SHARP STABILITY INEQUALITIES FOR PLANAR DOUBLE BUBBLES

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Abstract. Sharp stability inequalities for planar double bubbles are proved.

Contents

1. Introduction

1.1. **Overview.** In this paper we address the global stability problem for standard double bubbles in the plane. This is accomplished by combining the general stability theory for isoperimetric problems with multiple volume constraints developed in the companion paper [CLM12] with an ad-hoc analysis of the isoperimetric problem for planar double bubbles, which addresses the presence of singularities and the delicate interaction of multiple volume constraints in connection with stability issues.

1.2. **Standard double bubbles.** A standard double bubble in $\mathbb{R}^3$ is the familiar soap bubble configuration where three spherical caps meet at 120 degree angles along a circle; see Figure 1. From the mathematical point of view, standard double bubbles arise as solutions of isoperimetric problems with two volume constraints. Indeed, let us consider the geometric variational problems

$$\inf\Big\{P(\mathcal{E}) : \mathcal{E} = \{\mathcal{E}(1), \mathcal{E}(2)\}\,, \mathrm{vol}\,(\mathcal{E}) = (|\mathcal{E}(1)|, |\mathcal{E}(2)|) = (m_1, m_2)\Big\}\,, \qquad m_2 \ge m_1 > 0\,, \quad (1.1)$$

where $\mathcal{E}(1)$ and $\mathcal{E}(2)$ are two disjoint open sets with piecewise C^1-boundary in $\mathbb{R}^n$, $n \ge 2$, and the perimeter of $\mathcal{E}$ is defined as

$$P(\mathcal{E}) = \mathcal{H}^{n-1}\Big(\partial\mathcal{E}(1) \cap \partial\mathcal{E}(2)\Big) + \mathcal{H}^{n-1}\Big(\partial\mathcal{E}(1) \setminus \partial\mathcal{E}(2)\Big) + \mathcal{H}^{n-1}\Big(\partial\mathcal{E}(2) \setminus \partial\mathcal{E}(1)\Big)\,. \qquad (1.2)$$

(Here $|E|$ denotes the volume (Lebesgue measure) of $E \subset \mathbb{R}^n$, while $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$.) For every $m_2 \ge m_1 > 0$, there exists a unique way (up to isometries) to enclose volumes m_1 and m_2 in $\mathbb{R}^n$ by three $(n-1)$-dimensional spherical caps meeting at 120 degrees angles along a $(n-2)$-dimensional sphere. This construction leads to define *standard double bubbles* in every dimension and for every pair of prescribed volumes, and standard double bubbles are the unique minimizers in the variational problems (1.1). This has been first proved by Foisy, Alfaro, Brock, Hodges, and Zimba [FAB+93] in $\mathbb{R}^2$, by Hutchings, Morgan, Ritoré, and Ros [Hut97, HMRR02] in $\mathbb{R}^3$ with a major breakthrough, and, finally, by a delicate elaboration of the methods developed in [HMRR02], by Reichardt [Rei08] in $\mathbb{R}^n$, $n \ge 4$ (see also [RHLS03]). A common point of all these papers is the study of the stationarity (vanishing first variation) and stability (non-negative second variation) properties of standard double bubbles. In this direction, we also mention the paper by Morgan and Wichiramala

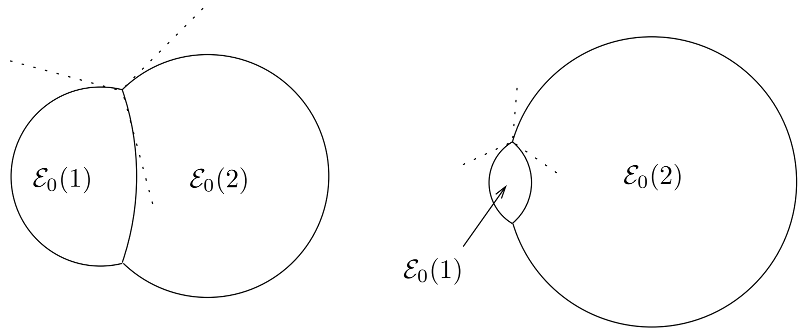

FIGURE 1. Standard double bubbles.

[MW02], where standard double bubbles are shown to be the unique double bubbles with non-negative second variation.

1.3. **Two global stability theorems for double bubbles.** Given a planar double bubble $\mathcal{E}$ with $\mathrm{vol}\,(\mathcal{E}) = (m_1, m_2)$, $m_2 \ge m_1 > 0$, let $\mathcal{E}_0$ denote a standard double bubble with $\mathrm{vol}\,(\mathcal{E}_0) = \mathrm{vol}\,(\mathcal{E})$, and define the *isoperimetric deficit* of $\mathcal{E}$ as

$$\delta(\mathcal{E}) = \frac{P(\mathcal{E})}{P(\mathcal{E}_0)} - 1\,, \qquad (\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0))\,.$$

By [FAB$^+$93], $\delta(\mathcal{E}) \ge 0$, and $\delta(\mathcal{E}) = 0$ if and only if, up to isometries, $\mathcal{E} = \mathcal{E}_0$. We seek to control, in terms of $\delta(\mathcal{E})$, the distance (modulo isometries) of $\mathcal{E}$ from $\mathcal{E}_0$. Given two planar double bubbles $\mathcal{E}$ and $\mathcal{F}$ with $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{F})$ we thus define the normalized L^1-distance between $\mathcal{E}$ and $\mathcal{F}$ as

$$\mathrm{d}(\mathcal{E}, \mathcal{F}) = \frac{|\mathcal{E}(1)\Delta\mathcal{F}(1)|}{m_1} + \frac{|\mathcal{E}(2)\Delta\mathcal{F}(2)|}{m_2}\,, \qquad (\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{F}))\,,$$

and, correspondingly, we consider the d-distance between $\mathcal{E}$ and $\mathcal{E}_0$ modulo isometries

$$\alpha(\mathcal{E}) = \inf\Big\{\mathrm{d}(\mathcal{E}, f(\mathcal{E}_0)) : f(\mathcal{E}_0) = \{f(\mathcal{E}_0(1)), f(\mathcal{E}_0(2))\}\,, f : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is an isometry}\Big\}\,.$$

In this way, $\delta(\mathcal{E}) = 0$ if and only if $\alpha(\mathcal{E}) = 0$. We call $\alpha(\mathcal{E})$ the *Fraenkel asymmetry* of the double bubble $\mathcal{E}$. Our main result provides a control on $\alpha(\mathcal{E})$ in terms of $\delta(\mathcal{E})$, with sharp decay rate of $\alpha(\mathcal{E})$ as $\delta(\mathcal{E}) \to 0$.

Theorem 1.1 (Global stability inequalities)**.** *If $m_2 \ge m_1 > 0$, then there exists $\kappa > 0$, depending on m_1/m_2 only, with the following property. If $\mathcal{E}_0$ is a standard double bubble in $\mathbb{R}^2$ with* $\mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$*, then*

$$\delta(\mathcal{E}) \ge \kappa\, \alpha(\mathcal{E})^2\,, \tag{1.3}$$

whenever $\mathcal{E}$ is a planar double bubble with $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0)$*. In other words, for every planar double bubble $\mathcal{E}$ with* $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$*, we have, up to isometries,*

$$P(\mathcal{E}) \ge P(\mathcal{E}_0)\Big\{1 + \kappa\Big(\frac{m_1}{m_2}\Big)\Big(\frac{|\mathcal{E}(1)\Delta\mathcal{E}_0(1)|}{m_1} + \frac{|\mathcal{E}(2)\Delta\mathcal{E}_0(2)|}{m_2}\Big)^2\Big\}\,.$$

Remark 1.1. We consider (1.3) as a global stability inequality, in the sense that no a priori constraint is assumed on the double bubble $\mathcal{E}$: for example, we do not assume $\mathcal{E}$ to be a local variation or a small global perturbation of $\mathcal{E}_0$. In fact, inequality (1.3) will be proved in the case $\mathcal{E}(1)$ and $\mathcal{E}(2)$ are merely sets of finite perimeter.

Remark 1.2 (Sharp decay rate)**.** Inequality (1.3) is sharp in the sense that the validity of $\delta(\mathcal{E}) \ge \varphi(\alpha(\mathcal{E}))$ for some $\varphi : [0, \infty) \to [0, \infty)$ implies the existence of $C \ge 0$ and $t_0 > 0$ such that $\varphi(t) \le C\, t^2$ for every $t \le t_0$. For a proof of this (in much greater generality), see [CLM12].

Remark 1.3 (Explicit constants)**.** Our approach to (1.3) does not produce an explicit stability constant κ. Although this does not look such an immediate task, an explicit value of κ could maybe be obtained by exploiting in a quantitative way the calibration proof of optimality of standard double bubbles by Dorff, Lawlor, Sampson, and Wilson [DLSW09].

Remark 1.4. In connection with the already mentioned result by Morgan and Wichiramala [MW02], Theorem 1.1 implies in particular that second variation of perimeter (with respect to vector-fields which preserve volume at first order) is strictly positive at standard double bubbles.

The typical situation in which we expect to observe double bubbles $\mathcal{E}$ whose perimeter $P(\mathcal{E})$ is close to that of a standard double bubble $\mathcal{E}_0$ with $\mathrm{vol}\,(\mathcal{E}_0) = \mathrm{vol}\,(\mathcal{E}_0)$, is when the double bubble $\mathcal{E}$ is the solution to a geometric variational problem sufficiently close to (1.1). In this direction, a natural family of problems to consider is

$$\inf\left\{P(\mathcal{E}) + \beta\int_{\mathcal{E}(1)\cup\mathcal{E}(2)} g(x)\,dx : \mathrm{vol}\,(\mathcal{E}) = (m_1, m_2)\right\}, \tag{1.4}$$

where the total free energy of the double bubble $\mathcal{E}$ is defined as the sum of its surface tension energy (perimeter) $P(\mathcal{E})$ plus a small potential energy term of the form

$$\beta\int_{\mathcal{E}(1)\cup\mathcal{E}(2)} g(x)\,dx\,.$$

Here, β is a suitably small positive constant, while the potential $g : \mathbb{R}^2 \to [0,\infty)$ is assumed to be coercive (that is, $g(x) \to \infty$ as $|x| \to \infty$) in order to trivialize existence issues, and to be smooth for the sake of simplicity. Under these assumptions, the regularity theory for planar minimizing clusters (see, for example, [CLM12, Section 6]) implies that the boundary $\partial\mathcal{E}_\beta = \partial\mathcal{E}_\beta(1)\cup\partial\mathcal{E}_\beta(2)$ of every minimizer $\mathcal{E}_\beta$ in (1.4) consists of finitely many $C^{1,1}$-curves, meeting at finitely many singular points, each singular point being the end-point of exactly three of these of curves. When β is small, however, we expect a minimizer $\mathcal{E}_\beta$ to closely resemble a standard double bubble $\mathcal{E}_0$. Indeed, by comparing $\mathcal{E}_\beta$ with $\mathcal{E}_0$, and knowing that both $\mathcal{E}_\beta$ and $\mathcal{E}_0$ are contained in a same ball of radius $R = R(m_1, m_2, g)$, we find that

$$P(\mathcal{E}_\beta) \le P(\mathcal{E}_0) + \beta\int_{(\mathcal{E}_0(1)\cup\mathcal{E}_0(2))\setminus(\mathcal{E}(1)\cup\mathcal{E}(2)))} g(x)\,dx \le P(\mathcal{E}_0) + \beta\sup_{B_R} g\,,$$

so that $\delta(\mathcal{E}_\beta) \le C\,\beta$ for some constant $C = C(m_1, m_2, g)$; therefore, by Theorem 1.1, $\alpha(\mathcal{E}_\beta)^2 \le (C/\kappa)\,\beta$, and $\mathcal{E}_\beta$ is close to $\mathcal{E}_0$ in L^1-sense. However, due to the minimizing property of $\mathcal{E}_\beta$, we expect proximity to $\mathcal{E}_0$ to hold true in a much stronger sense. Indeed, in [CLM12], we show the existence of some positive constant β_0 (depending on m_1, m_2, and g only) such that if $\beta \in (0, \beta_0)$, then there exist $C^{1,1}$-diffeomorphisms $f_\beta : \partial\mathcal{E}_0 \to \partial\mathcal{E}_\beta$ with $\|f_\beta - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)}$ vanishing as $\beta \to 0^+$. This result implies in particular that $\partial\mathcal{E}_\beta$ has the same topological structure of $\partial\mathcal{E}_0$, that is, it consists of three simple curves meeting in threes at two distinct singular points. (Moreover, in [CLM12], the (generalized) curvatures of these curves are shown to be uniformly close to the constant curvatures of the corresponding standard double bubble.) In this paper, exploiting Theorem 1.1 together with a suitable interpolation inequality, we provide a power law stability estimate for $\|f_\beta - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)}$ in terms of the difference between the surface tension energies of $\mathcal{E}_\beta$ and $\mathcal{E}_0$.

