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Discretization and approximation of surfaces using varifolds

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Abstract: We present some recent results on the possibility of extending the theory of varifolds to the realm of discrete surfaces of any dimension and codimension, for which robust notions of approximate curvatures, also allowing for singularities, can be defined. This framework has applications to discrete and computational geometry, as well as to geometric variational problems in discrete settings. We finally show some numerical tests on point clouds that support and confirm our theoretical findings.

Keywords: Varifold, approximate mean curvature, approximate second fundamental form, point cloud

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Introduction

One of the main issues in image processing and computer graphics is to extract geometric information from discrete data, that are provided in the form of polygonal (or polyhedral) meshes, level sets, point clouds, CAD models etc. We might classify the various discrete representations in two main classes: *structured* and *unstructured*. For instance, polyhedral meshes can be classified as structured, while point clouds are unstructured. In general, unstructured representations are characterized by the absence of (local, partial) information of topological surface type. This kind of discrete surfaces, like point clouds, have received a great attention in the last decades as they arise in many different contexts (medical imaging, shape modeling, object classification).

In order to reconstruct surface features (and in particular curvatures) it is often assumed that the discrete data refer to an unknown smooth surface, which needs to be first determined or characterized at least implicitly. This is the case, for instance, of the so-called Moving Least Squares (MLS) technique, especially proposed for the reconstruction of surface features from point cloud data, see [17]. More recently, some techniques based on integral geometry and geometric measure theory have been proposed by various authors, see [8, 9, 12, 13, 21, 25]. These methods are quite efficient for reconstructing curvatures in the smooth case as well as in presence of certain kinds of singularities, however major problems occur when more general singularities are present in the unknown surface. Moreover, convergence results are generally obtained under very strong regularity assumptions.

In the recent work [4] (see also [6, 7]) we propose a general approach based on a suitable adaptation of the theory of varifolds, which is based on a notion of approximate mean curvature associated with any d-dimensional varifold (thus in particular with the so-called *discrete varifolds*).

The main aim of our research is to provide the natural framework where different discrete surface models can be represented and analysed, also allowing for robust convergence results for the approximate curva-

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tures. This framework is very promising in respect of the numerical approximation of geometric variational problems, geometric flows, topological invariants, and so on.

At the same time, our method does not require the use of discrete surfaces of some special type. Basically, it can be applied with any discrete surface type, therefore its full potentials are mostly evidenced when we consider unstructured representations like point cloud surfaces, for which the identification of natural curvature estimators is a quite difficult problem.

The first step of our method consists in defining the *regularized first variation* of a varifold V. This is obtained by convolving the standard first variation δV (which is a distribution of order 1 acting on vector fields of class C^1 and compact support) with a regularizing kernel ρ_ε. By a similar regularization of the weight measure $\|V\|$ (obtained through another kernel ξ_ε) we arrive at the notion of *approximate mean curvature* vector field $H^V_{\rho,\xi,\varepsilon}$, defined as the ratio between the regularized first variation and the regularized weight (whenever the latter is positive). This allows us to define approximate mean curvature vector fields for *all* varifolds (even non-rectifiable ones). The parameter ε appearing in the above kernels should be understood as the scale at which the approximate mean curvature is evaluated. Of course, if the scale is too small, or too large, the approximate mean curvature might be far from what we expect to be naturally associated with the discrete varifold. The choice of ε is therefore crucially linked to some "intrinsic scale" associated with the varifold itself.

One might wonder if this approach can somehow include other previous (maybe classical) notions of discrete mean curvature proposed for instance in the case of polyhedral surfaces. In this sense it is possible to show that the classical *Cotangent Formula*, that is widely used for defining the mean curvature of a polyhedral surface $\mathcal{P}$ at a vertex v, can be simply understood as the first variation of the associated varifold $V_\mathcal{P}$ applied to any Lipschitz extension of the piecewise affine basis function φ_v that takes the value 1 on v and is identically zero outside the patch of triangles around v. In this sense, the Cotangent Formula can be understood as the regularization of $\delta V_\mathcal{P}$ by means of the finite family of piecewise affine kernels $\{\varphi_v(x) \; : \; v \text{ is a vertex of } \mathcal{P}\}$. See [4] for more details.

The approximate mean curvature defined as above satisfies some nice convergence properties, that are stated in Theorems 3.3, 3.4 and 3.6. These results rely on the notion of *Bounded Lipschitz distance* (see Section 1). We stress that our approximate mean curvature is of "variational nature" because it is obtained by mollifying the first variation of V, therefore it can be consistently defined also in presence of singularities.

Finally, in a forthcoming work we consider a weak notion of second fundamental form obtained as a slight variant of the one proposed by Hutchinson in [16], which has the nice feature of being easily regularized in the same spirit as done before in the case of the mean curvature. We thus obtain approximate second fundamental forms satisfying the same convergence results cited before. This opens the way to a number of possible applications to computational geometry, like for instance the definition of very general discrete geometric flows, as well as of general discrete equivalents of topological invariants that are related with curvatures (like in the Gauss-Bonnet theorem).

To show the consistency and robustness of our theory, some numerical experiment are shown in the last section, including in particular the computation of approximate mean curvatures of standard double bubbles in 2d and 3d, as well as of approximate Gaussian curvatures of a torus.

1 Preliminaries

Let $d, n \in \mathbb{N}$ with $1 \leq d < n$. Let ρ_1, ξ_1 be two symmetric mollifiers on $\mathbb{R}^n$, such that $\rho_1(x) = \rho(|x|)$ and $\xi_1(x) = \xi(|x|)$ for suitable one-dimensional, even profile functions ρ, ξ having compact support in $[-1, 1]$. We assume that

$$\int_{\mathbb{R}^n} \rho_1(x)\, dx = \int_{\mathbb{R}^n} \xi_1(x)\, dx = 1\,,$$

that is,

$$n\omega_n \int_0^1 \rho(t) t^{n-1}\, dt = n\omega_n \int_0^1 \xi(t) t^{n-1}\, dt = 1\,.$$

Given $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we set

$$\rho_\varepsilon(x) = \varepsilon^{-n}\rho_1(x/\varepsilon) \qquad \text{and} \qquad \xi_\varepsilon(x) = \varepsilon^{-n}\xi_1(x/\varepsilon)\,.$$

We assume at least that $\rho \in \mathrm{C}^1(\mathbb{R})$ and $\xi \in \mathrm{C}^0(\mathbb{R})$. At some point, some extra regularity will be required on ρ and ξ, namely that $\rho \in W^{2,\infty}$ and $\xi \in W^{1,\infty}$ (see Hypothesis 1).

