Locating critical regions by the Relevance Index

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Introduction

In this short communication, we summarise our recent findings on the use of the *Relevance Index* (RI) to identify critical states in complex systems.[1] The RI had been originally introduced to identify key features of the organization of complex dynamical systems, and it has proven able to provide useful results in various kinds of models, including e.g. those of gene regulatory networks and protein-protein interactions (Villani et al., 2015; Filisetti et al., 2015). The method can be applied directly to data and does not need to resort to models, possibly helping to uncover some non-trivial features of the underlying dynamical organization. In a nutshell, the RI is based upon Shannon entropies and can be used to identify groups of variables that change in a coordinated fashion, while they are less integrated with the rest of the system. The RI of a set of variables S is defined as $r(S) = \frac{I(S)}{M(S;U\setminus S)}$, where $I(S)$ is the *integration* of S and $M(\cdot)$ is the *mutual information* between S and the rest of the system. These groups of integrated variables may form the basis for an aggregate description of the system, at levels higher than that of the single variables and it can be applied also to networks, that are widespread in complex biological and social systems. We have found that the RI can also be used to identify critical states. In addition, it can also be used to detect situations that approach criticality, thus providing early warning signals that can be useful for controlling the behaviour of a system. Studying criticality through information-theoretical measures has already been documented in previous works, such as (Prokopenko et al., 2011; Wang et al., 2011), in which Fisher information (FI) is used to identify the critical state in both the Ising model and Boolean networks. Informally, FI measures the amount of knowledge an observation carries about an unknown parameter. FI is expected to be maximal where control parameters assume the critical value, because at phase transition systems are most sensitive to control parameters. Here, our use of information theory is considerably different: the RI measures the extent to which groups of variables are dynamically related; we experimentally observed that, even for small groups, this measure reaches its maximum at criticality. The reasons why it is able to capture critical states is subject of ongoing work. We also remark that the aim of this contribution is only to show that the RI can be effectively used to detect critical states and a comparison of different measures of criticality and the RI is planned for future work. In general, criticality refers to the existence of two qualitatively different behaviours that the same system can show, depending upon the values of some parameters. Criticality is then associated to parameter values that separate the qualitatively different behaviours. However, slightly different meanings of the word can be found in the literature, two major cases being *(a)* the one related to phase transitions and *(b)* dynamical criticality, sometimes called the "edge of chaos". In the former case, the different behaviours refer to equilibrium states that can be observed by varying the value of a macroscopic external parameter. In the latter case, the different behaviours are characterized by their dynamical properties: the attractors that describe the asymptotic behaviour of the system can be ordered states, like fixed points or limit cycles, or chaotic states. These two meanings are related but not identical.[2]

Results

In our experiments we considered two different kinds of systems, in which the term "criticality" takes different meanings: the Ising model for phase transitions, and the Random Boolean Network model for dynamical criticality. The two cases above do not differ only for the different kinds of criticality they show, but also for other important physical and mathematical properties: the Ising model is an ergodic system close to equilibrium, while the RBNs are dissipative, non ergodic systems. Moreover, the Ising model is inherently stochastic, while the RBN model is deterministic. For both the models, we computed the RI of randomly sampled groups of variables of varying size. Our main finding is that the RI is able to satisfactorily locate the critical points in both cases, notwithstanding their differences and evidenc-

[1]These results have been published in (Roli et al., 2017).

[2]See (Roli et al., 2015) for a more detailed discussion).

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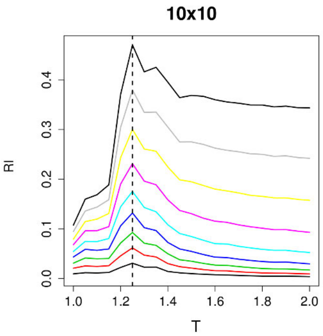

Figure 1: Plot of RI for 10×10 Ising lattice. The median of the average RI values for groups of size 2 to 10 is plotted against T. The curves shift up with group size. The peaks of RI correspond to the susceptibility peak (see Figure 2), which in turn corresponds to the critical value of the control parameter.

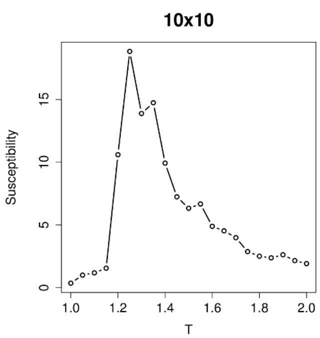

Figure 2: Plot of median susceptibility values for 10×10 Ising lattice. The peak of susceptibility empirically identifies the critical T value.

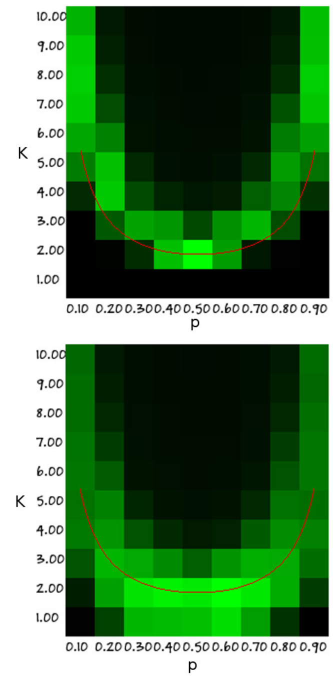

Figure 3: Heatmaps of the p–K diagram of RI indexes computed for group having different size: respectively, groups of size 2, 5. The superimposed red line denotes the position of the edge of chaos curve. This wide region has been sampled in 90 points by combining nine different biases and ten different connectivities; for each point we tested 100 different 40-nodes RBNs, each RBN being represented by the RI obtained by sampling the states of 200 different trajectories. Each pixel represents the median of the RI of 100 different RBNs sharing the same values of bias and connectivity.

ing that in these systems criticality involves groups of different size. An excerpt of the results is shown in Figures 1,2 and 3. Results show also that the RI behaves robustly against sampling effort and system size. Moreover, a detailed analysis shows that, while still detecting critical states, the RI varies across RBN critical ensembles, supporting the observation that critical RBNs exhibit a spectrum of different behaviours. Our analysis also supports the statement that the information provided by the dynamics of a RBN along its transients might differ from that of its attractors, providing a richer picture of system's dynamical behaviour. Detailed results can be found in (Roli et al., 2017).

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