

This is the peer reviewed version of the following article:

Infiltrations in immiscible fluids systems / Leonardi, Gian Paolo. - In: PROCEEDINGS OF THE ROYAL SOCIETY OF EDINBURGH. SECTION A. MATHEMATICS. - ISSN 0308-2105. - STAMPA. - 131:2(2001), pp. 425-436. [10.1017/S0308210500000937]

Royal Society of Edinburgh:22 24 George Street, Edinburgh EH2 2PQ United Kingdom:011 44 131
Terms of use:

The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. For all terms of use and more information see the publisher's website.

03/05/2026 07:44

(Article begins on next page)

Infiltrations in Immiscible Fluids Systems

Gian Paolo Leonardi

Università degli Studi di Trento

Dipartimento di Matematica

Abstract

In this paper we prove a certain regularity property of configurations of immiscible fluids, filling a bounded container Ω and locally minimizing the interface energy $\sum_{i < j} c_{ij} \|S_{ij}\|$, where S_{ij} represents the interface between fluid i and fluid j , $\|\cdot\|$ stands for *area* or more general area-type functional, and c_{ij} is a positive coefficient. More precisely, we show that, under *strict triangularity* of the c_{ij} 's, no *infiltrations* of other fluids are allowed between two main ones. A remarkable consequence of this fact is the almost-everywhere regularity of the interfaces. Our analysis is performed in general dimension $n \geq 2$ and with volume constraints on fluids.

1 Introduction

When a small quantity of oil is added to a glass of water, one sees at first some small oil drops floating on the water surface, that progressively tend to aggregate into bigger drops. This is a simple example of immiscible fluid system that tries to evolve towards a stable equilibrium configuration. In general one could consider mixtures of m fluids and ask whether some equilibrium configurations, or even configurations attaining the minimum of the total free energy, exist and (hopefully) have some regularity properties.

From a theoretical point of view, this corresponds to analyzing models of fluid systems, where the energies are essentially of *interface type*, i.e. depending upon the interfaces separating the various fluids in the mixture. The problem can thus be viewed as follows: given a container Ω , a set of *fluids* $\{f_1, \dots, f_m\}$ and prescribed volumes $v_i \geq 0$ so that $\sum v_i$ equals the volume of the container Ω , find an absolute minimizer of the *interface energy* subject to volume

constraints, i.e. a partition of Ω into regions F_i , $i = 1, \dots, m$ such that the volume of F_i equals v_i and the energy of the interface set $\{S_{ij} = \partial F_i \cap \partial F_j \cap \Omega\}$ is minimum.

Probably the easiest formulation of an energy model for immiscible fluids is

$$\mathbf{E}(\mathcal{F} = \{F_1, \dots, F_m\}) = \sum_{i < j} c_{ij} \cdot \text{area}(S_{ij}),$$

where $c_{ij} > 0$ for all $i < j$. This represents an isotropic surface energy, where each interface S_{ij} has a cost equal to c_{ij} times its (Euclidean) area, and thus depending upon the pair (f_i, f_j) of fluids touching along it. Of course, one could take into account other contributions to the total free energy of the system, such as gravity or other external forces; moreover, the energy could also depend upon surface orientation, as happens for crystals and polycrystals (see [6]).

In this paper we prove a regularity property for minimizing configurations of immiscible fluids, under *strict triangularity* of the interface coefficients c_{ij} (see Theorem 3.1). It has been shown by L.Ambrosio and A.Braides in [2], and independently by B.White in [22], that the triangle inequality $c_{ij} \leq c_{ik} + c_{kj}$ is necessary and sufficient for the lowersemicontinuity of the energy functional \mathbf{E} (see also the paper by F.Morgan [19]). White announces in [22] some regularity results obtained under strict triangularity of the c_{ij} 's, in particular an *elimination property* saying that, if any locally minimizing configuration is weakly close, in a ball B_r , to a configuration with only two fluids separated by a flat interface, then in a smaller ball it consists of exactly those two fluids: this means that no *infiltration* is permitted between two fluids that are shaped (at least locally) in an almost flat configuration.

Here we prove a stronger elimination-type result (Theorem 3.1), under the same hypothesis as in White's paper. This result points out that the flatness of the configuration does not really matter, the absence of infiltrations between two fluids being a pure consequence of energy minimization. The proof uses (1) a *Decay Lemma* (Lemma 3.2) that incorporates the elimination result (the original technique was developed by I.Tamanini and G.Congedo in [21]), (2) a technical *Balancing Lemma* (Lemma 3.3) where the main estimate needed to make (1) work is deduced from the "cooperation" of two weak energy estimates, and finally (3) a representation of the immiscible fluid configuration as a network (see [19]), on which we apply classical graph-theory results ("maxflow-mincut" and flow decomposition) to obtain one of the two weak energy estimates (the most hidden one). Thanks to the elimination property, one can then apply classical regularity results (see [15]) ensuring that the interface set is made of smooth surfaces with constant mean curvature, plus a *singular set* of zero $(n - 1)$ -dimensional Hausdorff measure.

