The Pythagorean theorem in mathematics laboratory

Michela Maschietto[1], Stefano Barbieri[2], Francesca Scorcioni[2]

[1]University of Modena e Reggio Emilia, Italy; michela.maschietto@unimore.it

[2]IC "Marconi", Castelfranco Emilia, Italy; stefano.barbieri@tin.it, scorcioni.francesca@gmail.com

This paper refers to the Pythagorean theorem and the use of physical artifacts (called mathematical machines), which are related to one of the proofs of the theorem. It aims to discuss the didactical use of these kinds of artifact, paying attention to students' work with them and the role of the teacher. It presents a laboratory approach to this theorem developed within the Theory of Semiotic Mediation in mathematics education for 13-year-old students in Italy. The analysis shows that manipulation of the machine only does not imply the emergence of the mathematical meanings embedded in the machine. It also pays attention to the different graphical representations of the artifact and their role in the learning process.

Keywords: Artifacts, geometry, laboratory, lower secondary school education, Pythagoras.

Introduction

The Pythagorean theorem is a traditional content in the mathematics curriculum of the secondary school, not only in Italian school (Moutsios-Rentzos, Spyrou & Peteinara, 2014). This theorem is often proposed in the geometrical domain at the beginning, and it is soon converted into formulas and related to algebraic calculations. There exist several proofs of this theorem[1], some of them are proposed as visual proofs. On this topic, Bardelle (2010) analyses how university students in approaching a visual proof of that theorem try to look for the algebraic relation among sides starting from their knowledge of the theorem rather than getting the relationships between the components of the given figure. On the other hand, different exhibits are constructed basing on this kind of proofs, and they are also associated with and spread as gadgets (Eaves, 1954). In our work, we ask if and how it is possible to approach the Pythagorean theorem starting from artifacts which embed one of its proofs (Rufus, 1975), taking into account the role of manipulation, with 7-grade students (*13-year old students*). At the same time, we are interested in reinforcing the geometrical meaning of equivalent figures, which makes this theorem a particular case.

In this paper, we introduce the theoretical background for the didactical use of physical artifacts (called mathematical machine[2], Maschietto & Bartolini Bussi, 2011), then we present the teaching experiment.

Theoretical framework

In this section, we outline the theoretical framework of our work, based on the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) and the cognitive processes in geometry fostered by the

[1] http://www.cut-the-knot.org/pythagoras/index.shtml Accessed 20th March 2017.

[2] Mathematics Laboratory at the University of Modena e Reggio Emilia: www.mmlab.unimore.it. Accessed 20th March 2017.

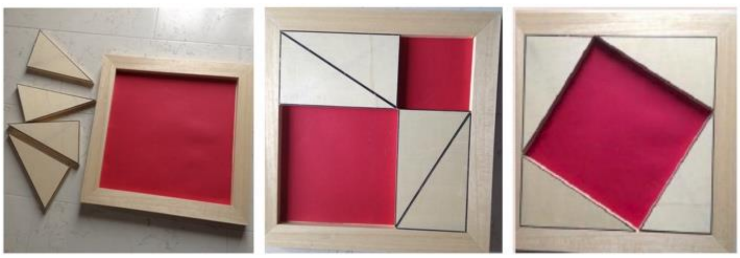
task of reproducing artifacts. The teaching experiment is designed according to the methodology of mathematics laboratory (Maschietto & Trouche, 2010) with different kinds of artifacts.

Mathematics laboratory

The teaching experiment is proposed and analyzed within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008, TSM), grounded in the Vygotskian notion of semiotic mediation and role of artifact in cognitive development. Following to the TSM, the teacher chooses the artifacts evoking particular mathematical meanings and uses them to mediate those meanings, proposing tasks to be accomplished by those artifacts. The tasks are organized in terms of didactical cycles with group work, individual work and collective discussions (mathematical discussions) orchestrated by the teacher. The cycle usually starts with the exploration of the chosen artifact, above all in small group work, structured following fundamental questions as: “How is the machine made?”, “What does the machine make?” and “Why does it make it?”. In general, the first two questions try to take in account students’ processes of instrumental genesis (Rabardel & Bourmaud, 2003). In the mathematics laboratory, students’ processes of formulation of conjectures and argumentation are strongly motivated and supported by the third question. The mathematical meanings emerge from the use of the artifacts, the interactions among peers and between peers and the teacher, who has the role of an expert guide. In all the activities, students are involved in a semiotic activity (producing gestures, words, drawings, called artifact signs) that the teacher makes evolving into mathematical signs (i.e., linked to mathematical contents) by the means of pivot signs. In this sense, the teacher uses the artifact as an instrument of mediation for mathematical meanings.