Theorem 1.2 (Perturbed minimizing clusters)**.** *If $m_2 \ge m_1 > 0$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with $g(x) \to \infty$ as $|x| \to \infty$, then there exist $\kappa_0 > 0$ and $\beta_0 > 0$, depending on m_1, m_2, and g only, with the following property. If $\mathcal{E}_\beta$ is a minimizer in the variational problem* (1.4) *for some $\beta \in (0, \beta_0)$, then there exists a standard double bubble $\mathcal{E}_0$ with $\mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$ and* *a $C^{1,1}$-diffeomorphism $f_\beta : \partial\mathcal{E}_0 \to \partial\mathcal{E}_\beta$ such that*

$$\frac{\beta}{\kappa_0} \ge P(\mathcal{E}_\beta) - P(\mathcal{E}_0) \ge \kappa_0 \, \|f_\beta - \mathrm{Id}\|^6_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} .$$

Remark 1.5. Ren and Wei [RW12] consider a similar problem, focusing on the case of a potential energy of non-local character, but limiting their analysis to the case $m_1 = m_2$.

1.4. **Background and method of proof.** After the pioneering contributions by Bernstein [Ber05] and Bonnesen [Bon24], the analysis of global stability problems has received a renewed attention in recent years, with the proof of the sharp stability inequality for the Euclidean isoperimetric problem [Fug89, Fug93, HHW91, Hal92, FMP08, CL12, FGP12], the Wulff isoperimetric problem [FMP10], the Gaussian isoperimetric problem [CFMP11, MN12], Plateau-type problems [DPM11], fractional isoperimetric problems [FVM10], and isoperimetric problems in higher codimension [BDF12]. (This list is probably incomplete, and it does not mention contributions to stability problems for functional inequalities.)

In [CLM12] we have started the study of stability issues for isoperimetric problems with multiple volume constraints. Among the various results proved in that paper, we have a reduction theorem for the global stability inequality (1.3). In the particular case of double bubbles, this result shows that, in proving (1.3), one can directly consider comparison double bubbles $\mathcal{E}$ with $\partial\mathcal{E} = f(\partial\mathcal{E}_0)$ for a standard double bubble $\mathcal{E}_0$ and a $C^{1,1}$-diffeomorphism $f : \partial\mathcal{E}_0 \to \partial\mathcal{E}$ with $\|f - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)}$ as small as wished.

This strategy of reduction to C^1-small diffeomorphic images of the minimizers has been introduced in [CL12, CL13] in the stability analysis of the Euclidean isoperimetric problem. In that case the solution of the "reduced" stability problem is achieved rather easily by a Fourier series argument originally introduced by Fuglede [Fug89, Fug93].

In the case of double bubbles the situation is much subtler, due to the presence of singularities and to the interaction between multiple volume constraints, which act as underlying constraints in the stability analysis of a "multiple" Poincaré-type inequality. We shall address this problem by combining Fourier series arguments in the spirit of Fuglede with the solution of certain one-dimensional variational problems, to proceed through a case by case analysis. Different cases will correspond to different behaviors of the perturbed interfaces, based for example on the relative size between their L^2-mean deviation and their L^2-distance from the corresponding interfaces of the reference standard double bubble. The resulting argument, although based on rather elementary mathematical tools, sheds light on the non-trivial interactions between single interfaces, on which the global stability of standard double bubbles ultimately depends. As an entirely analogous structure underlies the stability problem for standard double bubbles in higher dimensions, the methods employed in this paper should thus be useful also in the analysis of possible higher dimensional extensions, which we leave for future investigation.

We finally notice that, at present, there is only another instance of isoperimetric problem with multiple volume constraints whose minimizers are explicitly known. This is the case of the planar triple bubble problem, addressed by Wichiramala in [Wic04]. Considering that the preliminary analysis from [CLM12] applies to this case as well, by further exploiting the methods developed here it should be possible to obtain analogous results to Theorem 1.1 and Theorem 1.2 in the case of planar triple bubbles too.

1.5. **Organization of the paper.** In section 2 we introduce a notion of small perturbation of a standard double bubble, and then derive suitable bounds on the isoperimetric deficit and the Fraenkel asymmetry on such perturbations. In section 3 we prove a multiple Poincaré-type inequality which implies Theorem 1.1 on the class of perturbations introduced in section 2. Finally, in section 4, we prove Theorem 1.1 and Theorem 1.2.

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2. Small perturbations of standard double bubbles

A *double bubble* in $\mathbb{R}^n$ is a pair $\mathcal{E} = \{\mathcal{E}(1), \mathcal{E}(2)\}$ of sets of finite perimeter with $0 < |\mathcal{E}(h)| < \infty$, $h = 1, 2$, and $|\mathcal{E}(1) \cap \mathcal{E}(2)| = 0$. We call $\mathcal{E}(1)$ and $\mathcal{E}(2)$ the *chambers* of $\mathcal{E}$ (note that these sets are not assumed to be connected), and denote by $\mathcal{E}(0) = \mathbb{R}^n \setminus (\mathcal{E}(1) \cup \mathcal{E}(2))$ the *exterior chamber* of $\mathcal{E}$. We set $\mathrm{vol}\,(\mathcal{E}) = (|\mathcal{E}(1)|, |\mathcal{E}(2)|)$, and define the *perimeter* of $\mathcal{E}$ as

$$P(\mathcal{E}) = \frac{1}{2} \sum_{h=0}^{2} P(\mathcal{E}(h)) \,. \tag{2.1}$$

(Here, $P(E)$ denotes the distributional perimeter of $E \subset \mathbb{R}^n$, so that $P(E) = \mathcal{H}^{n-1}(\partial E)$ whenever E is an open set with piecewise C^1-boundary; see [Mag12, Example 12.7]. Thus, thanks to the factor $1/2$, (2.1) agrees with (1.2) whenever $\mathcal{E}(1)$ and $\mathcal{E}(2)$ are open set with piecewise C^1-boundary.) The analysis carried on in [CLM12], plus a trivial scaling argument, allows to reduce the proof of Theorem 1.1 for planar double bubbles whose chambers are generic sets of finite perimeter, to the proof of the following theorem.

Theorem 2.1 (Theorem 1.1 reduced)**.** *If $m_2 \ge m_1 > 0$, then there exist $\kappa > 0$ and $\eta > 0$, depending on m_1 and m_2 only, with the following property. If $\mathcal{E}_0$ is a standard double bubble in $\mathbb{R}^2$ with $\mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$, then*

$$\delta(\mathcal{E}) \ge \kappa\, \alpha(\mathcal{E})^2 \,, \tag{2.2}$$

whenever $\mathcal{E}$ is a planar double bubble with $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0)$ and $\partial\mathcal{E} = f(\partial\mathcal{E}_0)$ for a $C^{1,1}$-diffeomorphism $f : \partial\mathcal{E}_0 \to \partial\mathcal{E}$ with

$$\|f - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} < \eta \,.$$

Here we have set $\partial\mathcal{E} = \partial\mathcal{E}(1) \cup \partial\mathcal{E}(2)$.

We refer readers to [CLM12] for the proof of the fact that Theorem 2.1 implies Theorem 1.1. In this section, we introduce a class of small perturbations of standard double bubbles which shall be used later on to describe the clusters $\mathcal{E}$ appearing in Theorem 2.1, and, correspondingly, provide suitable estimates of $\delta(\mathcal{E})$ and $\alpha(\mathcal{E})$.

2.1. Circular arcs, circular sectors and their perturbations.

Let $B = \{x \in \mathbb{R}^2 : |x| < 1\}$. Given $\theta \in (0, \pi)$, we define a circular arc $A(\theta) \subset \partial B$ and a circular sector $S(\theta) \subset B$ by setting

$$\begin{aligned} A(\theta) &= \Big\{x \in \mathbb{R}^2 : |x| = 1 \,, x_1 > \cos\theta\Big\} \,, \\ S(\theta) &= \Big\{t\, x : x \in A(\theta) \,, 0 < t < 1\Big\} \,, \end{aligned}$$

while, given $u \in W^{1,2}_0(A(\theta))$ we denote by $A(\theta, u) \subset \mathbb{R}^2$ and $S(\theta, u) \subset \mathbb{R}^2$ the perturbed circular arc and perturbed circular sector defined as

$$\begin{aligned} A(\theta, u) &= \Big\{(1 + u(x))\, x : x \in A(\theta)\Big\} \,, \\ S(\theta, u) &= \Big\{(1 + t\, u(x))\, x : x \in A(\theta) \,, t \in (0,1)\Big\} \,; \end{aligned}$$

see Figure 2. (Notice that $A(\theta, 0) = A(\theta)$ and $S(\theta, 0) = S(\theta)$.) In the analysis of the case

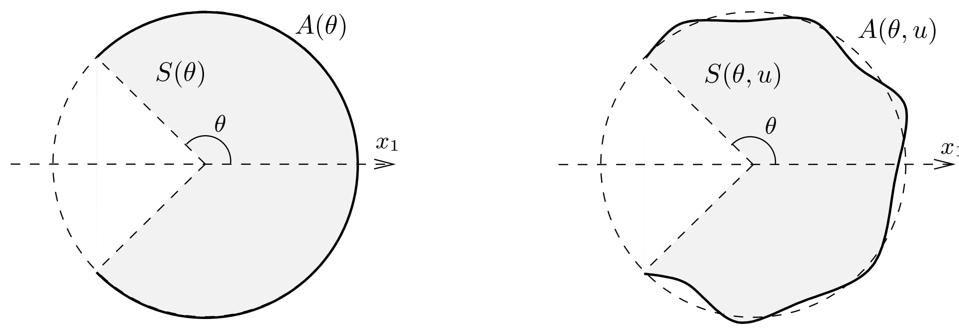

FIGURE 2. The circular arc $A(\theta)$, the circular sector $S(\theta)$, and their perturbations defined by $u \in W_0^{1,2}(A(\theta))$.