2 Basic definitions

By \mathbf{R}^n we denote the real Euclidean space of dimension n , and always take $n \geq 2$. $B_r(x)$ denotes the open Euclidean n -ball centered at $x \in \mathbf{R}^n$ with radius $r > 0$; B_r is used in place of $B_r(0)$. We denote by ω_n the volume (Lebesgue measure \mathcal{L}^n) of the unit ball of \mathbf{R}^n . Then, the volume of B_r is $|B_r| = \omega_n r^n$ (the notation $|A|$ is preferred to $\mathcal{L}^n(A)$). We also denote by \mathcal{H}^{n-1} the $(n-1)$ -dimensional Hausdorff measure in \mathbf{R}^n .

Given two sets A and B , we define their *symmetric difference* by

$$A\Delta B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

If Ω is an open set, we say that A is *relatively compact* in Ω (and write $A \Subset \Omega$) if the closure of A is a compact subset of Ω ; we say that A is a *compact variation* of B in Ω if $A\Delta B$ is relatively compact in Ω .

Given a Borel set $E \subset \mathbf{R}^n$ and $\alpha \in [0, 1]$, we define the *set of points of density* α of E as follows:

$$E(\alpha) := \left\{ x \in \mathbf{R}^n : \lim_{r \rightarrow 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = \alpha \right\}.$$

Clearly, $E(\alpha) \subset \partial E$ for all $\alpha \in (0, 1)$. Moreover, we have that $E(1)$ is the *Lebesgue set* of E , hence $|E(1)\Delta E| = 0$ (see e.g. [20]). Another remarkable density set is $E(\frac{1}{2})$, as will be better shown in the following.

We recall the notion of *Caccioppoli set*, i.e. set of (locally) finite perimeter, and of Caccioppoli partition. For E , $A \subset \mathbf{R}^n$, with A open and E Borel, the *perimeter* of E in A is defined as follows:

$$P(E, A) := \sup \left\{ \int_E \operatorname{div} g(x) dx : g \in C_0^1(A; \mathbf{R}^n), \|g\|_\infty \leq 1 \right\}.$$

It can be shown that $P(E, \cdot)$ extends to a Borel measure, by setting

$$P(E, B) := \inf \{ P(E, A) : B \subset A, A \text{ open} \}$$

for all Borel set $B \subset \mathbf{R}^n$. When $A = \mathbf{R}^n$, we use $P(E)$ instead of $P(E, \mathbf{R}^n)$. We say that E is a set of locally finite perimeter (or a Caccioppoli set) if $P(E, A) < \infty$ for every bounded open set $A \subset \mathbf{R}^n$.

We recall some properties of the perimeter of E in B :

- $P(E, B) = P(\mathbf{R}^n \setminus E, B)$;
- $P(E, B) = P(F, B)$ whenever $|(E\Delta F) \cap B| = 0$;

- $P(E\Delta F, B) \leq P(E, B) + P(F, B)$;
- $P(E \cup F, B) + P(E \cap F, B) \leq P(E, B) + P(F, B)$;

For these and additional properties we refer to [11] and [14].

Given a Borel set E , its *characteristic function* is denoted by $\chi_E(x)$, taking the value 1 if $x \in E$ and 0 otherwise. When E is a Caccioppoli set, we consider the *traces* (from outside and from inside) of E on ∂B_r (see [14] for general definitions and properties), denoted by χ_E^+ and χ_E^- , respectively. These are functions of $L^1(\partial B_r)$ and coincide for \mathcal{L}^1 -almost all $r > 0$, in which case they are simply denoted by χ_E . Moreover, we have

$$P(E, \partial B_r) = \int_{\partial B_r} |\chi_E^+(x) - \chi_E^-(x)| d\mathcal{H}^{n-1}(x)$$

for all $r > 0$.

We quote the well-known *isoperimetric inequality*. Let $E \subset \mathbf{R}^n$ be of finite perimeter in \mathbf{R}^n , then

$$\min(|E|, |\mathbf{R}^n \setminus E|)^{\frac{n-1}{n}} \leq c_n P(E), \quad (2.1)$$

where $c_n = (n\omega_n^{\frac{1}{n}})^{-1}$ (for the proof, see [9] or [17]).

Given a Caccioppoli set E , one can consider the so-called *reduced boundary* $\partial^* E$ of E , which is a subset of $E(\frac{1}{2})$ where a certain measure-theoretical, unit normal vector ν_E exists: for the precise definition, see e.g. [14].

The following properties are of crucial importance (for the proof, see [10], [14]):

- (a) $\partial^* E \subset E(\frac{1}{2})$ and $\mathcal{H}^{n-1}(E(\frac{1}{2}) \setminus \partial^* E) = 0$;
- (b) $P(E, B) = \mathcal{H}^{n-1}(\partial^* E \cap B) = \mathcal{H}^{n-1}(E(\frac{1}{2}) \cap B)$.