We recall here a few facts about varifolds, see for instance [22] for more details. Let us start with the general definition of varifold. Let $\Omega \subset \mathbb{R}^n$ be an open set. A d–varifold in Ω is a non-negative Radon measure on $\Omega \times G_{d,n}$, where $G_{d,n}$ denotes the Grassmannian manifold of d-dimensional unoriented subspaces of $\mathbb{R}^n$.

An important class is the one of *rectifiable varifolds*. Let M be a countably d–rectifiable set and θ be a non negative function with $\theta > 0$ $\mathcal{H}^d$–almost everywhere in M. The associated rectifiable varifold $V = v(M, \theta)$ is defined as $V = \theta\mathcal{H}^d_{|M} \otimes \delta_{T_xM}$, i.e.,

$$\int_{\Omega\times G_{d,n}} \varphi(x,T)\,dV(x,T) = \int_M \varphi(x,T_xM)\,\theta(x)\,d\mathcal{H}^d(x) \quad \forall\varphi \in \mathrm{C}^0_c(\Omega\times G_{d,n},\mathbb{R})\,,$$

where T_xM is the approximate tangent space at x, which exists $\mathcal{H}^d$–almost everywhere in M, and δ_{T_xM} is the Dirac delta at . The function θ is called the *multiplicity* of the rectifiable varifold. If additionally $\theta(x) \in \mathbb{N}$ for $\mathcal{H}^d$–almost every $x \in M$, we say that V is an *integral* varifold.

The *weight* (or mass) measure of a varifold V is the positive Radon measure defined by $\|V\|(B) = V(\pi^{-1}(B))$ for every $B \subset \Omega$ Borel, with $\pi : \Omega\times G_{d,n} \to \Omega$ defined by $\pi(x, S) = x$. In particular, the weight of a d–rectifiable varifold $V = v(M, \theta)$ is the measure $\|V\| = \theta\mathcal{H}^d_{|M}$.

The following result is well-known (see for instance [1]).

Proposition 1.1 (Young-measure representation). *Given a d–varifold V on Ω, there exists a family of probability measures $\{\nu_x\}_x$ on $G_{d,n}$ defined for $\|V\|$-almost all $x \in \Omega$, such that $V = \|V\| \otimes \{\nu_x\}_x$, that is,*

$$V(\varphi) = \int_{x\in\Omega}\int_{S\in G_{d,n}} \varphi(x,S)\,d\nu_x(S)\,d\|V\|(x)$$

for all $\varphi \in \mathrm{C}^0_c(\Omega\times G_{d,n})$.

We recall that a sequence $(\mu_i)_i$ of Radon measures defined on a locally compact metric space is said to weakly–$\star$ converge to a Radon measure μ (in symbols, $\mu_i \xrightarrow{\star} \mu$) if, for every $\varphi \in \mathrm{C}^0_c(\Omega)$, $\mu_i(\varphi) \to \mu(\varphi)$ as $i \to \infty$.

A sequence of d–varifolds $(V_i)_i$ weakly–$\star$ converges to a d–varifold V in Ω if, for all $\varphi \in \mathrm{C}^0_c(\Omega\times G_{d,n})$,

$$\langle V_i,\varphi\rangle = \int_{\Omega\times G_{d,n}} \varphi(x,P)\,dV_i(x,P) \xrightarrow[i\to\infty]{} \langle V,\varphi\rangle = \int_{\Omega\times G_{d,n}} \varphi(x,P)\,dV(x,P)\,.$$

We now recall the definition of *Bounded Lipschitz distance* between two Radon measures μ and ν defined on a locally compact metric space (X, d). We set

$$\Delta^{1,1}(\mu,\nu) = \sup\left\{\left|\int_X \varphi\,d\mu - \int_X \varphi\,d\nu\right| \;:\; \varphi \in \mathrm{Lip}_1(X),\ \|\varphi\|_\infty \le 1\right\}.$$

It is well-known that $\Delta^{1,1}$ defines a distance on the space of Radon measures on X.

The following fact is well-known (see [5, 24]).

Proposition 1.2. *Let μ, $(\mu_i)_i$, $i \in \mathbb{N}$, be Radon measures on a locally compact metric space (X, δ). Assume that $\mu(X) + \sup_i \mu_i(X) < +\infty$ and that there exists a compact set $K \subset X$ such that the supports of μ and of μ_i are contained in K for all $i \in \mathbb{N}$. Then $\mu_i \xrightarrow{\star} \mu$ if and only if $\Delta^{1,1}(\mu_i,\mu) \to 0$ as $i \to \infty$.*

For all $P \in G_{d,n}$ and $X = (X_1, \dots, X_n) \in \mathrm{C}^1_c(\Omega,\mathbb{R}^n)$ we set

$$\mathrm{div}_P X(x) = \sum_{j=1}^n \langle\nabla^P X_j(x), e_j\rangle = \sum_{j=1}^n \langle\Pi_P(\nabla X_j(x)), e_j\rangle$$

where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$. The definition of *first variation* of a varifold is due to Allard [2]. Given a varifold V on $\Omega \times G_{d,n}$, its first variation δV is the vector–valued distribution (of order 1) defined for any vector field $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$ as

$$\delta V(X) = \int_{\Omega \times G_{d,n}} \mathrm{div}_P X(x)\, dV(x, P)\,.$$

It is also useful to define the action of δV on a function $\varphi \in C_c^1(\Omega)$ as the vector

$$\delta V(\varphi) = \big(\delta V(\varphi\, e_1), \dots, \delta V(\varphi\, e_n)\big)\,.$$

We say that V has a *locally bounded first variation* if, for any fixed compact set $K \subset \Omega$, there exists a constant $c_K > 0$ such that for any vector field $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$ with $\operatorname{spt} X \subset K$ one has

$$|\delta V(X)| \le c_K \sup_K |X|\,.$$

In this case, by Riesz Theorem, there exists a vector–valued Radon measure on Ω (still denoted as δV) such that

$$\delta V(X) = \int_\Omega X \cdot \delta V \quad \text{for every } X \in \mathrm{C}_c^0(\Omega, \mathbb{R}^n)$$

Thanks to Radon-Nikodym Theorem, we can decompose δV as

$$\delta V = -H\|V\| + \delta V_s\,, \tag{1.1}$$

where $H \in \left(\mathrm{L}^1_{loc}(\Omega, \|V\|)\right)^n$ and δV_s is singular with respect to $\|V\|$. The function H is called the *generalized mean curvature vector*. By the divergence theorem, H coincides with the classical mean curvature vector if $V = v(M, 1)$, where M is a d-dimensional submanifold of class C^2.