$m_1 = m_2$, where the interface between the chambers is a segment, it is convenient to introduce as a reference domain the vertical open segment H and its perturbations $H(u)$ defined as

$$H = \Big\{x \in \mathbb{R}^2 : |x_2| < \frac{\sqrt{3}}{2},\, x_1 = 0\Big\}, \tag{2.3}$$

$$H(u) = \Big\{x + u(x)\, e_2 : x \in H\Big\}, \tag{2.4}$$

in correspondence of $u \in W_0^{1,2}(H)$. We sometime find it convenient to identify $A(\theta)$ with the interval $(-\theta, \theta)$ and H with the interval $(-\sqrt{3}/2, \sqrt{3}/2)$; correspondingly, we shall identify $W_0^{1,2}(A(\theta))$ with $W_0^{1,2}(-\theta, \theta)$ and $W_0^{1,2}(H)$ with $W_0^{1,2}(-\sqrt{3}/2, \sqrt{3}/2)$. The following elementary lemma provides useful formulas for the area of $S(\theta, u)$ and the length of $A(\theta, u)$ in the case that $\|u\|_{W^{1,2}(-\theta,\theta)}$ is small.

Lemma 2.2. *If $u \in W_0^{1,2}(-\theta, \theta)$, then*

$$|S(\theta, u)| - |S(\theta)| = \int_{-\theta}^{\theta} u + \frac{u^2}{2}, \tag{2.5}$$

$$\mathcal{H}^1(A(\theta, u)) - \mathcal{H}^1(A(\theta)) = \int_{-\theta}^{\theta} u + \frac{(u')^2}{2} + \mathrm{o}\Big(\int_{-\theta}^{\theta} u^2\Big) + \mathrm{o}\Big(\int_{-\theta}^{\theta} (u')^2\Big). \tag{2.6}$$

Moreover, if $|u| \le 1$, then

$$|S(\theta, u)\Delta S(\theta)| \le \frac{3}{2}\int_{-\theta}^{\theta} |u|\,. \tag{2.7}$$

Proof. Identity (2.5) is a straightforward consequence of

$$|S(\theta, u)| = \int_{-\theta}^{\theta} \frac{(1+u)^2}{2}\,.$$

Similarly, (2.7) follows as, provided $|u| \le 1$,

$$|S(\theta, u)\Delta S(\theta)| = \int_{-\theta}^{\theta} \Big|\frac{(1+u)^2 - 1}{2}\Big| \le \frac{3}{2}\int_{-\theta}^{\theta} |u|\,.$$

Concerning (2.6), we notice that $A(\theta, u) = T(A(\theta))$ where we have set $T : A(\theta) \to A(\theta, u)$, $T(x) = (1+u(x))x$, $x \in A(\theta)$. The Jacobian of T on $A(\theta)$ is $JT = \sqrt{(1+u)^2 + |u'|^2}$. Therefore, using the Taylor expansion $\sqrt{1+t} = 1 + (t/2) - (t^2/8) + \mathrm{o}(t^2)$ we find

$$JT = 1 + u + \frac{(u')^2}{2} + \mathrm{o}(u^2) + \mathrm{o}((u')^2)\,,$$

and (2.6) follows by the area formula. $\square$

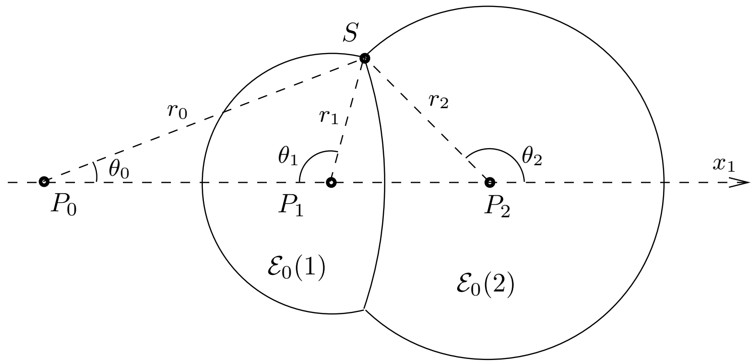

FIGURE 3. The reference standard double bubble $\mathcal{E}_0$.

2.2. **Reference planar standard double bubble.** Given $m_2 \geq m_1$, we now fix a reference standard double bubble $\mathcal{E}_0$ with $\mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$ by requiring that the two point singularities of $\mathcal{E}_0$ belong to the x_2-axis, and that their middle-point lies at the origin (indeed, these geometric requirements uniquely identify $\mathcal{E}_0$). In the case that $m_2 > m_1$, there exist $L_k : \mathbb{R}^2 \to \mathbb{R}^2$ isometries, $r_k > 0$, and $\theta_k \in (0, \pi)$, $k = 0, 1, 2$, such that

$$\begin{aligned} \partial\mathcal{E}_0(1) \cap \partial\mathcal{E}_0(2) &= L_0\, r_0\, A(\theta_0)\,, && (2.8)\\ \partial\mathcal{E}_0(1) \setminus \partial\mathcal{E}_0(2) &= L_1\, r_1\, A(\theta_1)\,, && (2.9)\\ \partial\mathcal{E}_0(2) \setminus \partial\mathcal{E}_0(1) &= L_2\, r_2\, A(\theta_2)\,. && (2.10) \end{aligned}$$

With reference Figure 3, we thus have

$$\begin{aligned} r_0 &= |S - P_0|\,, & \theta_0 &= (P_1 P_0 S)\,,\\ r_1 &= |S - P_1|\,, & \theta_1 &= (P_0 P_1 S)\,,\\ r_2 &= |S - P_2|\,, & \theta_2 &= \pi - (P_1 P_2 S)\,, \end{aligned}$$

and it holds

$$r_0 \sin\theta_0 = r_1 \sin\theta_1\,, \qquad r_0 \sin\theta_0 = r_2 \sin\theta_2\,. \tag{2.11}$$

By Plateau's laws (vanishing of first variation), the three circular arcs meet at 120 degrees angles,

$$\theta_1 + \theta_0 = \frac{2\pi}{3}\,, \qquad \theta_2 - \theta_0 = \frac{2\pi}{3}\,, \tag{2.12}$$

and, correspondingly, the following inequalities hold true

$$0 < \theta_0 < \frac{\pi}{3}\,, \qquad \frac{\pi}{3} < \theta_1 < \frac{2\pi}{3}\,, \qquad \frac{2\pi}{3} < \theta_2 < \pi\,. \tag{2.13}$$

Vanishing of first variation also implies the following "law of pressures",

$$\frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_0}\,. \tag{2.14}$$

Identities (2.11) and (2.12) provide four constraints on the six parameters r_k and θ_k, $k = 0, 1, 2$. Up to a scaling, which leaves the ratio m_2/m_1 invariant, we may add to (2.11) and (2.12) a fifth constraint by requiring that

$$r_2 = 1\,.$$

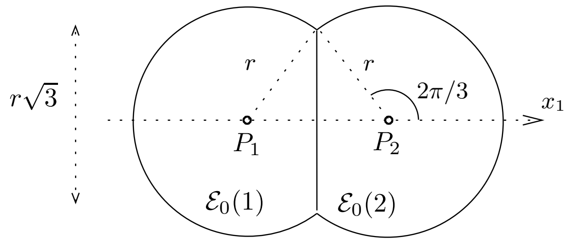

FIGURE 4. The reference standard double bubble $\mathcal{E}_0$ with $m_1 = m_2$.

This choice allows to express the remaining five parameters as functions of $r_1 \in (0,1)$, according to the following relations,

$$
\begin{aligned}
r_0 &= \frac{r_1}{1-r_1}\,, && (2.15)\\
\theta_0 &= \arctan\Big(\frac{1-r_1}{1+r_1}\sqrt{3}\Big)\,, && (2.16)\\
\theta_1 &= \frac{2\pi}{3} - \theta_0\,, && (2.17)\\
\theta_2 &= \frac{2\pi}{3} + \theta_0. && (2.18)
\end{aligned}
$$

Finally, in the case $m_1 = m_2$, we set $m = m_1 = m_2$, $r = r_1 = r_2$, we have

$$\theta_1 = \theta_2 = \frac{2\pi}{3}\,, \qquad \theta_0 = 0\,, \qquad r_0 = +\infty\,,$$

and describe the interfaces of the reference standard double bubble $\mathcal{E}_0$ as

$$
\begin{aligned}
\partial\mathcal{E}_0(1) \cap \partial\mathcal{E}_0(2) &= L_0\, r\, H, && (2.19)\\
\partial\mathcal{E}_0(1) \setminus \partial\mathcal{E}_0(2) &= L_1\, r\, A\left(\frac{2\pi}{3}\right), && (2.20)\\
\partial\mathcal{E}_0(2) \setminus \partial\mathcal{E}_0(1) &= L_2\, r\, A\left(\frac{2\pi}{3}\right), && (2.21)
\end{aligned}
$$

for some isometries $L_k : \mathbb{R}^2 \to \mathbb{R}^2$, $k = 0,1,2$; see Figure 4. Notice that (2.20) and (2.21) are obtained from (2.9) and (2.10) by setting $\theta_1 = \theta_2 = (2/3)\pi$, while (2.19) is not directly related to (2.8). Finally, we show the following useful formula for $P(\mathcal{E}_0)$ in terms of m_1, m_2, r_1, and r_2.

Lemma 2.3. *If $\mathcal{E}_0$ is the standard double bubble with $m_2 > m_1$, then*

$$
\begin{aligned}
P(\mathcal{E}_0) &= 2\Big(\frac{m_1}{r_1} + \frac{m_2}{r_2}\Big)\,, && (2.22)\\
m_1 &= \theta_1\, r_1^2 + \theta_0\, r_0^2 - \frac{\sqrt{3}}{2}\, r_0\, r_1\,, && (2.23)\\
m_2 &= \theta_2\, r_2^2 - \theta_0\, r_0^2 + \frac{\sqrt{3}}{2}\, r_0\, r_2\,. && (2.24)
\end{aligned}
$$

Moreover, (2.22) *holds true also when $m_2 = m_1 = m$, and in that case, we have*

$$m = \Big(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\Big)\, r^2\,. \qquad (2.25)$$

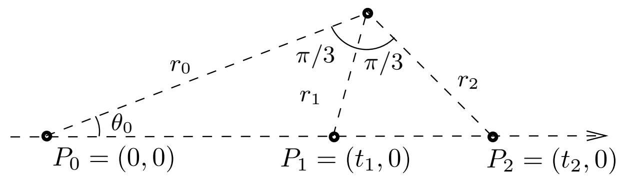

FIGURE 5. We have $t_1/\sin(\pi/3) = r_1/\sin\theta_0$ and $t_2/\sin(2\pi/3) = r_2/\sin\theta_0$.

