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Topology in colored tensor models via crystallization theory

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Abstract

The aim of this paper is twofold. On the one hand, it provides a review of the links between random tensor models, seen as quantum gravity theories, and the PL-manifolds representation by means of edge-colored graphs (*crystallization theory*). On the other hand, the core of the paper is to establish results about the topological and geometrical properties of the *Gurau-degree* (or *G-degree*) of the represented manifolds, in relation with the motivations coming from physics. In fact, the G-degree appears naturally in higher dimensional tensor models as the quantity driving their $1/N$ expansion, exactly as it happens for the genus of surfaces in the two-dimensional matrix model setting.

In particular, the G-degree of PL-manifolds is proved to be finite-to-one in any dimension, while in dimension 3 and 4 a series of classification theorems are obtained for PL-manifolds represented by graphs with a fixed G-degree. All these properties have specific relevance in the tensor models framework, showing a direct fruitful interaction between tensor models and discrete geometry, via crystallization theory.

Keywords: crystallization; regular genus; gem-complexity; Gurau degree; tensor models; quantum gravity

2010 Mathematics Subject Classification: 57Q15 - 57N10 - 57N13 - 57M15 - 57Q25 - 83E99.

1 Introduction

The problem of gravity quantization is a well-known and deeply investigated issue in the community of theoretical and mathematical physicists. There are dozen of approaches to solve the problem of Quantum Gravity (QG). While none of these approaches has been able to give a satisfactory theoretical and mathematical framework to QG yet, this topic attracts much activity for good reasons. It is indeed widely believed that, if such a framework exists, it contains answers to some of the most puzzling modern physics interrogations. Keywords to such questions are: black hole entropy, big bang singularity, background independence, problem of time... Even if progresses have been made thanks

to decades of research on these topics, we are still far from a good understanding.

On the bright side however, the study of QG yields new mathematics and mathematical-physics everyday; all approaches bring new ideas to geometry and push forward the study of older ideas. Calabi-Yau and geometric invariant theory, for instance, have been strongly moved thanks to string theory; connection theory and discrete geometry have been extensively used and improved by loop quantum gravity theorists; random geometry made incredible progresses in dimension two thanks to matrix models and Liouville gravity theory. This is, of course, a small number of examples that one can think of.

A recent line of development is the approach by tensor models. In some sense it aims at generalizing to higher dimensional cases the approach of matrix models which, in dimension two, has been very successful at providing a framework for QG. It also contains incredible mathematics, from moduli space invariants to Korteweg-de Vries and Kadomtsev-Petviashvili hierarchy of equations. The approach of matrix models can be very roughly described as follows.

- Compute the Einstein-Hilbert action on discrete (Piecewise-Linear = PL) 2-manifolds. This can be done for *any type of discretization*, although triangulations are generally preferred.
- Realize that discretizations of 2-manifolds can be seen as Feynman graphs of a 0-dimensional statistical field theory. Moreover, the exponential of the value of the Einstein-Hilbert action on each discretized 2-manifold can be obtained as the Feynman amplitudes of the underlying field theory by carefully choosing its dynamical variables and its parameters.
- The field variables of the theory need to be matrices.
- This field theory can be put in relation with Liouville gravity and topological gravity on any 2-manifold.

The approach of tensor models relies on the same idea, the main difference being that not any discretization of higher dimensional manifolds can do the job. In fact a field theory that generates PL manifolds can be constructed only if the PL structures are represented by “colored triangulations”; with this restriction, the fields encoding PL d -manifolds turn out to be rank d tensor variables. More precisely, colored triangulations are completely described by their dual 1-skeletons, which are regular bipartite edge-colored graphs arising as Feynman graphs of colored tensor models theory.

In this recent approach of tensor models, a lot of structures present in the matrix models framework can be generalized. One of the most striking generalization is the recovery of the so-called $\frac{1}{N}$ expansion in the tensor models setting. In matrix models, the $1/N$ expansion is a power series in the inverse of the size N of the matrix variables of the theory; this expansion is driven by the genera of the 2-manifolds represented by the Feynman graphs. In some sense, it classifies the 2-manifolds with respect to their possible mean scalar curvature; this is natural as indeed the Einstein-Hilbert action is nothing more than the integral of the scalar curvature over the manifold. In the higher dimensional case of tensor models, the $1/N$ expansion is driven by the so-called *G-degree* (that equals the genus in dimension two). The G-degree is a non-negative integer associated to edge-colored graphs via “regular embeddings” of the graph into surfaces (Definition 3 and Proposition 7). This gives rise, in any dimension, to a new manifold invariant, defined as the minimum G-degree among the graphs representing the manifold (Definition 6). However, while the properties of the genus of a surface are well-known, the mathematical properties of this new quantity are up to now mostly unknown. The main goal of this article is to lay the necessary foundations in order to understand the geometrical properties of the G-degree, in relation with the motivations coming from physics. Indeed, a deep grasp in the properties of the G-degree could allow us to establish connections between tensor models and others (continuum) theories of QG. With this aim, we need a better understanding of its $1/N$ expansion, and thus any geometric insight into the parameter driving it - the G-degree - can be useful.

It is worthwhile noting that, even if “classical” colored tensor models deal with complex tensor variables, giving rise to bipartite Feynman graphs, a real tensors version, involving also non-bipartite graphs, has been recently proposed: see [49]. For this reason, properties of the G-degree also in the

non-bipartite setting are welcome.

From a “geometric topology” point of view, the theory of manifold representation by means of edge-colored graphs (*GEM theory*) has been deeply studied since 1975: see the survey papers [26] and [14], together with their references. The great advantage of GEM theory is the possibility of representing, in any dimension, every PL d -manifold by means of a totally combinatorial tool. Indeed, each bipartite (resp. non-bipartite) $(d + 1)$ -colored graph encodes a colored triangulation P of an orientable (resp. non-orientable) d -pseudomanifold: the vertices of the graph represent the d -simplices of P and the colored edges of the graph describe the pairwise gluing in P of the $(d - 1)$ -faces of its maximal simplices (the graph thus becomes the dual 1-skeleton of P). In this framework, many results have been achieved during the last 40 years; noteworthy are the classification results obtained in dimension 3 and 4 with respect to the PL-manifold invariants *regular genus* and *gem-complexity*, specifically introduced and investigated in GEM theory with geometric topology aims (see for example [10] for the 3-dimensional case, [12] and [14] for the 4-dimensional one). In the present paper we show that the G-degree, which arises with physics motivations, can be linked with both these invariants: thanks to known results about them, new ideas are obtained about the meaning of the G-degree.

As far as the arbitrary dimension d is concerned, a relevant achievement allows to state that all bipartite $(d + 1)$ -colored graphs with G-degree less than $d!/2$ do represent the PL d -sphere (Proposition 9). Since the G-degree is always a multiple of $\frac{(d-1)!}{2}$ (Proposition 7), this implies that only graphs encoding the d -sphere contribute to the d most significant terms of the above mentioned $1/N$ expansion. On the contrary, in the non-bipartite case, no $(d + 1)$ -colored graph is proved to exist, with G-degree less than $\lceil d!/4 \rceil$ (Proposition 8). This lower bound is of interest in the real tensors version of the theory, since it implies that also the first $\lceil d/2 \rceil$ terms of the analogue of the $1/N$ expansion involve only d -spheres.

Another important outcome is that, despite its similarity with the regular genus (which coincides with the Heegaard genus in dimension three), the invariant G-degree is a finite-to-one quantity in any dimension (Theorem 14). All these properties have specific importance in the tensor models framework (Subsection 3.3, Theorem 6).

Of particular interest, also for applications to physics, are the dimensions three and four. In this paper we show that the G-degree of a closed 3-manifold is nothing but its gem-complexity (Theorem 16): this allows to obtain many classification results for 3-manifolds with respect to the G-degree (Subsection 5.1). In dimension four we prove that, due to the existence of infinitely many PL structures on the same topological manifold, the G-degree is not additive with respect to connected sum of manifolds (Proposition 31). Furthermore, we show that in the 4-dimensional case, the G-degree splits into two summands, one being a topological invariant, the second being a PL invariant (Corollary 24). From a physical standpoint, this leads to wonder whether or not the PL part comes from the local degree of freedom present in the gravity theory in dimension four. As in dimension three, the relationship between G-degree and gem-complexity allows to obtain a lot of classification results for 4-manifolds with respect to G-degree (Subsection 6.3).

As already pointed out, edge-colored graphs represent pseudomanifolds, not necessarily manifolds: in the $1/N$ expansion context, it should be useful to distinguish graphs encoding manifolds. In this direction, Corollary 23 gives a strong property: 4-manifolds (and “singular” 4-manifolds) only appear if the G-degree is congruent to zero mod 6.

In order to make the paper self-contained for specialists in both the involved research fields (i.e. geometric topology via GEM theory and QG via tensor models), we include in the first sections basic notions about Gaussian Integrals and Feynman graphs (Section 2), colored graphs and represented pseudomanifolds (Subsection 3.1) and colored tensors (Subsections 3.2 and 3.3). The central sections of the paper contain the original results, concerning G-degree in arbitrary dimension (Section 4), and in the 3-dimensional and 4-dimensional setting (Sections 5 and 6 respectively).

Mutual connections between GEM theory and colored tensor models theory, with a particular focus on the properties of the G-degree, seem to be a context in which geometric topology and quantum gravity can fruitfully cooperate: trends for further investigations in this direction are sketched in the

final Section 7.

2 Gaussian Integrals and Feynman Graphs

Feynman graphs are often seen as a non-rigorous technical tool used by physicists. There is however one notable exception, when the integrals under consideration are not path integrals but usual finite dimensional integrals. In this section we consider Gaussian integration on \mathbb{R}^d .

2.1 Gaussian correlations

For any positive definite symmetric bilinear form $C : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ (whose representative matrix is also denoted by C) and for any element $S \in \mathbb{R}^d$, we define:¹

$$\begin{aligned} Z_0[C] &:= \int_{\mathbb{R}^d} dx \exp(-\tfrac{1}{2}\langle x, Cx \rangle) = \left(\det \frac{C}{2\pi}\right)^{-1/2}, \\ Z[C, S] &:= \int_{\mathbb{R}^d} dx \exp(-\tfrac{1}{2}\langle x, Cx \rangle + \langle S, x \rangle) = Z_0[C] e^{-\tfrac{1}{2}\langle S, C^{-1}S \rangle}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ represents the canonical scalar product on \mathbb{R}^d .

For each collection i_1, \dots, i_m of (possibly not distinct) indices in $\mathbb{N}_d = \{1, \dots, d\}$, let us consider the following *correlation* (i.e., the mean value of a product of Gaussianly distributed random variables):

$$\langle x_{i_1}, \dots, x_{i_m} \rangle := Z_0[C]^{-1} \int_{\mathbb{R}^d} dx x_{i_1} x_{i_2} \dots x_{i_m} \exp(-\tfrac{1}{2}\langle x, Cx \rangle). \quad (1)$$

Wick's theorem [48] allows us to expand any correlation as a sum of products of correlations between pairs of variables.

Theorem 1 (Wick expansion)

$$\langle x_{i_1}, \dots, x_{i_m} \rangle = \begin{cases} 0, & \forall m \text{ odd}, \\ \sum_{\sigma \in \mathcal{P}} \prod_{(r,s) \in \sigma} \langle x_{i_r}, x_{i_s} \rangle & \forall m \text{ even} \end{cases}$$

where \mathcal{P} is the set (of cardinality $(m-1)!!$) of pairings of the elements of \mathbb{N}_m .

Hence the computation of formula (1) can be effectively performed since, as it is easy to check,

$$\langle x_i, x_j \rangle = C_{ij}^{-1}.$$

Example: We consider the following simple case,

$$\langle x_1, x_2, x_3, x_4 \rangle = Z_0[C]^{-1} \int_{\mathbb{R}^d} dx x_1 x_2 x_3 x_4 \exp(-\tfrac{1}{2}\langle x, Cx \rangle) \quad (2)$$

What Wick's theorem tells us is

$$\langle x_1, x_2, x_3, x_4 \rangle = C_{12}^{-1} C_{34}^{-1} + C_{13}^{-1} C_{24}^{-1} + C_{14}^{-1} C_{23}^{-1},$$

where each summand corresponds to one of the three pairings of the elements of \mathbb{N}_4 .

In the next subsection we will show how to represent each summand in the Wick expansion by a Feynman graph.

¹In physics literature C^{-1} is often called the *propagator*, while S is often called a *source*.

2.2 Feynman Graphs

For our purpose we first describe a procedure to yield graphs, which is motivated by the way Feynman graphs arise in computation.

Definition 1 A half-edges graph is a triplet $G = (\mathcal{H}, \mathcal{V}, \alpha)$ such that \mathcal{H} is a set of even cardinality, called the half-edges set, \mathcal{V} is a partition of \mathcal{H} and $\alpha : \mathcal{H} \rightarrow \mathcal{H}$ is an involution on \mathcal{H} without fixed points.

To each half-edges graph $G = (\mathcal{H}, \mathcal{V}, \alpha)$, it is naturally associated a pseudograph² with vertex set \mathcal{V} and edge set \mathcal{E} consisting of all unordered couples $\{i, \alpha(i)\}$, $\forall i \in \mathcal{H}$: each edge is obtained by gluing the half-edges i and $\alpha(i)$. Note that, in general, many half-edge graphs have the same associated (pseudo)graph. However, with slight abuse of notation, we will denote the associated (pseudo)graph with the same symbol: $G = (\mathcal{V}, \mathcal{E})$.

Example:

Consider $\mathcal{H} = \{a, b, c, d, e, f\}$ and the partition $\mathcal{V} = \{\{a, b, c\}, \{d, e, f\}\}$; then set, for instance, $\alpha(a) = b$, $\alpha(c) = d$, $\alpha(e) = f$. This defines a half-edges graph, whose associated (order two) pseudograph is depicted in Fig. 1.



Figure 1: Pseudograph described in the example.

Given a correlation $\langle x_{i_1}, \dots, x_{i_m} \rangle$ with even m and $\#\{i_1, \dots, i_m\} = l$, let us consider m half-edges e_1, \dots, e_m and l vertices v_1, \dots, v_l such that the set of half-edges incident to the vertex v_k ($1 \leq k \leq l$) is $\{e_j / i_j = k\}$.

Each pairing $\sigma \in \mathcal{P}$ of \mathbb{N}_m defines (see [48]) a Feynman graph G_σ with half-edges set $\{e_1, \dots, e_m\}$, vertex set $\{v_1, \dots, v_l\}$ and the following involution α_σ :

for each pair $(r, s) \in \sigma$, $\alpha_\sigma(e_r) = e_s$ and $\alpha_\sigma(e_s) = e_r$ (i.e. the half-edges e_r and e_s are glued).

Therefore the graph G_σ represents in the Wick expansion of $\langle x_{i_1}, \dots, x_{i_m} \rangle$ the summand

$$\prod_{(r,s) \in \sigma} \langle x_{i_r}, x_{i_s} \rangle = \prod_{(r,s) \in \sigma} C_{i_r i_s}^{-1}.$$

Note that distinct pairings may give rise to the same Feynman graph, i.e. some summands may coincide.