1.1 Discrete varifolds

Every time a varifold is defined by a finite set of parameters, we shall call it *discrete varifold*. As anticipated in the Introduction, we shall mainly focus on “unstructured” discrete varifolds (discrete volumetric varifolds or point cloud varifolds, see below). Note that all definitions and results of Sections 2 and 3 hold in particular for all sequences of discrete varifolds, including those of polyhedral type. Concerning the results of Section 4, the construction of approximations V_i of a rectifiable varifold V, such that $\Delta^{1,1}(V_i, V)$ is infinitesimal, as shown in Theorem 4.4, seems to be quite delicate in the polyhedral setting, since the tangent directions are prescribed by the directions of the cells of the polyhedral surface, which are not necessarily converging when the polyhedral surfaces converge to a smooth one in Hausdorff distance. Nevertheless, the construction of such approximations is much simpler in the case of volumetric and point cloud varifolds.

Let $\Omega \subset \mathbb{R}^n$ be an open set. A *mesh* of Ω is a countable partition $\mathcal{K}$ of Ω, that is, a collection of pairwise disjoint subdomains (“cells”) of Ω such that $\{K \in \mathcal{K} \,:\, K \cap B \neq \emptyset\}$ is finite for any bounded set $B \subset \Omega$ and

$$\Omega = \bigsqcup_{K \in \mathcal{K}} K\,.$$

Here, no other assumptions on the geometry of the cells $K \in \mathcal{K}$ are needed. We shall often refer to the *size* of the mesh $\mathcal{K}$, denoted by

$$\delta = \sup_{K \in \mathcal{K}} \operatorname{diam} K\,.$$

We come to the definition of discrete volumetric varifold (see [7]).

Definition 1.3 (Discrete volumetric varifold). *Let $\mathcal{K}$ be a mesh of Ω and let $\{(m_K, P_K)\}_{K \in \mathcal{K}} \subset \mathbb{R}_+ \times G_{d,n}$. We set*

$$V_{\mathcal{K}}^{vol} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K}\,, \qquad \textit{where } |K| = \mathcal{L}^n(K)$$

and call it discrete volumetric varifold. *We remark that discrete volumetric d–varifolds are typically not d–rectifiable (indeed their support is n–rectifiable, while $d < n$).*

We can similarly define point cloud varifolds.

Definition 1.4 (Point cloud varifolds). *Let $\{x_i\}_{i=1\dots N} \subset \mathbb{R}^n$ be a point cloud, weighted by the masses $\{m_i\}_{i=1\dots N}$ and provided with directions $\{P_i\}_{i=1\dots N} \subset G_{d,n}$. We associate the collection of triplets $\{(x_i, m_i, P_i) : i = 1, \dots, N\}$ with the* point cloud d–varifold

$$V^{pt} = \sum_{i=1}^{N} m_i \, \delta_{x_i} \otimes \delta_{P_i} \, .$$

Of course, a point cloud varifold is not d-rectifiable as its support is zero-dimensional.

The idea behind these "unstructured" types of discrete varifolds is that they can be used to discretize more general varifolds. For instance, given a d–varifold V, and defining

$$m_K = \|V\|(K) \text{ and } P_K \in \operatorname*{arg\,min}_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - S\| \, dV(x, S) \, ,$$

one obtains a volumetric approximation of V. Similarly one can construct a point cloud approximation of V. The possibility of switching between discrete volumetric varifolds and point cloud varifolds, up to a controlled error depending on the size of a given mesh, is shown in the following proposition.

Proposition 1.5 ([4]). *Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider a mesh $\mathcal{K}$ of Ω of size $\delta = \sup_{K \in \mathcal{K}} \operatorname{diam} K$ and a family $\{x_K, m_K, P_K\}_{K \in \mathcal{K}} \subset \mathbb{R}^n \times \mathbb{R}_+ \times G_{d,n}$ such that $x_K \in K$, for all $K \in \mathcal{K}$. Define the volumetric varifold $V_{\mathcal{K}}^{vol}$ and the point cloud varifold $V_{\mathcal{K}}^{pt}$ as*

$$V_{\mathcal{K}}^{vol} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \text{ and } V_{\mathcal{K}}^{pt} = \sum_{K \in \mathcal{K}} m_K \delta_{x_K} \otimes \delta_{P_K} \, .$$

Then, for any open set $U \subset \Omega$ one obtains

$$\Delta_U^{1,1}(V_{\mathcal{K}}^{vol}, V_{\mathcal{K}}^{pt}) \leq \delta \min \left(\|V_{\mathcal{K}}^{vol}\|(U^\delta), \|V_{\mathcal{K}}^{pt}\|(U^\delta) \right)$$

Proof. Let $\varphi \in \operatorname{Lip}_1(\mathbb{R}^n \times G_{d,n})$ such that $\operatorname{spt} \varphi \subset U$, then

$$\begin{aligned}
\left| \int_{U \times G_{d,n}} \varphi \, dV_{\mathcal{K}}^{vol} - \int_{U \times G_{d,n}} \varphi \, dV_{\mathcal{K}}^{pt} \right| &= \left| \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \int_K \varphi(x, P_K) \mathcal{L}^n(x) - \sum_{K \in \mathcal{K}} m_K \varphi(x_K, P_K) \right| \\
&\leq \sum_{\substack{K \in \mathcal{K} \\ U \cap K \neq \emptyset}} m_K \fint_K \left| \varphi(x, P_K) - \varphi(x_K, P_K) \right| \, d\mathcal{L}^n(x) \\
&\leq \sum_{\substack{K \in \mathcal{K} \\ U \cap K \neq \emptyset}} \fint_K \underbrace{\operatorname{lip}(\varphi)}_{\leq 1} |x - x_K| \, d\mathcal{L}^n(x) \leq \delta \sum_{\substack{K \in \mathcal{K} \\ U \cap K \neq \emptyset}} m_K \\
&= \delta \, \|V_{\mathcal{K}}^{vol}\| \left(\bigcup_{U \cap K \neq \emptyset} K \right) = \delta \, \|V_{\mathcal{K}}^{pt}\| \left(\bigcup_{U \cap K \neq \emptyset} K \right) \\
&\leq \delta \, \min \left(\|V_{\mathcal{K}}^{vol}\|(U^\delta), \|V_{\mathcal{K}}^{pt}\|(U^\delta) \right) \, ,
\end{aligned}$$

which concludes the proof. □

Remark 1.6. We note that the first variation of a point cloud varifold is not a measure but only a distribution: indeed it is obtained by directional differentiation of a weighted sum of Dirac deltas. On the other hand, the first variation of a discrete volumetric varifold is bounded as soon as the cells in $\mathcal{K}$ have a boundary with $\mathcal{H}^{n-1}$ finite measure (or even as soon as the cells in $\mathcal{K}$ have finite perimeter), but its total variation typically blows up as the size of the mesh goes to zero (see for instance Example 6 in [7]). Nevertheless, this bad behavior of the first variation, when applied to discrete varifolds, can be somehow controlled via regularization, as described in Section 2.