Proof. We apply the divergence theorem on the chamber $\mathcal{E}_0(1)$ to the vector field $x - P_1$, and on the chamber $\mathcal{E}_0(2)$ to the vector field $x - P_2$, to find that

$$2\,m_1 = 2\theta_1\,r_1^2 + \int_{\partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)} (x - P_1)\cdot\nu_{\mathcal{E}_0(1)}(x)\,d\mathcal{H}^1(x)\,, \tag{2.26}$$

$$2\,m_2 = 2\theta_2\,r_2^2 + \int_{\partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)} (x - P_2)\cdot(-\nu_{\mathcal{E}_0(1)}(x))\,d\mathcal{H}^1(x)\,. \tag{2.27}$$

(Here, $\nu_{\mathcal{E}_0(1)}$ denotes the outer unit normal to $\mathcal{E}_0(1)$.) In the case $m_2 > m_1$, we set the origin at P_0 (see Figure 3), and parameterize $\partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)$ as $\{r_0\,e^{i\theta} : |\theta| < \theta_0\}$. In this way, see Figure 5, we have $P_1 = (t_1, 0)$ and $P_2 = (t_2, 0)$, where

$$\frac{t_1}{\sin(\pi/3)} = \frac{r_1}{\sin\theta_0}\,, \qquad \frac{t_2}{\sin(2\pi/3)} = \frac{r_2}{\sin\theta_0}\,,$$

and, correspondingly

$$\begin{aligned}
\int_{\partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)} (x - P_1)\cdot\nu_{\mathcal{E}_0(1)}(x)\,d\mathcal{H}^1(x) &= \int_{-\theta_0}^{\theta_0} (r_0\,e^{i\theta} - (t_1, 0))\cdot e^{i\,\theta}\,r_0\,d\theta \\
&= 2\theta_0\,r_0^2 - 2\sin\theta_0\,r_0\,t_1 = 2\theta_0\,r_0^2 - \sqrt{3}\,r_0\,r_1\,, \\
\int_{\partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)} (P_2 - x)\cdot\nu_{\mathcal{E}_0(1)}(x)\,d\mathcal{H}^1(x) &= \int_{-\theta_0}^{\theta_0} ((t_2, 0) - r_0\,e^{i\theta})\cdot e^{i\,\theta}\,r_0\,d\theta \\
&= -2\theta_0\,r_0^2 + 2\sin\theta_0\,r_0\,t_2 = -2\theta_0\,r_0^2 + \sqrt{3}\,r_0\,r_2\,.
\end{aligned}$$

We plug these identities into (2.26) and (2.27) to find (2.23) and (2.24); moreover, dividing (2.23) and (2.24) by r_1 and r_2 respectively, by adding up the resulting inequalities, and by (2.14),

$$2\Big(\frac{m_1}{r_1} + \frac{m_2}{r_2}\Big) = 2\theta_1\,r_1 + 2\theta_2\,r_2 + 2\theta_0\Big(\frac{r_0^2}{r_1} - \frac{r_0^2}{r_2}\Big) = 2\theta_1\,r_1 + 2\theta_2\,r_2 + 2\theta_0\,r_0 = P(\mathcal{E}_0)\,,$$

that is (2.22). In the case $m_2 = m_1$, $\nu_{\mathcal{E}_0(1)}(x) = e_1$ and $(x - P_1)\cdot e_1 = (P_2 - x)\cdot e_1 = \ell$ for every $x \in \partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)$, where, by Pythagoras' theorem, $\ell = r/2$. Therefore, (2.26) gives

$$2\,m = 2\,\frac{2\pi}{3}\,r^2 + \ell\,\mathcal{H}^1(\partial\mathcal{E}_0(1)\cap\partial\mathcal{E}_0(2)) = \frac{4\pi}{3}\,r^2 + \frac{\sqrt{3}}{2}\,r^2 = \frac{P(\mathcal{E}_0)}{2}\,r\,,$$

and (2.22) holds true when $m_2 = m_1$ too. □

2.3. (ε, σ)**-perturbations of the reference standard double bubble.** Let us consider the reference standard double bubble $\mathcal{E}_0$ defined in section 2.2, and introduce a useful class of perturbations of $\mathcal{E}_0$. For example, in the case that $m_2 > m_1$, these perturbations are obtained by first replacing the circular arcs $r_k\,L_k\,A(\theta_k)$ $(k = 0, 1, 2)$ with perturbed circular arcs $r_k\,L_k\,A(\theta_k, u_k)$ $(k = 0, 1, 2)$ associated to functions $u_k \in C^1_0(A(\theta_k))$ with $\|u_k\|_{C^1}$ suitably small (note that this operation leaves the singularities of $\mathcal{E}_0$ fixed), and then by dilating the resulting configuration by a uniform factor $1+\sigma$ (an operation that moves the singularities by a distance $|\sigma|$). We now state this precisely. In the case $m_2 > m_1$, one says that a double bubble $\mathcal{E}$ is an (ε, σ)-perturbation of $\mathcal{E}_0$ (ε and σ are small constants, ε is positive, σ with no sign restriction), if

$$\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$$

and if there exist functions $u_k \in C^1_0(A(\theta_k))$ with $\|u_0\|_{C^1} \le \varepsilon$ ($k = 0, 1, 2$), such that (compare with (2.8), (2.9), and (2.10)),

$$\partial\mathcal{E}(1) \setminus \partial\mathcal{E}(2) = (1+\sigma)\, L_1\, r_1\, A(\theta_1, u_1)\,, \tag{2.28}$$

$$\partial\mathcal{E}(2) \setminus \partial\mathcal{E}(1) = (1+\sigma)\, L_2\, r_2\, A(\theta_2, u_2)\,, \tag{2.29}$$

$$\partial\mathcal{E}(1) \cap \partial\mathcal{E}(2) = (1+\sigma)\, L_0\, r_0\, A(\theta_0, u_0)\,. \tag{2.30}$$

In the case $m_2 = m_1$, we say that $\mathcal{E}$ is an (ε, σ)-perturbation of $\mathcal{E}_0$ provided there exist functions $v_0 \in C^1_0(H)$, and $u_k \in C^1_0(A(\theta_k))$, $\|v_0\|_{C^1} \le \varepsilon$ and $\|u_k\|_{C^1} \le \varepsilon$ ($k = 1, 2$), such that (2.28) and (2.29) hold true for u_1 and u_2, and, moreover (compare with (2.19)),

$$\partial\mathcal{E}(1) \cap \partial\mathcal{E}(2) = (1+\sigma)\, L_0\, r\, H(v_0)\,.$$

Remark 2.1. The double bubbles considered in Theorem 2.1 are, up to isometries, (ε, σ)-perturbations of $\mathcal{E}_0$. Indeed, if $\mathcal{E}$ is a planar double bubble with $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0)$ and $\partial\mathcal{E} = f(\partial\mathcal{E}_0)$ for a C^1-diffeomorphism $f : \partial\mathcal{E}_0 \to \partial\mathcal{E}$ with $\|f - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} < \eta$, then up to a rotation (turning the segment joining the singularities of $\mathcal{E}$ until it becomes parallel to the segment joining the singularities of $\mathcal{E}_0$) and a translation (that makes the middle-points of these two segments coincide), then $\mathcal{E}$ is an (ε, σ)-perturbation of $\mathcal{E}_0$ for some $\varepsilon = \varepsilon(\eta)$ and $\sigma = \sigma(\eta)$ such that $\varepsilon(\eta) \to 0^+$ and $\sigma(\eta) \to 0$ as $\eta \to 0^+$. More precisely, corresponding to the value of σ and to the functions u_0, u_1, and u_2 (in case $m_2 > m_1$) or v_0, u_1, and u_2 (in case $m_2 = m_1$) that make $\mathcal{E}$ an (ε, σ)-perturbation of $\mathcal{E}_0$, we may define a second C^1-diffeomorphism $g : \partial\mathcal{E}_0 \to \partial\mathcal{E}$ with

$$\|g - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} \le C\Big(|\sigma| + \max_{k=0,1,2} \|u_k\|_{C^1(-\theta_k,\theta_k)}\Big) \tag{2.31}$$

$$\|g - \mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} \le C\Big(|\sigma| + \|v_0\|_{C^1(-\sqrt{3}/2,\sqrt{3}/2)} + \max_{k=1,2} \|u_k\|_{C^1(-2\pi/3,2\pi/3)}\Big)\,,$$

depending on whether $m_2 > m_1$ or $m_2 = m_1$. Therefore, it will suffice to prove Theorem 2.1 in the case of (ε, σ)-perturbations of $\mathcal{E}_0$ with ε and σ sufficiently small (depending on m_1 and m_2).

Next, we introduce the following notation: in the case $m_2 > m_1$ we set

$$\mathrm{o}\,(u^2) = \mathrm{o}\Big(\sum_{k=0}^{2} \int_{-\theta_k}^{\theta_k} u_k^2\Big)\,, \qquad \mathrm{o}\,((u')^2) = \mathrm{o}\Big(\sum_{k=0}^{2} \int_{-\theta_k}^{\theta_k} (u_k')^2\Big)\,,$$

and, in the case $m_2 = m_1$,

$$\mathrm{o}\,(u^2) = \mathrm{o}\Big(\sum_{k=1}^{2} \int_{-2\pi/3}^{2\pi/3} u_k^2\Big) + \mathrm{o}\Big(\int_{-\sqrt{3}/2}^{\sqrt{3}/2} v_0^2\Big)\,,$$

$$\mathrm{o}\,((u')^2) = \mathrm{o}\Big(\sum_{k=1}^{2} \int_{-2\pi/3}^{2\pi/3} (u_k')^2\Big) + \mathrm{o}\Big(\int_{-\sqrt{3}/2}^{\sqrt{3}/2} (v_0')^2\Big)\,.$$

We now consider an (ε, σ)-perturbation $\mathcal{E}$ of $\mathcal{E}_0$, and provide the second order Taylor expansion of $P(\mathcal{E})$ in terms of σ, $\mathrm{o}\,(u^2)$, and $\mathrm{o}\,((u')^2)$.

Lemma 2.4. *If $\mathcal{E}$ is an (ε, σ)-perturbation of $\mathcal{E}_0$ and $m_2 > m_1$, then*

$$\frac{P(\mathcal{E}) - P(\mathcal{E}_0)}{1+\sigma} = \sum_{k=0}^{2} r_k \int_{-\theta_k}^{\theta_k} \Big(\frac{(u_k')^2}{2} - \frac{u_k^2}{2}\Big) + \frac{\sigma^2}{2} P(\mathcal{E}_0) + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2) + \mathrm{o}\,(\sigma^2)\,. \tag{2.32}$$

If, otherwise, $m_2 = m_1$ (and we set $r_1 = r_2 = r$), then we have

$$\begin{aligned}\frac{P(\mathcal{E}) - P(\mathcal{E}_0)}{1+\sigma} &= r\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{(v_0')^2}{2} + r\sum_{k=1}^{2}\int_{-2\pi/3}^{2\pi/3}\Big(\frac{(u_k')^2}{2} - \frac{u_k^2}{2}\Big) + \frac{\sigma^2}{2}P(\mathcal{E}_0) \\ &\quad + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2) + \mathrm{o}\,(\sigma^2)\,. \end{aligned} \tag{2.33}$$

Proof. We prove the statement in the case $m_2 > m_1$, the proof in the other case being the same up to minor changes. By (2.6), (2.28), (2.29) and (2.30), we find that

$$\begin{aligned} P(\mathcal{E}) - P((1+\sigma)\mathcal{E}_0) &= (1+\sigma)\sum_{k=0}^{2} r_k\Big(\mathcal{H}^1(A(\theta_k,u_k)) - \mathcal{H}^1(A(\theta_k))\Big)\,, \\ &= (1+\sigma)\sum_{k=0}^{2} r_k\int_{-\theta_k}^{\theta_k}\left(\frac{(u_k')^2}{2} + u_k\right) + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2). \end{aligned}$$

Therefore we may write

$$\begin{aligned}\frac{P(\mathcal{E}) - P(\mathcal{E}_0)}{1+\sigma} &= \sum_{k=0}^{2} r_k\int_{-\theta_k}^{\theta_k}\left(\frac{(u_k')^2}{2} + u_k\right) + (\sigma - \sigma^2)\,P(\mathcal{E}_0) + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2) + \mathrm{o}\,(\sigma^2) \\ &= \sum_{k=0}^{2} r_k\int_{-\theta_k}^{\theta_k}\left(\frac{(u_k')^2}{2} - \frac{u_k^2}{2}\right) + \sum_{k=0}^{2} r_k\int_{-\theta_k}^{\theta_k}\left(\frac{u_k^2}{2} + u_k\right) \\ &\quad + (\sigma - \sigma^2)P(\mathcal{E}_0) + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2) + \mathrm{o}\,(\sigma^2)\,. \end{aligned} \tag{2.34}$$