The teaching experiment on the Pythagorean theorem is carried out with the use of two mathematical machines (M1 and M2 in Figure 1)[3]. They were analyzed in terms of their semiotic potential (Bartolini Bussi & Mariotti, 2008), corresponding to a semiotic relationship between an artifact and: on the one hand the personal meanings emerging from its use to accomplish a task; on the other hand, the mathematical meanings evoked by its use.

The analysis of the semiotic potential considers three components: mathematical content, historical references and utilization schemes (Rabardel & Bourmaud, 2003). This kind of analysis is essential for the choice of the artifact and the identification of mathematical meanings evoked by it.

M1: 

M2: 

Figure 1: The mathematical machines proposed to the classes (M1 on the left, M2 on the right)

[3] http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=162&Itemid=243&lang=it. Accessed 20th March 2017.

Semiotic potential of the artifacts

The mathematical machine M1 (Figure 1, on the left) is a wooden artifact, composed of a square frame and four triangular prisms, with right triangles as the base that are congruent each other. The fundamental relationship between the prisms and the square inside the frame (red square in Figure 1) is that the sum of the legs of the right triangles (base of the prism) is equal to the side of the square frame. This artifact shows a proof of the theorem (Rufus, 1975). For making evident the interior squares as figures, we have added a red paper into the frame.

The scheme of use of this mathematical machine is quite simple: shift the prisms into the square frame, without raising them from the base and without superposing them (this condition is evident because of the height of the prisms and the frame). The mathematical meanings involved in this artifact are: geometrical figures as right triangle and square, the area of those figures, and equivalence of area by addition/subtraction of congruent parts. The property of the triangles to be right-angled is obtained by the support of the square frame, and that represents the hypothesis of the theorem embedded in the machine itself. The movement of the prisms is bound by the frame, which ensures the invariance of the sum of the areas of the triangles and the squares or, in other words, the invariance of the area of the squares, whatever it is. Two tasks can be proposed: the first one is to place the prisms for obtaining square hole(s), the second one is to pass to a configuration (M1 in Figure 1, in the center) to the other one (M1 in Figure 1, on the right).

In our experiment, we asked the students to reproduce 1:1 the first mathematical machine on paper (four triangles and square corresponding to the interior of the frame), after its manipulation and description. This choice was due to the fact that we had only one wooden model in the classroom and we wanted to propose the task about the configurations with a model for each small group. In such a way, the students constructed a new artifact. We want to pay attention to the two elements that characterize the semiotic potential of the reproduction of the machine: the negligible thickness for all the components of the machine and the lack of the frame. The first element can force the students to transfer implicit constraints of the manipulation of the wooden machine into a control of the reciprocal position of the right triangles to avoid their superposition (Figure 2, on the right). The second element fosters to make evident the range of the movement of the right triangles on the big square base (Figure 2, on the left). In this way, making explicit the mathematical components of the utilization schemes is supposed to reinforce the link to mathematics evoked by the machine.