As an example, let us consider $\langle x_1, x_1, x_1, x_2 \rangle$. Wick's theorem yields

$$\langle x_1, x_1, x_1, x_2 \rangle = C_{11}^{-1} C_{12}^{-1} + C_{11}^{-1} C_{12}^{-1} + C_{12}^{-1} C_{11}^{-1} = 3C_{11}^{-1} C_{12}^{-1},$$

where we used the symmetry of C for the last equality.

On the other hand, all Feynman graphs associated to $\langle x_1, x_1, x_1, x_2 \rangle$ have a vertex v_1 with three half-edges e_1, e_2, e_3 and a vertex v_2 with only one half-edge e_4 . Then, the three pairings of the elements of \mathbb{N}_4 (see formula (2)) correspond to three involutions on $\{e_1, e_2, e_3, e_4\}$, giving rise to three half-edges graphs, with the same associated pseudograph: see Fig. 2. Hence, the correlation $\langle x_1, x_1, x_1, x_2 \rangle$ may be computed by taking three times the term $C_{11}^{-1} C_{12}^{-1}$ associated to this Feynman (pseudo)graph.

Note that the value of a correlation $\langle x_{i_1}, \dots, x_{i_m} \rangle$ does not depend on the ordering of the x_{i_j} 's; therefore in the following we will simply write $\langle x_{i_1} \dots x_{i_m} \rangle$. For example, the correlation $\langle x_1, x_1, x_1, x_2 \rangle$ in the above example will be written as $\langle x_1^3 x_2 \rangle$.

²The term *pseudograph* means that multiple edges and loops are allowed.

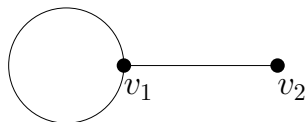


Figure 2: Feynman graph associated to $\langle x_1, x_1, x_1, x_2 \rangle$.

2.3 Non-Gaussian correlations

Let us now consider the *non-Gaussian case*, i.e. the following integral, called a *partition function*:

$$\hat{Z}[C, \lambda] := \int_{\mathbb{R}^d} dx \exp\left(-\frac{1}{2}\langle x, Cx \rangle\right) \exp(\lambda U(x)),$$

where $U : \mathbb{R}^d \rightarrow \mathbb{R}$ and λ is a real parameter.

In order to have an estimation of the above integral, physicists perform an *asymptotic* (or *perturbative*) *expansion* around the zero value of the parameter λ . Hence, what they really compute are the terms of the following series, which is called a *formal*, or *perturbative* partition function:

$$Z_f[C, \lambda] := \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int dx (U(x))^k \exp\left(-\frac{1}{2}\langle x, Cx \rangle\right).$$

The series $Z_f[C, \lambda]$ is also called the *formal integral* associated to $\hat{Z}[C, \lambda]$.

We warn that, in general, $\hat{Z}[C, \lambda] \neq Z_f[C, \lambda]$; however, in some cases, it is possible to show that $Z_f[C, \lambda]$ contains enough information to re-construct $\hat{Z}[C, \lambda]$, see for instance [46].

In case $U(x)$ has a polynomial form (which is the most encountered case in physics), each term of the perturbative expansion may be expressed by means of Gaussian correlations and hence can be computed by applying Wick's theorem and by using Feynman graphs as a way to recall which are the pairings and which are their possible values.

3 Tensor Models and colored Feynman graphs

In this section we recall the definition of colored graphs and Graph Encoded Manifolds (GEM) and then define tensor models with respect to them.

We warn the reader that throughout the paper - with the exception of the first part of Subsection 6.3 - we will work in the Piecewise Linear (PL) setting and we will consider only the case of compact spaces with empty boundary; therefore in the following all manifolds are assumed to be PL and closed. Moreover, all graphs will be assumed to be finite and connected, unless otherwise stated.

3.1 Colored Graphs and Pseudomanifolds

Definition 2 Consider $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$ a regular $d+1$ valent multigraph ($d \geq 2$); a coloration of Γ is a map $\gamma : \mathcal{E}(\Gamma) \rightarrow \Delta_d = \{0, \dots, d\}$ that is injective on adjacent edges.³ The pair (Γ, γ) is called a $(d+1)$ -colored graph.

For every $\mathcal{B} \subseteq \Delta_d$ let $\Gamma_{\mathcal{B}}$ be the subgraph obtained from (Γ, γ) by deleting all the edges with colors not belonging to \mathcal{B} . The connected components of $\Gamma_{\mathcal{B}}$ are called \mathcal{B} -residues or, if $\#\mathcal{B} = h$, h -residues of Γ .

³According to basic notions of graph theory, a *multigraph* can contain multiple edges, but no loops. On the other hand, the existence of a coloration implies that loops are not allowed. However, note that not any $(d+1)$ -regular multigraph admits a coloration.

In particular, if $\mathcal{B} = \Delta_d - \{i\}$ (resp. $\mathcal{B} = \{i, j, k\}$) (resp. $\mathcal{B} = \{i, j\}$), we write Γ_i (resp. Γ_{ijk}) (resp. Γ_{ij}) instead of $\Gamma_{\mathcal{B}}$. Furthermore, the number of connected components of Γ_i (resp. Γ_{ijk}) (resp. Γ_{ij}) is denoted by g_i (resp. g_{ijk}) (resp. g_{ij}).

A d -dimensional pseudocomplex $K(\Gamma)$ associated to a $(d + 1)$ -colored graph (Γ, γ) can be constructed in the following way:

- for each vertex of Γ let us consider a d -simplex and label its vertices by the elements of Δ_d ;
- for each pair of c -adjacent vertices of Γ ($c \in \Delta_d$), the corresponding d -simplices are glued along their $(d - 1)$ -dimensional faces opposite to the c -labeled vertices, the gluing being determined by the identification of equally labeled vertices.

Note that, as a consequence of the above construction, $K(\Gamma)$ is endowed with a vertex-labeling by Δ_d that is injective on any simplex. Moreover, Γ can be visualized as the dual 1-skeleton of $K(\Gamma)$. The duality establishes a bijective correspondence between the h -residues of Γ colored by any subset \mathcal{B} of Δ_d and the $(d - h)$ -simplices of $K(\Gamma)$ whose vertices are labeled by $\Delta_d - \mathcal{B}$ (see for example [9] for details). In particular, for each color $i \in \Delta_d$ there is a bijective correspondence between the connected components of Γ_i and the vertices of $K(\Gamma)$ labeled by i .

Moreover, $K(\Gamma)$ is orientable if and only if Γ is bipartite. As a side remark, notice that the number of vertices of a colored graph is even and this does not depend on the bipartiteness.

In general $|K(\Gamma)|$ is a d -pseudomanifold and (Γ, γ) is said to represent it⁴; if $|K(\Gamma)|$ is a d -dimensional PL manifold M^d , then (Γ, γ) is called a GEM of M^d . In particular, the following theorem holds:

Theorem 2 Any PL d -manifold admits a GEM representation.

A characterization of GEMs among colored graphs is stated in the following proposition.

Proposition 3 A $(d + 1)$ -colored graph (Γ, γ) is a GEM of a PL d -manifold iff for each color $i \in \Delta_d$ the connected components of Γ_i represent $(d - 1)$ -spheres.

A GEM (Γ, γ) of a d -manifold M^d is called a crystallization of M^d iff, for each $i \in \Delta_d$, the subgraph Γ_i is connected. By duality this is equivalent to requiring that the pseudocomplex $K(\Gamma)$ has exactly $d + 1$ vertices.

An r -dipole ($1 \leq r \leq d$) of colors c_1, c_2, \dots, c_r in a $(d + 1)$ -colored graph (Γ, γ) is a subgraph of Γ made by r parallel edges colored by c_1, c_2, \dots, c_r , whose endpoints belong to different connected components of $\Gamma_{\mathcal{B}}$, with $\mathcal{B} = \Delta_d - \{c_1, c_2, \dots, c_r\}$.

An r -dipole can be eliminated from Γ by deleting the subgraph and welding the remaining hanging edges according to their colors; in this way another $(d + 1)$ -colored graph (Γ', γ') is obtained. If (Γ, γ) is a GEM, then (Γ, γ) and (Γ', γ') represent the same d -manifold (see [24], where r -dipole eliminations and their inverse process are identified as dipole moves).

The next important result by Pezzana establishes crystallization theory as a representation theory for (PL) manifolds of arbitrary dimension.

Theorem 4 ([44]) Any PL d -manifold admits a crystallization.

Proof. Let (Γ, γ) be a GEM of a d -manifold M^d . If Γ is not a crystallization, then there exists at least one color i such that the subgraph Γ_i is not connected. Hence Γ contains a 1-dipole of color i ; by eliminating this dipole we obtain a $(d + 1)$ -colored graph (Γ', γ') , still representing M^d and such that Γ'_i has one connected component less than Γ_i . By repeating the same argument, after a finite sequence of 1-dipole eliminations, we get a crystallization of M^d . □

⁴A d -pseudomanifold is a pure, non-branching and strongly connected pseudocomplex ([47]). However, throughout the paper we will use the term “pseudomanifold” both for the pseudocomplex $K(\Gamma)$ and for the topological space $|K(\Gamma)|$.

To any bipartite (resp. non bipartite) $(d + 1)$ -colored graph a particular set of embeddings into orientable (resp. non orientable) surfaces can be associated.

Theorem 5 ([32]) *Let (Γ, γ) be a bipartite (resp. non-bipartite) $(d + 1)$ -colored graph of order $2p$. Then for each cyclic permutation $\varepsilon = (\varepsilon_0, \dots, \varepsilon_d)$ of Δ_d there exists a cellular embedding, called regular, of (Γ, γ) into an orientable (resp. non-orientable) closed surface $F_\varepsilon(\Gamma)$ of Euler characteristic*

$$\chi(F_\varepsilon(\Gamma)) = \sum_{j \in \mathbb{Z}_{d+1}} g_{\varepsilon_j \varepsilon_{j+1}} + (1 - d)p.$$

such that the regions of the embeddings are bounded by the images of the $\{\varepsilon_j, \varepsilon_{j+1}\}$ -colored cycles, for each $j \in \mathbb{Z}_{d+1}$. Moreover, ε^{-1} induces the same embedding.

No regular embeddings of (Γ, γ) exist into non-orientable (resp. orientable) surfaces.

As a consequence, there are exactly $d!/2$ regular embeddings (also called *Jackets* in the tensor models context) and each one comes with a genus $\rho_\varepsilon(\Gamma)$, which is defined in the bipartite (resp. non bipartite) case as the genus⁵ (resp. half the genus) of the orientable (resp. non orientable) surface $F_\varepsilon(\Gamma)$. Hence, if Γ is bipartite (resp. non-bipartite), then for each ε we have $\rho_\varepsilon(\Gamma) \in \mathbb{Z}$ (resp. $2\rho_\varepsilon(\Gamma) \in \mathbb{Z}$), while $\chi(F_\varepsilon(\Gamma)) = 2 - 2\rho_\varepsilon(\Gamma)$ holds in both cases.

The *Gurau degree* (often called *degree* in the tensor models literature) and the *regular genus* of a colored graph are defined in terms of the embeddings of Theorem 5.

Definition 3 Let (Γ, γ) be a $(d + 1)$ -colored graph. If $\{\varepsilon^{(1)}, \varepsilon^{(2)}, \dots, \varepsilon^{(\frac{d!}{2})}\}$ is the set of all cyclic permutations of Δ_d (up to inverse), the *Gurau degree* (or *G-degree* for short) of Γ , denoted by $\omega_G(\Gamma)$, is defined as

$$\omega_G(\Gamma) = \sum_{i=1}^{\frac{d!}{2}} \rho_{\varepsilon^{(i)}}(\Gamma).$$

and the *regular genus* of Γ , denoted by $\rho(\Gamma)$, is defined as

$$\rho(\Gamma) = \min \left\{ \rho_{\varepsilon^{(i)}}(\Gamma) / i = 1, \dots, \frac{d!}{2} \right\}.$$

As a consequence of the definition of regular genus of a colored graph and of Theorem 2, a PL invariant for d -manifolds can be defined:

Definition 4 Let M^d be a d -dimensional manifold ($d \geq 2$). The *regular genus* of M^d is defined as

$$\mathcal{G}(M^d) = \min\{\rho(\Gamma) \mid (\Gamma, \gamma) \text{ is a GEM of } M^d\}.$$

As regards dimension 2, it is well-known that any bipartite (resp. non-bipartite) 3-colored graph (Γ, γ) represents an orientable (resp. non-orientable) surface $|K(\Gamma)|$ and $\rho_\varepsilon(\Gamma)$ is exactly the genus (resp. half the genus) of $|K(\Gamma)|$.

On the other hand, for $d \geq 3$ the regular genus is proved to be an *integer* PL manifold invariant (see [20, Proposition A]), which extends to arbitrary dimension the Heegaard genus of a 3-manifold. An analogous definition of a PL manifold invariant based on the notion of G-degree will be introduced in Section 4 (Definition 6).

Another PL invariant that will play an important rôle in the paper is the *gem-complexity* $k(M^d)$ of a d -manifold M^d , defined as the integer $p - 1$, where $2p$ is the minimum order of a GEM of M^d . Both regular genus and gem-complexity of a d -manifold are always realized by a crystallization.

⁵The use of the letter g for the number of connected components of the residues of a colored graph is standard within crystallization theory; this is why the genus of a surface (and the regular genus of a graph) is here denoted by ρ , instead of using the usual symbol g .

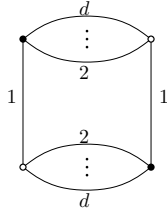


Figure 3: d -colored graph representing $Q_{m,1}(T, \bar{T})$.

3.2 Invariants of tensors and their Gaussian Integrals

In this subsection we sketch the construction of invariants of tensors. Let V be a \mathbb{C} -vector space of finite dimension N . There is a natural action of $GL(N)$ on V and this action extends to a natural action of $GL(N)^{\times d}$ on the tensor product $E = V^{\otimes d}$ and on its dual E^* .

Given a basis $\{e_i\}$ of V , each $T \in E$ and $\bar{T} \in E^*$ can be written as

$$T = \sum_{i_1, \dots, i_d=1}^N T_{i_1 \dots i_d} e_{i_1} \otimes \dots \otimes e_{i_d}$$

$$\bar{T} = \sum_{i_1, \dots, i_d=1}^N \bar{T}_{i_1 \dots i_d} e_{i_1}^* \otimes \dots \otimes e_{i_d}^*,$$

where $\{e_i^*\}$ denotes the basis of V^* dual to $\{e_i\}$.