Again by (2.28), (2.29) and (2.30) we find that

$$\begin{aligned} |\mathcal{E}(1)| - (1+\sigma)^2|\mathcal{E}_0(1)| &= (1+\sigma)^2 r_1^2\Big(|S(\theta_1,u_1)| - |S(\theta_1)|\Big) \\ &\quad + (1+\sigma)^2 r_0^2\Big(|S(\theta_0,u_0)| - |S(\theta_0)|\Big)\,, \end{aligned} \tag{2.35}$$

$$\begin{aligned} |\mathcal{E}(2)| - (1+\sigma)^2|\mathcal{E}_0(2)| &= (1+\sigma)^2 r_2^2\Big(|S(\theta_2,u_2)| - |S(\theta_2)|\Big) \\ &\quad - (1+\sigma)^2 r_0^2\Big(|S(\theta_0,u_0)| - |S(\theta_0)|\Big)\,. \end{aligned} \tag{2.36}$$

Since $\mathrm{vol}\,(\mathcal{E}) = \mathrm{vol}\,(\mathcal{E}_0) = (m_1, m_2)$, by (2.5), (2.35) and (2.36) we infer

$$\Big(\frac{1}{(1+\sigma)^2} - 1\Big)\,m_1 = r_1^2\int_{-\theta_1}^{\theta_1}\left(u_1 + \frac{u_1^2}{2}\right) + r_0^2\int_{-\theta_0}^{\theta_0}\left(u_0 + \frac{u_0^2}{2}\right)\,, \tag{2.37}$$

$$\Big(\frac{1}{(1+\sigma)^2} - 1\Big)\,m_2 = r_2^2\int_{-\theta_2}^{\theta_2}\left(u_2 + \frac{u_2^2}{2}\right) - r_0^2\int_{-\theta_0}^{\theta_0}\left(u_0 + \frac{u_0^2}{2}\right)\,. \tag{2.38}$$

We now divide (2.37) and (2.38) by r_1 and r_2 respectively and sum the resulting identities to find that

$$\begin{aligned}\Big(\frac{1}{(1+\sigma)^2} - 1\Big)\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) &= r_1\int_{-\theta_1}^{\theta_1}\left(u_1 + \frac{u_1^2}{2}\right) + r_2\int_{-\theta_2}^{\theta_2}\left(u_2 + \frac{u_2^2}{2}\right) \\ &\quad + \left(\frac{1}{r_1} - \frac{1}{r_2}\right) r_0^2\int_{-\theta_0}^{\theta_0}\left(u_0 + \frac{u_0^2}{2}\right) + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2)\,. \end{aligned}$$

Taking into account (2.14) and (2.22) we conclude that

$$\Big(\frac{1}{(1+\sigma)^2} - 1\Big)\,\frac{P(\mathcal{E}_0)}{2} = \sum_{k=0}^{2} r_k\int_{-\theta_k}^{\theta_k}\left(u_k + \frac{u_k^2}{2}\right) + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2)\,.$$

Plugging this relation into (2.34) we find

$$\begin{aligned}\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{(1+\sigma)} &= \sum_{k=0}^{2} r_k \int_{-\theta_k}^{\theta_k}\Big(\frac{(u_k')^2}{2}-\frac{u_k^2}{2}\Big) \\ &+\Big(\Big(\frac{1}{(1+\sigma)^2}-1\Big)+2(\sigma-\sigma^2)\Big)\frac{P(\mathcal{E}_0)}{2}+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)+\mathrm{o}\,(\sigma^2)\,.\end{aligned} \tag{2.39}$$

Since $((1+\sigma)^{-2}-1)+2(\sigma-\sigma^2)=\sigma^2+\mathrm{o}\,(\sigma^2)$ we finally have

$$\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{(1+\sigma)}=\sum_{k=0}^{2} r_k \int_{-\theta_k}^{\theta_k}\Big(\frac{(u_k')^2}{2}-\frac{u_k^2}{2}\Big)+\frac{\sigma^2}{2}P(\mathcal{E}_0)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)+\mathrm{o}\,(\sigma^2)\,.$$

The lemma is proved. □

We now provide an upper bound on the relative asymmetry of an (ε,σ)-perturbation of $\mathcal{E}_0$.

Lemma 2.5. *There exists a constant C (depending on m_1/m_2 only) with the following property. If $\mathcal{E}$ is an (ε,σ)-perturbation of $\mathcal{E}_0$ with $\varepsilon\le 1/C$ and $|\sigma|<1/2$, then, in case $m_2>m_1$,*

$$\alpha(\mathcal{E})^2\le C\Big(\sigma^2+\sum_{k=0}^{2}\frac{r_k^4\theta_k}{m_1^2}\int_{-\theta_k}^{\theta_k}u_k^2\Big)\,, \tag{2.40}$$

while, in case $m_2=m_1=m$, setting $r_1=r_2=r$,

$$\alpha(\mathcal{E})^2\le C\left(\sigma^2+\frac{r^4}{m^2}\sum_{k=1}^{2}\int_{-2\pi/3}^{2\pi/3}u_k^2+\frac{r^4}{m^2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0^2\right).$$

Proof. Once again, we directly focus on the case $m_2>m_1$. In the following, the symbol C will be used to denote generic constants, possibly depending on m_1/m_2. By definition of asymmetry, since $\mathrm{vol}\,(\mathcal{E})=\mathrm{vol}\,(\mathcal{E}_0)$ and $(a+b)^2\le 2(a^2+b^2)$, we have

$$\alpha(\mathcal{E})^2\le 2\left(\frac{|\mathcal{E}(1)\Delta\mathcal{E}_0(1)|}{m_1}\right)^2+2\left(\frac{|\mathcal{E}(2)\Delta\mathcal{E}_0(2)|}{m_2}\right)^2.$$

If $\varepsilon\le 1/C$ for C large enough, then

$$|\mathcal{E}(1)\Delta(1+\sigma)\mathcal{E}_0(1)|=(1+\sigma)^2\sum_{k=0}^{1}r_k^2\,|S(\theta_k,u_k)\Delta S(\theta_k)|\,,$$

and thus, by the triangular inequality

$$|\mathcal{E}(1)\Delta\mathcal{E}_0(1)| \le (1+\sigma)^2\sum_{k=0}^{1}r_k^2\,|S(\theta_k,u_k)\Delta S(\theta_k)|+\Big|(1+\sigma)\mathcal{E}_0(1)\Delta\mathcal{E}_0(1)\Big|\,,$$

By [FM11, Lemma 4], if $|\sigma|<1/2$ and $E\subset B_R\subset\mathbb{R}^n$ we have

$$|E\Delta(1+\sigma)E|\le C(n)\,R\,|\sigma|\,P(E)\,,$$

(for a constant depending on the ambient space dimension n only). Since, by scaling, and for suitable values of $C=C(m_1/m_2)$, we have $\mathcal{E}_0(1)\subset B_{C\sqrt{m_1}}$ and $P(\mathcal{E}_0(1))\le C\sqrt{m_1}$, we find

$$\Big|(1+\sigma)\mathcal{E}_0(1)\Delta\mathcal{E}_0(1)\Big|\le C\,m_1\,|\sigma|\,.$$

Thus, by $(1+\sigma)^2\le 9/4$ (recall that $|\sigma|<1/2$), we conclude

$$\frac{|\mathcal{E}(1)\Delta\mathcal{E}_0(1)|}{m_1}\le C\Big(\sum_{k=0}^{1}\frac{r_k^2}{m_1}\int_{-\theta_k}^{\theta_k}|u_k|+|\sigma|\Big)\le C\Big(\sum_{k=0}^{1}\frac{r_k^2\theta_k^{1/2}}{m_1}\Big(\int_{-\theta_k}^{\theta_k}u_k^2\Big)^{1/2}+|\sigma|\Big)\,,$$

where (2.7) was also taken into account. Thus,

$$\Big(\frac{|\mathcal{E}(1)\Delta\mathcal{E}_0(1)|}{m_1}\Big)^2\le C\Big(\sum_{k=0}^{1}\frac{r_k^4\theta_k}{m_1^2}\int_{-\theta_k}^{\theta_k}u_k^2+\sigma^2\Big)\,.$$

By arguing similarly with $\mathcal{E}(2)$ in place of $\mathcal{E}(1)$, and since $m_2>m_1$, we obtain (2.40). □

3. A multiple Poincaré-type inequality

The results of the previous section indicates that in order to prove (1.3) on (ε,σ)-perturbation (say, in the case $m_2>m_1$) we have to provide a control over

$$\sum_{k=0}^{2}\int_{-\theta_k}^{\theta_k}u_k^2 \tag{3.1}$$

in terms of

$$\sum_{k=0}^{2}\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\,. \tag{3.2}$$

The difficulty here is that the single term

$$\int_{-\theta}^{\theta}(u')^2-u^2\,, \tag{3.3}$$

is not L^2-coercive on $W^{1,2}_0(-\theta,\theta)$, unless $\theta<\pi/2$. Indeed, we easily see that

$$\inf\Big\{\int_{-\theta}^{\theta}(u')^2:u\in W^{1,2}_0(-\theta,\theta)\,,\int_{-\theta}^{\theta}u^2=1\Big\}=\Big(\frac{\pi}{2\theta}\Big)^2\,,\qquad\forall\theta>0\,,$$

so that the best control over $\|u\|^2_{L^2(-\theta,\theta)}$ in terms of $\|u'\|^2_{L^2(-\theta,\theta)}$ is

$$\int_{-\theta}^{\theta}(u')^2\ge\Big(\frac{\pi}{2\theta}\Big)^2\int_{-\theta}^{\theta}u^2\,,\qquad\forall u\in W^{1,2}_0(-\theta,\theta)\,. \tag{3.4}$$

In other words, if $\theta>\pi/2$, then

$$\inf\Big\{\int_{-\theta}^{\theta}(u')^2-u^2:u\in W^{1,2}_0(-\theta,\theta)\Big\}=-\infty\,.$$

Taking into account that θ_1 and θ_2 may possibly range on $(\pi/2,\pi)$, see (2.13), we conclude that in order to control (3.1) in terms of (3.2) we necessarily have to exploit the interaction between the single perturbations u_k through the multiple volume constraints. This will be achieved in section 3.2 through a careful application of the two Poincaré-type inequalities discussed in the next section.

3.1. **Two Poincaré-type inequalities.** We start by addressing the minimization of (3.3) under a constraint on the mean value of u.

Lemma 3.1. *If $\theta\in(0,\pi)$ and $s\in\mathbb{R}$, then*

$$\inf\Big\{\int_{-\theta}^{\theta}(u')^2-u^2:u\in W^{1,2}_0(-\theta,\theta)\,,\int_{-\theta}^{\theta}u=s\Big\}=\frac{s^2\,\cos\theta}{2(\sin\theta-\theta\,\cos\theta)}\,. \tag{3.5}$$

Notice that $\sin\theta-\theta\cos\theta$ defines an increasing function on $(0,\pi)$, with values in $(0,\pi)$. Thus the right-hand side of (3.5) decreases from $+\infty$ to 0 as $\theta\in(0,\pi/2)$, is equal to 0 for $\theta=\pi/2$, and decreases from 0 to $-s^2/2\pi$ as $\theta\in(\pi/2,\pi)$.