A finite *Caccioppoli partition* of an open set Ω is a finite collection $\mathcal{F} = (F_i)_{i=1}^m$ of Borel subsets of Ω , such that

- (i) F_i has locally finite perimeter in Ω ;
- (ii) $|F_i \cap F_j| = 0$ whenever $i \neq j$;
- (iii) $|\Omega \setminus \bigcup_{i=1}^m F_i| = 0$.

Theorem 2.1 (Structure) *Let $\mathcal{F} = (F_i)_{i=1}^m$ be a Caccioppoli partition of Ω . Then*

$$\mathcal{H}^{n-1} \left[\Omega \setminus \left(\bigcup_i F_i(1) \cup \bigcup_{i < j} [F_i(\frac{1}{2}) \cap F_j(\frac{1}{2})] \right) \right] = 0.$$

For the proof, see [8].

Summing up, according to Theorem 2.1, a Caccioppoli partition of Ω yields a decomposition of Ω into “solid components” $F_i(1) \cap \Omega$, along with a set of “interfaces” $S_{ij} = F_i(\frac{1}{2}) \cap F_j(\frac{1}{2}) \cap \Omega$ separating F_i and F_j inside Ω ($i \neq j$), and a \mathcal{H}^{n-1} -negligible set containing e.g. “multiple points” where three or more components meet. Moreover, we are allowed to redefine the interface S_{ij} to be $\partial^* F_i \cap \partial^* F_j \cap \Omega$, owing to property (a) above, this becoming our default setting from now on. Finally, to simplify the notation, we shall write $\|S\|$ instead of $\mathcal{H}^{n-1}(S)$.

3 Statement of the problem and preliminary lemmas

Let $\Omega \subset \mathbf{R}^n$ be a nonempty, bounded open set and let $\mathcal{F} = (F_i)_{i=1}^{m+2}$ be a Caccioppoli partition of Ω with $(m+2)$ components of finite perimeter in Ω (here, $m \geq 1$). Hence (recall Section 2) the F_i 's are Borel subsets of Ω , such that

- (i) F_i has finite perimeter in Ω for all i ;
- (ii) $|F_i \cap F_j| = 0$ whenever $i \neq j$;
- (iii) $|\Omega \setminus \bigcup_{i=1}^{m+2} F_i| = 0$.

We shall refer to such \mathcal{F} as a generic *configuration* of fluids. We also prescribe the volume of each fluid, that is, we fix $v_1, v_2, \dots, v_{m+2} \geq 0$ so that $|\Omega| = \sum v_i$, and call \mathcal{F} *admissible* if $|F_i| = v_i$ for all i . We will be concerned about regularity properties of configurations \mathcal{G} minimizing the energy functional

$$\mathbf{E}(\mathcal{F}) = \sum_{i < j} c_{ij} \|S_{ij}\| \tag{3.1}$$

among all admissible configurations \mathcal{F} . Here, $c_{ij} \geq 0$ represents the energy density associated with the interface $S_{ij} = \partial^* F_i \cap \partial^* F_j \cap \Omega$ separating fluids

F_i and F_j . We also consider the localized energy

$$\mathbf{E}(\mathcal{F}, O) = \sum_{i < j} c_{ij} \|S_{ij}^O\|,$$

where O is a generic open subset of Ω and $S_{ij}^O = S_{ij} \cap O$. In case $O = B_r$, we write S_{ij}^r instead of $S_{ij}^{B_r}$.

Let us start by localizing within a fixed ball $B_R \subset \Omega$ and by considering a fixed pair G_1, G_2 of fluids of a minimizing configuration. Our aim is to see if *infiltrations* can be excluded by imposing suitable hypotheses on the coefficients c_{ij} . We would eventually like to prove the following result: if there exists a radius $r < R$ such that G_1 and G_2 fill a sufficiently high percentage of B_r , then G_1 and G_2 completely fill $\frac{1}{2}B_r$.

This result can be viewed as a “2 against m ” version of an elimination-type theorem proved by I.Tamanini and G.Congedo in [21] (see also [16]): in that paper, the authors consider “general” Caccioppoli partitions (even with countably many components) minimizing the simple *perimeter functional* ($c_{ij} = 1$, for all $i \neq j$) plus higher-order volume-type terms. The technique of [21] fails when, as in our case, more general hypotheses on the c_{ij} ’s are assumed, and has to be integrated with additional results involving, among other things, some tools from graph theory.

A similar result, with extra assumptions on the minimizing configuration, has been announced by B. White in [22], for an immiscible fluid energy like (3.1). More precisely, White considers the following property (P): if a minimizing configuration is weakly close, in a small ball $B(x, r)$, to a configuration consisting of fluid i and fluid j separated by a hyperplane H through x , then in a smaller ball $B(x, r/2)$ the configuration consists exactly of fluid i and fluid j separated by a smooth hypersurface. Then, he claims that (P) holds if and only if a *strict triangularity* hypothesis is assumed on the coefficients c_{ij} , in addition to the following standard requirements:

- (i) $c_{ij} \geq 0$ and $c_{ij} = 0$ if and only if $i = j$;
- (ii) $c_{ij} = c_{ji}$.

