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(Article begins on next page)

On generalized null polarities

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Abstract

Assuming that a linear complex of planes without singular lines exists, the properties of the related generalized polarity are investigated.

Keyword: Linear complex, Grassmannian, linear mapping, polarity

1 Introduction

1.1 Linear complexes

Let F be a commutative field and let $PG(n, F)$ denote the n -dimensional projective space coordinatized by the field F . With a d -subspace of $PG(n, F)$ we will mean a subspace of dimension d of the projective space $PG(n, F)$. The *Grassmann space* $\Gamma(n, h+1, F)$, $-1 \leq h \leq n-2$, is the geometry whose points are the $(h+1)$ -subspaces of $PG(n, F)$, and whose lines are the pencils of $(h+1)$ -subspaces, where a *pencil* is the set of all $(h+1)$ -subspaces between a given h -subspace X and a given $(h+2)$ -subspace Y , $X \subset Y$. A *linear complex of $(h+1)$ -subspaces* in $PG(n, F)$ is the set of all $(h+1)$ -subspaces whose Grassmann coordinates satisfy a non-trivial linear equation. Equivalently, a linear complex of $(h+1)$ -subspaces is a geometric hyperplane of $\Gamma(n, h+1, F)$ [7, 9]. The classification of linear complexes of lines reduces to the classification of non-zero alternating matrices. For the case $h > 0$ the results in the ancient literature only concern the field of complex numbers.

In [8] linear complexes of $(h+1)$ -subspaces are dealt with from an incidence-geometric point of view. More specifically, linear complexes of $(h+1)$ -subspaces are constructed by the notion of a *generalized null polarity*.

A *singular h -subspace* for a linear complex K of $(h+1)$ -subspaces in $PG(n, F)$ is an h -subspace X , such that every $(h+1)$ -subspace of $PG(n, F)$ containing X belongs to K . It seems to be hard to establish whether there

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exists or not a linear complex of $(h + 1)$ -subspaces having no singular h -subspaces, for $h > 0$. If there exists such a linear complex, then there also exists a linear complex of planes without singular lines. Only the case $F = \mathbb{C}$ has been settled: for $h > 0$, every linear complex of $(h + 1)$ -subspaces has a singular h -subspace, [1].

We assume that there exists a linear complex K of planes without singular lines and in Section 2 we prove some of its geometric properties, also with respect to the related generalized null polarity. In particular, in our main Theorem we will focus on the intersection of a 4-subspace of $PG(n, F)$ with the linear complex K .

In Section 3 we will see that a generalized global null polarity of index one in $PG(n, F)$ induces a map χ' of $\Gamma(n, 2, F)$ in $\Gamma(n, n - 3, F)$. Some properties of χ' are deduced. In particular, χ' is not linear.

Section 4 is devoted to *line partitions* of $PG(n, q)$. A line partition Ω of $PG(n, q)$ is a partition of the line set into line spreads of hyperplanes consisting of exactly one line spread for each hyperplane. To each line partition it is associated a surjective map π_Ω of the set of lines onto the dual space $PG^*(n, q)$, assigning to each line l the unique hyperplane containing the equivalence class of l . If π_Ω is a *linear mapping*, i.e. maps pencils of lines into pencils of hyperplanes, then Ω is said to be a *linear line partition*.

There is a close relation between linear line partitions and linear complexes of planes having no singular lines. More specifically, in [8, Th. 18] it has been proved that if Ω is a linear line partition of $PG(n, F)$, then the set K of all planes ε containing a line l with the property that $\varepsilon \subseteq l^{\pi_\Omega}$ is a linear complex of planes without singular lines. Also the converse is true. More precisely, if K is such a linear complex, then by mapping each line l into the union of all planes ε such that $l \subseteq \varepsilon \in K$ one obtains a linear mapping, and a linear line partition. If this happens, then π_Ω is the generalized null polarity related to K .

A computer search based on [8, Th. 18] allows to say that the known line partitions give rise to non-linear maps of lines into hyperplanes. So they are not of the kind which is related to the object of our investigation.