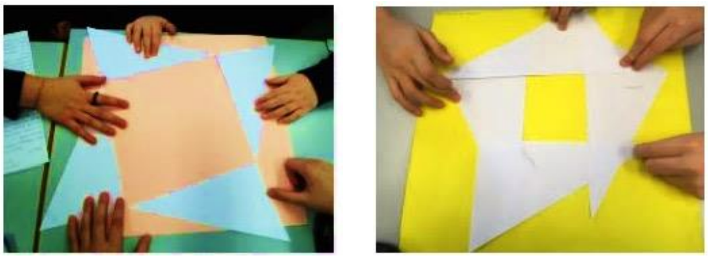


Figure 2: configurations by manipulating the paper machine

Drawings and geometrical figures

In the first activities with the artifact, the students are asked to answer the question "how the machine is made", with the request of representing it. As we have written above, in this case, the students had to physically reproduce 1:1 the machine (while, in general, they should draw the machine in their homework or worksheet, which often is not squared paper). In the TSM framework, drawing the artifact corresponds to individual production of artifact signs, strictly dependent on student's knowledge and his interpretation of the artifact. However, with respect to the TSM, we aim to pay more attention to our request of drawing. Following Duval (2005), this is a task of geometrical construction involving student's visualization and how geometrical properties are identified (see also Vendeira & Coutat, 2017). Our tasks involve the two kinds of visualizations that Duval distinguishes as iconic and non-iconic:

> A visualisation is iconic when, for instance, it represents positions or shape of real-world. It is non-iconic while it is organised to internal constraints and gives access to all cases possible. (Duval, 2008, p.49)

Concerning the role of visualization as an argument in proof, Duval (2005) analyzes the proof of the Pythagorean theorem corresponding to our first mathematical machine (as given by Rufus, 1975). He claims that the visualization is not complete if it only considers the two configurations (see Figure 1), because the relationship between the big square and the hypotenuse of the right triangles on one hand, and the two other squares and the legs of the same right triangles on the other hand are supposed known for the reader. This is grounded on the relationship between a conjecture and a figure. But if an arrow from left to right, for instance, connects the two representations, the transformation from one representation to another is realized. Nevertheless, the comparison of the areas of the squares is not directly possible, but it has to consider a computation (i.e., the difference between the big square and the four triangles) for paying attention to invariant elements in that transformation. In our machines, the transformation of representations corresponds to the movement of the four triangles, nevertheless with the loss of their simultaneous view.

Research questions

In this paper, we are interested in the didactical use of the mathematical machines for the Pythagorean theorem. Our research questions are:

1. Is it possible, and how, to approach the Pythagorean theorem with the mathematical machines described above?

2. Does the sequence of movements with the machines give a sufficient representation of the theorem for its understanding?

3. Which kinds of visualization are related to the tasks of drawing M1?

Methodology

According to our theoretical framework, the didactical methodology is the mathematics laboratory with artifacts. The tasks for students are organized in didactic cycles (Bartolini Bussi & Mariotti, 2008), consisting of small group work (GW), individual activities (IW), and collective mathematical discussions (CW). In the classrooms, other technologies are available, such as the Interactive Whiteboard with its software for making animations of the machines, and the simulations of the second machine made with Dynamic Geometry Software from the web. In the specific case of two classes involved in the experiments, the platform Edmodo was used. Therefore, the teaching experiment proposes a learning environment in which material and digital technologies are present. In general, it is structured in three phases, as follows:

Phase A: 1) GW: Exploration of the first mathematical machine M1 (Figure 1); 2) CW: sharing of the description of the M1; 3) GW: construction of the M1 by paper; 4) GW: study of the possible configurations of the four triangles of M1 (Figure 2); 5) IW: representation of M1 on workbook; 6) CW: identification of relationships (invariants) between the components of M1.

Phase B: 7) History of the Pythagorean theorem and Pythagorean triples; 8) GW: Generalization of the theorem by different puzzles.

Phase C: 9) CW: Exploration of the second mathematical machine M2 and its reproduction with paper; 10) GW: Preparation of posters on the two mathematical machines.

The teaching experiments have started in 2013, and have involved six Italian classes of 13-years old students and two teachers, co-authors of this paper.

The analysis is carried out on students' worksheets, videos, photos and IWB files.

Findings

In this section, we refer to phases A focusing on the task of drawing the machine M1.

Steps 1-3. Work with the material model in small group and its reproduction

During the first three steps, the students worked in small group with the task of describing the machine M1 and collecting the elements (for instance, the types of triangles, the length of the sides) useful for its reproduction with colored paper. Before the reproduction, a collective discussion allowed students sharing their explorations and agreeing on a written description of the machine, with the measure of its sides. In particular, the right triangles were described as equivalent and some students recalled the Tangram game. Then, the students obtained the reproduction scale 1:1 by measuring and using tools for drawing (above all, rules and set square).