Strict triangularity is simply the following:

$$c_{ij} < c_{ih} + c_{hj} \quad \forall i, j \quad \text{and} \quad \forall k \neq i, j \quad (\text{ST})$$

or, equivalently:

$$c_{ij} \leq c_{ik} + c_{kj} - \delta \quad \forall i, j \quad \text{and} \quad \forall k \neq i, j \quad (\text{ST2})$$

for a suitable constant $\delta > 0$.

Let us first consider a simple example: suppose that three fluids f_1, f_2, f_3 fill some container Ω (we always assume absence of external forces and of surface tension with walls of Ω) and that two of them (say, f_1 and f_2) meet along some flat interface. If the tension coefficients c_{ij} did not satisfy a strict triangle inequality, and in particular if even $c_{12} > c_{13} + c_{23}$, then it would be energetically advantageous to let a thin layer of fluid f_3 flow between f_1 and f_2 , in such a way that these two fluids do not touch anymore. Clearly, this corresponds to a loss of lower semicontinuity of the energy functional, which could be macroscopically observed as a “relaxation” of the energy densities c_{ij} (i.e. the fluid system behaves as if $c_{12} = c_{13} + c_{23}$). Absence of infiltrations appears to be quite related to lower semicontinuity of the energy functional: indeed, it is shown in [2], as well as in [22], that the (simple) triangle inequality is necessary and sufficient for the lowersemicontinuity of the fluid energy (3.1) (for a more general and detailed study of the semicontinuity of fluid-type energies, see [2], [3] and [19]). This justifies, in some sense, the choice of (ST) as a *sharp* condition to prevent infiltrations between pairs of fluids.

White also announces in [22] results regarding both the regularity of the interfaces S_{ij} and the estimate of the dimension of the “singular set” of a minimizing configuration of fluids. Some other problems remain unsolved, like the validity of a lower volume-density estimate for each fluid at every boundary point (see [7]).

Here we state our main result for a pair (G_1, G_2) of fluids in a locally minimizing configuration (of course, any other pair will do the same): its proof will need some intermediate lemmas and, as anticipated before, the use of graph theory techniques.

Theorem 3.1 (Elimination). *Suppose $\mathcal{G} = (G_i)_{i=1}^{m+2}$ is a locally minimizing configuration of immiscible fluids inside $B_R \subset \Omega$, and that the coefficients c_{ij} verify the previous standard requirements (i), (ii) along with the strict triangularity condition (ST). Then \mathcal{G} has the elimination property (EP), i.e. there exists a positive constant η and a radius $r_0 < R$ such that, if $V = G_3 \cup \dots \cup G_{m+2}$ (the so-called “infiltration”) and $0 < \rho < r_0$, then*

$$|V \cap B_\rho| \leq \eta \rho^n \quad \Rightarrow \quad |V \cap B_{\rho/2}| = 0.$$

The need of performing compact variations of a given configuration of fluids (i.e. variations that are localized in small balls and preserve boundary data) suggests us to define the concept of *cut*. A cut \mathcal{K} (relative to the pair (F_1, F_2)) is a bipartition (K_1, K_2) of the index set $\{1, \dots, m+2\}$ such that $i \in K_i$,

$i = 1, 2$. It gives a way of “refilling” Ω by using the first two fluids (F_1, F_2) . Precisely, starting with some configuration \mathcal{F} , some cut \mathcal{K} , and a fixed radius $0 < r < R$, we can define a new configuration $\mathcal{F}^r \equiv \mathcal{F}^{\mathcal{K}, r}$ as follows:

$$F_i^r = \begin{cases} F_i \setminus B_r & \text{if } i > 2, \\ F_i \cup \bigcup_{j \in K_i} (F_j \cap B_r) & \text{otherwise.} \end{cases}$$

Clearly, \mathcal{F}^r is a compact variation of \mathcal{F} inside B_R . At this point, we introduce some more notation: by $\Delta \mathbf{E}$ we mean the energy change $\mathbf{E}(\mathcal{F}^r, B_R) - \mathbf{E}(\mathcal{F}, B_R)$ and by $\Delta_r \mathbf{E}$ we mean $\mathbf{E}(\mathcal{F}^r, B_r) - \mathbf{E}(\mathcal{F}, B_r)$. Finally, we define $A_r^{\mathcal{K}}$ (improperly called the *area of \mathcal{K} inside B_r*) as follows:

$$A_r^{\mathcal{K}} = \sum_{\substack{h \in K_1 \\ t \in K_2 \\ (h, t) \neq (1, 2)}} \|S_{ht}^r\|,$$

where as usual $S_{ht}^r = \partial^* F_h \cap \partial^* F_t \cap B_r$.