Unfortunately, the properties we found do not finish the problem of the existence.

1.2 Notions

Here below we write the concepts which are frequently used in the rest of the paper and give some notations. We refer to [8] for more details about the notions involved in the study of a generalized null polarity. If U and W are subspaces of $PG(n, F)$ we will denote by $U \vee W$ the smallest subspace of $PG(n, F)$ containing both U and W .

A *semilinear space* is a pair $\Sigma = (\mathcal{P}, \mathcal{B})$, where \mathcal{P} is a non-empty set of *points*, and $\mathcal{B} \subseteq 2^{\mathcal{P}}$ is a set of *lines* satisfying the following axioms:

- (i) $|l| \geq 2$ for each $l \in \mathcal{B}$;
- (ii) for every $P \in \mathcal{P}$ there exists $l \in \mathcal{B}$ such that $P \in l$;
- (iii) $|l \cap l'| \leq 1$ for every $l, l' \in \mathcal{B}$, with $l \neq l'$.

Let P, Q be points of \mathcal{P} . We say that P, Q are *collinear*, and we write $P \sim Q$, if there is a line $l \in \mathcal{B}$ such that $P, Q \in l$. Otherwise we say that P, Q are *non-collinear* and we write $P \not\sim Q$. If P and Q are distinct and collinear, then we will denote by PQ the unique line $l \in \mathcal{B}$ such that $P, Q \in l$.

The h -th Grassmannian of $PG(n, F)$, with $0 \leq h \leq n - 1$, is the semi-linear space $\Gamma(n, h, F) = (\mathcal{P}, \mathcal{B})$, where \mathcal{P} is the set of all h -subspaces of $PG(n, F)$ and \mathcal{B} is the set of all pencils of h -subspaces. We have that $\Gamma(n, h, F)$ is isomorphic to $\Gamma(n, n - h - 1, F)$, since F is a commutative field.

Let $\mathcal{P}, \mathcal{P}'$ be sets. We say that χ is a *partial map* of \mathcal{P} into \mathcal{P}' if there exists a subset $\mathbb{D}(\chi)$ of \mathcal{P} such that $\chi : \mathbb{D}(\chi) \rightarrow \mathcal{P}'$ is a map in the usual sense. The set $\mathbb{D}(\chi)$ is called the *domain* of χ while the set $\mathbb{A}(\chi) = \mathcal{P} \setminus \mathbb{D}(\chi)$ is called the *exceptional subset* of χ and its elements are called *exceptional points* of χ . If $\mathbb{A}(\chi)$ is empty, then χ is said to be *global*. Also when χ is not global we will write $\chi : \mathcal{P} \rightarrow \mathcal{P}'$.

Let $\Sigma = (\mathcal{P}, \mathcal{B})$ and $\Sigma' = (\mathcal{P}', \mathcal{B}')$ be semilinear spaces. A partial map $\chi : \mathcal{P} \rightarrow \mathcal{P}'$ is called a *linear mapping* if for each $l \in \mathcal{B}$ one of the following holds:

- (i) $l^x \in \mathcal{B}'$ and χ maps l bijectively onto l^x ;
- (ii) $l^x = \{P'\}$ where $P' \in \mathcal{P}'$ and $|l \cap \mathbb{A}(\chi)| = 1$;
- (iii) $l \subseteq \mathbb{A}(\chi)$.

An injective linear mapping χ in a projective space is said to be a *full projective embedding*. Linear mappings in projective spaces are described in [2] and in [7], where are also characterized linear mappings of Grassmann spaces into projective spaces.

Let $\Sigma = (\mathcal{P}, \mathcal{B})$ be a semilinear space. A *prime* of Σ is a proper subset L of \mathcal{P} with the property that for each $\phi \in \mathcal{B}$ either $\phi \subseteq L$ or $|\phi \cap L| = 1$. As remarked in [8], each linear complex of $(h + 1)$ -subspaces is a prime of $\Gamma(n, h + 1, F)$, and conversely.