Proof. Given $u \in W_0^{1,2}(-\theta,\theta)$ with $\int_{-\theta}^{\theta} u = s$, let $v(t) = u(t\theta/\pi)$. Thus $v \in W_0^{1,2}(-\pi,\pi)$,

$$\int_{-\theta}^{\theta} v = s\,\frac{\pi}{\theta} \tag{3.6}$$

and

$$\int_{-\theta}^{\theta} (u')^2 - u^2 = \int_{-\pi}^{\pi} \frac{\pi}{\theta}\,(v')^2 - \frac{\theta}{\pi}\,v^2.$$

Let $\{\phi_k\}_{k\in\mathbb{N}} \subset L^2(-\pi,\pi)$ be the orthonormal basis of trigonometric functions with $\phi_0 = (2\pi)^{-1/2}$, and let $c_k = \int_{-\pi}^{\pi} v\,\phi_k$ the k-th Fourier coefficient of v. We have

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\pi}{\theta}\,(v')^2 - \frac{\theta}{\pi}\,v^2 &= \Big(\frac{\pi}{\theta} - \frac{\theta}{\pi}\Big)\int_{-\pi}^{\pi} (v')^2 - \frac{\theta}{\pi}\int_{-\pi}^{\pi} v^2 - (v')^2 \\
&\ge \Big(\frac{\pi}{\theta} - \frac{\theta}{\pi}\Big)\int_{-\pi}^{\pi} (v')^2 + \frac{\theta}{\pi}\Big(\sum_{k=1}^{\infty} k^2\,c_k^2 - \sum_{k=0}^{\infty} c_k^2\Big) \\
&\ge \Big(\frac{\pi}{\theta} - \frac{\theta}{\pi}\Big)\int_{-\pi}^{\pi} (v')^2 - \frac{\theta}{\pi}\,c_0^2 \\
&= \Big(\frac{\pi}{\theta} - \frac{\theta}{\pi}\Big)\int_{-\pi}^{\pi} (v')^2 - \frac{s^2}{2\theta}\,,
\end{aligned}$$

where in the last equality we used (3.6) to compute c_0. We have thus proved that

$$\int_{-\theta}^{\theta} (u')^2 - u^2 \ge \Big(1 - \Big(\frac{\theta}{\pi}\Big)^2\Big)\int_{-\theta}^{\theta} (u')^2 - \frac{1}{2\theta}\Big(\int_{-\theta}^{\theta} u\Big)^2, \qquad \forall u \in W_0^{1,2}(-\theta,\theta)\,,$$

which immediately lead to prove the existence of minimizers in (3.5) by a standard application of the Direct Method. We may thus consider a minimizer u in (3.5), that has to be a smooth solution to the Euler-Lagrange equation

$$\begin{cases} u'' + u = c\,, \\ u(\theta) = u(-\theta) = 0\,, \end{cases} \tag{3.7}$$

for some $c \in \mathbb{R}$. If $\theta = \pi/2$, then $u(t) = \cos(t)$ solves (3.7) (with $c = 0$), and, correspondingly, the infimum in (3.5) is equal to zero. If, instead, $\theta \neq \pi/2$, then (3.7) has solution

$$u(t) = c\Big(1 - \frac{\cos t}{\cos\theta}\Big)\,, \qquad |t| < \theta\,.$$

A simple computation then gives,

$$s = \int_{-\theta}^{\theta} u = 2c\Big(\theta - \tan\theta\Big)\,, \qquad \text{that is} \qquad c = \frac{s}{2(\theta - \tan\theta)}\,.$$

Therefore, again by direct computation,

$$\int_{-\theta}^{\theta} (u')^2 - u^2 = \frac{-s^2}{2(\theta - \tan\theta)} = \frac{s^2\,\cos\theta}{2(\sin\theta - \theta\,\cos\theta)}\,.$$

□

Lemma 3.2. *For every $\theta \in (0,\pi)$ there exists $M = M(\theta)$ such that, if $u \in W_0^{1,2}(-\theta,\theta)$ with*

$$\Big(\int_{-\theta}^{\theta} u\Big)^2 \le \frac{1}{M}\int_{-\theta}^{\theta} u^2\,, \tag{3.8}$$

then

$$\int_{-\theta}^{\theta} (u')^2 - u^2 \ge \frac{1}{4}\Big(1 - \frac{\theta^2}{\pi^2}\Big)\int_{-\theta}^{\theta} (u')^2 + \frac{1}{2}\Big(\frac{\pi^2}{\theta^2} - 1\Big)\int_{-\theta}^{\theta} u^2\,. \tag{3.9}$$

A possible value for $M = M(\theta)$ is

$$M = \frac{1}{\theta}\,\frac{2\pi^2}{\pi^2-\theta^2}\,. \tag{3.10}$$

Proof. Given $u \in W_0^{1,2}(-\theta,\theta)$, define $v \in W_0^{1,2}(-\pi,\pi)$ as $v(t) = u(t\,\theta/\pi)$. By (3.8),

$$\left(\int_{-\pi}^{\pi} v\right)^2 \le \frac{\pi}{\theta\,M}\int_{-\pi}^{\pi} v^2\,, \tag{3.11}$$

Let ϕ_k and c_k be defined as in the proof of Lemma 3.1. For every $\lambda \in (0,1)$ we have

$$\begin{aligned}(1-\lambda)\int_{-\theta}^{\theta}(u')^2 - \int_{-\theta}^{\theta} u^2 &= \frac{\pi}{\theta}(1-\lambda)\sum_{k=1}^{\infty}k^2 c_k^2 - \frac{\theta}{\pi}\sum_{k=0}^{\infty}c_k^2\\ &\ge \left(\frac{\pi}{\theta}(1-\lambda)-\frac{\theta}{\pi}\right)\sum_{k=0}^{\infty}c_k^2 - \frac{\pi}{\theta}(1-\lambda)c_0^2\\ &\ge \frac{\pi}{\theta}\left(\frac{\pi}{\theta}(1-\lambda)-\frac{\theta}{\pi}-\frac{\pi(1-\lambda)}{2\theta^2 M}\right)\int_{-\theta}^{\theta}u^2, \end{aligned} \tag{3.12}$$

where we have estimated c_0 thanks to (3.8) as follows,

$$c_0^2 = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} v\right)^2 \le \frac{1}{2\theta M}\int_{-\pi}^{\pi} v^2 = \frac{\pi}{2\theta^2 M}\int_{-\theta}^{\theta} u^2\,.$$

Let us now rearrange (3.12) as

$$\int_{-\theta}^{\theta}(u')^2 - u^2 \ge \lambda\int_{-\theta}^{\theta}(u')^2 + \left(\frac{\pi^2}{\theta^2}\left(1-\frac{1}{2\theta\,M}\right)(1-\lambda)-1\right)\int_{-\theta}^{\theta}u^2\,.$$

We prove (3.9) by choosing M as in (3.10), by setting

$$\lambda = \frac{1}{4}\Big(1-\frac{\theta^2}{\pi^2}\Big) = \frac{1}{4}\frac{\theta^2}{\pi^2}\Big(\frac{\pi^2}{\theta^2}-1\Big)\,,$$

and finally noticing that

$$\frac{\pi^2}{\theta^2}\left(1-\frac{1}{2\theta\,M}\right)(1-\lambda)-1 \ge \frac{\pi^2}{\theta^2}-1-\frac{\pi^2}{\theta^2}\Big(\lambda+\frac{1}{2\theta M}\Big) = \frac{1}{2}\Big(\frac{\pi^2}{\theta^2}-1\Big)\,.$$

□

3.2. **A stability inequality on (ε,σ)-perturbations.** We now come to the crucial estimate of the paper.

Theorem 3.3. *For every $m_2 \ge m_1 > 0$, there exist positive constants ε_1, σ_1, and κ_1 (depending on m_1/m_2 only) with the following property. If $\mathcal{E}$ is an (ε,σ)-perturbation of $\mathcal{E}_0$ with $\mathrm{vol}\,(\mathcal{E}_0) = (m_1,m_2)$, and if $\varepsilon < \varepsilon_1$ and $|\sigma| < \sigma_1$, then, in the case $m_2 > m_1$*

$$P(\mathcal{E}) - P(\mathcal{E}_0) \ge \kappa_1\left(\sigma^2 + \sum_{k=0}^{2} r_k\int_{-\theta_k}^{\theta_k} u_k^2\right), \tag{3.13}$$

while, in the case $m_2 = m_1$ (and $r_2 = r_1 = r$),

$$P(\mathcal{E}) - P(\mathcal{E}_0) \ge \kappa_1\left(\sigma^2 + r\int_{-\sqrt{3}/2}^{\sqrt{3}/2} v_0^2 + \sum_{k=1}^{2} r\int_{-2\pi/3}^{2\pi/3} u_k^2\right). \tag{3.14}$$

Proof. Step one: Let $\theta \in (0,\pi)$, and let $M(\theta)$ be as in (3.10). We notice that for every $\theta \in (0,\pi)$ there exists $\varepsilon(\theta) > 0$ such that if

$$\|u\|_{C^0(-\theta,\theta)} \le \varepsilon(\theta)\,, \qquad \Big(\int_{-\theta}^{\theta} u + \frac{u^2}{2}\Big)^2 \le \frac{1}{2\,M(\theta)} \int_{-\theta}^{\theta} u^2\,,$$

then

$$\Big(\int_{-\theta}^{\theta} u\Big)^2 \le \frac{1}{M(\theta)} \int_{-\theta}^{\theta} u^2\,.$$

In the rest of the proof, given m_1 and m_2, and thus fixed θ_1 and θ_2 according to (2.17) and (2.18), we shall assume to work with (ε,σ)-perturbations of $\mathcal{E}_0$ with $\varepsilon < \min\{\varepsilon(\theta_1),\varepsilon(\theta_2)\}$.

Step two: We start considering the case $m_2 > m_1$. If $\mathcal{E}$ is an (ε,σ)-perturbation of $\mathcal{E}_0$ with functions u_0, u_1, and u_2, then, for $t > 0$, $t\,\mathcal{E}$ is an (ε,σ)-perturbation of $t\,\mathcal{E}_0$ with the same functions u_0, u_1, and u_2. Therefore, without loss of generality, in the following we may assume that $r_2 = 1$. For the sake of symmetry (and, thus, of clarity) we shall keep writing r_2 in place of 1 in the following formulas, until we exploit this scaling assumption. Let us now set

$$I_k = \int_{-\theta_k}^{\theta_k} u_k + \frac{u_k^2}{2}\,, \qquad k = 0,1,2\,,$$

so that the volume constraints (2.35) and (2.36) take the form

$$I_0 = -\Big(\frac{r_1}{r_0}\Big)^2 I_1 + \frac{m_1}{r_0^2}\Big(\frac{1}{(1+\sigma)^2} - 1\Big)\,, \tag{3.15}$$

$$I_0 = \Big(\frac{r_2}{r_0}\Big)^2 I_2 - \frac{m_2}{r_0^2}\Big(\frac{1}{(1+\sigma)^2} - 1\Big)\,. \tag{3.16}$$

Multiplying (3.15) by $m_2/(m_1+m_2)$, (3.16) by $m_1/(m_1+m_2)$, and then adding up, we find

$$I_0 = \frac{m_1}{m_1+m_2}\Big(\frac{r_2}{r_0}\Big)^2 I_2 - \frac{m_2}{m_1+m_2}\Big(\frac{r_1}{r_0}\Big)^2 I_1\,. \tag{3.17}$$

Similarly, multiplying both (3.15) and (3.16) by r_0^2, and then subtracting the resulting identities, we come to $r_1^2\,I_1 + r_2^2\,I_2 = (m_1+m_2)((1+\sigma)^{-2} - 1)$, which gives

$$\sigma^2 + \mathrm{o}\,(\sigma^2) = \frac{(r_1^2 I_1 + r_2^2 I_2)^2}{4(m_1+m_2)^2} + \mathrm{o}\,(u^2)\,. \tag{3.18}$$

By (3.18) we deduce that

$$\sigma^2 + \mathrm{o}\,(\sigma^2) \le \frac{r_1^4 I_1^2 + r_2^4 I_2^2}{2(m_1+m_2)^2} + \mathrm{o}\,(u^2)\,, \tag{3.19}$$

and, since $I_k \le C\int_{-\theta_k}^{\theta_k} u_k^2$, that $\mathrm{o}\,(\sigma^2) = \mathrm{o}\,(u^2)$. (This is a reflection of the fact that if the u_k's are all zero, then, by the volume constraint, we necessarily have $\sigma = 0$.) Thus (2.32) gives

$$2\,\frac{P(\mathcal{E}) - P(\mathcal{E}_0)}{1+\sigma} = \sum_{k=0}^{2} r_k \int_{-\theta_k}^{\theta_k} (u_k')^2 - u_k^2 + P(\mathcal{E}_0)\,\sigma^2 + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2)\,. \tag{3.20}$$