After this, the students had to fill a worksheet with the properties of the two figures, square and right triangle, constituting the machine. The manipulation of this new paper machine was guided by the task of looking for "square holes". But this task requires being conscious of the two schemes of use: the triangles must remain in the big square and do not overlap each other (Figure 2). During students' work, the configuration with the two square holes (Figure 1, M1 in the center) often appears first with respect to the configuration with the square alone (Figure 1, M1 on the right). This could be because the sides of the square are not parallel to the side of the square frame.

Individual Work for representing the two configurations in paper and pencil (Step 5)

Although the students had correctly described the congruence of the four right triangles (and constructed those in the previous step) into the square, several representations were not correctly drawn. We summarize some elements of students' drawings:

1) Square base is not equal in the two configurations (Figure 3, on the right);

2) All the four right triangles are not all congruent: a) in one confirmation itself (Figure 3, left, drawing on the left); b) between the two configurations (Figure 4);

3) The "square with the hypotenuse as side" is not a square (Figures 3 and 4).

The review of all the representations shows an important invariant of the machine was not taken into account by the students: the side of the square base is equal to the sum of the two legs.

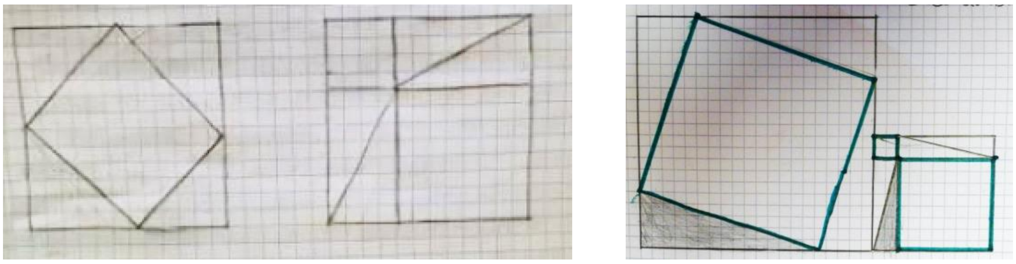


Figure 3: Students' representations of the two configurations of M1 on their workbooks

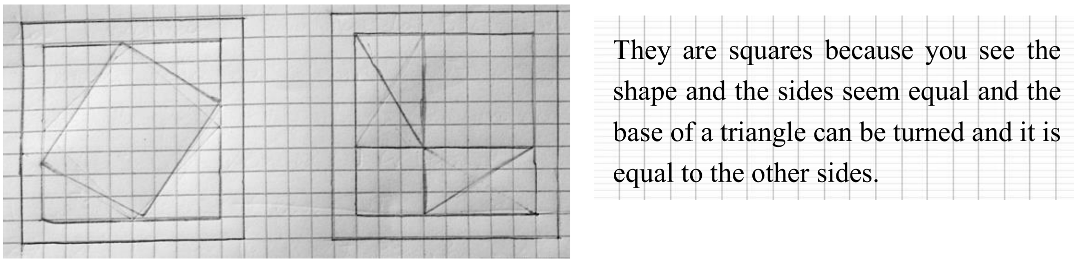


Figure 4: Student' representations of the two configurations of M1 on his workbook

Collective discussion with IWB

The collective discussion had two phases: the teacher paid attention to the wrong representations of the configurations; he took into account the passage from acting on the machine (both wooden and paper) to identify the relationship between the two configurations. First, the teacher used a checklist with the geometrical properties of the components of the machine that had been shared in the previous discussion for comparing the different representations. After, he asked to make new representations on the workbooks.