A linear complex K of $(h + 1)$ -subspaces in a projective space $PG(n, F)$ is said to be *degenerate* if there exists an $(n - h - 2)$ -subspace M such that for each $(h + 1)$ -subspace X there holds $X \in K$ if and only if $X \cap M \neq \emptyset$.

A *generalized polarity* [8] of index h is a linear mapping χ of $\Gamma(n, h, F)$ into the dual space $PG^*(n, F)$ of $PG(n, F)$, such that for every $U_1, U_2 \in \Gamma(n, h, F)$, with $U_1 \sim U_2$, the following holds:

$$U_1 \subseteq U_2^x \Rightarrow U_2 \subseteq U_1^x.$$

Here it is meant that if $X \in \mathbb{A}(\chi)$, then X^χ is the whole space $PG(n, F)$. From now on, when we will speak of “polarities” we will always mean generalized polarities. In [8, Th. 4] it is proved that a linear mapping $\chi : \Gamma(n, h, F) \rightarrow PG^*(n, F)$ such that $U \subseteq U^\chi$ for each h -subspace U is a polarity. The property above is called the *null property*, and a polarity satisfying the null property is called a *null polarity*.

There is a one-to-one correspondence between generalized null polarities of index h and linear complexes of $(h + 1)$ -subspaces. More specifically, to each generalized null polarity χ of index h it is possible to associate a linear complex K of $(h + 1)$ -subspaces, which is characterized by the following property: $W \in K$ if and only if there exists an h -subspace $X \subseteq W$ such that $W \subseteq X^\chi$. In this case, $W \subseteq Y^\chi$ for every h -subspace $Y \subseteq W$ [8].

Conversely, if K is a linear complex of $(h + 1)$ -subspaces, then it is possible to associate a generalized null polarity $\chi : \Gamma(n, h, F) \rightarrow PG^*(n, F)$ defined by:

$$X^\chi = \bigcup_{X \subseteq U \in K} U, \text{ for every } X \in \Gamma(n, h, F)$$

By this correspondence one can see that to ask for the existence of a linear complex of $(h + 1)$ -subspaces, having no singular h -subspace, is equivalent to ask for the existence of a global generalized null polarity of index h .

Let U be a subspace of $PG(n, F)$ and let K be a linear complex of $(h + 1)$ -subspaces. We set $\Gamma^i(U)$ to be the set of all i -subspaces of U . We denote by $K \cap U$ the set of all $(h + 1)$ -subspaces of K which are contained in U . Note that $K \cap U$ is not the intersection set between K and U .

2 Global Generalized Null Polarities

From now on we assume that there exists in $PG(n, F)$ a global generalized null polarity χ of index 1. By [8], prop. 12, n is even. The related linear complex of planes K is given by all the planes ε of $PG(n, F)$ possessing a line l such that $\varepsilon \subseteq l^\chi$. By the remarks in the previous section, if $\varepsilon \in K$, then $\varepsilon \subseteq l^\chi$ for every line $l \subseteq \varepsilon$.

A d -subspace U of $PG(n, F)$, with $d \geq 2$, is said to be *special* (with respect to χ) if $\Gamma^2(U) = K \cap U$. If this is not the case, then $K \cap U$ is a prime of $\Gamma^2(U)$. Hence $K \cap U$ is a linear complex of planes of the projective space U .

Observe that a plane ε of $PG(n, F)$ is special if and only if $\varepsilon \in K$.

In the next statements we restrict the generalized global null polarity χ to 3 and 4-subspaces of $PG(n, F)$.

Proposition 1. *Let S be a 3-subspace of $PG(n, F)$. Then exactly one of the following properties holds:*

- (i) S is special;
- (ii) S is non-special and $K \cap S$ is a degenerate linear complex of planes of the projective space S .

Proof. If S is non-special, then $K \cap S$ is a prime of $\Gamma^2(S)$, and $\Gamma^2(S)$ is a three-dimensional projective space. Hence $K \cap S$ is the set of all planes through a fixed point. \square

For a subspace U of $PG(n, F)$, with $1 \leq \dim U \leq n - 1$, we denote by $L(U)$ the set of all lines $l \subseteq U$ such that $U \subseteq l^\chi$, and by $C(U)$ the set of all points $P \in U$ such that for every line l passing through P it holds $l \in L(U)$.