We now claim that, for a suitable constant C (depending on $\mathcal{E}_0$) we have

$$C\,(P(\mathcal{E}) - P(\mathcal{E}_0)) \ge r_1\,I_1^2 + r_2\,I_2^2 + \mathrm{o}\,(u^2) + \mathrm{o}\,((u')^2)\,. \tag{3.21}$$

To this end, let us set for the sake of brevity

$$g(\theta) = \frac{\cos\theta}{2(\sin\theta - \theta\cos\theta)}\,, \qquad 0 < \theta < \pi\,. \tag{3.22}$$

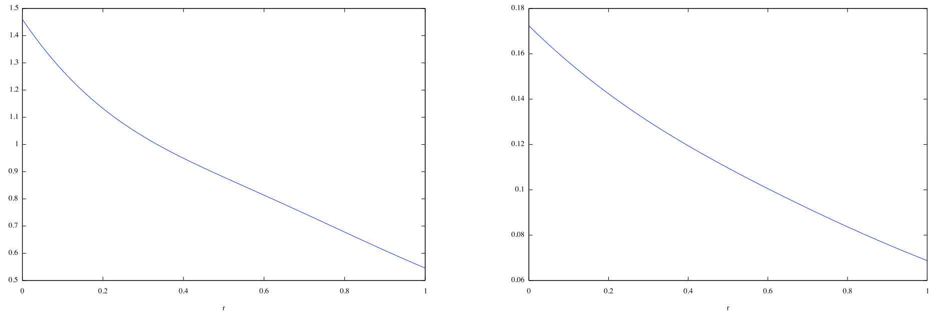

FIGURE 6. Plotting of $\beta_1(r)$ (left) and of $(\beta_1(r)\beta_2(r)-\beta_3(r)^2)/r$ (right) for $r\in(0,1)$. In particular, $\beta_1(r)\beta_2(r)-\beta_3(r)^2\approx r$ for r small. The plots have been drawn by Maxima v.5.28.0 (http://maxima.sourceforge.net) starting from equations $r_2=1$, $r_1=r\in(0,1)$, (2.15), (2.16), (2.17), (2.18), (2.22), (2.23), (2.24), (3.22), (3.25), (3.26), and (3.27).

By Lemma 3.1, for $k=0,1,2$ we have

$$\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\ge g(\theta_k)\left(I_k-\int_{-\theta_k}^{\theta_k}\frac{u_k^2}{2}\right)^2=g(\theta_k)\,I_k^2+\mathrm{o}\,(u^2), \tag{3.23}$$

and thus, by inserting (3.18) and (3.23) into (3.20), and since $\mathrm{o}\,(\sigma^2)=\mathrm{o}\,(u^2)$, we obtain

$$\begin{aligned} 2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma} &\ge \sum_{k=0}^{2} r_k\,g(\theta_k)I_k^2+\frac{P(\mathcal{E}_0)(r_1^2I_1+r_2^2I_2)^2}{4(m_1+m_2)^2}+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\\ &= \beta_1\,r_1\,I_1^2+\beta_2\,r_2\,I_2^2+2\beta_3\sqrt{r_1r_2}\,I_1\,I_2+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,. \end{aligned} \tag{3.24}$$

Here, by taking into account (3.17), we have set

$$\beta_1 = g(\theta_0)\,\frac{r_1^3}{r_0^3}\,\frac{m_2^2}{(m_1+m_2)^2}+g(\theta_1)+\frac{r_1^3}{4}\,\frac{P(\mathcal{E}_0)}{(m_1+m_2)^2}\,, \tag{3.25}$$

$$\beta_2 = g(\theta_0)\,\frac{r_2^3}{r_0^3}\,\frac{m_1^2}{(m_1+m_2)^2}+g(\theta_2)+\frac{r_2^3}{4}\,\frac{P(\mathcal{E}_0)}{(m_1+m_2)^2}\,, \tag{3.26}$$

$$\beta_3 = -g(\theta_0)\,\frac{r_1^{3/2}\,r_2^{3/2}}{r_0^3}\,\frac{m_1\,m_2}{(m_1+m_2)^2}+\frac{r_1^{3/2}r_2^{3/2}}{4}\,\frac{P(\mathcal{E}_0)}{(m_1+m_2)^2}\,. \tag{3.27}$$

The quadratic form in $(\sqrt{r_1}\,I_1,\sqrt{r_2}\,I_2)$ on the right-hand side (3.24) is coercive: indeed, it suffices to show the existence of $\beta_*>0$ (depending on m_1/m_2 only) such that

$$\min\{\beta_1,\beta_1\beta_2-\beta_3^2\}\ge\beta_*\,. \tag{3.28}$$

To this end, let us note that, having set $r_2=1$, it turns out that r_0, θ_0, θ_1, θ_2, m_1, and m_2 are all explicit functions of $r_1\in(0,1)$ according to equations (2.15), (2.16), (2.17), (2.18), (2.23), and (2.24). Correspondingly, the coefficients β_k can be easily expressed as functions of $r_1\in(0,1)$, and the validity of (3.28) can be deduced by a numerical plot; see Figure 6. As a consequence of (3.28), and up to decrease the value of β_*, we find

$$\beta_1\,r_1\,I_1^2+\beta_2\,r_2\,I_2^2+2\beta_3\sqrt{r_1r_2}\,I_1\,I_2\ge\beta_*(r_1\,I_1^2+r_2\,I_2^2)\,.$$

We combine this inequality with (3.24) to prove (3.21), as claimed. Now, by (3.19) and (3.21),

$$C\,(P(\mathcal{E})-P(\mathcal{E}_0))\ge\sigma^2+r_1\,I_1^2+r_2\,I_2^2+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,. \tag{3.29}$$

By the choice of ε performed in step one, we now notice that, if for some $k=1,2$ we have

$$I_k^2\le\frac{1}{2M(\theta_k)}\int_{-\theta_k}^{\theta_k}u_k^2\,,$$

then, by Lemma 3.2,

$$\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\geq\frac{1}{4}\Big(1-\frac{\theta_k^2}{\pi^2}\Big)\int_{-\theta_k}^{\theta_k}(u_k')^2+\frac{1}{2}\Big(\frac{\pi^2}{\theta_k^2}-1\Big)\int_{-\theta_k}^{\theta_k}u_k^2\,. \tag{3.30}$$

Therefore, for $k=1,2$, either (3.30) holds true, or

$$I_k^2\geq\frac{1}{2M(\theta_k)}\int_{-\theta_k}^{\theta_k}u_k^2\,. \tag{3.31}$$

Concerning u_0, let us notice that, by the sharp Poincaré inequality (3.4), and since $\theta_0<\pi/3$,

$$\int_{-\theta_0}^{\theta_0}(u_0')^2\geq\Big(\frac{\pi}{2\theta_0}\Big)^2\int_{-\theta_0}^{\theta_0}u_0^2\geq\frac{9}{4}\int_{-\theta_0}^{\theta_0}u_0^2\,,$$

which gives

$$\int_{-\theta_0}^{\theta_0}(u_0')^2-u_0^2\geq\frac{1}{3}\int_{-\theta_0}^{\theta_0}(u_0')^2+\Big(\frac{3}{2}-1\Big)\int_{-\theta_0}^{\theta_0}u_0^2\,. \tag{3.32}$$

We are now going to use (3.30), (3.31), and (3.32) together with (3.29) to prove that, for some constant C depending on $\mathcal{E}_0$, we always have

$$C\,(P(\mathcal{E})-P(\mathcal{E}_0))\geq\sigma^2+\sum_{k=0}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2+u_k^2\,. \tag{3.33}$$

We divide the argument in three cases:

Case one: We assume that (3.30) holds true for $k=1,2$. By this assumption, (3.20), and (3.32),

$$C\,(P(\mathcal{E})-P(\mathcal{E}_0))\geq\sigma^2+\sum_{k=0}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2+u_k^2+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,. \tag{3.34}$$

Then, up to decrease the value of ε and up to increase the value of C, we may absorb $\mathrm{o}\,(u^2)$ and $\mathrm{o}\,((u')^2)$ in the other terms on the right hand side, and deduce (3.33).

Case two: We assume that (3.31) holds true for $k=1,2$. In this case, by (3.20) we obtain

$$\begin{aligned}
2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma}\ &\geq\ \tau\Big(\sum_{k=0}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\Big)+(1-\tau)\,2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma}\\
&\qquad+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\\
\text{(by (3.32))}\qquad &\geq\ \tau\Big(\frac{r_0}{3}\int_{-\theta_0}^{\theta_0}(u_0')^2+\frac{r_0}{2}\int_{-\theta_0}^{\theta_0}u_0^2\Big)+\tau\sum_{k=1}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\\
\text{(by (3.29))}\qquad &\qquad+\frac{1-\tau}{C}\Big(\sigma^2+r_1\,I_1^2+r_2\,I_2^2\Big)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\\
&\geq\ \tau\Big(\frac{r_0}{3}\int_{-\theta_0}^{\theta_0}(u_0')^2+\frac{r_0}{2}\int_{-\theta_0}^{\theta_0}u_0^2\Big)+\tau\sum_{k=1}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\\
\text{(by (3.31) for } k=1,2) \qquad &\qquad+\frac{1-\tau}{C}\Big(\sigma^2+\sum_{k=1}^{2}\frac{r_k}{2M(\theta_k)}\int_{-\theta_k}^{\theta_k}u_k^2\Big)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\\
&\geq\ \tau\Big(\frac{r_0}{3}\int_{-\theta_0}^{\theta_0}(u_0')^2+\frac{r_0}{2}\int_{-\theta_0}^{\theta_0}u_0^2\Big)+\tau\sum_{k=1}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2\\
&\qquad+\frac{1-\tau}{2C}\Big(\sigma^2+\sum_{k=1}^{2}\frac{r_k}{2M(\theta_k)}\int_{-\theta_k}^{\theta_k}u_k^2\Big)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,,
\end{aligned}$$

where in the last inequality we have absorbed the negative terms in u_k^2, $k=1,2$, by choosing τ so small to have

$$\tau \le \frac{1-\tau}{4\,C}\min_{k=1,2}\frac{1}{M(\theta_k)}\,.$$

We have thus proved (3.34), and thus (3.33), up to suitably choose ε and C.

Case three: We assume that (3.30) holds true for $k=1$, while (3.31) holds true for $k=2$. By arguing as in case two we find, for any $\tau\in(0,1)$,

$$\begin{aligned}2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma} \quad&\ge\quad \tau\Big(\frac{r_0}{3}\int_{-\theta_0}^{\theta_0}(u_0')^2+\frac{r_0}{2}\int_{-\theta_0}^{\theta_0}u_0^2\Big)+\tau\sum_{k=1}^{2}r_k\int_{-\theta_k}^{\theta_k}(u_k')^2-u_k^2\\&\quad+\frac{1-\tau}{C}\Big(\sigma^2+r_1\,I_1^2+r_2\,I_2^2\Big)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\end{aligned}$$

By using (3.30) for $k=1$ and (3.31) for $k=2$, and discarding some positive terms, we find

$$\begin{aligned}2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma} \quad&\ge\quad \tau\,c\left(r_0\int_{-\theta_0}^{\theta_0}\Big((u_0')^2+u_0^2\Big)+r_1\int_{-\theta_1}^{\theta_1}\Big((u_1')^2+u_1^2\Big)+r_2\int_{-\theta_2}^{\theta_2}(u_2')^2\right)\\&\quad+\frac{1-\tau}{C}\Big(\sigma^2+\frac{r_2}{2M(\theta_2)}\int_{-\theta_2}^{\theta_2}u_2^2\Big)-\tau\,r_2\int_{-\theta_2}^{\theta_2}u_2^2+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,,\end{aligned}$$

for some positive constant c depending on $\mathcal{E}_0$. As in case two, we may choose τ small enough to have the negative term in u_2^2 absorbed by its positive counterpart, and come to prove (3.34). Finally, when (3.30) holds true for $k=2$ and (3.31) holds true for $k=1$ (note that, formally, this is a fourth different case, as $m_2>m_1$), then we just repeat the very same argument. Summarizing, we have proved the validity of (3.33), which of course implies (3.13). The theorem is proved in the case $m_2>m_1$.