We want to construct quantities that are invariant under the action of $GL(N)^{\times d}$ on both E and E^* . This is done as follows. The action of an element $g = (g_1, \dots, g_d) \in GL(N)^{\times d}$ changes the components of the contravariant tensor T under $g^{-1} = (g_1^{-1}, \dots, g_d^{-1})$, while the ones of the covariant tensor \bar{T} are changed under g . Hence any quantity constructed out of contractions of indices of components of T, \bar{T} respecting their ordering is an invariant $B(T, \bar{T})$ of tensors.

Indeed, it has been proved in [36] that any invariant of tensors can be represented as a linear combinations of such $B(T, \bar{T})$'s.

Well-known examples of invariants are

$$\bar{T} \cdot T := \sum_{i_1, \dots, i_d=1}^N \bar{T}_{i_1 \dots i_d} T_{i_1 \dots i_d}, \quad (3)$$

$$Q_{m,1}(T, \bar{T}) := \sum_{\substack{i_1, \dots, i_d=1 \\ j_1, \dots, j_d=1}}^N \bar{T}_{i_1 i_2 \dots i_d} T_{j_1 i_2 \dots i_d} \bar{T}_{j_1 j_2 \dots j_d} T_{i_1 j_2 \dots j_d}. \quad (4)$$

The first one is the only quadratic invariant, while the second is a quartic invariant (in fact $Q_{m,1}$ is the first non-trivial element of a family of tensor invariants called *melonics*).

Any invariant $B(T, \bar{T})$ of rank d tensors can be encoded in a bipartite d -colored graph (B, b) as follows:

- take a white vertex for each T appearing in the formula of $B(T, \bar{T})$ and a black vertex for each \bar{T} .
- Each time the c^{th} index of a T is contracted with the c^{th} index of a \bar{T} , join the two corresponding vertices by a c -colored edge.

The colored graph representing the invariant $Q_{m,1}$ is pictured in Fig. 3.

Note that $Q_{m,1}$ can also be written as

$$Q_{m,1}(T, \bar{T}) = \sum_{\substack{i_h, j_h, l_h, k_h=1 \\ \forall h \in \mathbb{N}_d}}^N \delta_{i_1 k_1} \delta_{j_1 l_1} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \bar{T}_{i_1 \dots i_d} T_{j_1 \dots j_d} \bar{T}_{l_1 \dots l_d} T_{k_1 \dots k_d},$$

where, for each summand and for each $c \in \{1, \dots, d\}$, the Kronecker deltas with subindex c correspond to the c -colored edges of the associated graph.

By analogy, the generic invariant $B(T, \bar{T})$ may be expressed as:

$$B(T, \bar{T}) = \sum_{\substack{i_h^{(l)}=1 \\ \forall h \in \mathbb{N}_d, \forall l \in \mathbb{N}_{2p}}}^N \delta_B \left(\prod_{l=1}^p T_{i_1^{(2l-1)} \dots i_d^{(2l-1)}} \right) \left(\prod_{l=1}^p \bar{T}_{i_1^{(2l)} \dots i_d^{(2l)}} \right), \quad (5)$$

where $2p$ is the order of the associated d -colored graph (B, b) and δ_B is the product of all Kronecker deltas corresponding to contractions of indices involved in $B(T, \bar{T})$ (which give rise to the colored edges of (B, b)).

Gaussian integrals of tensor invariants

Given an invariant $B(T, \bar{T})$ of rank d tensors, let us consider its mean value

$$\langle B(T, \bar{T}) \rangle = \int \frac{dT d\bar{T}}{(2\pi)^{N^d}} B(T, \bar{T}) \exp(-N^{d-1} \bar{T} \cdot T),$$

where the integral is done over \mathbb{C}^{N^d} .

Wick's theorem allows to compute $\langle B(T, \bar{T}) \rangle$ in terms of correlations between pairs of components of T and \bar{T} .

In fact, since

$$\langle T_{i_1 \dots i_d} T_{j_1 \dots j_d} \rangle = \int \frac{dT d\bar{T}}{(2\pi)^{N^d}} T_{i_1 \dots i_d} T_{j_1 \dots j_d} \exp(-N^{d-1} \bar{T} \cdot T) = 0$$

and

$$\langle \bar{T}_{i_1 \dots i_d} \bar{T}_{j_1 \dots j_d} \rangle = \int \frac{dT d\bar{T}}{(2\pi)^{N^d}} \bar{T}_{i_1 \dots i_d} \bar{T}_{j_1 \dots j_d} \exp(-N^{d-1} \bar{T} \cdot T) = 0,$$

the linearity of the integral yields the following expansion for the mean value of the invariant of equation (5):

$$\langle B(T, \bar{T}) \rangle = \sum_{\sigma \in \mathcal{S}_p} \left[\sum_{\substack{i_h^{(l)}=1 \\ \forall h \in \mathbb{N}_d, \forall l \in \mathbb{N}_{2p}}}^N \delta_B \left(\prod_{(r,s) \in \sigma} \langle T_{i_1^{(2r-1)} \dots i_d^{(2r-1)}} \bar{T}_{i_1^{(2s)} \dots i_d^{(2s)}} \rangle \right) \right], \quad (6)$$

where \mathcal{S}_p denotes the set of all possible pairings of \mathbb{N}_{2p} (obviously corresponding to the set of pairings in \mathbb{N}_{2p} whose pairs consist in an odd and an even integer).

Each summand of the Wick expansion can be represented by a bipartite Feynman graph, in a similar way as in section 2, starting from the d -colored graph (B, b) representing the invariant $B(T, \bar{T})$: for each pair (r, s) of corresponding elements in the permutation σ , add a 0-labelled edge between the white vertex associated to $T_{i_1^{(2r-1)} \dots i_d^{(2r-1)}}$ and the black vertex associated to $\bar{T}_{i_1^{(2s)} \dots i_d^{(2s)}}$. Hence, in this case, the Feynman graphs are $(d+1)$ -colored graphs.

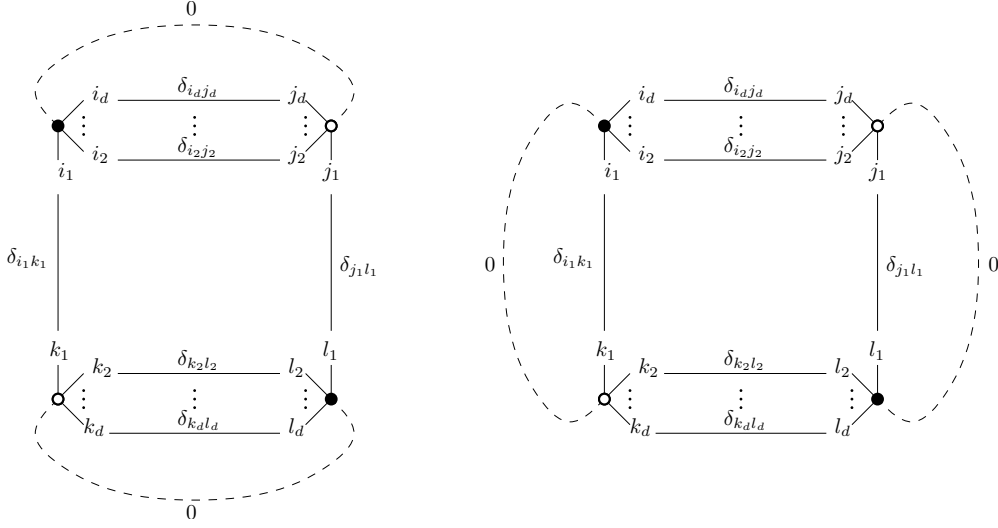


Figure 4: Feynman graphs associated to $\langle Q_{m,1}(T, \bar{T}) \rangle$.

For example, the correlation associated to the quartic invariant $Q_{m,1}$ yields, via Wick's theorem:

$$\begin{aligned}
\langle Q_{m,1}(T, \bar{T}) \rangle &= \sum_{\substack{i_h, j_h, l_h, k_h=1 \\ \forall h \in \mathbb{N}_d}}^N \delta_{i_1 k_1} \delta_{j_1 l_1} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \langle \bar{T}_{i_1 \dots i_d} T_{j_1 \dots j_d} \bar{T}_{l_1 \dots l_d} T_{k_1 \dots k_d} \rangle \\
&= \sum_{\substack{i_h, j_h, l_h, k_h=1 \\ \forall h \in \mathbb{N}_d}}^N \delta_{i_1 k_1} \delta_{j_1 l_1} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \\
&\quad [\langle \bar{T}_{i_1 \dots i_d} T_{j_1 \dots j_d} \rangle \cdot \langle \bar{T}_{l_1 \dots l_d} T_{k_1 \dots k_d} \rangle + \langle \bar{T}_{i_1 \dots i_d} T_{k_1 \dots k_d} \rangle \cdot \langle \bar{T}_{l_1 \dots l_d} T_{j_1 \dots j_d} \rangle]. \quad (7)
\end{aligned}$$

Fig. 4 shows the two $(d+1)$ -colored graphs obtained by adding 0-colored edges to the d -colored graph representing $Q_{m,1}(T, \bar{T})$, according to the Wick pairings (see Fig.3). More precisely the graphs pictured in Fig. 4 represent the summands appearing in equation (7).

The effective computation of the mean value $\langle B(T, \bar{T}) \rangle$ may be performed by recalling the following formula, concerning the correlation between a pair of components of T and \bar{T} :

$$\langle \bar{T}_{i_1 \dots i_d} T_{j_1 \dots j_d} \rangle = \int \frac{dT d\bar{T}}{(2\pi)^{Nd}} \bar{T}_{i_1 \dots i_d} T_{j_1 \dots j_d} \exp(-N^{d-1} \bar{T} \cdot T) = \frac{1}{N^{d-1}} \prod_{q=1}^d \delta_{i_q j_q}. \quad (8)$$

The Feynman graphs allow to easily visualize the final result of the above computation. In fact, via formulas (6) and (8), it is not difficult to check that:

$$\begin{aligned}
\langle B(T, \bar{T}) \rangle &= \sum_{\sigma \in \mathcal{S}_p} \left[\sum_{\substack{i_h^{(l)}=1 \\ \forall h \in \mathbb{N}_d, \forall l \in \mathbb{N}_{2p}}}^N \delta_B \left(\prod_{(r,s) \in \sigma} \frac{1}{N^{d-1}} \prod_{c=1}^d \delta_{i_c^{2r-1} i_c^{2s}} \right) \right] \\
&= \sum_{\sigma \in \mathcal{S}_p} \left[\frac{1}{N^{p(d-1)}} \sum_{\substack{i_h^{(l)}=1 \\ \forall h \in \mathbb{N}_d, \forall l \in \mathbb{N}_{2p}}}^N \delta_B \left(\prod_{(r,s) \in \sigma} \prod_{c=1}^d \delta_{i_c^{2r-1} i_c^{2s}} \right) \right] \\
&= \sum_{\sigma \in \mathcal{S}_p} \left[\frac{\prod_{c=1}^d N^{g_{0,c}^\sigma}}{N^{p(d-1)}} \right] = \sum_{\sigma \in \mathcal{S}_p} N^{-p(d-1) + \sum_{c=1}^d g_{0,c}^\sigma}, \quad (9)
\end{aligned}$$

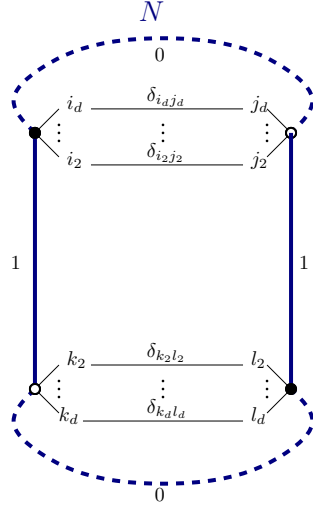


Figure 5: The marked $\{0,1\}$ -cycle corresponds to a factor N in the first summand of equation (11).

where $g_{0,c}^\sigma$ denotes the number of $\{0,c\}$ -colored cycles in the Feynman graph associated to the Wick pairing σ .

The third equality in the above equation, involving the number of $\{0,c\}$ -colored cycles ($c \in \mathbb{N}_d$) in the Feynman graphs, may be understood via the example of the computation of $\langle Q_{m,1}(T, \bar{T}) \rangle$ by means of the two graphs depicted in Fig. 4.

In fact, by applying equation (8) to equation (7), we have:

$$\begin{aligned} \langle Q_{m,1}(T, \bar{T}) \rangle &= \sum_{\substack{i_h, j_h, l_h, k_h=1 \\ \forall h \in \mathbb{N}_d}}^N \delta_{i_1 k_1} \delta_{j_1 l_1} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \\ &\quad \left[\frac{1}{N^{d-1}} \left(\prod_{q=1}^d \delta_{i_q j_q} \right) \frac{1}{N^{d-1}} \left(\prod_{q=1}^d \delta_{l_q k_q} \right) + \frac{1}{N^{d-1}} \left(\prod_{q=1}^d \delta_{i_q k_q} \right) \frac{1}{N^{d-1}} \left(\prod_{q=1}^d \delta_{l_q j_q} \right) \right]. \end{aligned} \quad (10)$$

In order to perform the computation, we note that, in the sum, the first index of each tensor variable plays a special rôle: then we write

$$\begin{aligned} \prod_{q=1}^d \delta_{i_q j_q} &= \delta_{i_1 j_1} \left(\prod_{q=2}^d \delta_{i_q j_q} \right), & \prod_{q=1}^d \delta_{l_q k_q} &= \delta_{l_1 k_1} \left(\prod_{q=2}^d \delta_{l_q k_q} \right), \\ \prod_{q=1}^d \delta_{i_q k_q} &= \delta_{i_1 k_1} \left(\prod_{q=2}^d \delta_{i_q k_q} \right), & \prod_{q=1}^d \delta_{l_q j_q} &= \delta_{l_1 j_1} \left(\prod_{q=2}^d \delta_{l_q j_q} \right). \end{aligned}$$

Replacing in equation (10), we obtain

$$\begin{aligned} \langle Q_{m,1}(T, \bar{T}) \rangle &= \sum_{\substack{i_h, j_h, l_h, k_h=1 \\ \forall h \in \mathbb{N}_d}}^N \left(\frac{\delta_{i_1 k_1} \delta_{j_1 l_1} \delta_{i_1 j_1} \delta_{l_1 k_1}}{N^{2(d-1)}} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \left(\prod_{q=2}^d \delta_{i_q j_q} \right) \left(\prod_{q=2}^d \delta_{l_q k_q} \right) + \right. \\ &\quad \left. + \frac{\delta_{i_1 k_1} \delta_{j_1 l_1} \delta_{i_1 k_1} \delta_{l_1 j_1}}{N^{2(d-1)}} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \left(\prod_{q=2}^d \delta_{i_q k_q} \right) \left(\prod_{q=2}^d \delta_{l_q j_q} \right) \right). \end{aligned} \quad (11)$$

Consider now the summation over the first indices i_1, j_1, k_1, l_1 . Paying attention to the first summand of equation (11) and hence to the leftmost Feynman graph of Figure 4, an easy calculation

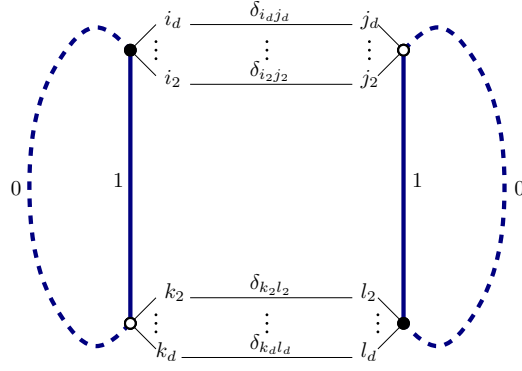


Figure 6: The two marked $\{0, 1\}$ -cycles correspond to two factors N in the second summand of equation (11).

shows that:

$$\begin{aligned}
& \sum_{k_1=1}^N \sum_{l_1=1}^N \sum_{j_1=1}^N \sum_{i_1=1}^N (\delta_{i_1 k_1} \delta_{j_1 l_1} \delta_{i_1 j_1} \delta_{l_1 k_1}) = \sum_{k_1=1}^N \sum_{l_1=1}^N \sum_{j_1=1}^N (\delta_{j_1 k_1} \delta_{j_1 l_1} \delta_{l_1 k_1}) = \\
& = \sum_{k_1=1}^N \sum_{l_1=1}^N (\delta_{l_1 k_1} \delta_{l_1 k_1}) = \sum_{k_1=1}^N \delta_{k_1 k_1} = N.
\end{aligned} \tag{12}$$

On the other hand, as previously pointed out, the Kronecker delta $\delta_{i_1 k_1}$ (resp. $\delta_{j_1 l_1}$) corresponds to the 1-colored edge gluing i_1 with k_1 (resp. j_1 with l_1) in the graph; moreover, the Kronecker delta $\delta_{i_1 j_1}$ (resp. $\delta_{l_1 k_1}$) comes from the Wick pairing of the T and \bar{T} variables and corresponds to the 0-colored edge between the uppermost (resp. lowermost) vertices of the graph.