2 Regularized First Variation

Given a sequence of varifolds $(V_i)_i$ weakly–$\star$ converging to a varifold V, a sufficient condition for V to have locally bounded first variation, i.e. for δV to be a Radon measure, is

$$\sup_i \|\delta V_i\| < +\infty \,. \tag{2.1}$$

However, the typical sequences of discrete varifolds that have been introduced in Section 1.1 may not have uniformly bounded first variations, or it may even happen that the first variations themselves are not measures, as in the case of point cloud varifolds (see Remark 1.6). Nevertheless, δV_i are distributions of order 1 converging to δV (in the sense of distributions). The idea is to compose the first variation operator δ with convolutions defined by a sequence of regularizing kernels $(\rho_{\varepsilon_i})_{i\in\mathbb{N}}$ as in Definition 2.1 below, and then to require a uniform control on the L^1-norm of $\delta V_i \star \rho_{\varepsilon_i}$.

We also point out that the parameter ε_i may be viewed as a "scale parameter".

Besides some technical results, that will be used in the next sections, we prove in Theorem 2.6 a compactness and rectifiability result, which relies on the assumption that $\delta V_i \star \rho_{\varepsilon_i}$ is uniformly bounded in L^1.

As we are going to regularize the first variation of a varifold V in Ω by convolution, we conveniently extend δV to a linear and continuous form on $\mathrm{C}^1_c(\mathbb{R}^n,\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with mass $\|V\|(\Omega) < +\infty$. First of all, we notice that $(x,S) \mapsto \mathrm{div}_S X(x)$ is continuous and bounded, and that V is a finite Radon measure, thus we set

$$\delta V(X) = \int_{\Omega\times G_{d,n}} \mathrm{div}_S X(x)\, dV(x,S), \qquad \forall X \in \mathrm{C}^1_c(\mathbb{R}^n,\mathbb{R}^n). \tag{2.2}$$

For more simplicity, in (2.2) the extended first variation is denoted as the standard first variation. We immediately obtain

$$\delta V(X) \le \|X\|_{\mathrm{C}^1}\, \|V\|(\Omega)\,, \qquad \forall X \in \mathrm{C}^1_c(\mathbb{R}^n,\mathbb{R}^n)\,,$$

which means that the linear extension is continuous with respect to the C^1-norm. Notice that the extended first variation coincides with the standard first variation whenever $X \in \mathrm{C}^1_c(\Omega,\mathbb{R}^n)$ but may contain additional boundary information if the support of $\|V\|$ is not relatively compact in Ω.

For the reader's convenience we recall from Section 1 that ρ denotes a non-negative kernel profile, such that $\rho_1(x) = \rho(|x|)$ is of class C^1, has compact support in $B_1(0)$, and satisfies $\int \rho_1(x)\,dx = 1$. Then, given $\varepsilon > 0$ we set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho_1(x/\varepsilon)$.

Definition 2.1 (regularized first variation)**.** *Given a vector field $X \in \mathrm{C}^1_c(\mathbb{R}^n,\mathbb{R}^n)$, for any $\varepsilon > 0$ we define*

$$\delta V \star \rho_\varepsilon(X) := \delta V(X \star \rho_\varepsilon) = \int_{\Omega\times G_{d,n}} \mathrm{div}_S(X \star \rho_\varepsilon)(y)\, dV(y,S)\,. \tag{2.3}$$

We generically say that $\delta V \star \rho_\varepsilon$ is a regularized first variation *of V.*

Of course (2.3) defines $\delta V \star \rho_\varepsilon$ in the sense of distributions. The following, elementary proposition shows that $\delta V \star \rho_\varepsilon$ is actually represented by a smooth vector field with bounded L^1-norm.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with finite mass $\|V\|(\Omega)$. Then $\delta V \star \rho_\varepsilon$ is represented by the continuous vector field*

$$\delta V \star \rho_\varepsilon(x) = \int_{\Omega\times G_{d,n}} \nabla^S \rho_\varepsilon(y-x)\, dV(y,S) = \frac{1}{\varepsilon^{n+1}} \int_{\Omega\times G_{d,n}} \nabla^S \rho_1\left(\frac{y-x}{\varepsilon}\right) dV(y,S) \tag{2.4}$$

and moreover one has $\delta V \star \rho_\varepsilon \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. Taking into account (2.3), for every $y \in \mathbb{R}^n$ we find $\operatorname{div}_S(X * \rho_\varepsilon)(y) = X * \nabla^S \rho_\varepsilon(y) := \sum_{i=1}^n X_i * \partial_i^S \rho_\varepsilon(y)$, thus by Fubini-Tonelli's theorem we get

$$\begin{aligned}
\delta V * \rho_\varepsilon(X) &= \int_{\Omega \times G_{d,n}} (X * \nabla^S \rho_\varepsilon)(y)\, dV(y,S) \\
&= \int_{\Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} X(x) \cdot \nabla^S \rho_\varepsilon(y-x)\, d\mathcal{L}^n(x)\, dV(y,S) \\
&= \int_{x \in \mathbb{R}^n} X(x) \cdot \left(\int_{\Omega \times G_{d,n}} \nabla^S \rho_\varepsilon(y-x)\, dV(y,S) \right) d\mathcal{L}^n(x) ,
\end{aligned}$$

which proves (2.4). The fact that $\delta V * \rho_\varepsilon \in \mathrm{L}^1(\mathbb{R}^n;\mathbb{R}^n)$ is an immediate consequence of the fact that $\nabla \rho_\varepsilon$ is bounded on $\mathbb{R}^n$. □

Remark 2.3. We stress that $\delta V * \rho_\varepsilon$ is in $L^1(\mathbb{R}^n)$ even when δV is not locally bounded.