Step three: We now address the case $m_2=m_1$. In this case we set $r=r_1=r_2$, $m=m_1=m_2$, and $\theta=\theta_1=\theta_2=2\pi/3$. Once again, up to scaling, we may assume that $r=1$, so that

$$m=\frac{2\pi}{3}+\frac{\sqrt{3}}{4}\,,\qquad P(\mathcal{E}_0)=4\,m=\frac{8\,\pi}{3}+\sqrt{3}\,.$$

The volume constraints now take the form

$$\Big((1+\sigma)^{-2}-1\Big)\,m=I_1+\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0=I_2-\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0\,,$$

so that, by arguing as in step one, we find, in analogy to (3.17) and (3.18),

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0=\frac{I_2-I_1}{2}\,,\qquad \sigma^2+o(\sigma^2)=\frac{(I_1+I_2)^2}{4\,m^2}\,. \tag{3.35}$$

By Lemma 3.1 we have (3.23) for $k=1,2$, and, similarly,

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2}(v_0')^2\ge\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0^2+g\Big(\frac{\sqrt{3}}{2}\Big)\Big(\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0\Big)^2=\int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0^2+g\Big(\frac{\sqrt{3}}{2}\Big)\,\frac{(I_2-I_1)^2}{4}\,. \tag{3.36}$$

(Notice that $\sqrt{3}/2 < \pi/2$, thus $g(\sqrt{3}/2)$ is positive.) By (3.35) and (3.36), and since $\mathrm{o}\,(\sigma^2) = \mathrm{o}\,(u^2)$, from (2.33) we deduce

$$
\begin{aligned}
2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma} &= \int_{-\sqrt{3}/2}^{\sqrt{3}/2}(v_0')^2+\sum_{k=1}^{2}\int_{-2\pi/3}^{2\pi/3}(u_k')^2-u_k^2+\frac{\sigma^2}{2}\,P(\mathcal{E}_0)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\\
&\geq \int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0^2+g\Big(\frac{\sqrt{3}}{2}\Big)\,\frac{(I_2-I_1)^2}{4}+g\Big(\frac{2\pi}{3}\Big)(I_1^2+I_2^2)+\frac{(I_1+I_2)^2}{2\,m}\\
&\quad+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\\
&\geq \int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0^2+\alpha_1\,I_1^2+\alpha_2\,I_2^2+2\alpha_3\,I_1\,I_2\,,+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,,
\end{aligned}
$$

provided we set

$$
\begin{aligned}
\alpha_1=\alpha_2 &= \frac{1}{4}\,g\Big(\frac{\sqrt{3}}{2}\Big)+g\Big(\frac{2\pi}{3}\Big)+\frac{1}{2m}\\
\alpha_3 &= -\frac{1}{4}\,g\Big(\frac{\sqrt{3}}{2}\Big)+\frac{1}{2m}\,.
\end{aligned}
$$

By direct evaluation we see that $\alpha_1 > 0$ and $\alpha_1\alpha_2-\alpha_3^2 > 0$. Therefore there exists $\alpha_* > 0$ such that $\alpha_1\,I_1^2+\alpha_2\,I_2^2+2\alpha_3\,I_1\,I_2 \geq \alpha_*(I_1^2+I_2^2)$, and thus

$$
2\,\frac{P(\mathcal{E})-P(\mathcal{E}_0)}{1+\sigma} \geq \int_{-\sqrt{3}/2}^{\sqrt{3}/2}v_0^2+\alpha_*\,(I_1^2+I_2^2)+\mathrm{o}\,(u^2)+\mathrm{o}\,((u')^2)\,. \tag{3.37}
$$

We conclude the proof exactly as in step two, with (3.37) playing the role of (3.21), and with

$$
\int_{-\sqrt{3}/2}^{\sqrt{3}/2}(v_0')^2 \geq \frac{1}{2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}(v_0')^2+v_0^2 \tag{3.38}
$$

playing the role of (3.32). (Note that (3.38) follows trivially from (3.36).) This completes the proof of Theorem 3.3. □

4. Proofs of the main theorems

Proof of Theorem 1.1. By the theory developed in [CLM12] and by a scaling argument, Theorem 1.1 follows immediately from Theorem 2.1. □

Proof of Theorem 2.1. By Remark 2.1, it suffices to show that for every $m_2 \geq m_1 > 0$ there exist positive constants κ, ε_1 and σ_1 such that $\delta(\mathcal{E}) \geq \kappa\alpha(\mathcal{E})^2$ for every (ε,σ)-perturbation $\mathcal{E}$ of $\mathcal{E}_0$ with $\varepsilon < \varepsilon_1$ and $|\sigma| < \sigma_1$. This follows immediately from Theorem 3.3 and Lemma 2.5. □

Proof of Theorem 1.2. We directly focus on the case $m_2 > m_1$, the case $m_2 = m_1$ being entirely analogous. As shown in [CLM12], there exists $\beta_0 > 0$ such that if $\mathcal{E}_\beta$ is a minimizer in (1.4) for $\beta \in (0,\beta_0)$, then there exists a standard double bubble $\mathcal{E}_0$ with $\mathrm{vol}\,(\mathcal{E}_0) = (m_1,m_2)$ and a $C^{1,1}$-diffeomorphism $f_\beta : \partial\mathcal{E}_0 \to \partial\mathcal{E}_\beta$ with $\|f_\beta-\mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} \to 0$ as $\beta \to 0$. In particular, by Remark 2.1, up to further decrease the value of β_0, we may assume that if $\beta < \beta_0$, then $\mathcal{E}_\beta$ is an $(\varepsilon_\beta,\sigma_\beta)$-perturbation of $\mathcal{E}_0$ for $\varepsilon_\beta \leq \varepsilon_1$ and $|\sigma_\beta| \leq \sigma_1$, where ε_1 and σ_1 are the constants defined in Theorem 3.3. If we denote by $g_\beta : \partial\mathcal{E}_0 \to \partial\mathcal{E}_\beta$ the diffeomorphism associated to σ_β and to the functions $\{u_{\beta,k}\}_{k=0,1,2}$, then by Theorem 3.3 and by (2.31) we find

$$
P(\mathcal{E}_\beta)-P(\mathcal{E}_0) \;\geq\; \kappa\left(\sigma_\beta^2+\sum_{k=0}^{2}\int_{-\theta_k}^{\theta_k}u_{\beta,k}^2\right), \tag{4.1}
$$

$$
\|g_\beta-\mathrm{Id}\|_{C^1(\partial\mathcal{E}_0;\mathbb{R}^2)} \;\leq\; C\left(|\sigma|+\max_{k=0,1,2}\|u_{\beta,k}\|_{C^1(-\theta_k,\theta_k)}\right). \tag{4.2}
$$

Since the curvatures of the interfaces $\partial\mathcal{E}_\beta$ are converging to the corresponding values of $\partial\mathcal{E}_0$ (see, again, [CLM12]), we have that

$$\sup_{\beta<\beta_0} \|u''_{\beta,k}\|_{L^\infty(-\theta_k,\theta_k)} \leq \Lambda\,,$$

for a suitable constant Λ. Thus, by (4.1) and by Lemma 4.1 below, it follows that

$$\begin{aligned}\|u'_{\beta,k}\|_{C^0(-\theta_k,\theta_k)} &\leq 2\,\Lambda^{2/3}\|u_{\beta,k}\|^{1/3}_{L^1(-\theta_k,\theta_k)} + \frac{1}{\theta_k^2}\,\|u_{\beta,k}\|_{L^1(-\theta_k,\theta_k)} \\ &\leq C\,\|u_{\beta,k}\|^{1/3}_{L^2(-\theta_k,\theta_k)} \leq C\Big(P(\mathcal{E}_\beta)-P(\mathcal{E}_0)\Big)^{1/6}\,. \end{aligned} \tag{4.3}$$

Since $u_{\beta,k}(\pm\theta_k)=0$, we conclude the proof by combining (4.2) and (4.3). □

Lemma 4.1 (An interpolation inequality)**.** *If $u\in C^{1,1}([0,s])$, then*

$$\|u'\|_{C^0(0,s)} \leq 2\,\|u\|^{1/3}_{L^1(0,s)}\|u''\|^{2/3}_{L^\infty(0,s)} + \frac{4}{s^2}\,\|u\|_{L^1(0,s)}\,.$$

Proof. By scaling, we can assume that $s=1$. Up to change u with $-u$ we may assume that

$$\ell = \|u'\|_{L^\infty(0,1)} = \max_{[0,1]}|u'| = u'(x_0) > 0\,.$$

Notice that this last assumption (that is, the maximum of u' is achieved at a point where u' has positive value), as well as the various norms of u, u', and u'' considered in the statement, are preserved if we change $u(x)$ with $-u(1-x)$. We set $\Lambda = \|u''\|_{L^\infty(0,1)}$.

Case one: Assume that $\ell < 2\Lambda$. Up to change $u(x)$ with $-u(1-x)$ we may assume that $x_0 < 1-(\ell/2\Lambda)$. In this way, $u'(x_0+t)\geq \ell/2$ whenever $0<t<\ell/2\Lambda$. If $|u(x_0)|\geq \ell^2/4\Lambda$, then

$$\|u\|_{L^1(0,1)} \geq \int_{x_0}^{x_0+\ell/4\Lambda}|u| \geq \int_{x_0}^{x_0+\ell/4\Lambda}\Big(|u(x_0)|-\ell\,t\Big)\,dt \geq \int_0^{\ell/4\Lambda}\Big(\frac{\ell^2}{4\Lambda}-\ell\,t\Big)\,dt = \frac{\ell^3}{8\,\Lambda^2}\,;$$

if $|u(x_0)| < \ell^2/4\Lambda$, then again

$$\|u\|_{L^1(0,1)} \geq \int_{x_0}^{x_0+\ell/4\Lambda}|u| \geq \int_{x_0}^{x_0+\ell/4\Lambda}\Big(\ell\,t-|u(x_0)|\Big)\,dt \geq \int_0^{\ell/4\Lambda}\Big(\ell\,t-\frac{\ell^2}{4\Lambda}\Big)\,dt = \frac{\ell^3}{8\,\Lambda^2}\,.$$

Case two: Assume that $2\Lambda\leq\ell$. In this case $u'\geq \ell/2$ on $[0,1]$. If $u(0)\geq 0$ then $\|u\|_{L^1(0,1)}\geq \ell/2$ and we are done. If $u(0)<0$, but $u(1)\geq 0$, then, up to switch $u(x)$ with $-u(1-x)$ we reduce to the previous case. The only possibility left is that $u(0)$ and $u(1)$ are both negative. In this case, of course, $u(x)\leq(\ell/2)(1-x)$, and thus we find $\|u\|_{L^1(0,1)}\geq \ell/4$. □

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