Hence, in the first summand of equation (11) the factor N obtained in equation (12) corresponds to the (unique) $\{0, 1\}$ -cycle of the graph (see Fig. 5).

The analogous computation in the second summand of equation (11) gives:

$$\begin{aligned}
& \sum_{k_1=1}^N \sum_{l_1=1}^N \sum_{j_1=1}^N \sum_{i_1=1}^N (\delta_{i_1 k_1} \delta_{j_1 l_1} \delta_{i_1 k_1} \delta_{l_1 j_1}) = \sum_{k_1=1}^N \sum_{l_1=1}^N \sum_{j_1=1}^N (\delta_{k_1 k_1} \delta_{j_1 l_1} \delta_{l_1 j_1}) = \\
& = \sum_{k_1=1}^N \sum_{l_1=1}^N (\delta_{k_1 k_1} \delta_{l_1 l_1}) = \sum_{l_1=1}^N N \delta_{l_1 l_1} = N^2.
\end{aligned}$$

Here, the two factors N correspond to the two $\{0, 1\}$ -cycles in the corresponding Feynman graph (see Fig. 6).

From these computations we deduce that

$$\begin{aligned}
\langle Q_{m,1}(T, \bar{T}) \rangle &= \sum_{\substack{i_h, j_h, l_h, k_h=1 \\ \forall h \in \mathbb{N}_d}}^N \left(\frac{N}{N^{2(d-1)}} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \left(\prod_{q=2}^d \delta_{i_q j_q} \right) \left(\prod_{q=2}^d \delta_{l_q k_q} \right) \right) + \\
&+ \frac{N^2}{N^{2(d-1)}} \left(\prod_{c=2}^d \delta_{i_c j_c} \right) \left(\prod_{c=2}^d \delta_{l_c k_c} \right) \left(\prod_{q=2}^d \delta_{i_q k_q} \right) \left(\prod_{q=2}^d \delta_{l_q j_q} \right).
\end{aligned}$$

However, from the former computation we learn that we just need to count cycles of colors $\{0, i\}$, $\forall i \in \{1, \dots, d\}$ to deduce the number of factors of N . Applying this idea to our example, we end up with

$$\langle Q_{m,1}(T, \bar{T}) \rangle = \frac{N^{2d-1}}{N^{2(d-1)}} + \frac{N^{d+1}}{N^{2(d-1)}} = N + N^{3-d}.$$

Non-Gaussian integration of tensor models

We can define the corresponding non-Gaussian models in the same way than in the vector case we investigated in subsection 2.3. In the present case, the equivalent of the former $U(x)$ is assumed to be a polynomial of tensor invariants, i.e. we consider a potential $\sum_B \alpha_B B(T, \bar{T})$, where the α_B 's are formal variables and the sum over invariants B is finite (there is only a finite number of non-zero α_B).

Let us denote by $\mathcal{CG}(d)$ the set of bipartite d -colored graphs.

By using the graphical techniques exposed above, it is easy to see that the potential can be represented by the disjoint union of (a suitable number of) d -colored graphs $(B, b) \in \mathcal{CG}(d)$. From now on, with a slight abuse, we identify each tensor invariant with the d -colored graph representing it, so that the above sum can be thought as indexed on $\mathcal{CG}(d)$.

We define the *partition function* associated to the above potential as

$$\hat{Z}[N, \{\alpha_B\}_{B \in \mathcal{CG}(d)}] := \int \frac{dT d\bar{T}}{(2\pi)^{N^d}} \exp(-N^{d-1} \bar{T} \cdot T + \sum_B \alpha_B B(T, \bar{T})) .$$

A *tensor model* is a priori an element of $\mathbb{C}[[\{\alpha_B\}]]$, the set of formal series with ‘‘counting’’ variables $\{\alpha_B\}$.

Definition 5 A $(d+1)$ -dimensional colored tensor model is a formal partition function written as

$$\mathcal{Z}[N, \{\alpha_B\}_{B \in \mathcal{CG}(d)}] := \int_f \frac{dT d\bar{T}}{(2\pi)^{N^d}} \exp(-N^{d-1} \bar{T} \cdot T + \sum_B \alpha_B B(T, \bar{T})) , \quad (13)$$

where T belongs to $(\mathbb{C}^N)^{\otimes d}$ and \bar{T} to its dual.

The formal integral means, as in subsection 2.3, that $\exp(\sum_B \alpha_B B(T, \bar{T}))$ is expanded in power series and the integration is commuted with the sum. More precisely

$$\mathcal{Z}[N, \{\alpha_B\}_{B \in \mathcal{CG}(d)}] = \sum_{n \geq 0} \frac{1}{n!} \int \frac{dT d\bar{T}}{(2\pi)^{N^d}} \left(\sum_B \alpha_B B(T, \bar{T}) \right)^n \exp(-N^{d-1} \bar{T} \cdot T).$$

As a consequence, $\mathcal{Z}[N, \{\alpha_B\}_{B \in \mathcal{CG}(d)}] \in \mathbb{C}[[\{\alpha_B\}_{B \in \mathcal{CG}(d)}]]$.

Once again $\hat{Z} \neq \mathcal{Z}$, as indeed the formal series \mathcal{Z} is not a priori convergent.

Therefore, in order to evaluate $\mathcal{Z}[N, \{\alpha_B\}]$, it is necessary to compute the Gaussian mean values of the powers of $\sum_B \alpha_B B(T, \bar{T})$. Indeed, expanding $\sum_B \alpha_B B(T, \bar{T})$ will lead to compute quantities of the form $\langle \prod_i B_i(T, \bar{T})^{q_i} \rangle$ for some $\{q_i\}$. Again, each product $\prod_i B_i(T, \bar{T})^{q_i}$ can be represented by the disjoint union of (a suitable number of) copies of the d -colored graphs B_i . Then, $\langle \prod_i B_i(T, \bar{T})^{q_i} \rangle$ can be obtained by looking at all the $(d+1)$ -colored bipartite graphs that can be formed by adding edges of color 0 on the (disconnected) d -colored graph representing the product $\prod_i B_i(T, \bar{T})^{q_i}$.

In the next section, we will add constraints on the value of α_B in order to obtain that the value of a term indexed by a given Feynman graph Γ precisely encodes the value of the Einstein-Hilbert action discretized on the pseudo-manifold represented by Γ . The value of a term indexed by a Feynman graph Γ is often called its *weight* $W(\Gamma)$.

Note that, in the case of the Gaussian mean value of a single invariant of tensors, the previous formula (9) proves that the weight of each Feynman graph $\Gamma^{(\sigma)}$ obtained by the Wick expansion is

$$W(\Gamma^{(\sigma)}) = N^{-p(d-1) + \sum_{c=1}^d g_{0,c}^\sigma} . \quad (14)$$

3.3 $1/N$ expansion of Tensor Models

From a physical point of view, tensor models are used as tentative partition functions for $d \geq 2$ dimensional discrete QG in the Euclidean setting. This idea relies on the discretization of the Einstein-Hilbert action on d -manifolds endowed with a PL triangulation. This approach is called Regge calculus [45]. When performed on equilateral triangulations, the curvature term is encoded in the number of $(d - 2)$ -simplices, while the volume (cosmological constant) term is encoded in the number of d -simplices.

More information on these facts can be found in [4], [45] and references therein. In the path integral framework, quantizing gravity may be thought of as summing over all Riemannian manifolds with summands weighted by the Einstein-Hilbert action. Tensor models are an attempt to do so in a combinatorial/PL setting.

Let us explain why we consider not only manifolds but also pseudo-manifolds. In the approach of tensor models we sum over weighted (pseudo)-manifolds by summing over Feynman graphs representing them. We do not know of a way to quantize geometry and topology using a formal integrals approach without pseudo-manifolds contributing to the physical processes. Of course one could discard the contributions of pseudo-manifolds by hand, but by doing so, one would violate unitarity⁶. We could also consider other models that are not representable with the help of formal integrals, but this would deprive us of the tools and concepts coming with formal integrals and quantum field theories. Moreover, there are no strong physical arguments against the presence of pseudo-manifolds in the models⁷, at least as long as they do not contribute much to the physical processes (or more precisely, to the classical limit of the physical processes).

From a mathematical standpoint, the study of colored tensor models reduces to the study of generating series of PL triangulations counting the number of top simplices and $(d - 2)$ -simplices.

We consider now a $(d + 1)$ -dimensional colored tensor model corresponding to a particular choice of the α_B 's; with regard to the related notations, we point out that by an *automorphism* of colored graphs, we mean a graph automorphism that preserves colors⁸. Moreover, we denote by $|\text{Aut}(B)|$ the order of the automorphism group of a colored graph (B, b) .

In [5] the following theorem is proved.

Theorem 6 *The $(d + 1)$ -dimensional colored tensor model $\mathcal{Z}[N, \{t_B\}_{B \in \mathcal{CG}(d)}] := \mathcal{Z}[N, \{\alpha_B\}_{B \in \mathcal{CG}(d)}]$ with*

$$\alpha_B = N^{d-1-\frac{2}{(d-2)!}\omega_G(B)} \frac{t_B}{|\text{Aut}(B)|}$$

is a (not convergent) generating series of bipartite $(d + 1)$ -colored graphs whose $\hat{0}$ -residues B are counted by (the exponents of) the formal variables t_B .

The free energy $\frac{1}{N^d} \log \mathcal{Z}[N, \{t_B\}]$ is also a formal series in N^{-1} ; more precisely,

$$\frac{1}{N^d} \log \mathcal{Z}[N, \{t_B\}] = \sum_{\omega_G \geq 0} N^{-\frac{2}{(d-1)!}\omega_G} F_{\omega_G}[\{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]] \quad (15)$$

where the coefficients $F_{\omega_G}[\{t_B\}]$ are convergent generating series (i.e. generating functions) of connected bipartite $(d + 1)$ -colored graphs with fixed G -degree ω_G .

More details on the notions of generating series and functions can be found in [27].

The non-trivial part of the theorem is that the quantity $\frac{1}{N^d} \log \mathcal{Z}[N, \{t_B\}] \in \mathbb{C}[[N^{-1}, \{t_B\}]]$ is a formal series in solely N^{-1} and the t_B 's. Apart from arguments related to convergence problems, the

⁶Not mentioning that it would also be a tedious computational problem in high dimensions.

⁷Indeed physicists have no way to tell if our space is actually a manifold. Physicists just know that up to some level of precision, that is limited by the precision of experiments, our space looks locally like a manifold at small energy scales.

⁸We warn the reader that the concept of automorphism of colored graphs presented here is different from that usually considered in crystallization theory (see [15]).

proof relies on the weight of a Feynman graph associated to a single tensor invariant (formula (14)): in fact, with the chosen value of α_B , the main steps consist of the application of the combinatorial formula (16) (which is already known in the literature for the case of bipartite graphs: see [5]) both to the d -colored graphs B and to the $(d+1)$ -colored graphs Γ having B as $\hat{0}$ -residues.

Remark 1 An analogous result can be shown for tensor models involving real tensor variables $T \in (\mathbb{R}^N)^{\otimes d}$, but taking into account non-bipartite colored graphs, too. This case has not been worked out in detail in the literature, nevertheless these models appear in the study of toy models for physicists AdS/CFT correspondence: see [49].

The choice to fix $\alpha_B = N^{d-1-\frac{2}{(d-2)!}\omega_G(B)} \frac{t_B}{|\text{Aut}(B)|}$ comes from the fact that the G-degree appears naturally as the quantity that allows to enforce the weights of the $(d+1)$ -colored graphs to encode the discretized Einstein-Hilbert action on equilateral triangulations. However, this is not enough: it is also necessary to set $t_B = g^{p(B)}$, where $p(B)$ is the half number of vertices of B and g is a parameter that depends on the Newton gravitational constant and the cosmological constant. An explicit relation is given for instance in [4]. Yet it is convenient to use the coupling constants t_B as parameters, since indeed it allows one to keep track of the $\hat{0}$ -residues structures of the different Feynman graphs of the theory.

It is easy to show that all graphs of G-degree $\omega_G = 0$ are spheres (this was claimed in [4]). In a more general setting, it is important to understand which are the manifolds and pseudo-manifolds that can be represented by colored graphs of a given degree and their possible geometrical meaning. Indeed, in the case of 3- and 4-dimensional tensor models, these graphs represent the possible states of the physical quantum space.

In the next section, we study general properties of the G-degree for colored graphs and GEMS. Then, in sections 5 and 6, we focus on what can be said respectively in dimensions three and four.

4 General properties of G-degree

As regards dimension 2, the definition of G-degree ensures that $\omega_G(\Gamma)$ equals the genus (resp. half the genus) of the surface $|K(\Gamma)|$ for any bipartite (resp. non-bipartite) 3-colored graph (Γ, γ) . Hence all properties of the G-degree for $d = 2$ are well-known.

In this section, we will take into account the higher dimensions, i.e. $d \geq 3$.

First of all, we note that it is easy to compute the G-degree directly from the combinatorial properties of the edge-colored graph, without restrictions related to bipartition or non-bipartition.