Remark 2.4. If the support of $\|V\|$ is compactly contained in Ω then using the extended or the standard first variation in the convolution $\delta V * \rho_\varepsilon$ is equivalent up to choosing ε small enough. In general, the same equivalence holds up to restricting the distribution $\delta V * \rho_\varepsilon$ to $\mathrm{C}_c^1(\Omega_\varepsilon, \mathbb{R}^n)$, where $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, which amounts to restricting the function $\delta V * \rho_\varepsilon$ to Ω_ε.

In the next proposition we show that the classical first variation of a varifold V is the weak–* limit of regularized first variations of V, under the assumption that δV is a bounded measure. This will immediately follow from the basic estimate (2.5), which is true for all varifolds.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a varifold in Ω with $\|V\|(\Omega) < +\infty$. Then for any $X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\left|\delta V * \rho_\varepsilon(X) - \delta V(X)\right| \le \|V\|\Big(\Omega \cap \big(\operatorname{spt} X + B_\varepsilon(0)\big)\Big)\, \|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \xrightarrow[\varepsilon \to 0]{} 0 . \tag{2.5}$$

Moreover, if V has bounded extended first variation then

$$\delta V * \rho_\varepsilon \xrightarrow[\varepsilon \to 0]{*} \delta V . \tag{2.6}$$

Proof. Let $X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$. Since $\delta V * \rho_\varepsilon(X) = \delta V(\rho_\varepsilon * X)$ we obtain

$$\left|\delta V * \rho_\varepsilon(X) - \delta V(X)\right| = \left|\delta V(\rho_\varepsilon * X - X)\right| \le \|V\|(\Omega)\, \|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} .$$

On observing that $\|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \xrightarrow[\varepsilon \to 0]{} 0$ we get (2.5). If in addition V has bounded extended first variation, then for all $X \in \mathrm{C}_c^0(\mathbb{R}^n, \mathbb{R}^n)$ we obtain

$$\left|\delta V * \rho_\varepsilon(X) - \delta V(X)\right| \le \|\delta V\|\, \|\rho_\varepsilon * X - X\|_\infty \xrightarrow[\varepsilon \to 0]{} 0 ,$$

which proves (2.6). □

The next theorem is a partial generalization of Allard's compactness theorem for rectifiable varifolds. It shows that, given a sequence $(\varepsilon_i)_i$ of positive numbers and a sequence of d-varifolds $(V_i)_i$ with uniformly bounded total masses, such that $\delta V_i * \rho_{\varepsilon_i}$ satisfies a uniform boundedness assumption, there exists a subsequence of V_i that weakly-* converges to a limit varifold V with bounded first variation. If in addition $\|V_i\|(B_r(x)) \ge \theta_0 r^d$ for $\|V\|$-almost every x and for $\beta_i \le r \le r_0$, with $(\beta_i)_{i\in\mathbb{N}}$ an infinitesimal sequence, then the limit varifold V is rectifiable. We stress that V_i is required neither to have bounded first variation, nor to be rectifiable. Notice also the appearance of the scale parameters β_i providing infinitesimal lower bounds on the radii to be used for approximate density estimates.

Theorem 2.6 (compactness and rectifiability). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(V_i)_i$ be a sequence of d-varifolds. Assume that there exists a positive, decreasing and infinitesimal sequence $(\varepsilon_i)_i$, such that*

$$M := \sup_{i\in\mathbb{N}} \left\{ \|V_i\|(\Omega) + \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} \right\} < +\infty \,. \tag{2.7}$$

Then there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$\star$ converging in Ω to a d-varifold V with bounded first variation, such that $\|V\|(\Omega)+|\delta V|(\Omega) \le M$. Moreover, if we further assume the existence of an infinitesimal sequence $\beta_i \downarrow 0$ and $\theta_0, r_0 > 0$ such that, for any $\beta_i < r < r_0$ and for $\|V_i\|$-almost every $x \in \Omega$,

$$\|V_i\|(B_r(x)) \ge \theta_0 r^d \,, \tag{2.8}$$

then V is rectifiable.

Proof. Since M is finite, there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$\star$ converging in Ω to a varifold V. By Proposition 2.5, for any $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$ we obtain

$$\begin{aligned}
\left|\delta V_{\varphi(i)} * \rho_{\varepsilon_{\varphi(i)}}(X) - \delta V(X)\right| &\le \left|\delta V_{\varphi(i)} * \rho_{\varepsilon_{\varphi(i)}}(X) - \delta V_{\varphi(i)}(X)\right| + \left|\delta V_{\varphi(i)}(X) - \delta V(X)\right| \\
&\le \underbrace{\|V_i\|(\Omega)}_{\le C < +\infty} \left\|X * \rho_{\varepsilon_{\varphi(i)}} - X\right\|_{\mathrm{C}^1} + \left|\delta V_{\varphi(i)}(X) - \delta V(X)\right| \\
&\xrightarrow[i\to\infty]{} 0 \,.
\end{aligned}$$

Consequently, for any $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$ one has $\left|\delta V(X)\right| \le \sup_i \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} \; \|X\|_\infty$. We conclude that δV extends to a continuous linear form in $\mathrm{C}^0_c(\Omega, \mathbb{R}^n)$ whose norm is bounded by $\sup_i \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1}$, thus $\|V\|(\Omega)+|\delta V|(\Omega) \le M$.

Assuming the additional hypothesis (2.8), it is not difficult to pass to the limit and prove the same inequality for $\|V\|$–a.e. x and for all $0 < r < r_0$. We refer to Proposition 3.3 in [7] for more details on this point. By Theorem 5.5(1) in [2] we obtain the last part of the claim. □

3 Approximate Mean Curvature

3.1 Definition and convergence

We now introduce the notion of *approximate mean curvature* associated with V, in a consistent way with the notion of regularized first variation. We refer to Section 1 for the notations and the basic assumptions on the kernel profiles ρ, ξ. We also set

$$C_\rho = d\,\omega_d \int_0^1 \rho(r) r^{d-1}\, dr \,, \qquad C_\xi = d\,\omega_d \int_0^1 \xi(r) r^{d-1}\, dr \,. \tag{3.1}$$

Definition 3.1 (approximate mean curvature). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω. For every $\varepsilon > 0$ and $x \in \Omega$, such that $\|V\| * \xi_\varepsilon(x) > 0$, we define*

$$H^V_{\rho,\xi,\varepsilon}(x) = -\frac{C_\xi}{C_\rho} \cdot \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \xi_\varepsilon(x)} \,, \tag{3.2}$$

where C_ρ and C_ξ are as in (3.1). *We generically say that the vector $H^V_{\rho,\xi,\varepsilon}(x)$ is an* approximate mean curvature *of V at x.*

