) = \frac{2}{3}n^3 + n^2 - \frac{2}{3}n$.

Proof.

$$\begin{aligned}
r_{3,n} &= r_{2,n} + r_{2,n-1} + r_{2,n-2} + \cdots + r_{2,2} + r_{3,1} \\
&= (n^2+2) + ((n-1)^2+2) + \cdots + (4+2) + 5 \\
&= \sum_{i=1}^{n} i^2 + 2n + 2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{13}{6}n + 2 \\
s_{3,n} &= (n(n+1)+2) + ((n-1)n+2) + \cdots + (6+2) + 6 \\
&= \sum_{i=1}^{n} i(i+1) + 2n + 2 \\
&= \frac{1}{3}n^3 + n^2 + \frac{8}{3}n + 2 \\
u_{3,n} &= (n^2 - n + 4) + ((n-1)^2 - (n-1) + 4) + \cdots + (4 - 2 + 4) + 4 \\
&= \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i + 4n \\
&= \frac{1}{3}n^3 + \frac{11}{3}n
\end{aligned}$$

The result follows directly from Proposition 3. □

Hence we have the following:

Corollary 5. *For each* $n \geq 3$, $\mathcal{G}(\mathbb{S}^3 \times \mathbb{S}^n) \leq \frac{2}{3}n^3 + n^2 - \frac{2}{3}n$.

Remark 4. Again by [1, Corollary 1], the statement of Corollary 5 is an equality for $n = 2$.

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