Proposition 7 *If (Γ, γ) is a $(d+1)$ -colored graph of order $2p$, then*

$$\omega_G(\Gamma) = \frac{(d-1)!}{2} \left(d + \frac{d}{2}(d-1)p - \sum_{r,s \in \Delta_d} g_{rs} \right). \quad (16)$$

As a consequence, the G-degree of any $(d+1)$ -colored graph ($d \geq 3$) is a non-negative integer multiple of $\frac{(d-1)!}{2}$.

Proof. Let $\varepsilon^{(i)}$ be a cyclic permutation of Δ_d ; then, by Theorem 5 and by definition, $\rho_{\varepsilon^{(i)}}(\Gamma)$ satisfies the following relation:

$$2 - 2\rho_{\varepsilon^{(i)}}(\Gamma) = \sum_{j \in \mathbb{Z}_{d+1}} g_{\varepsilon_j^{(i)} \varepsilon_{j+1}^{(i)}} + (1-d)p.$$

Summing over all cyclic permutations of Δ_d yields:

$$\sum_{i=1}^{\frac{d!}{2}} [2 - 2\rho_{\varepsilon^{(i)}}(\Gamma)] = (d-1)! \sum_{r,s \in \Delta_d} g_{rs} + \frac{d!}{2} \cdot (1-d)p,$$

from which the statement follows. □

Remark 2 It would be interesting to know whether all non-negative integer multiples of $\frac{(d-1)!}{2}$ are realized as G-degree of $(d+1)$ -colored graphs⁹ or if something may be stated about certain multiples. As a partial result note that, if d is even, the G-degrees of two $(d+1)$ -colored graphs obtained from each other by dipole moves (and hence representing, in the GEM case, the same PL d -manifold) differ by an even multiple of $\frac{(d-1)!}{2}$, i.e. by a multiple of $(d-1)!$. In fact, as proved in [37], if (Γ, γ) is a $(d+1)$ -colored graph and (Γ', γ') is obtained from Γ by eliminating an r -dipole ($1 \leq r \leq d$), then:

$$\omega_G(\Gamma) = \frac{(d-1)!}{2}(r-1)(d-r) + \omega_G(\Gamma').$$

By the definitions of G-degree and regular genus of a $(d+1)$ -colored graph Γ , the following inequality obviously holds:

$$\omega_G(\Gamma) \geq \frac{d!}{2} \cdot \rho(\Gamma). \quad (17)$$

The following Proposition yields a lower bound for the G-degree of a non-bipartite graph, where for any $x \in \mathbb{Q}$, we denote by $\lceil x \rceil$ the ceiling of x (i.e. the least integer that is greater than or equal to x). For this type of results see also [6].

Proposition 8 *No non-bipartite $(d+1)$ -colored graph (Γ, γ) exists with $\omega_G(\Gamma) < \lceil \frac{d!}{4} \rceil$.*

Proof. Suppose (Γ, γ) is a non-bipartite $(d+1)$ -colored graph, then by Theorem 5 it cannot be regularly embedded into an orientable surface and hence $\rho_\varepsilon(\Gamma) \geq \frac{1}{2}$ for each cyclic permutation ε of Δ_d . As a consequence, by inequality (17), $\omega_G(\Gamma) \geq \frac{d!}{4}$. Now the claim easily follows, since by Proposition 7 the G-degree must be an integer. □

The well-known characterization of PL spheres as the only PL manifolds with regular genus zero allows to prove the following proposition.

Proposition 9 *If (Γ, γ) is a bipartite $(d+1)$ -colored graph such that $\omega_G(\Gamma) < \frac{d!}{2}$, then $|K(\Gamma)| \cong_{PL} \mathbb{S}^d$.*

Proof. Let (Γ, γ) be a bipartite $(d+1)$ -colored graph; if $\omega_G(\Gamma) < \frac{d!}{2}$, then, by inequality (17), there exists a cyclic permutation ε of Δ_d such that $\rho_\varepsilon(\Gamma) = 0$.

We will prove by induction on $d \geq 2$ that $\rho_\varepsilon(\Gamma) = 0$ implies that $|K(\Gamma)|$ is a PL d -sphere.

If $d = 2$ the statement is trivially true, since $\rho_\varepsilon(\Gamma)$ coincides with the genus of the surface $|K(\Gamma)|$.

Suppose now $d > 2$; given $i \in \mathbb{Z}_{d+1}$, let us denote by ε_i the cyclic permutation $(\varepsilon_0, \dots, \varepsilon_{i-1}, \varepsilon_{i+1}, \dots, \varepsilon_d)$. Each connected component Ξ of Γ_{ε_i} is a d -colored graph and it is not difficult to prove that $\rho_{\varepsilon_i}(\Xi) \leq \rho_\varepsilon(\Gamma)$ (see [20, Lemma 4.1]).

Therefore, by induction, $|K(\Xi)|$ is a PL $(d-1)$ -sphere; since the result obviously holds for any d -residue of Γ , then, by Proposition 3, Γ is a gem of a PL d -manifold. Now, the main theorem of [25] ensures that a (bipartite) gem of regular genus zero always represents a PL sphere. □

Remark 3 Note that, as a consequence of the above Proposition 9, the coefficients of the terms of powers greater than $-d$ in the $\frac{1}{N}$ expansion of the free energy (Theorem 6) count only colored graphs representing PL spheres. Moreover, as a consequence of Proposition 8, the same situation occurs in the first $\lceil d/2 \rceil$ coefficients (i.e. the coefficients of N^{-k} with $0 \leq k < \lceil d/2 \rceil$) of the real tensors $\frac{1}{N}$ expansion, that involves also non-bipartite graphs.

⁹In a work appeared in the ArXiv after the submission of the present paper, it is proved that, under suitable hypotheses, not all multiples of $\frac{(d-1)!}{2}$ are actually allowed: see [17].

Let now (Γ, γ) and (Γ', γ') be two $(d + 1)$ -colored graphs. If $v \in V(\Gamma)$ and $v' \in V(\Gamma')$, the *graph connected sum* of Γ and Γ' with respect to the vertices v, v' (denoted by $(\Gamma \#_{vv'} \Gamma', \gamma \# \gamma')$) is defined as the graph obtained from Γ and Γ' by deleting v and v' and welding the “hanging” edges of the same color. A basic result in crystallization theory ensures that, if $(\Gamma, \gamma), (\Gamma', \gamma')$ are assumed to be GEMs of the PL d -manifolds M, M' respectively, then $(\Gamma \#_{vv'} \Gamma', \gamma \# \gamma')$, for each pair (v, v') , is a GEM of a connected sum of M and M' .¹⁰

It is not difficult to check that the G-degree of edge-colored graphs is additive with respect to graph connected sum.

Proposition 10 *Let (Γ, γ) and (Γ', γ') be two $(d + 1)$ -colored graphs. Then:*

$$\omega_G(\Gamma \#_{vv'} \Gamma') = \omega_G(\Gamma) + \omega_G(\Gamma')$$

for each $v \in V(\Gamma)$ and $v' \in V(\Gamma')$.

Proof. It is sufficient to notice that, when erasing the vertices and reconnecting the edges of the edge-colored graphs, one also performs a connected sum of the embedding of them for each choice of a cyclic permutation. Moreover the genus ρ of a surface is additive with respect to connected sum (*i.e.*, if Σ and Σ' are surfaces, then $\rho(\Sigma \# \Sigma') = \rho(\Sigma) + \rho(\Sigma')$). The conclusion follows from the definition itself of the G-degree. □

Let us now introduce a further PL invariant based on the G-degree of colored graphs.

Definition 6 The *Gurau degree* (or *G-degree*, for short) of a PL d -dimensional manifold M^d is the integer defined as

$$\mathcal{D}_G(M^d) = \min\{\omega_G(\Gamma) \mid (\Gamma, \gamma) \text{ is a GEM of } M^d\}.$$

Remark 4 Note that, as it happens for regular genus and gem-complexity, the G-degree of a PL d -manifold is *always* realized by crystallizations:

$$\mathcal{D}_G(M^d) = \min\{\omega_G(\Gamma) \mid (\Gamma, \gamma) \text{ is a crystallization of } M^d\}.$$

In fact, as it is easy to check, $\omega_G(\Gamma)$ is not affected by 1-dipole elimination. Hence, the proof of Theorem 4 proves the assertion.

Definition 6 can be easily generalized to any d -pseudomanifold representable by $(d + 1)$ -colored graphs. A rôle analogous to crystallizations is played in that context by *contracted* $(d + 1)$ -colored graphs (Γ, γ) , for which either Γ_i is connected or none of its connected components represents \mathbb{S}^{d-1} ; see [16], where, in particular, the case of the so called “singular manifolds” is taken into account¹¹.

The following statement directly follows from inequality (17), together with known properties of the regular genus of PL-manifolds.

Proposition 11 *For each PL d -manifold M^d ,*

$$\mathcal{D}_G(M^d) \geq \frac{d!}{2} \cdot \mathcal{G}(M^d) \geq \frac{d!}{2} \cdot rk(\pi_1(M^d)) \geq \frac{d!}{2} \cdot \beta_1(M^d).$$

¹⁰Note that the connected sum of two given d -manifolds is, in general, not uniquely defined; however, if two distinct connected sums of M and M' exist, they both may be represented via graph connected sum of Γ and Γ' , by a suitable choice of v, v' .

¹¹The definition of singular manifold is given at the beginning of subsection 6.1.

Proof. The first inequality is a direct consequence of inequality (17); as regards the other ones, it is sufficient to recall that inequalities $\beta_1(M^d) \leq rk(\pi_1(M^d)) \leq \rho(\Gamma)$ hold for each gem Γ of M^d . \square

Let \mathcal{M}_d denote the set of all *PL* d -dimensional manifolds.

The additivity of the G-degree with respect to graph connected sum has the following consequence.

Corollary 12 \mathcal{D}_G induces a filtration of the monoid $(\mathcal{M}_d, \#)$.

Proof. First, notice that $\mathcal{D}_G(M^d) \geq 0$ trivially holds for each $M^d \in \mathcal{M}_d$. Moreover, as a direct consequence of Proposition 10, we have:

$$\mathcal{D}_G(M \# M') \leq \mathcal{D}_G(M) + \mathcal{D}_G(M').$$

Let us now define, for each non-negative integer \bar{S} ,

$$\mathcal{M}_{d, \bar{S}} = \{M^d \in \mathcal{M}_d \mid \mathcal{D}_G(M^d) \leq \bar{S}\}.$$

We obviously have $\mathcal{M}_{d, \bar{S}} \subset \mathcal{M}_{d, \bar{S}'}$ if $\bar{S} \leq \bar{S}'$, while we have that

$$\mathcal{M}_{d, \bar{S}} \# \mathcal{M}_{d, \bar{S}'} \subseteq \mathcal{M}_{d, \bar{S} + \bar{S}'}. \quad \square$$

Let us now face the finiteness problem about the G-degree. First, we recall that in [37, Lemma 4.2] Gurau and Ryan obtain a formula that allows to compute the G-degree of a bipartite $(d+1)$ -colored graph Γ by making use of the G-degrees of its d -residues. Actually, it is easy to see that the proof of the result does not depend on the bipartiteness of the graph; therefore we can state the following lemma.

Lemma 13 For each $(d+1)$ -colored graph Γ of order $2p$,

$$\omega_G(\Gamma) = \frac{(d-1)!}{2} \left(p + d - \sum_{i \in \Delta_d} g_i \right) + \sum_{i \in \Delta_d} \omega_G(\Gamma_i),$$

where, for each $i \in \Delta_d$, $\omega_G(\Gamma_i)$ denotes the sum of the G-degrees of the connected components of Γ_i . \square

Theorem 14 For each fixed non-negative integer \bar{S} , only a finite number of $(d+1)$ -colored graphs (Γ, γ) with $g_i = 1$ for each $i \in \Delta_d$ ¹² exists, with $\omega_G(\Gamma) = \bar{S}$.

Hence, the filtration induced by the G-degree on \mathcal{M}_d is finite-to-one.

Proof. If (Γ, γ) is assumed to have $g_i = 1$ for each $i \in \Delta_d$, Lemma 13 directly ensures

$$\omega_G(\Gamma) = \frac{(d-1)!}{2} (p-1) + \sum_{i \in \Delta_d} \omega_G(\Gamma_i);$$

hence, $\omega_G(\Gamma) = \bar{S}$ implies $p \leq 1 + \frac{2\bar{S}}{(d-1)!}$.

The finiteness property of G-degree for such a class of graphs is now easily proved as a consequence of the fact that, for each fixed $p \geq 1$, only a finite number of $(d+1)$ -colored graphs (Γ, γ) exists, with $\#V(\Gamma) = 2p$.

Moreover, by Theorem 4, each PL d -manifold M^d admits a crystallization, i.e. a $(d+1)$ -colored graph representing M^d and satisfying the hypothesis $g_i = 1$ for each $i \in \Delta_d$. In virtue of Remark 4,

¹² $(d+1)$ -colored graphs with connected d -residues are called *combinatorial core graphs* in [37, Definition 5.1].

this proves that G-degree is finite to one on \mathcal{M}_d . □

The above theorem implies that only a finite number of PL manifolds is represented by the colored graphs appearing in each term of the $\frac{1}{N}$ expansion.

It is to be noted that the Gura degree shares with the gem-complexity the finiteness property stated in the above theorem, while for the regular genus the same property does not hold for $d = 3$ and it is unknown in higher dimension.

Actually, though the definition of Gura degree is strictly connected with the regular genus, nevertheless, as we will see in Subsections 5.1 and 6.1, the Gura degree of a 3- or 4-manifold turns out to be closer to gem-complexity.

5 G-degree: the 3-dimensional case

5.1 G-degree for 4-colored graphs and 3-manifolds

Proposition 15 *If (Γ, γ) is an order $2p$ 4-colored graph, then*

$$\omega_G(\Gamma) = p - 1 - \sum_{i \in \Delta_3} (g_i - 1) + \chi(K(\Gamma)) \quad (18)$$

where $\chi(K(\Gamma))$ is the Euler characteristic of the 3-dimensional pseudomanifold $K(\Gamma)$.