Example 3.2 (approximate mean curvature of a point cloud varifold). Let us consider a point cloud varifold $V = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$. We remark that δV is not a measure. An approximate mean curvature of V is given by the formula

$$H^V_{\rho,\xi,\varepsilon}(x) = -\frac{C_\xi}{C_\rho}\cdot\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \xi_\varepsilon(x)} = \frac{C_\xi}{C_\rho \varepsilon}\cdot\frac{\displaystyle\sum_{x_j\in B_\varepsilon(x)\setminus\{x\}} m_j\,\rho'\left(\frac{|x_j-x|}{\varepsilon}\right)\Pi_{P_j}\frac{x_j-x}{|x_j-x|}}{\displaystyle\sum_{x_j\in B_\varepsilon(x)} m_j\,\xi\left(\frac{|x_j-x|}{\varepsilon}\right)}\,. \tag{3.3}$$

The formula is well-defined for instance when $x = x_i$ for some $i = 1, \dots, N$. The choice of ε here is crucial: it must be large enough to guarantee that the ball $B_\varepsilon(x)$ contains points of the cloud different from x, but not too large to avoid over-smoothing.

If δV is locally bounded then we recall the Radon-Nikodym-Lebesgue decomposition (1.1), which says that $\delta V = -H\|V\| + \delta V_s$, where $H = H(x)$ is the generalized mean curvature of V. Note that the approximate mean curvature introduced in Definition 3.1 can be equivalently defined as the Radon–Nikodym derivative of the regularized first variation with respect to the regularized mass of V. When V is rectifiable, it turns out that formula (3.2) gives a pointwise $\|V\|$–almost everywhere approximation of $H(x)$, as proved by the following result.

Theorem 3.3 (Convergence I). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold with locally bounded first variation in Ω. Then for $\|V\|$–almost all $x \in \Omega$ we have*

$$H^V_{\rho,\xi,\varepsilon}(x) \xrightarrow[\varepsilon\to 0]{} H(x)\,. \tag{3.4}$$

Proof sketch. The proof consists of estimating the difference $|H_{\rho,\xi,\varepsilon}(x) - H(x)|$ when x is a differentiability point for δV with respect to $\|V\|$ and, at the same time, the approximate tangent plane to M at x is well-defined. Obviously $\|V\|$-almost all $x \in \Omega$ satisfy these two properties. Then, by the relations

$$\begin{aligned}
\varepsilon^{-d}\|V\|(B_\varepsilon(x)) &\xrightarrow[\varepsilon\to 0]{} \omega_d\theta(x)\,,\\
\varepsilon^{n-d}\|V\| * \rho_\varepsilon(x) &\xrightarrow[\varepsilon\to 0]{} \theta(x)\int_{T_xM}\rho(y)\,d\mathcal{H}^d(y) = C_\rho\theta(x),\\
\varepsilon^{n-d}\|V\| * \xi_\varepsilon(x) &\xrightarrow[\varepsilon\to 0]{} \theta(x)\int_{T_xM}\xi(y)\,d\mathcal{H}^d(y) = C_\xi\theta(x)
\end{aligned}$$

and by

$$\frac{|\delta V_s|(B_\varepsilon(x))}{\|V\|(B_\ell(x))} \xrightarrow[\varepsilon\to 0]{} 0\,,$$

one can show that the above difference is infinitesimal as $\varepsilon \to 0$, thus proving (3.4). See [4] for more details. □

Given a rectifiable d-varifold V with locally bounded first variation and a sequence of generic d-varifolds $(V_i)_i$ weakly–* converging to V, our goal is now to determine an infinitesimal sequence of *regularization scales* $(\varepsilon_i)_i$, in dependence of an infinitesimal sequence $(d_i)_i$ measuring how well the V_i's are locally approximating V, in order to derive an asymptotic, quantitative control of the error between the approximate mean curvatures of V_i and V. In this spirit we obtain two convergence results, Theorem 3.4 and Theorem 3.6 that we describe hereafter.

In Theorem 3.4 we extend the basic convergence property proved in Theorem 3.3. More specifically we show the pointwise convergence of $H^{V_i}_{\rho,\xi,\varepsilon_i}$ to H as $i \to \infty$, up to an infinitesimal offset and for a suitable choice of $\varepsilon_i > 0$ tending to zero as $i \to \infty$. The presence of an offset in the evaluation of $H^{V_i}_{\rho,\xi,\varepsilon_i}$ and H (that is, we compare $H^{V_i}_{\rho,\xi,\varepsilon_i}(z_i)$ with $H(x)$, where z_i is a sequence of points converging to x) is motivated by the fact that we do not have spt $\|V_i\| \subset$ spt $\|V\|$ in general. Moreover, in typical applications one first constructs the varifold V_i (which for instance could be a varifold solving some "discrete approximation" of a geometric variational problem or PDE) and then, by possibly applying Theorem 2.6, one infers the existence of a limit varifold V of the sequence $(V_i)_i$, up to extraction of a subsequence. In this sense, V_i is typically explicit while V is not. We also provide in (3.7) an asymptotic, quantitative estimate of the gap between $H^{V_i}_{\rho,\xi,\varepsilon_i}(z_i)$ and $H^{V}_{\rho,\xi,\varepsilon_i}(x)$ (notice that for this estimate we take the ε_i-regularized mean curvatures for both varifolds V_i and V) in terms of the parameters ε_i, d_i and of the offset $|x-z_i|$. We stress that the regularity of V that is assumed in Theorem 3.4 is in some sense minimal (for instance the singular part δV_s of the first variation may not be zero). The price to pay for such a generality is a non-optimal convergence rate, which can be improved under stronger regularity assumptions on V and by using a modified notion of approximate mean curvature (see Definition 3.5 and Theorem 3.6).

From now on we require a few extra regularity on the pair of kernel profiles (ρ, ξ), according to the following hypothesis.

Hypothesis 1. *We say that the pair of kernel profiles (ρ, ξ) satisfy Hypothesis 1 if ρ, ξ are as specified at the beginning of Section 1 and, moreover, ρ is of class $W^{2,\infty}$ while ξ is of class $W^{1,\infty}$.*

The next result represents a quantitative improvement of Theorem 3.3, in that it provides an explicit estimate of the error between the approximate mean curvature of a member V_i of a sequence of varifolds converging to a limit varifold V, and the approximate mean curvature of V. The quantification takes into account a suitable estimate on the localized $\Delta^{1,1}$ distance between V_i and V, as well as an estimate on an offset $|x - z_i|$. Here the choice of the sequence of regularization scales ε_i appears to be deeply linked to the previous estimates. The proof (that we do not recall here, but we refer to [4] for all the details) is more technical than the one of Theorem 3.3, even though some similarities appear at various points.