Proposition 2. $C(U)$ is a subspace of U .

Proof. Let P, Q be distinct points of $C(U)$. We show that every point $R \in PQ$ lies in $C(U)$. Let $R \in PQ$ and let l denote an arbitrary line of U passing through R . If $l = PQ$, then $U \subseteq l^\chi$. Now we assume $l \neq PQ$. We can choose a point, say A , lying in l but not in PQ . Consider the lines AP, AQ . Since $P, Q \in C(U)$, we have that $U \subseteq AP^\chi$ and $U \subseteq AQ^\chi$. Then $U \subseteq AR^\chi = l^\chi$, since χ is linear. So, $R \in C(U)$. \square

Proposition 3. Let S be a 3-subspace of $PG(n, F)$. Then exactly one of the following properties hold:

- (i) S is special if and only if $C(S) = S$;
- (ii) S is non-special if and only if $C(S)$ is a point;
- (iii) given a line $l \subseteq S$, then $S \subseteq l^\chi$ if and only if $l \cap C(S) \neq \emptyset$. \square

Main Theorem. Let W be a 4-subspace of $PG(n, F)$. Then exactly one of the following properties holds:

- (i) W is special;
- (ii) W is non-special and $K \cap W$ is a degenerate linear complex of planes of the projective space W ;
- (iii) W is non-special and there exists a unique 3-subspace $S \subseteq W$ containing all singular lines of $K \cap W$. Such lines form a non-degenerate linear complex of S .

We divide the proof of the Theorem above into the following steps.

Step 1. Let $S \subseteq W$ be a 3-subspace and let $P \in C(S)$ be a point. In S there exists at least one pencil of lines with center P such that $W \subseteq l^\chi$ for every line l belonging to the pencil.

Proof. Let ϕ be a pencil of lines in S with center P . By definition of $C(S)$ we have that $S \subseteq l^\chi$ for every line $l \in \phi$. Hence ϕ contains at least one line m such that $W \subseteq m^\chi$, since ϕ^χ covers the whole space $PG(n, F)$.

If ϕ contains another line, say r , such that $W \subseteq r^\chi$, then from the linearity of χ it follows that $W \subseteq l^\chi$ for every line $l \in \phi$.

We have thus proved that the set $L_P = \{l : l \text{ line, } P \in l \subseteq S, l \in L(W)\}$ is either a pencil, or the set of all lines of S passing through P . \square

Step 2. *Let P and m be a point and a line of W , respectively, with $P \in m$ and such that $W \subseteq m^\chi$. Then there exists at least one pencil of lines with center P such that $W \subseteq l^\chi$ for every line l of the pencil.*

Proof. Let $r \subseteq W$ be a line distinct from m and passing through P and let ϕ denote the pencil of lines of W containing both m and r . From the assumption $r \subseteq m^\chi$. Since $m \sim r$ and χ is a polarity, we have that $m \subseteq r^\chi$. If $W \subseteq r^\chi$, then for every line l of the pencil ϕ we have that $W \subseteq l^\chi$. If $W \not\subseteq r^\chi$, then $S = r^\chi \cap W$ is a 3-subspace. Observe that $m \vee r \subseteq S$, hence $P \in C(S)$. The assertion follows since we can apply Step 1 and Prop. 3. \square

Step 3. *Let $P \in W$ be a point and l_1, l_2, l_3 be three independent lines passing through P such that $W \subseteq l_i^\chi$ for every $i = 1, 2, 3$. Then $P \in C(W)$.*