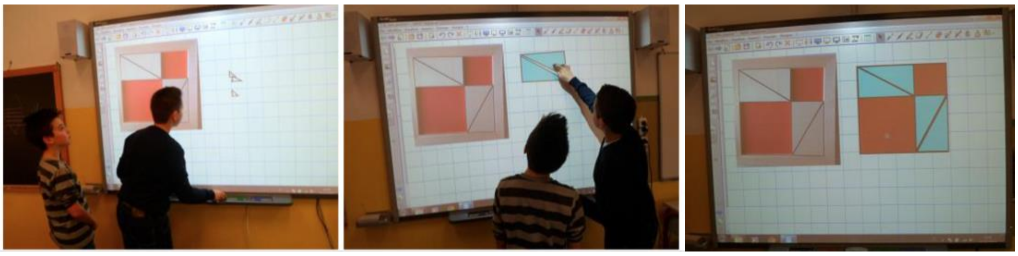


Figure 5: Collective work on IWB

Then the machine is represented on the IWB from a photo (Fig. 5, on the left). The use of the IWB enables a new collective manipulation of the machine, in which the students passed from one configuration to another one by dragging the right triangles as they made with the material machine.

An important part of the discussion focused on the argumentation that the holes were squares (Figure 6, on the left). The collective use of digital machine allows students linking the manipulation of the triangles to the manipulation of Tangram pieces (Figure 6, on the right) and, so, emphasizing the conservation of the areas of the holes. The Pythagorean theorem becomes a particular case in the equivalence of areas.

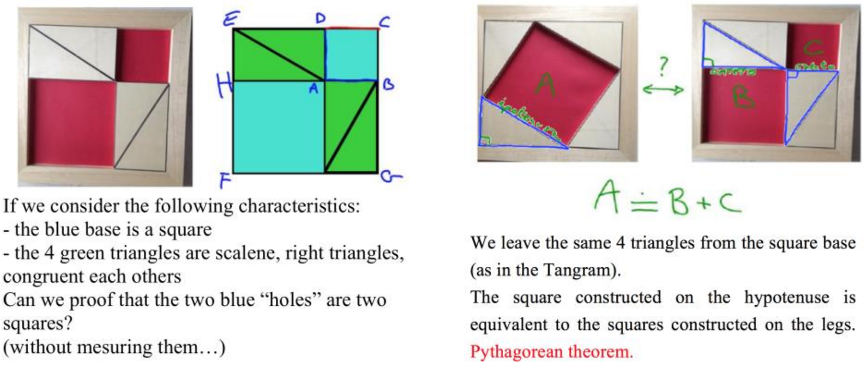


Figure 6: Question on proof and conclusion of the collective work on IWB (screenshots)

Discussion and concluding remarks

This paper aims to study the approach to the Pythagorean theorem using some physical artifacts that are material representations of that theorem. Students' answers to different task seem to confirm the assumption that the manipulation carried out by the students on the first mathematical machine is not enough for the emergence of mathematical meanings embedded in the machine. About our first research question, the analysis shows that those tasks allow fostering the production of signs, according to the theoretical framework of the TSM, and representations that can be used by the teacher for the mediation of mathematical meanings.

The scheme of use of shifting triangles for obtaining different configurations can support the emergence of personal signs and show the Pythagorean theorem in the context of equivalence of areas. For instance, in the first task of describing M1, some students recall the Tangram. If this meaning is not available for the students, the teacher has to focus on areas through a written, and/or symbolic calculation. With respect to our second research question on the feasibility of approaching the theorem with artifacts, we can argue that the Tangram, or meaning related to it, can be considered a prerequisite. In this case, Tangram means equivalent areas and manipulation of pieces for obtaining equivalent figures.

The comparison between the resolution of the tasks of making M1 by paper and representing M1 on workbook pays attention that the two tasks foster two different visualizations, as we have asked in our third research question. The first task solicits an iconic visualization of the two configurations, in which the shapes are drawn, but not their relationships inside the same configurations and between the two configurations. The second task seems to support a non-iconic visualization, because the students have to choose the measures of the sides of the figures (that are the parameters of M1) and make links between them. This choice has the potential of giving access to generalization to all the right triangles. However, it is not enough to draw twice a square and four triangles but the students have to represent their relationship, that is, an iconic visualization does not support the resolution as the wrong representations on workbooks show. Moreover, the students do not use the previous description of the components of M1.

Within the TSM framework, when the teacher proposes the discussion about those representations, the students' drawings are pivot signs for him. They are signs related to the artifact, but they are used for identifying and representing geometrical properties and invariants of M1. The potential of giving access to generalization is exploited by the teacher.

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