Theorem 3.4 (Convergence II). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold in Ω with bounded first variation. Let (ρ, ξ) satisfy Hypothesis 1. Let $(V_i)_i$ be a sequence of d–varifolds, for which there exist two positive, decreasing and infinitesimal sequences $(\eta_i)_i$, $(d_i)_i$, such that for any ball $B \subset \Omega$ centered in* spt $\|V\|$, *one has*

$$\Delta^{1,1}_B(V, V_i) \le d_i \min\left(\|V\|(B^{\eta_i}), \|V_i\|(B^{\eta_i})\right) . \tag{3.5}$$

For $\|V\|$–almost any $x \in \Omega$ and for any sequence $(z_i)_i$ tending to x, let $(\varepsilon_i)_i$ be a positive, decreasing and infinitesimal sequence such that

$$\frac{d_i + |x - z_i|}{\varepsilon_i^2} \xrightarrow[i\to\infty]{} 0 \quad \textit{and} \quad \frac{\eta_i}{\varepsilon_i} \xrightarrow[i\to\infty]{} 0 . \tag{3.6}$$

Then we have

$$\left| H^{V_i}_{\rho,\xi,\varepsilon_i}(z_i) - H^{V}_{\rho,\xi,\varepsilon_i}(x) \right| \le C\|\rho\|_{W^{2,\infty}} \frac{d_i + |x - z_i|}{\varepsilon_i^2} \qquad \textit{for i large enough}, \tag{3.7}$$

$$H^{V_i}_{\rho,\xi,\varepsilon_i}(z_i) \xrightarrow[i\to\infty]{} H(x) . \tag{3.8}$$

Below we quote a third, pointwise convergence result where an even better convergence rate shows up when the limit varifold is (locally) a manifold M of class C^2 with multiplicity $= 1$. First we notice that $H^{V}_{\rho,\xi,\varepsilon}(x)$ is obtained as an integration of tangential vectors, while the (classical) mean curvature of M is a normal vector. This means that even small errors affecting the mass distribution of the approximating varifolds V_i might lead to non-negligible errors in the tangential components of the approximate mean curvature. A workaround for this is, then, to project $H^{V}_{\rho,\xi,\varepsilon}(x)$ onto the normal space at x. In order to properly define the orthogonal component of the mean curvature of a general varifold V, we recall Proposition 1.1:

$$V(\varphi) = \int_{x\in\Omega} \int_{P\in G_{d,n}} \varphi(x, P)\, d\nu_x(P)\, d\|V\|(x), \qquad \forall\, \varphi \in C^0_c(\Omega \times G_{d,n}) .$$

Now we introduce the following definition.

Definition 3.5 (orthogonal approximate mean curvature). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω. For $\|V\|$-almost every x an orthogonal approximate mean curvature of V at x is defined as*

$$H^{V,\perp}_{\rho,\xi,\varepsilon}(x) = \int_{P\in G_{d,n}} \Pi_{P^\perp}\left(H^V_{\rho,\xi,\varepsilon}(x)\right)\, dv_x(P)\,. \tag{3.9}$$

Theorem 3.6 below establishes a better convergence rate under stronger regularity assumptions on V and sufficient accuracy in the approximation of V by V_i. For its proof we refer the reader to [4].

Theorem 3.6 (Convergence III). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $M \subset \Omega$ be a d–dimensional submanifold of class C^2 without boundary, and let $V = v(M, 1)$ be the rectifiable d–varifold in Ω associated with M, with multiplicity 1. Let us extend T_yM to a C^1 map $\widetilde{T_yM}$ defined in a tubular neighbourhood of M. Let $(V_i)_i$ be a sequence of d–varifolds in Ω. Let (ρ, ξ) satisfies Hypothesis 1. Let $x \in M$ and let $(z_i)_i \subset \Omega$ be a sequence tending to x and such that $z_i \in \operatorname{spt}\|V_i\|$. Assume that there exist positive, decreasing and infinitesimal sequences $(\eta_i)_i$, $(d_{1,i})_i$, $(d_{2,i})_i$, $(\varepsilon_i)_i$, such that for any ball $B \subset \Omega$ centered in $\operatorname{spt}\|V\|$ and contained in a neighbourhood of x, one has*

$$\Delta^{1,1}_B(\|V\|, \|V_i\|) \le d_{1,i} \min\left(\|V\|(B^{\eta_i}), \|V_i\|(B^{\eta_i})\right)\ , \tag{3.10}$$

and, recalling the decomposition $V_i = \|V_i\| \otimes v^i_x$,

$$\sup_{\{y\in B_{\varepsilon_i+|x-z_i|}(x)\cap \operatorname{spt}\|V_i\|\}} \int_{S\in G_{d,n}} \|\widetilde{T_yM} - S\|\, dv^i_y(S) \le d_{2,i}\,. \tag{3.11}$$

Then, there exists $C > 0$ such that

$$\left|H^{V_i,\perp}_{\rho,\xi,\varepsilon_i}(z_i) - H^{V,\perp}_{\rho,\xi,\varepsilon_i}(x)\right| \le C\frac{d_{1,i} + d_{2,i} + |x - z_i|}{\varepsilon_i}\,. \tag{3.12}$$

Moreover, if we also assume that $d_{1,i} + d_{2,i} + \eta_i + |x - z_i| = o(\varepsilon_i)$ as $i \to \infty$, then

$$H^{V_i,\perp}_{\rho,\xi,\varepsilon_i}(z_i) \xrightarrow[i\to\infty]{} H(x)\,.$$

4 Approximating a varifold by discrete varifolds

In this section, we show that the family of discrete volumetric varifolds and the family of point cloud varifolds approximate well the space of rectifiable varifolds in the sense of weak–$\star$ convergence, or $\Delta^{1,1}$ metric. Moreover, we give a way of quantifying this approximation in terms of the mesh size and the mean oscillation of tangent planes. Our construction starts with the following result.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω. Let $(\mathcal{K}_i)_{i\in\mathbb{N}}$ be a sequence of meshes of Ω, and set*

$$\delta_i = \sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \quad \forall\, i \in \mathbb{N}\,.$$

Then, there exists a sequence of discrete (point cloud or volumetric) varifolds $(V_i)_i$ such that for any open set $U \subset \Omega$,

$$\Delta^{1,1}_U(V, V_i) \le \delta_i\|V\|(U^{\delta_i}) + \sum_{K\in\mathcal{K}_i} \min_{P\in G_{d,n}} \int_{(U^{\delta_i}\cap K)\times G_{d,n}} \|P - S\|\, dV(x, S)\,. \tag{4.1}$$