Let \mathcal{G} be an admissible fluid configuration which is “locally minimizing” inside a ball B_R , that is, $\mathbf{E}(\mathcal{G}) \leq \mathbf{E}(\mathcal{F})$ for all compact variations \mathcal{F} of \mathcal{G} inside B_R . Actually, we should be more careful when considering compact variations of a fluid configuration, since most of such variations will not preserve volumes and will thus not be admissible anymore. However, thanks to an argument originally due to F. Almgren (see [1] VI.2(3), and [18] Lemma 13.5), any (small) volume change Δv can be adjusted at a cost proportional to Δv itself (hence, infinitesimal of higher order when compared with an area-type change). This fact is extremely important, since it allows us to virtually ignore the problem of adjusting volumes after a small change of the fluid configuration.

The following decay lemma incorporates the main engine that gives rise to the elimination property (EP):

Lemma 3.2 (Decay). *Suppose $\mathcal{G} = (G_i)_{i=1}^{m+2}$ is locally minimizing inside B_R and that there exists a constant $C > 0$ such that, for almost all $0 < r < R$, there is at least one cut \mathcal{K} for which*

$$\Delta_r \mathbf{E} \leq -C P(V, B_r), \tag{3.2}$$

where $V = \bigcup_{j>2} G_j$ is the infiltration. Then \mathcal{G} has the elimination property (EP), i.e. there exists a positive constant η and a radius $r_0 < R$ such that, if $0 < \rho < r_0$ and $|V \cap B_\rho| \leq \eta \rho^n$, then $|V \cap B_{\rho/2}| = 0$; moreover, we can choose $\eta = \omega_n \left(\frac{C}{2(C_1+C)} \right)^n$, with $C_1 = \max_{i,j} c_{ij}$.

PROOF. Define $\alpha(r) = |V \cap B_r|$. The function $\alpha(r)$ is monotone increasing, with

$$\alpha'(r) = \int_{\partial B_r} \chi_V(y) d\mathcal{H}^{n-1}(y) \quad (3.3)$$

for almost all $r \in (0, R)$. Since for almost all r and all i, j it holds $\|\partial B_r \cap S_{ij}\| = 0$, we can write

$$\mathbf{E}(\mathcal{G}, B_R) = \mathbf{E}(\mathcal{G}, B_R \setminus \overline{B_r}) + \mathbf{E}(\mathcal{G}, B_r) \quad (3.4)$$

for almost all $r \in (0, R)$. Fix now $r_0 = R$ (had we taken into account the small cost due to possible volume adjustments, r_0 should have been chosen sufficiently small and η changed a little bit...) and take $\rho \in (0, r_0)$ such that $\alpha(\rho) \leq \eta\rho^n$.

By contradiction, suppose that $\alpha(\rho/2) > 0$, so that in particular $\alpha(r) > 0$ for all $\rho/2 < r < \rho$. Given $r \in (\rho/2, \rho)$ for which (3.3) and (3.4) are verified, we obtain

$$\begin{aligned} 0 &\leq \Delta \mathbf{E} = \mathbf{E}(\mathcal{G}^r, B_R) - \mathbf{E}(\mathcal{G}, B_R) \\ &\leq \Delta_r \mathbf{E} + C_1 \alpha'(r), \end{aligned} \quad (3.5)$$

thanks to the minimality of \mathcal{G} . The isoperimetric inequality (2.1) gives

$$P(V, B_r) = P(V \cap B_r) - \alpha'(r) \geq n\omega_n^{\frac{1}{n}} \alpha^{\frac{n-1}{n}}(r) - \alpha'(r),$$

thus, by (3.2) and (3.5),

$$\frac{C\omega_n^{\frac{1}{n}}}{C_1 + C} \leq \frac{1}{n} \alpha^{\frac{1}{n}-1}(r) \alpha'(r) = \left(\alpha^{\frac{1}{n}} \right)'(r)$$

for almost all r , hence by integration between $\rho/2$ and ρ we get

$$\alpha^{\frac{1}{n}}(\rho/2) \leq \alpha^{\frac{1}{n}}(\rho) - \frac{C\omega_n^{\frac{1}{n}}}{2(C_1 + C)} \rho \leq \eta^{\frac{1}{n}} \rho - \frac{C\omega_n^{\frac{1}{n}}}{2(C_1 + C)} \rho = 0,$$

that is, a contradiction. □

The next lemma shows how to get (3.2) starting from the weaker estimate (3.6).

Lemma 3.3 (Balancing). *Suppose there exists a positive constant $\delta > 0$ and a cut \mathcal{K} , such that*

$$\Delta_r \mathbf{E} \leq -\delta A_r^{\mathcal{K}}, \quad (3.6)$$

where $A_r^{\mathcal{K}}$ is the ‘‘area of the cut inside B_r ’’ (see above). Then there exists a positive constant $C = C(\delta, C_0, C_1)$, with $C_0 = \min_{i < j} c_{ij}$, $C_1 = \max_{i < j} c_{ij}$, such that

$$\Delta_r \mathbf{E} \leq -C P(V, B_r),$$

where $V = \bigcup_{j > 2} G_j$ is the infiltration.