Proof. We have to prove that $W \subseteq l^\chi$ for every line $l \subseteq W$ passing through P . If $l \subseteq l_1 \vee l_2 \vee l_3$, this follows from the linearity of χ . Otherwise, assume $W = l \vee l_1 \vee l_2 \vee l_3$. From the assumptions it follows that $l \subseteq l_i^\chi$, for every $i = 1, 2, 3$. Since χ is a polarity and $l \sim l_i$, we have that $l_i \subseteq l^\chi$, for every $i = 1, 2, 3$. By the null property $l \subseteq l^\chi$. Hence $W \subseteq l^\chi$. \square

Step 4. *W contains at least one special 3-subspace.*

Proof. If W is a special subspace, then the assertion is clear. So, we shall assume that W is not special. For each point $P \in W$ set $L_P = \{l : l \text{ line, } P \in l, l \in L(W)\}$. One can easily check that L_P is a projective space embedded in $\Gamma(n, 1, F)$. By Step 1, there exists a point $Q \in W$ such that $L_Q \neq \emptyset$.

We separately deal with the cases $\dim L_Q > 1$ and $\dim L_Q \leq 1$.

Assume that $\dim L_Q > 1$. Let m_1, m_2, m_3 denote three independent lines belonging to L_Q . By Step 3 we have that $Q \in C(W)$.

Take an arbitrary line $m_0 \in L_Q$. Assume firstly that every line $l \subseteq W$ such that $W \subseteq l^\chi$ intersects m_0 .

Let $s \subseteq W$ be a line incident with m_0 , $s \neq m_0$, and let R denote a point of s not lying on m_0 . We have that $W \subseteq QR^\chi$, since we have proved before that $Q \in C(W)$. Nevertheless, by Step 2 we know that there exists a pencil ϕ of lines with center R such that $W \subseteq l^\chi$ for every line $l \in \phi$. By the assumptions, ϕ lies on the plane $m_0 \vee R$. Hence $W \subseteq s^\chi$. Since s is

an arbitrary line incident with m_0 , we have that $m_0 \subseteq C(W)$. Therefore, every 3-subspace $S \subseteq W$ containing m_0 is special, since $m_0 \subseteq C(W)$ implies $m_0 \subseteq C(S)$.

Next, assume there exists a line $l_1 \subseteq W$ such that $W \subseteq l_1^\chi$ and $m_0 \cap l_1 = \emptyset$. Then the 3-subspace S given by $m_0 \vee l_1$ is special. In fact, $S \subseteq m_0^\chi$, $S \subseteq l_1^\chi$, but m_0, l_1 are disjoint (see Proposition 3, (ii) and (iii)).

The arguments above prove that if $\dim L_Q > 1$, then W contains a special 3-subspace.

We consider the case $\dim L_Q \leq 1$. By Step 2 we have that L_Q is a pencil of lines. Denote by ε the plane containing all the lines of L_Q . Let $S' \subseteq W$ be a 3-subspace such that $S' \cap \varepsilon = m_0$. If S' is special, then the assertion follows. Otherwise, $C(S') \in m_0$. Observe that $C(S') \neq Q$. In fact, if $C(S') = Q$ then, by Step 1, in S' there exists a pencil of lines lying in L_Q , a contradiction. Hence $C(S') \neq Q$ so, again by Step 1, there is a line $m \subseteq S'$ such that $Q \notin m$ and $W \subseteq m^\chi$. Denote by S'' the 3-subspace $\varepsilon \vee m$. Then $S'' \subseteq m^\chi$, since $W \subseteq m^\chi$, but also $S'' \subseteq r^\chi$ for every line $r \in L_Q$. By Proposition 3, (ii) and (iii) we have that S'' is special. \square

Step 5. *If $C(W)$ is empty, then the following properties hold:*

- (i) *W contains a unique special 3-subspace, say S ;*
- (ii) *each $l \in L(W)$ is contained in S ;*
- (iii) *$L(W)$ is a non-degenerate linear complex of lines in S .*

Proof. By Step 4 we have that W contains a special 3-subspace, say S . By Step 1 and Step 3, for every point $P \in W$ the set L_P of lines in W through P and belonging to $L(W)$ is a pencil ϕ_P contained in S . So, $L(W) = \bigcup_{P \in S} \phi_P$ is a prime of $\Gamma^1(S)$ (the Grassmannian of lines of S), and a non-degenerate linear complex of lines in S . \square

The end of the proof of the main Theorem.