Furthermore, if (Γ, γ) is a crystallization of a 3-manifold, then

$$\omega_G(\Gamma) = p - 1.$$

Proof. The duality between the 4-colored graph Γ and the pseudocomplex $K(\Gamma)$ allows to compute the Euler characteristic of $K(\Gamma)$ by means of the number of the h -residues of Γ ($h = 0, 1, 2, 3$):

$$\chi(K(\Gamma)) = \sum_{i \in \Delta_3} g_i - \sum_{i, j \in \Delta_3} g_{ij} + \frac{4 \cdot 2p}{2} - 2p$$

Hence

$$\sum_{i, j \in \Delta_3} g_{ij} = \sum_{i \in \Delta_3} g_i + 2p - \chi(K(\Gamma))$$

By substituting the value of $\sum_{i, j \in \Delta_3} g_{ij}$ in the formula of Proposition 7 for $d = 3$, we have:

$$\omega_G(\Gamma) = 3 + 3p - \sum_{i \in \Delta_3} g_i - 2p + \chi(K(\Gamma)) = p - 1 - \sum_{i \in \Delta_3} (g_i - 1) + \chi(K(\Gamma)) \quad (19)$$

The second part of the statement follows easily from the fact that $K(\Gamma)$ is a (closed) 3-manifold iff its Euler characteristic is zero ([47]) and from the assumption that Γ is a crystallization, and hence $g_i = 1$ for each $i \in \Delta_3$. □

Remark 5 Easy arguments of geometric topology allow to prove that, for each 4-colored graph Γ :

$$\chi(K(\Gamma)) = \sum_{i \in \Delta_3} \omega_G(\Gamma_i).$$

Hence, formula (18) can be also written as

$$\omega_G(\Gamma) = \left(p + 3 - \sum_{i \in \Delta_3} g_i \right) + \sum_{i \in \Delta_3} \omega_G(\Gamma_i),$$

which is exactly the formula of Lemma 13 in the particular case $d = 3$.

The following theorem proves that, in the case of 3-manifolds, G-degree and gem-complexity actually coincide.

Theorem 16 *For each 3-manifold M^3*

$$\mathcal{D}_G(M^3) = k(M^3)$$

Proof. Note that $\mathcal{D}_G(M^3)$ has to be realized by a crystallization of M^3 , as pointed out in Remark 4. Then, the thesis follows from the previous result, together with the definition of gem-complexity. \square

The coincidence between G-degree and gem-complexity of a 3-manifold established by Theorem 16 allows to obtain classification results according to the G-degree from the existing catalogues of crystallizations of orientable (resp. non-orientable) 3-manifolds up to gem-complexity 15, [10], [11] (resp. up to gem-complexity 14, [1]). The catalogues can be found at the WEB page <http://cdm.unimo.it/home/matematica/casali.mariarita/CATALOGUES.htm>.

Remark 6 It must be pointed out that the above catalogues could fail to present the 4-colored graphs of minimal order (and so also of minimal G-degree) only in the case of manifolds containing handles, i.e. manifolds that can be decomposed into a connected sum with $\mathbb{S}^1 \times \mathbb{S}^2$ or $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ (the orientable or non-orientable \mathbb{S}^2 -bundle over \mathbb{S}^1).

Nevertheless, for low values of the G-degree, the catalogues yield the classification for any 4-colored graph (Γ, γ) representing a 3-manifold M^3 as follows:¹³

- $\omega_G(\Gamma) \leq 2 \quad \Rightarrow \quad M^3 \cong \mathbb{S}^3$
- $\omega_G(\Gamma) \leq 5 \quad \Rightarrow \quad M^3 \in \{\mathbb{S}^3, \mathbb{S}^2 \times \mathbb{S}^1, \mathbb{S}^1 \tilde{\times} \mathbb{S}^2, L(2, 1), L(3, 1)\}.$

The above catalogues also allow to obtain information about the geometry of 3-manifolds, in Thurston's sense, up to G-degree 14.

Proposition 17 *Let M^3 be a prime orientable 3-manifold; then,*

- $\mathcal{D}_G(M^3) \leq 10 \Rightarrow$ either $M^3 \cong \mathbb{S}^1 \times \mathbb{S}^2$ or M^3 has spherical geometry.
- $\mathcal{D}_G(M^3) \leq 13 \Rightarrow M^3$ is not hyperbolic (in particular, $\mathcal{D}_G(M^3) \leq 11 \Rightarrow$ either $M^3 \cong \mathbb{S}^1 \times \mathbb{S}^2$ or M^3 is flat or it is spherical).
- If M^3 is the Matveev-Fomenko-Weeks manifold¹⁴, then $\mathcal{D}_G(M^3) = 14$.

5.2 Relationship between G-degree and regular genus, for $d = 3$

If $d = 3$, inequality (17) gives

$$\omega_G(\Gamma) \geq 3 \cdot \rho(\Gamma)$$

for any 4-colored graph Γ , and Proposition 11 yields

$$\mathcal{D}_G(M^3) \geq 3 \cdot \mathcal{G}(M^3)$$

for any 3-manifold M^3 .

In the following proposition we investigate the gap between the two quantities $3\rho(\Gamma)$ and $\omega_G(\Gamma)$, for any crystallization¹⁵ Γ of a 3-manifold.

¹³Other results can also be obtained from Proposition 11 (case $d = 3$) and known classification results in terms of regular/Heegaard genus. For example, if $\omega_G(\Gamma) \leq 8$ and Γ is bipartite, then M^3 is a 2-fold branched covering of \mathbb{S}^3 .

¹⁴We recall that the Matveev-Fomenko-Weeks manifold is the (closed) hyperbolic 3-manifold with smallest volume ([30])

¹⁵Since both the regular genus and G-degree are not affected by 1-dipole elimination, the restriction to crystallizations does not cause loss of generality (see Remark 4).

Proposition 18 *If (Γ, γ) is an order $2p$ crystallization of a 3-manifold M^3 , then*

$$\omega_G(\Gamma) - 3\rho(\Gamma) = p + 2 - 3 \cdot \min\{g_{ij} / i, j \in \Delta_3\}.$$

Proof. From Proposition 15, we have $\omega_G(\Gamma) = p - 1$; on the other hand, [31, Corollary 16] proves that $\rho(\Gamma) = \min\{g_{ij} - 1 / i, j \in \Delta_3\}$ holds for any crystallization Γ of a 3-dimensional manifold. Hence, the statement directly follows. \square

Let us now take into account the case of equality between the two quantities.

Proposition 19 *If (Γ, γ) is an order $2p$ crystallization of a 3-manifold M^3 , then:*

$$\omega_G(\Gamma) = 3\rho(\Gamma) \iff g_{ij} = \frac{p+2}{3} \quad \forall i, j \in \Delta_3.$$

Proof. By Proposition 18, $\omega_G(\Gamma) = 3\rho(\Gamma)$ if and only if $\min\{g_{ij} / i, j \in \Delta_3\} = \frac{p+2}{3}$. On the other hand, for any crystallization of a 3-manifold the relation $g_{ij} + g_{jk} + g_{ki} = p + 2$ holds $\forall i, j, k \in \Delta_3$ (see [31, Corollary 16]); hence, the existence of a pair $\bar{i}, \bar{j} \in \Delta_3$ such that $g_{\bar{i}\bar{j}} = \frac{p+2}{3}$ implies $g_{ij} = \frac{p+2}{3} \forall i, j \in \Delta_3$. The statement now directly follows. \square

In order to discuss the case of equality between $\mathcal{D}_G(M^3)$ and $3 \cdot \mathcal{G}(M^3)$, let us introduce a class of 3-manifolds that has already been studied in [8].

Definition 7 A 3-manifold M^3 is called *minimal* if $k(M^3) = 3\mathcal{G}(M^3)$ or, equivalently, if $k(M^3) = 3\mathcal{H}(M^3)$, where $\mathcal{H}(M^3)$ is the Heegaard genus of M^3 .

Corollary 20

- (a) *Let M^3 be a minimal 3-manifold and (Γ, γ) a crystallization of M^3 realizing gem-complexity; then $\omega_G(\Gamma) = 3\rho(\Gamma)$.*
- (b) *If M^3 is a 3-manifold M^3 , then*

$$\mathcal{D}_G(M^3) = 3 \cdot \mathcal{G}(M^3) \iff M^3 \text{ is a minimal 3-manifold.}^{16}$$

Proof. As proved in [8, Proposition 5], if M^3 is a minimal 3-manifold and Γ is a crystallization of M^3 realizing gem-complexity (i.e. $\#V(\Gamma) = 2(k(M^3) + 1)$), then $\rho_\varepsilon(\Gamma) = \mathcal{G}(M^3)$ for any cyclic permutation ε of Δ_3 . Statement (a) now easily follows.

In order to prove statement (b) it is sufficient to note that, by Theorem 16, condition $\mathcal{D}_G(M^3) = 3 \cdot \mathcal{G}(M^3)$ is equivalent to condition $k(M^3) = 3 \cdot \mathcal{G}(M^3)$, which characterizes minimal 3-manifolds. \square

6 G-degree: the 4-dimensional case

6.1 G-degree for 5-colored graphs and 4-manifolds

With regard to the 4-dimensional case, we restrict our attention to 5-colored graphs representing singular 4-manifolds. We recall that a *singular (PL) d -manifold* ($d > 1$) is a compact connected

¹⁶It has been proved that, if $\mathcal{G}(M^3) \leq 4$ is assumed (or $k(M^3) \leq 14$), then the minimal 3-manifolds are exactly $L(2, 1)$, $\mathbb{S}^1 \times \mathbb{S}^2$, $\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ and their connected sums; moreover, the same characterization is conjectured to hold in the general case, too (see [8]).

d -dimensional polyhedron admitting a simplicial triangulation where the links of vertices are closed connected $(d - 1)$ -manifolds, while the links of all h -simplices with $h > 0$ are PL $(d - h - 1)$ -spheres.

By the duality between colored graphs and their associated pseudocomplexes, it is not difficult to see that, given a $(d + 1)$ -colored graph (Γ, γ) , then $|K(\Gamma)|$ is a singular d -manifold iff each $(d - 1)$ -residue of Γ represents the $(d - 2)$ -sphere¹⁷. In particular, if $d = 4$, then (Γ, γ) represents a singular 4-manifold iff all its 3-residues have genus zero.

The following lemma will be useful in order to establish relations concerning the G-degree of 5-colored graphs representing singular 4-manifolds.¹⁸

Lemma 21 *Let (Γ, γ) be an order $2p$ 5-colored graph representing a singular 4-manifold, then*

$$2 \sum_{r,s,t \in \Delta_4} g_{rst} = 3 \sum_{r,s \in \Delta_4} g_{rs} - 10p \quad (20)$$

Proof. For each $i, j \in \Delta_4$ let $f_k(i, j)$ denote the number of k -simplices of $K(\Gamma)$ containing an edge whose endpoints are labeled by i and j .

Given an edge e of $K(\Gamma)$, let us consider the regular neighborhood of e made by all d -simplexes of the first barycentric subdivision of $K(\Gamma)$ having an edge contained in e : the boundary of this neighborhood is called the *disjoint link*, $lkd(e, K(\Gamma))$, of e .

Since $K(\Gamma)$ is a singular manifold, the disjoint link of any edge e is a 2-sphere; hence it is not difficult to see that

$$2 = \chi(lkd(e, K(\Gamma))) = f_2(e) - f_3(e) + f_4(e)$$

where $f_k(e)$ is the number of k -simplices of $K(\Gamma)$ containing e .

By summing over all edges of $K(\Gamma)$ having endpoints labeled by i and j , we obtain

$$2 g_{rst} = f_2(i, j) - f_3(i, j) + f_4(i, j) = g_{rt} + g_{rs} + g_{st} - 3p + 2p = g_{rt} + g_{rs} + g_{st} - p,$$

where $\{r, s, t\} = \Delta_4 - \{i, j\}$.

By summing again over all choices of i and j , we have

$$2 \sum_{r,s,t \in \Delta_4} g_{rst} = 3 \sum_{r,s \in \Delta_4} g_{rs} - 10p.$$

□

Theorem 22 *If (Γ, γ) is an order $2p$ 5-colored graph representing a singular 4-manifold, then the following relations hold:*

$$\omega_G(\Gamma) = 3 \left(6(p - 1) - \sum_{r,s \in \Delta_4} (g_{rs} - 1) \right);$$

$$\omega_G(\Gamma) = 8(p - 1) - 2 \sum_{r,s,t \in \Delta_4} (g_{rst} - 1);$$

$$\omega_G(\Gamma) = 6 \left((p - 1) - \sum_{i \in \Delta_4} (g_i - 1) + (\chi(K(\Gamma)) - 2) \right).$$

Proof. The first relation comes directly from Proposition 7 (and hence it holds for any 5-colored graph, with no restriction on the represented pseudomanifold).

In order to prove the second relation, it is sufficient to apply Lemma 21 to the first one.

With regard to the third relation, let us consider the computation of the Euler characteristic of $K(\Gamma)$ in terms of the number of h -residues of Γ ($h = 1, 2, 3, 4$) and use Lemma 21.

¹⁷Note that any 4-colored graph represents a singular 3-manifold, while in dimension $d \geq 4$ not any $(d + 1)$ -colored graph does represent a singular d -manifold.

¹⁸Lemma 21 extends to general 5-colored graphs an analogous relation obtained in [19, Lemma 1] in the particular case of crystallizations of 4-manifolds.

$$\chi(K(\Gamma)) = \sum_{i \in \Delta_4} g_i - \sum_{r,s,t \in \Delta_4} g_{rst} + \sum_{r,s \in \Delta_4} g_{rs} - 3p = \sum_{i \in \Delta_4} g_i - \frac{1}{2} \sum_{r,s \in \Delta_4} g_{rs} + 2p$$

Hence, by substituting $\sum_{r,s \in \Delta_4} g_{rs} = 2 \sum_{i \in \Delta_4} g_i + 4p - 2\chi(K(\Gamma))$ in the first relation we get the third one. □

As a trivial consequence of the third relation of Theorem 22, we obtain a strong and unexpected property of G-degrees of 5-colored graphs representing (singular) PL 4-manifolds.

This fact is remarkable especially with regard to the $\frac{1}{N}$ expansion of Theorem 6: in fact, all terms corresponding to a G-degree not congruent to zero mod 6 turn out NOT to represent (singular) 4-manifolds.

Corollary 23 *If (Γ, γ) is a 5-colored graph representing a singular 4-manifold, then*

$$\omega_G(\Gamma) \equiv 0 \pmod{6}.$$

□

Another consequence of the third relation of Theorem 22 is the possibility of computing the G-degree of a PL 4-manifold M^4 directly from its gem-complexity and Euler characteristic. With respect to the $\frac{1}{N}$ expansion of Theorem 6, it is worthwhile noting that the G-degree of a PL 4-manifold may be written as the sum of a TOP-addendum (depending only on the Euler characteristic of M^4 , and hence on its second Betti number in the simply-connected case) and a PL-addendum (proportional to the gem-complexity of M^4):

Corollary 24 *For each PL 4-manifold M^4*

$$\mathcal{D}_G(M^4) = 6 \left(k(M^4) + (\chi(M^4) - 2) \right).$$

In particular:

- *if M^4 is assumed to be orientable,*

$$\mathcal{D}_G(M^4) = 6 \left(k(M^4) + (\beta_2(M^4) - 2\beta_1(M^4)) \right);$$

- *if M^4 is assumed to be simply-connected,*

$$\mathcal{D}_G(M^4) = 6 \left(k(M^4) + \beta_2(M^4) \right).$$

Proof. As already pointed out, the general statement is a direct consequence of the third relation of Theorem 22. The statements regarding particular cases trivially follow from the general one. □

In order to discuss the effective computation of the G-degree for a large class of PL 4-manifolds, let us recall two particular types of crystallizations introduced and studied in [3], [13] and [2]: they are proved to be “minimal” both with respect to the gem-complexity and to the regular genus.