PROOF. Let us estimate

$$\begin{aligned} \Delta_r \mathbf{E} &= c_{12} A_r^{\mathcal{K}} - \sum_{\substack{i < j \\ (i, j) \neq (1, 2)}} c_{ij} \|S_{ij}^r\| \\ &\leq \sum_{\substack{h \in K_1 \\ t \in K_2 \\ (h, t) \neq (1, 2)}} (c_{12} - c_{ht}) \|S_{ht}^r\| - \sum_{j \in K_1} c_{1j} \|S_{1j}^r\| - \sum_{j \in K_2} c_{2j} \|S_{2j}^r\| \\ &\leq (C_1 - C_0) A_r^{\mathcal{K}} - C_0 \left(\sum_{j \in K_1} \|S_{1j}^r\| + \sum_{j \in K_2} \|S_{2j}^r\| \right). \end{aligned} \quad (3.7)$$

On the other hand, it is not difficult to check that, by Theorem 2.1,

$$P(V, B_r) \leq A_r^{\mathcal{K}} + \sum_{j \in K_1} \|S_{1j}^r\| + \sum_{j \in K_2} \|S_{2j}^r\|,$$

that is,

$$- \left(\sum_{j \in K_1} \|S_{1j}^r\| + \sum_{j \in K_2} \|S_{2j}^r\| \right) \leq A_r^{\mathcal{K}} - P(V, B_r).$$

Therefore, from (3.7) we deduce

$$\Delta_r \mathbf{E} \leq C_1 A_r^{\mathcal{K}} - C_0 P(V, B_r). \quad (3.8)$$

At this point we use the following inequality (its proof is straightforward). Let δ, C_0, C_1 be positive constants, then for any $A, P \in \mathbf{R}$

$$\min \{ -\delta A, C_1 A - C_0 P \} \leq -\frac{\delta C_0}{C_1 + \delta} P. \quad (3.9)$$

By combining (3.9), (3.8) and (3.6), we conclude that

$$\Delta_r \mathbf{E} \leq -\frac{\delta C_0}{C_1 + \delta} P(V, B_r) = -C P(V, B_r),$$

as was to be proved. □

4 Network Representation

In this section we will use graphs to represent fluid configurations and employ typical graph-theory results to deal with region-merging procedures and to get estimates on the energy changes. First, we recall some basic definitions and results about *directed graphs* and *networks*.

A directed graph G is a finite set of *vertices* (or *nodes*) v_i , $i = 1, \dots, n$ that are connected by oriented arcs. The arc going from node v_i to node v_j is represented by the ordered pair (v_i, v_j) , or e_{ij} for short: in this case, v_i is called the *tail* and v_j the *head* of e_{ij} . The set E of all arcs of G is, in general, a subset of $G \times G$. A graph G is said to be *weighted* if a certain non-negative coefficient p_{ij} is associated with each arc e_{ij} , representing its *capacity*. This last kind of graph is quite often called a *network*.

We are particularly interested in networks where each node has no connection (arc) to itself and where each pair of nodes $v_i \neq v_j$ is connected by both arcs e_{ij} and e_{ji} . Therefore, the arc set E coincides with $G \times G \setminus \Delta$ (where Δ is the diagonal of $G \times G$), and hence our networks are *completely connected*, that is, each pair of different nodes has exactly two connecting arcs (with opposite *orientation*); on the other hand, the capacities p_{ij} are chosen to be symmetric ($p_{ij} = p_{ji}$) and are allowed to be zero.

Definition 4.1 *Given a network (G, p) , and chosen a pair of nodes s, t , we say that a function $f : E \rightarrow [0, \infty)$ is a “flow” from the source s to the sink t if*

- (1) **(capacity constraint)** $f(e_{ij}) \leq p_{ij}$ for all $i \neq j$;
- (2) **(conservation condition)** if $\phi : G \rightarrow \mathbf{R}$ is the “node function” associated to f , i.e.

$$\phi(v_i) = \sum_{j \neq i} f(e_{ij}) - f(e_{ji}),$$

then $\phi(v) = 0$ for each $v \neq s, t$ and $\phi(s) = -\phi(t) \geq 0$. We will also denote by $\|f\|$ the “intensity” of the flow f , that is, $\|f\| = \phi(s)$.

Definition 4.2 *Let s and t be, respectively, the source and the sink in a network (G, p) . A bipartition $\mathcal{K} = (K_1, K_2)$ of G is called a “cut” if the source s and the sink t are contained, respectively, in K_1 and K_2 (compare with the definition given in Section 3). The “size” of the cut is then defined as*

$$\sigma(\mathcal{K}) = \sum_{\substack{i \in K_1 \\ j \in K_2}} p_{ij}.$$

Now some results follow (for their proof, see [5]):

Theorem 4.3 *If f is a flow between s and t , and \mathcal{K} is a cut with respect to s and t , then*

$$\|f\| \leq \sigma(\mathcal{K}).$$

Let us say that f is a *maximum flow* if there is no other flow f' such that $\|f'\| > \|f\|$. Symmetrically, let us say that \mathcal{K} is a *minimum cut* if there is no other cut \mathcal{K}' such that $\sigma(\mathcal{K}') < \sigma(\mathcal{K})$. We recall that an algorithm has been developed by Ford and Fulkerson (see [12]) to find a maximum flow in a network. This algorithm is essentially contained in the proof of the following fundamental result:

Theorem 4.4 (Max Flow – Min Cut). *If f is a maximum flow and \mathcal{K} is a minimum cut, then*

$$\|f\| = \sigma(\mathcal{K}).$$

We now recall a flow-decomposition result, better known as *conformal decomposition* (see [4]), saying that any flow can be decomposed as a “sum” of flows along paths. First of all, we give a definition of “path”.