Let W be a 4-subspace. If $C(W) = \emptyset$, then by Step 5, (iii) of the Theorem holds.

Now assume that a point P exists such that $P \in C(W)$. Let S be a 3-subspace not containing P . By Step 1 we have that S contains a pencil $\phi \subseteq L(W)$. Let Q be the center of ϕ and let $l_1, l_2 \in \phi$ be distinct lines. We have that $PQ, l_1, l_2 \in L(W)$, hence $Q \in C(W)$ (see Step 3). So, $PQ \in L(W)$ and $K \cap W$ contains the degenerate linear complex, say K' , of planes in W , consisting of all planes intersecting PQ . Therefore, either W is a special 4-subspace of $PG(n, F)$, or $K \cap W = K'$ is a degenerate linear complex. \square

Proposition 4. *In $PG(n, F)$ there is a 4-subspace W with $C(W) = \emptyset$.*

Proof. First of all, there exists a non-special 3-subspace: an example can be obtained by taking a line l and a 3-subspace S such that $S \not\subseteq l$.

Now take a line m such that $C(S) \in m \subseteq S$, a point $P \notin m^\chi$, and define $W = S \vee P$. The 4-subspace W is clearly non-special.

Assume that $K \cap W$ is a degenerate linear complex. Then $C(S) \in C(W)$. However, by the reciprocity property, $P \notin m^\chi$ implies $m \not\subseteq C(S)P$, hence $C(S) \notin C(W)$, a contradiction. The assertion follows from the main Theorem. \square

Proposition 5. *Let W be a 4-subspace such that $C(W) = \emptyset$ and let S denote the unique special 3-subspace contained in W . Furthermore, let π denote the polarity in S associated with $L(W)$ (cf. main Theorem). Then, for every line $l \not\subseteq S$, we have $l^\chi \cap W = l \vee (l \cap S)^\pi$.*

Proof. By the null property we have that $l \subseteq l^\chi$, hence it suffices to prove that $(l \cap S)^\pi \subseteq l^\chi$.

Let ϕ be the pencil of lines of S with center $P = l \cap S$ and plane P^π . Every line $l \in \phi$ belongs to $L(W)$. Let $l_1, l_2 \in \phi$ be distinct lines. For $i = 1, 2$ we have that $l \sim l_i$, $l \subseteq l_i^\chi$, hence $l_i \subseteq l^\chi$. Whence $P^\pi \subseteq l^\chi$. \square

By Proposition 5, $K \cap W$ is uniquely determined by $L(W)$ in each case.

3 The map induced by χ

A global generalized null polarity χ of index 1, associated with a linear complex of planes K , induces a map

$$\chi' : \Gamma(n, 2, F) \rightarrow \Gamma(n, n-3, F)$$

defined by

$$\varepsilon^{\chi'} = \bigcap_{l \subseteq \varepsilon} l^\chi.$$

The following properties hold.

Proposition 6. *Let ε be a plane not in K . Then*

- (i) $\varepsilon \cap \varepsilon^{\chi'} = \emptyset$, and
- (ii) $l^\chi = l \vee \varepsilon^{\chi'}$ for every line l of ε .

Proof. We prove (i). Suppose that $\varepsilon \cap \varepsilon^{\chi'}$ is non-empty. Let P be a point in $\varepsilon \cap \varepsilon^{\chi'}$ and let m be a line of ε not passing through P . Since $P \in \varepsilon^{\chi'}$ we have that $P \in m^\chi$. But $m \subseteq m^\chi$, since χ is a null polarity. Hence $\varepsilon = P \vee m \subseteq m^\chi$, that is $\varepsilon \in K$, a contradiction.