Definition 8 A crystallization of a PL 4-manifold M^4 with $rk(\pi_1(M)) = m$ ($m \geq 0$) is called a *semi-simple crystallization of type m* if the 1-skeleton of the associated colored triangulation contains exactly $m + 1$ 1-simplices for each pair of 0-simplices.

Semi-simple crystallizations of type 0 are called *simple crystallizations*: the 1-skeleton of their associated colored triangulation equals the 1-skeleton of a single 4-simplex.

Proposition 25 *If (Γ, γ) is an order $2p$ crystallization of a PL 4-manifold M^4 , with $rk(\pi_1(M^4)) = m$, then:*

$$\omega_G(\Gamma) \leq 8(p-1) - 20m.$$

Moreover:

$$\omega_G(\Gamma) = \mathcal{D}_G(M^4) = 8(p-1) - 20m \iff (\Gamma, \gamma) \text{ is a semi-simple crystallization};$$

$$\omega_G(\Gamma) = \mathcal{D}_G(M^4) = 8(p-1) \iff (\Gamma, \gamma) \text{ is a simple crystallization}.$$

In particular, if (Γ, γ) is a simple crystallization of a (simply-connected) PL 4-manifold M^4 , then $\omega_G(\Gamma) = \mathcal{D}_G(M^4) = 8 \cdot k(M^4) = 24 \cdot \beta_2(M^4)$.

Proof. As regards $\omega_G(\Gamma)$, the first and second statements are direct consequence of the second relation of Theorem 22, together with the property $g_{rst} = 1 + rk(\pi_1(M^4)) \forall r, s, t \in \Delta_4$, which is - by duality - the characterization of (simple and) semi-simple crystallizations of PL 4-manifolds. Moreover, the equality between $\omega_G(\Gamma)$ and $\mathcal{D}_G(M^4)$, in case Γ being a (simple or) semi-simple crystallization, follows from the fact that (simple and) semi-simple crystallizations always realize the gem-complexity of the represented PL 4-manifold.

The last statement is a consequence of the property $p = 1 + 3\beta_2(M^4)$, which holds for each simple crystallization of M^4 (and from which $k(M^4) = 3\beta_2(M^4)$ follows for any PL 4-manifold admitting simple crystallizations): see [13]. □

In [3] (resp. [2]), simple (resp. semi-simple) crystallizations of \mathbb{S}^4 , $\mathbb{C}\mathbb{P}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and the $K3$ -surface $K3$ (resp. of $\mathbb{R}\mathbb{P}^4$ and both the orientable and non-orientable \mathbb{S}^3 -bundles over \mathbb{S}^1) are presented; moreover, the class of PL 4-manifolds admitting simple (resp. semi-simple) crystallizations is proved to be closed under connected sum. Hence, all PL 4-manifolds of type

$$N(p, p', q, r, s, t) \cong_{PL} (\#_p \mathbb{C}\mathbb{P}^2) \# (\#_{p'} (-\mathbb{C}\mathbb{P}^2)) \# (\#_q (\mathbb{S}^2 \times \mathbb{S}^2)) \# (\#_r (\mathbb{S}^1 \odot \mathbb{S}^3)) \# (\#_s \mathbb{R}\mathbb{P}^4) \# (\#_t K3),$$

with $p, p', q, r, s, t \geq 0$, where $\mathbb{S}^1 \odot \mathbb{S}^3$ denotes either the orientable or non-orientable \mathbb{S}^3 -bundle over \mathbb{S}^1 , and $\mathbb{C}\mathbb{P}^2, -\mathbb{C}\mathbb{P}^2$ are two copies of the complex projective plane with opposite orientations, turn out to admit simple or semi-simple crystallizations.

As a consequence, we are able to compute their G-degree, too:

Corollary 26

$$\mathcal{D}_G(N(p, p', q, r, s, t)) = 12 \cdot [2(p + p' + 2q + 22t) + r + 3s].$$

Proof. According to [14, Proposition 5.9], we have:

$$k(N(p, p', q, r, s, t)) = 3(p + p' + 2q + 22t) + 4r + 7s.$$

Hence, in order to prove the statement, it is sufficient to apply the suitable formula of Proposition 24, by making use of the well-known values of the Euler characteristic (and/or of the Betti numbers) of each summand involved in the connected sum. □

Remark 7 Note that - in virtue of Proposition 25 - the G-degree \mathcal{D}_G turns out to be additive on the large class of PL 4 manifolds admitting simple or semi-simple crystallizations. The general property, however, does not hold: see Proposition 31.

In virtue of the proof of [2, Theorem 1], it is known that any crystallization (Γ, γ) of a PL 4-manifold M^4 with $rk(\pi_1(M^4)) = m$ has order $2p = 2(\bar{p} + q)$, where $\bar{p} = 3\chi(M^4) + 5(2m - 1)$ (the hypothetical half order of a crystallization of M^4 , which is attained if and only if M^4 admits semi-simple crystallizations) and $q \geq 0$. As a consequence, we can obtain another way to decompose the G-degree of M^4 into the sum of a TOP-addendum and a PL-addendum:

Proposition 27 *With the above notations, the following relations hold:*

$$\omega_G(\Gamma) = 12 \cdot (2\chi(M^4) + 5m - 4) + 6q;$$

$$\mathcal{D}_G(M^4) = 12 \cdot (2\chi(M^4) + 5m - 4) + 6 \cdot \min\{q / \Gamma \text{ gem of } M^4\}.$$

Proof. Starting from the second relation of Theorem 22, and making use of the notation $g_{rst} = 1 + m + t_{rst}$ ($t_{rst} \geq 1$ and $\sum_{r,s,t \in \Delta_4} t_{rst} = q$) used in the proof of [2, Theorem 1], we have:

$$\begin{aligned} \omega_G(\Gamma) &= 8(p-1) - 2 \sum_{r,s,t \in \Delta_4} (g_{rst} - 1) = \\ &= 8(\bar{p} + q - 1) - 2 \sum_{r,s,t \in \Delta_4} (t_{rst} + m) = \\ &= 8(\bar{p} - 1) + 8q - 2q - 20m = \\ &= 8(3\chi(M^4) + 5(2m - 1) - 1) - 20m + 6q = \\ &= 4[6\chi(M^4) + 10(2m - 1) - 2] - 5m + 6q = \\ &= 12[2\chi(M^4) + 5m - 4] + 6q. \end{aligned}$$

The second formula trivially follows from the first one. □

6.2 Relationship between G-degree and regular genus, for $d = 4$

If $d = 4$, inequality (17) gives

$$\omega_G(\Gamma) \geq 12 \cdot \rho(\Gamma)$$

for any 5-colored graph Γ , and Proposition 11 yields

$$\mathcal{D}_G(M^4) \geq 12 \cdot \mathcal{G}(M^4)$$

for any PL 4-manifold M^4 .

In the following proposition we investigate the gap between the two quantities $12\rho(\Gamma)$ and $\omega_G(\Gamma)$, for any 5-colored graph Γ .

Proposition 28 *If (Γ, γ) is an order $2p$ 5-colored graph, then*

$$\omega_G(\Gamma) - 12\rho(\Gamma) = 3 \left(\sum_{i \in \mathbb{Z}_5} g_{\bar{\varepsilon}_i \bar{\varepsilon}_{i+1}} - \sum_{i \in \mathbb{Z}_5} g_{\bar{\varepsilon}_i \bar{\varepsilon}_{i+2}} \right),$$

where $\bar{\varepsilon}$ is the cyclic permutation of Δ_4 such that $\rho(\Gamma) = \rho_{\bar{\varepsilon}}(\Gamma)$.

Proof. From Theorem 22, we have $\omega_G(\Gamma) = 3 \left(6(p-1) - \sum_{r,s \in \Delta_4} (g_{rs} - 1) \right)$, while $\rho_{\varepsilon}(\Gamma) = 1 + \frac{3}{2}p - \frac{1}{2} \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j \varepsilon_{j+1}}$.

Hence, if $\bar{\varepsilon}$ denotes the cyclic permutation of Δ_4 such that $\rho_{\bar{\varepsilon}}(\Gamma) = \rho(\Gamma)$, we have:

$$\begin{aligned} \omega_G(\Gamma) - 12\rho(\Gamma) &= 3 \left(6(p-1) - \sum_{r,s \in \Delta_4} (g_{rs} - 1) \right) - 6 \left(2 + 3p - \sum_{j \in \mathbb{Z}_5} g_{\varepsilon_j \varepsilon_{j+1}} \right) = \\ &= 3 \left(\sum_{i \in \mathbb{Z}_5} g_{\bar{\varepsilon}_i \bar{\varepsilon}_{i+1}} - \sum_{i \in \mathbb{Z}_5} g_{\bar{\varepsilon}_i \bar{\varepsilon}_{i+2}} \right), \end{aligned}$$

according to the statement. □

Let us now take into account the case of equality between the two quantities.

Proposition 29

(a) If (Γ, γ) is an order $2p$ 5-colored graph, then:

$$\omega_G(\Gamma) = 12 \cdot \rho(\Gamma) \implies \text{a cyclic permutation } \bar{\varepsilon} \text{ of } \Delta_4 \text{ exists, so that } \sum_{i \in \Delta_4} g_{\bar{\varepsilon}_i \bar{\varepsilon}_{i+1}} = \sum_{i \in \Delta_4} g_{\bar{\varepsilon}_i \bar{\varepsilon}_{i+2}}.$$

(b) If M^4 is a PL 4-manifold M^4 , then:

$$\mathcal{D}_G(M^4) = 12 \cdot \mathcal{G}(M^4) \iff k(M^4) = 2\mathcal{G}(M^4) - \chi(M^4) + 2.$$

Proof. Statement (a) is a trivial consequence of Proposition 28.

On the other hand, by making use of the first statement of Corollary 24, we have: $\mathcal{D}_G(M^4) = 12 \cdot \mathcal{G}(M^4)$ if and only if $6 \left(k(M^4) + (\chi(M^4) - 2) \right) = 12 \cdot \mathcal{G}(M^4)$, i.e. $k(M^4) + (\chi(M^4) - 2) = 2 \cdot \mathcal{G}(M^4)$. Statement (b) directly follows. □

Corollary 30

(a) If (Γ, γ) is a semi-simple (resp. simple) crystallization of a PL 4-manifold (resp. of a simply-connected PL 4-manifold), then $\omega_G(\Gamma) = 12 \cdot \rho(\Gamma)$.

(b) If M^4 is PL 4-manifold (resp. a simply-connected PL 4-manifold) admitting semi-simple (resp. simple) crystallizations, then $\mathcal{D}_G(M^4) = 12 \cdot \mathcal{G}(M^4)$.

Proof. By definition (see [3] and [2]), (Γ, γ) is a semi-simple (resp. simple) crystallization of a PL 4-manifold (resp. a simply-connected PL 4-manifold) M^4 if $g_{ijk} = 1 + m \forall i, j, k \in \Delta_4$ where $m = rk(\pi_1(M^4))$ (resp. $g_{ijk} = 1 \forall i, j, k \in \Delta_4$). Moreover, as proved in Proposition 3.6 of [13] and Proposition 8 of [2], both simple and semi-simple crystallizations (Γ, γ) satisfy the property: $\rho_\varepsilon(\Gamma) = \mathcal{G}(M^4)$ for any cyclic permutation ε of Δ_4 . Hence, both statement (a) and statement (b) directly follow. □

6.3 TOP and PL classification of PL 4-manifolds via G-degree

It is a classical result of geometric topology that any topological 3-manifold admits a PL-structure which is unique up to PL-isomorphisms, and that each PL-structure on a 3-manifold is smoothable in a unique way up to diffeomorphisms: so, the categories TOP of topological manifolds (and homeomorphisms), PL of PL manifolds (and PL-isomorphisms) and DIFF of smooth manifolds (and diffeomorphisms) turn out to coincide in dimension three.

On the contrary, in dimension four, the situation is quite different, since PL and DIFF categories still coincide, but TOP and PL don't. In fact, each PL-structure on a 4-manifold is smoothable in a unique way up to diffeomorphisms, but it is well-known that there exist topological 4-manifolds admitting no smooth structures (an example is the so-called E_8 -manifold) and that there can be non-diffeomorphic smooth structures on the same topological 4-manifold: see [28].

We recall also that, in the simply-connected case, the complete topological classification has been long established by Freedman and it is mainly determined by the intersection form. On the other hand, although the important work by Donaldson [23] (improved quite recently by Furuta [29]) yields restrictions on the possible intersection forms of PL simply-connected 4-manifolds, there is no classification of the PL structures on any given simply-connected triangulable topological 4-manifold.

Furthermore, unlike what happens in all other dimensions, the different PL structures on the same topological 4-manifold may be infinitely many. Actually, this kind of situation has been proved to exist for several (simply-connected) topological manifolds, among which the one with the smallest

second Betti number is $\mathbb{C}\mathbb{P}^2 \#_2(-\mathbb{C}\mathbb{P}^2)$, while it is still an open problem the existence of different PL-structures on \mathbb{S}^4 , $\mathbb{C}\mathbb{P}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ or $\mathbb{C}\mathbb{P}^2 \#(-\mathbb{C}\mathbb{P}^2)$.

As a consequence of the existence of infinitely many PL-structures on certain TOP 4-manifolds and of the finiteness-to-one of the G-degree \mathcal{D}_G , the G-degree is proved not to satisfy the additivity property, within the whole set \mathcal{M}_4 of PL 4-manifolds.

Proposition 31 *PL 4-manifolds N and N' exist, so that*

$$\mathcal{D}_G(N \# N') \neq \mathcal{D}_G(N) + \mathcal{D}_G(N').$$

Proof. Let us consider a TOP 4-manifold \bar{M} which is known to admit infinitely many PL-structures (for example, $\mathbb{C}\mathbb{P}^2 \#_2(-\mathbb{C}\mathbb{P}^2)$). Since the G-degree is finite-to-one on the set \mathcal{M}_4 of PL 4-manifolds (Theorem 14), not all PL-structures on \bar{M} have the same G-degree. Let \bar{M}_1 and \bar{M}_2 be two different (i.e. not PL-homeomorphic) PL 4-manifolds, with $\bar{M}_i \equiv_{TOP} \bar{M}$ for each $i = 1, 2$ and $\mathcal{D}_G(\bar{M}_1) \neq \mathcal{D}_G(\bar{M}_2)$. Now, by the well-known Wall's theorem, a non-negative integer l exists, so that $\bar{M}_1 \#_l(\mathbb{S}^2 \times \mathbb{S}^2)$ and $\bar{M}_2 \#_l(\mathbb{S}^2 \times \mathbb{S}^2)$ are PL-homeomorphic. If the G-degree were additive in \mathcal{M}_4 , then $\mathcal{D}_G(\bar{M}_1) + l\mathcal{D}_G(\mathbb{S}^2 \times \mathbb{S}^2) = \mathcal{D}_G(\bar{M}_2) + l\mathcal{D}_G(\mathbb{S}^2 \times \mathbb{S}^2)$ would follow, which is obviously a contradiction. \square

Remark 8 As regards the G-degrees of the (possible) different PL-structures on the same TOP 4-manifold, the following facts may be pointed out.