Definition 4.5 *An ordered n -tuple of arcs $\gamma = (e_{j_0j_1}, e_{j_1j_2}, \dots, e_{j_{n-1}j_n})$ is called a “path” from v to w if the vertices v_{j_i} are all distinct, with $v_{j_0} = v$ and $v_{j_n} = w$.*

Theorem 4.6 (Flow decomposition). *Given a flow f on a network (G, p) between s and t , there exist paths $\gamma_1, \dots, \gamma_h$ from s to t and non-negative constants f_1, \dots, f_h representing the flow through each path, such that*

$$(i) \quad \|f\| = \sum_{i=1}^h f_i;$$

$$(ii) \quad \sum_{i: e_{rk} \in \gamma_i} f_i \leq f(e_{rk}) \leq p_{rk}, \text{ for all arc } e_{rk}.$$

PROOF. (Sketch). If $\|f\| = 0$ there is nothing to do, since any path γ from s to t together with the zero flow gives the required family, so that we suppose $\|f\| > 0$. Let Γ be the set of all paths γ from s to t such that the minimum value of f over the arcs belonging to γ is greater than zero. If Γ were empty, then one could show that the flow f is a “null flow”, i.e. $\|f\| = 0$, but this contradicts our assumption, hence $\Gamma \neq \emptyset$. By induction, if Γ has only one element γ , then the pair $(\gamma, \min_{\gamma} f)$ necessarily gives the decomposition, otherwise we suppose that such a decomposition can be found whenever Γ has

at most d elements, and prove that this is also true when Γ has $d + 1$ elements (recall that, by the definition of path, Γ is always a finite set): indeed, consider a path $\gamma_{d+1} \in \Gamma$ and define f_{d+1} as the minimum (positive) value of f over γ_{d+1} . By subtracting f_{d+1} from f over the arcs of γ_{d+1} , we obtain a new flow f' such that its corresponding set Γ has at most d elements. Finally the decomposition of f' plus the pair (γ_{d+1}, f_{d+1}) gives the required family. \square

Proof of the main result

We can represent a configuration of fluids as a network, with nodes corresponding to fluids, arcs representing the separating interfaces, and with capacity equal to interfacial area. Then a suitable use of the above results will let us prove the elimination theorem, under hypothesis (ST2) of strict triangularity of the coefficients c_{ij} .

Proof of Theorem 3.1. We only need to show that the hypotheses of Lemma 3.3, and hence of Lemma 3.2, are all satisfied. To do this, we represent the configuration inside B_r as a network, where node v_i corresponds to fluid F_i and arc capacity p_{ij} equals $\|S_{ij}^r\|$ (area of the interface S_{ij}^r separating fluid F_i from fluid F_j inside B_r). Choose v_1 as source and v_2 as sink, then take a minimum cut \mathcal{K} of size $\sigma(\mathcal{K}) = A_r^{\mathcal{K}} + \|S_{12}^r\|$ and consider a maximum flow f (whose existence is guaranteed by the Ford-Fulkerson algorithm, see [12] or [5]). It is easy to see that this maximum flow can be decomposed by Theorem 4.6, in such a way that the path γ_1 consists of the single arc e_{12} , and the flow f_1 along γ_1 coincides with the full capacity p_{12} . Now, we estimate the corresponding energy change by using (ST2), Theorem 4.4 and Theorem 4.6:

$$\begin{aligned}
\mathbf{E}(\mathcal{G}; B_r) &= c_{12}(A_r^{\mathcal{K}} + \|S_{12}^r\|) = c_{12}\|f\| = c_{12} \sum_i f_i \\
&\leq c_{12}f_1 + \sum_{i>1} \left(\sum_{h,k:e_{hk} \in \gamma_i} c_{hk} - \delta \right) f_i \\
&= c_{12}f_1 + \sum_{i>1} \sum_{h,k:e_{hk} \in \gamma_i} c_{hk} f_i - \delta(\|f\| - f_1) \\
&= \sum_{h,k} c_{hk} \sum_{i:e_{hk} \in \gamma_i} f_i - \delta(\sigma(\mathcal{K}) - \|S_{12}^r\|) \\
&\leq \sum_{h<k} c_{hk} p_{hk} - \delta A_r^{\mathcal{K}} = \mathbf{E}(\mathcal{G}; B_r) - \delta A_r^{\mathcal{K}},
\end{aligned}$$

hence we obtain

$$\Delta_r \mathbf{E} \leq -\delta A_r^{\mathcal{X}},$$

and, by means of Lemma 3.3 and Lemma 3.2, the proof is achieved. □

Remark 4.7 - The above results are still true when $\|\cdot\|_\phi$ is used in place of $\|\cdot\|$. Here, ϕ represents a generic norm on \mathbf{R}^n and

$$\|S\|_\phi = \int_S \phi(\nu(x)) d\mathcal{H}^{n-1}(x),$$

where $\nu(x)$ is any unit normal vector to S at x . This gives rise in general to non-isotropic energies, modeling, for instance, polycrystals.