We prove (ii). Let $l \subseteq \varepsilon$ be a line. By (i) we have that $l \cap \varepsilon^{\chi'} = \emptyset$, therefore $\dim(l \vee \varepsilon^{\chi'}) = n-1$. Nevertheless $l \vee \varepsilon^{\chi'} \subseteq l^\chi$, since $l, \varepsilon^{\chi'} \subseteq l^\chi$, hence $l^\chi = l \vee \varepsilon^{\chi'}$. \square

Proposition 7. *Let $\varepsilon_1, \varepsilon_2$ be two distinct planes which are collinear in $\Gamma(n, 2, F)$. Assume that $\varepsilon_1 \notin K$. Then $W = \varepsilon_1^{X'} \vee \varepsilon_2^{X'}$ is an m -subspace such that $m \geq n - 1$.*

Proof. Suppose that $m < n - 1$. We denote by r the line $\varepsilon_1 \cap \varepsilon_2$. We distinguish two cases.

Case n.1: $\varepsilon_2 \notin K$. At least one hyperplane among the hyperplanes of $PG(n, F)$ passing through W does not contain r , otherwise $r \subseteq W$, whence $r \cap \varepsilon_i^{X'} \neq \emptyset$, a contradiction since $\varepsilon_i \cap \varepsilon_i^{X'} = \emptyset$ for $i = 1, 2$. Let H be a hyperplane such that $W \subseteq H$ and $r \not\subseteq H$.

For $i = 1, 2$ we set $l_i = H \cap \varepsilon_i$. We have that $l_1 \neq l_2$. In fact, if $l_1 = l_2$, then $l_1, l_2 \subseteq \varepsilon_i$, for $i = 1, 2$. But $\varepsilon_1 \cap \varepsilon_2 = r$. Hence $l_1 = l_2 = r$, a contradiction since $r \not\subseteq H$.

We show that $l_1^X = l_2^X$. From Proposition 6 we have that $l_i^X = l_i \vee \varepsilon_i^{X'}$ for $i = 1, 2$. Since l_i and $\varepsilon_i^{X'}$ are both contained in H , we have that $H = l_i^X$ for $i = 1, 2$. Hence $l_1^X = l_2^X$.

We have thus proved that there exist two distinct collinear lines, namely l_1, l_2 , such that $l_1^X = l_2^X$. That yields a contradiction since χ is global. Hence $m \geq n - 1$.

Case n.2: $\varepsilon_2 \in K$. In this case $\varepsilon_1 \cap \varepsilon_1^{X'} = \emptyset$, and $\varepsilon_2 \subseteq \varepsilon_2^{X'}$. Hence, r and $\varepsilon_1^{X'}$ are disjoint subspaces of W whose dimension are 1 and $n - 3$, respectively, a contradiction. \square

As a consequence, there exists a line in $\Gamma(n, 2, F)$ whose image is a set of pairwise non-collinear points in $\Gamma(n, n - 3, F)$:

Proposition 8. *Let Φ be a line of $\Gamma(n, 2, F)$ which is not contained in K . Then χ' maps distinct elements of Φ to non-collinear elements of $\Gamma(n, n - 3, F)$. \square*

4 Line partitions

Line partitions of finite projective spaces $PG(n, q)$ have been studied in [3, 4, 5, 6] and in [10].

In [8] it was proved that for $n < 8$ the projective space $PG(n, q)$ admits no linear line partitions. Such result allowed to say that the line partitions in [4] and in [10] are not linear.

In [5] some examples of line partitions of $PG(2n, 2)$, for $n = 1, 2, 3, 4, 5$, are given. As regards $n = 4$ and $n = 5$, for each line partition Ω we found by a computer search in each space three collinear lines, say l_1, l_2, l_3 , such that the intersection of the related hyperplanes, $l_1^{\pi\Omega}, l_2^{\pi\Omega}, l_3^{\pi\Omega}$, is a $(2n - 3)$ -subspace. As a consequence, the map π_Ω is not a linear mapping, and Ω is not a linear line partition. So, it seems that all explicit examples of line partitions in the literature are non-linear. By the equivalence between

linear line partitions and linear complexes of planes without singular lines, explained in the introduction, the known examples of line partitions are not of the kind related to null polarities and linear complexes.

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