- There exist (infinite families of) different PL 4-manifolds with the same underlying TOP manifold, and with different G-degree. As already seen in order to prove Proposition 31, it is sufficient to take into account a TOP 4-manifold which admits infinitely many PL-structures, and to make use of the finiteness-to-one of the G-degree.
- There exist 5-colored graphs Γ and Γ' which encode the same underlying TOP manifold and have the same G-degree (and the same gem-complexity and regular genus, too), but it is an open problem whether their PL-structure is the same or not. For example: the two simple crystallizations of $K3$ (obtained from the 16- and 17-vertex triangulations of the $K3$ -surface) mentioned in [3] and [13].
- Concrete examples of 5-colored graphs (actually, simple or semi-simple crystallizations) exist, encoding different PL 4-manifolds with the same underlying TOP manifold and having the same G-degree (and the same gem-complexity and regular genus, too). One of these examples is based on a result by Kronheimer and Mrowka (see [40]) stating that the two simply-connected 4-manifolds $M_1 = K3 \#(-\mathbb{C}\mathbb{P}^2)$ and $M_2 = \#_3(\mathbb{C}\mathbb{P}^2) \#_{20}(-\mathbb{C}\mathbb{P}^2)$ are not PL-homeomorphic, though they are TOP-homeomorphic, since they have the same intersection form. The unique simple crystallization of $\mathbb{C}\mathbb{P}^2$, of order 8, was first introduced in [34], while a simple crystallization of the $K3$ -surface¹⁹ is depicted in [3]. By performing graph connected sums of a suitable number of copies of these graphs, we obtain two simple crystallizations of M_1 and M_2 respectively. Since the G-degree is additive within the class of manifolds admitting simple crystallizations, an easy computation yields $\mathcal{D}_G(M_1) = \mathcal{D}_G(M_2)$. For the case of non-simply connected manifolds, a result by Kreck [39] ensures that $\mathbb{R}\mathbb{P}^4 \# K3 \not\cong_{PL} \mathbb{R}\mathbb{P}^4 \#_{11}(\mathbb{S}^2 \times \mathbb{S}^2)$, while the two manifolds are TOP homeomorphic. Again, by using known simple and semi-simple crystallizations of the involved manifolds and performing graph connected sums, we obtain the required example.

¹⁹This crystallization of $K3$ has 134 vertices; a numerical “code” encoding its combinatorial structure can be obtained on request from the authors of the present paper.

The formulas of the previous subsections, establishing relationships among the G-degree and both the gem-complexity and the regular genus in dimension 4, enable to “translate” all known results about the (TOP or PL) classification of PL 4-manifolds via regular genus and/or gem-complexity into results concerning the G-degree.

As far as the TOP classification is concerned, the following statement holds:

Proposition 32 *Let (Γ, γ) be a gem of a simply-connected PL 4-manifold M^4 . If $\omega_G(\Gamma) \leq 527$, then M^4 is TOP-homeomorphic to*

$$(\#_r \mathbb{C}\mathbb{P}^2) \# (\#_{r'} (-\mathbb{C}\mathbb{P}^2)) \quad \text{or} \quad \#_s (\mathbb{S}^2 \times \mathbb{S}^2),$$

where $r + r' = \beta_2(M^4)$ and $s = \frac{1}{2}\beta_2(M^4)$, with $\beta_2(M^4) \leq \frac{1}{24} \cdot \omega_G(\Gamma)$.

Proof. Within crystallization theory it is well-known that the inequality $\mathcal{G}(M^4) \geq 2\beta_2(M^4)$ holds for any simply-connected PL 4-manifold M^4 : see, for example, [14, Theorem 3.1] or [2]. On the other hand, since $\omega_G(\Gamma) \geq 12 \cdot \rho(\Gamma)$ holds for any 5-colored graph, in the simply-connected case we have:

$$\omega_G(\Gamma) \geq 24 \cdot \beta_2(M^4).$$

Hence, $\omega_G(\Gamma) \leq 527$ easily implies $\beta_2(M^4) < 22$. The thesis now easily follows by making use of the up-to-date results about topological classification of simply connected PL 4-manifolds (see [23] and [29]), exactly as in the proof of [12, Proposition 23] or [14, Theorem 3.5]: in fact, only forms of type $r[1] \oplus r'[-1]$ or $s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ can occur as intersection forms of a simply-connected smooth 4-manifold with $\beta_2 < 22$. □

Remark 9 Right now, we point out that, if M^4 satisfies the hypotheses of the above Proposition and $\omega_G(\Gamma) \leq 59$, then TOP and PL classifications coincide: see Proposition 34.

With regard to the PL classification of PL 4-manifolds, the following statements collect some classifying results involving the G-degree: in particular, Proposition 33 (resp. Proposition 35) provides the complete list of all orientable (resp. non-orientable) PL 4-manifolds which appear in the $1/N$ expansion of Theorem 6 (resp. of its real tensors version: see Remark 1) up to G-degree 42 (resp. 35).

Proposition 33 *Let (Γ, γ) be a bipartite 5-colored graph representing an orientable PL 4-manifold M^4 . Then:*

- (a) $\omega_G(\Gamma) \in \{0, 6\} \quad \Rightarrow \quad M^4 \cong \mathbb{S}^4.$
- (b) $\omega_G(\Gamma) \in \{12, 18\} \quad \Rightarrow \quad M^4 \in \{\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1\}.$
- (c) $\omega_G(\Gamma) \in \{24, 30\} \quad \Rightarrow \quad M^4 \in \{\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1, \mathbb{C}\mathbb{P}^2, \#_2(\mathbb{S}^3 \times \mathbb{S}^1)\}.$
- (d) $\omega_G(\Gamma) \in \{36, 42\} \quad \Rightarrow \quad M^4 \in \{\mathbb{S}^4, \mathbb{S}^3 \times \mathbb{S}^1, \mathbb{C}\mathbb{P}^2, \#_2(\mathbb{S}^3 \times \mathbb{S}^1), \#_3(\mathbb{S}^3 \times \mathbb{S}^1), (\mathbb{S}^3 \times \mathbb{S}^1) \# \mathbb{C}\mathbb{P}^2\}.$

Proof. First of all, recall that - by Corollary 23 - $\omega_G(\Gamma) \equiv 0 \pmod{6}$ for each gem Γ of a PL 4-manifold M^4 . On the other hand, $\omega_G(\Gamma) \leq 11$ (resp. $\omega_G(\Gamma) \leq 23$) (resp. $\omega_G(\Gamma) \leq 35$) (resp. $\omega_G(\Gamma) \leq 47$) obviously implies $\rho(\Gamma) = 0$ (resp. $\rho(\Gamma) \leq 1$) (resp. $\rho(\Gamma) \leq 2$) (resp. $\rho(\Gamma) \leq 3$) via Proposition 11 (case $d = 4$). Statement (a) (resp. (b)) (resp. (c)) (resp. (d)) now directly follows by the well-known PL classification of orientable PL 4-manifolds with regular genus 0 (resp. 1) (resp. 2) (resp. 3): see for example Prop. 4.2(a) of the survey paper [14]. □

Proposition 34 *Let M^4 be a simply-connected PL 4-manifold. Then:*

- (a) $\mathcal{D}_G(M^4) = 0 \iff M^4 \cong \mathbb{S}^4;$
- (b) $\mathcal{D}_G(M^4) = 24 \iff M^4 \cong \mathbb{C}\mathbb{P}^2;$
- (c) $\mathcal{D}_G(M^4) = 48 \iff M^4 \in \{\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2, \mathbb{C}\mathbb{P}^2 \# (-\mathbb{C}\mathbb{P}^2)\}.$

No other simply-connected PL 4-manifold M^4 exists, with $\mathcal{D}_G(M^4) \leq 59$.

Proof. The last formula of Proposition 24 ensures that, for each simply-connected PL 4-manifold M^4 , $\mathcal{D}_G(M^4) = 6(k(M^4) + \beta_2(M^4))$. Since both addenda on the right side are non-negative, $\mathcal{D}_G(M^4) = 0$ (resp. $\mathcal{D}_G(M^4) = 24$) (resp. $\mathcal{D}_G(M^4) = 48$) trivially implies $k(M^4) = 0$ (resp. $k(M^4) \leq 4$) (resp. $k(M^4) \leq 8$). Statement (a) (resp. (b)) (resp. (c)) now directly follows by the PL classification of orientable PL 4-manifolds with gem-complexity $k(M^4) \leq 2$ (resp. $k(M^4) \leq 5$) (resp. $k(M^4) \leq 8$): see Proposition 29 of [12] or, equivalently, Theorem 4.6 of the survey paper [14].

Finally, in order to prove the last statement, note that $\mathcal{D}_G(M^4) \leq 59$ implies $k(M^4) + \beta_2(M^4) \leq 9$. Hence, either $k(M^4) \leq 8$ (and so the previous cases occur) or $k(M^4) = 9$ with $\beta_2(M^4) = 0$; however, Theorem 4.6 of [14] ensures that no simply-connected PL 4-manifold satisfies the second hypothesis (in fact, any PL 4-manifold with $k(M^4) = 9$ turns out to be simply-connected with second Betti number equal to three). □

Proposition 35 *Let (Γ, γ) be a non-bipartite 5-colored graph representing a non-orientable PL 4-manifold M^4 . Then:*

$$\omega_G(\Gamma) \leq 35 \implies M^4 \in \{\mathbb{S}^3 \tilde{\times} \mathbb{S}^1, \#_2(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)\}.$$

Proof. Since $\omega_G(\Gamma) \leq 35$ obviously implies $\rho(\Gamma) \leq 2$ via Proposition 11 (case $d = 4$), the thesis directly follows by the well-known PL classification of non-orientable PL 4-manifolds up to regular genus 2: see for example Prop. 4.2(b) of the survey paper [14]. □

Finally, we point out that - via the formula in Lemma 13 - it is possible to translate known results concerning the regular genus of the subgraph $\Gamma_{\hat{i}}$ of a crystallization of a PL 4-manifold into classifying results by means of the G-degree of Γ or of $\Gamma_{\hat{i}}$.

Proposition 36 *Let (Γ, γ) be an order $2p$ crystallization of a PL 4-manifold M^4 . Then:*

- (a) *If there exists a color $i \in \Delta_4$ so that $\omega_G(\Gamma_{\hat{i}}) \leq 2$, then either $M^4 \cong \#_{\rho}(\mathbb{S}^3 \times \mathbb{S}^1)$ or $M^4 \cong \#_{\rho}(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)$, with $\rho = \mathcal{G}(M^4) \geq 0$.*
- (b) *If $\omega_G(\Gamma) \leq 3(p - 1) + 14$, then either $M^4 \cong \#_{\rho}(\mathbb{S}^3 \times \mathbb{S}^1)$ or $M^4 \cong \#_{\rho}(\mathbb{S}^3 \tilde{\times} \mathbb{S}^1)$, with $\rho = \mathcal{G}(M^4) \geq 0$.*

Proof. It is easy to check that $\omega_G(\Gamma_{\hat{i}}) \leq 2$ implies $\rho(\Gamma_{\hat{i}}) = 0$ ($= \rho_{\varepsilon}(\Gamma_{\hat{i}})$ for any permutation ε of Δ_4). Statement (a) is now a direct consequence of [18, Proposition 2], yielding the PL classification of all PL 4-manifolds admitting a crystallization for which a color $i \in \Delta_4$ exists such that the regular genus of $\Gamma_{\hat{i}}$ is zero.

On the other hand, the 4-dimensional case of the formula in Lemma 13, applied to a crystallization of a PL 4-manifold, yields:

$$\omega_G(\Gamma) = 3(p - 1) + \sum_{i \in \Delta_4} \omega_G(\Gamma_{\hat{i}}).$$

Hence, statement (b) follows from statement (a), since the hypothesis $\omega_G(\Gamma) \leq 3(p - 1) + 14$ easily implies the existence of a color $i \in \Delta_4$ such that $\omega_G(\Gamma_{\hat{i}}) \leq 2$. □

7 Conclusion and research trends

In this paper we have explored several properties of the Gurau degree, which is a natural quantity appearing in tensor models, driving their main physical behavior. These models are introduced as models for QG. QG models often bring insights into geometry and good geometric understanding of these models allows to progress on the problem of QG. These fruitful exchanges between geometry and QG models were the first motivation of this work.

Several research directions thus open. Concerning the G-degree, many results are obtained here, mostly in dimension 3 and 4, relating this invariant with regular genus and gem complexity. With this starting point, a natural trend would be to investigate the link connecting the G-degree with other notions of complexity (such as Matveev complexity [42], or its higher dimensional extensions [22] [41]).

In tensor models manifolds and pseudo-manifolds are (almost) on the same footing, since they constitute the class of polyhedra represented by the (edge-colored) Feynman graphs arising within tensor models theory. Most of the results obtained in this paper concern the manifold case; nevertheless, the structure of the $1/N$ expansion makes significant the theme of the classification of all pseudomanifolds represented by graphs of a given G-degree. Indeed, the main physical motivations for such classification is to get insight into the physical processes involved in the quantum fluctuations of geometry. Therefore, it seems to be fruitful in this framework to look for classifications results concerning all pseudomanifolds, or at least singular manifolds (subsection 6.1). The recently introduced representation theory for 3-manifolds with boundary (and their naturally associated singular manifolds) via regular 4-colored graphs (see [21]), if suitably extended to higher dimensions, might be a significant tool for this purpose.

Other questions arise. We stress that, in [38], efficient combinatorial techniques allow to describe the possible colored graphs appearing at a given Gurau degree²⁰. Therefore, it would be interesting to shed a new light on the topology (and geometry) of the pseudomanifolds represented by the graphs arising in this way. Another research trend arises in dimension 4 from the existence of infinitely many different PL structures on the same topological 4-manifold. It would be of interest to find significant examples of colored graphs encoding different *PL* 4-manifolds, with different G-degree but with the same underlying TOP manifold (see Remark 8); such a result would hint at the ability of tensor models to accurately reflect geometric degrees of freedom of QG (which is non-trivial as many QG models are actually only topological).

Finally, tensor models can be seen as toy models of other QG models called Group Field Theory (GFT). In these GFT, the colored graphs are endowed with an additional structure that can be seen as a discrete *G*-connection on the corresponding (pseudo)-manifolds where *G* is a Lie group, generally supposed to be compact. In these models there are quantities [35, 7], that play the same rôle as the G-degree. Since they are built from graphs that contain more geometric information, it would be interesting to study the properties of such quantities, thus allowing a better insight into the topology (and geometry) of the underlying (pseudo)-manifolds.

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²⁰Note that the problem is related to the question posed in Remark 2: are all multiples of $\frac{(d-1)!}{2}$ allowed as the G-degree of a $(d+1)$ -colored graph? See [17] for some partial results.

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