Corollary 4.8 (Regularity). *Let ϕ be a norm on \mathbf{R}^n and $\mathcal{G} = \{G_1, \dots, G_m\}$ be a locally-minimizing configuration for the energy*

$$\mathbf{E}(\mathcal{F}) = \sum_{i < j} c_{ij} \|S_{ij}\|_\phi.$$

Then

- (a) *the density-1 set $G_i(1)$ is open in Ω for all i ;*
- (b) *\mathcal{H}^{n-1} -almost every “boundary” point x_0 has a neighborhood with only two fluids inside;*
- (c) *if ϕ is the Euclidean norm (or, more generally, a smooth, uniformly convex norm) then the interface set is made of smooth surfaces with constant mean curvature, plus an \mathcal{H}^{n-1} -negligible singular set.*

PROOF. (a) follows immediately from the definition of $G_i(1)$ and Theorem 3.1. To prove (b), simply observe that, thanks to Theorem 2.1, \mathcal{H}^{n-1} -almost every “boundary” point x_0 belongs to the reduced boundaries of exactly two fluids, hence x_0 is a “zero-density point” for all other fluids. Therefore, by Theorem 3.1, there exists a neighborhood of x_0 with only those two fluids inside. Finally, (c) is a consequence of well known results about the regularity of area-minimizing boundaries with volume constraint (see [1] IV, [15] and [13]). □

References

- [1] F. ALMGREN, *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. A.M.S., 4, No. 165 (1976).
- [2] L. AMBROSIO AND A. BRAIDES, *Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization*, J. Math. Pures Appl. (9), 69 (1990), pp. 307–333.
- [3] L. AMBROSIO, N. FUSCO, AND D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, Oxford U.P. (to appear).
- [4] D. P. BERTSEKAS, *Linear network optimization: algorithms and codes*, MIT Press, Cambridge, MA, 1991.
- [5] J. A. BONDY AND U. S. R. MURTY, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.
- [6] D. G. CARABALLO, *A Variational Scheme for the Evolution of Polycrystals by Curvature*, PhD thesis, Princeton University, 1996.
- [7] C. CHAN, *Structure of the Singular Set in Energy-Minimizing Partitions and Area-minimizing Surfaces in \mathbf{R}^n* , PhD thesis, Stanford University, 1997.
- [8] G. CONGEDO AND I. TAMANINI, *On the existence of solutions to a problem in multidimensional segmentation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 8 (1991), pp. 175–195.
- [9] E. DE GIORGI, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Memorie Acc. Naz. Lincei, Serie VIII, 5 (1958), pp. 33–44.
- [10] E. DE GIORGI, F. COLOMBINI, AND L. PICCININI, *Frontiere orientate di misura minima e questioni collegate*, Scuola Norm. Sup. Pisa, (1972).
- [11] L. C. EVANS AND R. F. GARIEPY, *Lecture Notes on Measure Theory and Fine Properties of Functions*, Studies in Advanced Math., CRC Press, Ann Harbor, 1992.
- [12] J. FORD, L. R. AND D. R. FULKERSON, *Maximal flow through a network*, Canad. J. Math., 8 (1956), pp. 399–404.

- [13] C. GERHARDT, *On the capillarity problem with constant volume*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 2 (1975), pp. 303–320.
- [14] E. GIUSTI, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Boston-Basel-Stuttgart, 1984.
- [15] E. GONZALEZ, U. MASSARI, AND I. TAMANINI, *On the regularity of boundaries of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J., 32 (1983), pp. 25–37.
- [16] G. P. LEONARDI, *Optimal Subdivisions of n -dimensional Domains*, PhD thesis, Università di Trento, 1998.
- [17] U. MASSARI AND M. MIRANDA, *Minimal surfaces of codimension one*, Notas de Matemática, North Holland, Amsterdam, 1984.
- [18] F. MORGAN, *Geometric measure theory – A beginner’s guide*, Academic Press Inc., San Diego, CA, second ed., 1995, third ed., 2000.
- [19] ———, *Lowersemicontinuity of energy clusters*, Proc. Roy. Soc. Edinburgh Sect. A, 127 (1997), pp. 819–822.
- [20] W. RUDIN, *Real and Complex Analysis*, Mc Graw-Hill Book Co., Singapore, 1987.
- [21] I. TAMANINI AND G. CONGEDO, *Optimal segmentation of unbounded functions*, Rend. Sem. Mat. Univ. Padova, 95 (1996), pp. 153–174.
- [22] B. WHITE, *Existence of least-energy configurations of immiscible fluids*, J. Geom. Anal., 6 (1996), pp. 151–161.