Proof. We define V_i as either the volumetric varifold

$$V_i = \sum_{K\in\mathcal{K}_i} \frac{m^i_K}{|K|}\mathcal{L}^n \otimes \delta_{P^i_K}\,,$$

or the point cloud varifold

$$V_i = \sum_{K\in\mathcal{K}_i} m^i_K \delta_{x^i_K} \otimes \delta_{P^i_K}\,,$$

with

$$m_K^i = \|V\|(K), \quad x_K^i \in K \quad \text{and} \quad P_K^i \in \operatorname*{arg\,min}_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - S\| \; dV(x, S)\,.$$

Let us now explain the proof for the case of volumetric varifolds, as it is completely analogous in the case of point cloud varifolds. For any open set $U \subset \Omega$ and $\varphi \in \mathrm{Lip}_1(\mathbb{R}^n \times G_{d,n})$ with $\operatorname{spt} \varphi \subset U \times G_{d,n}$, we set

$$\Delta_i(\varphi) = \int_{\Omega \times G_{d,n}} \varphi \, dV_i - \int_{\Omega \times G_{d,n}} \varphi \, dV$$

and obtain

$$\begin{aligned}
|\Delta_i(\varphi)| &= \left| \sum_{K \in \mathcal{K}_i} \int_{K \cap U} \varphi(x, P_K^i) \frac{\|V\|(K)}{|K|} \, d\mathcal{L}^n(x) - \sum_{K \in \mathcal{K}_i} \int_{(K \cap U) \times G_{d,n}} \varphi(y, T) \, dV(y, T) \right| \\
&\le \sum_{\substack{K \in \mathcal{K}_i \\ K \cap U \neq \emptyset}} \fint_{x \in K} \int_{(y,T) \in K \times G_{d,n}} \underbrace{\left| \varphi(x, P_K^i) - \varphi(y, T) \right|}_{\le (|x-y| + \|P_K^i - T\|)} \, dV(y, T) \, d\mathcal{L}^n(x) \\
&\le \delta_i \sum_{\substack{K \in \mathcal{K}_i \\ K \cap U \neq \emptyset}} \|V\|(K) + \sum_{\substack{K \in \mathcal{K}_i \\ K \cap U \neq \emptyset}} \int_{K \times G_{d,n}} \left\| P_K^i - T \right\| \, dV(y, T) \\
&\le \delta_i \, \|V\|(U^{\delta_i}) + \sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(U^{\delta_i} \cap K) \times G_{d,n}} \|P - T\| \; dV(y, T)\,,
\end{aligned}$$

which concludes the proof up to taking the supremum of $\Delta_i(\varphi)$ over φ. □

In Theorem 4.4 below we show that rectifiable varifolds can be approximated by discrete varifolds. Moreover we get explicit convergence rates under the following regularity assumption.

Definition 4.2 (piecewise $C^{1,\beta}$ varifold). *Let S be a d-rectifiable set, θ be a positive Borel function on S, and $\beta \in (0, 1]$. We say that the rectifiable d–varifold $V = v(S, \theta)$ is* piecewise $C^{1,\beta}$ *if there exist $R > 0$, $C \ge 1$ and a closed set $\Sigma \subset S$ such that the following properties hold:*

- *(Ahlfors-regularity of S) for all $x \in S$ and $0 < r < R$*

$$C^{-1} r^d \le \mathcal{H}^d(S \cap B(x, r)) \le C r^d\,; \tag{4.2}$$

- *(Ahlfors-regularity of Σ) for all $z \in \Sigma$ and $0 < r < R$*

$$C^{-1} r^{d-1} \le \mathcal{H}^{d-1}(\Sigma \cap B(z, r)) \le C r^{d-1}\,; \tag{4.3}$$

- *($C^{1,\beta}$ regularity of $S \setminus \Sigma$) the function*

$$\tau(r) = \sup\{\|T_y S - T_z S\| : \; y, z \in S \cap B(x, r), \; x \in S \text{ with } \operatorname{dist}(x, \Sigma) > Cr\}$$

  satisfies

$$\tau(r) \le C\, r^\beta \qquad \forall\, 0 < r < R\,; \tag{4.4}$$

- *for all $0 < r < \varepsilon < R$ and all $z \in \Sigma$*

$$C^{-1} r \, \mathcal{H}^{d-1}(\Sigma \cap B(z, \varepsilon)) \le \mathcal{H}^d(S \cap [\Sigma]_r \cap B(z, \varepsilon)) \le C\, r\, \mathcal{H}^{d-1}(\Sigma \cap B(z, \varepsilon))\,. \tag{4.5}$$

- *for $\mathcal{H}^d$-almost all $x \in S$ we have*

$$C^{-1} \le \theta(x) \le C\,. \tag{4.6}$$

Remark 4.3. We note that varifolds associated with Almgren's $(\mathbf{M}, \varepsilon, \delta)$-minimal sets of dimension 1 and 2 in $\mathbb{R}^3$ are piecewise $C^{1,\beta}$, as a consequence of Taylor's regularity theory [23]. See also [10, 14, 18]. Of course, the family of rectifiable varifolds in $\mathbb{R}^3$ that are piecewise $C^{1,\beta}$ is much larger than $(\mathbf{M}, \varepsilon, \delta)$-minimal sets.

Hereafter we prove an approximation result for rectifiable d–varifolds, that becomes quantitative as soon as the varifolds are assumed to be piecewise $C^{1,\beta}$ in the sense of Definition 4.2. In order to avoid a heavier, localized form of Definition 4.2 we take $\Omega = \mathbb{R}^n$. We also choose to provide the full proof (taken from [4]) as this results has potential applications to geometric variational problems.

Theorem 4.4. *Let $(\mathcal{K}_i)_{i\in\mathbb{N}}$ be a sequence of meshes of $\mathbb{R}^n$, set $\delta_i = \sup_{K\in\mathcal{K}_i} \operatorname{diam}(K)$ for all $i \in \mathbb{N}$ and assume that $\delta_i \to 0$ as $i \to \infty$. Let $V = v(M, \theta)$ be a rectifiable d–varifold in $\mathbb{R}^n$ with $\|V\|(\mathbb{R}^n) < +\infty$. Then there exists a sequence of discrete (volumetric or point cloud) varifolds $(V_i)_i$ with the following properties:*
(i) $\Delta^{1,1}(V_i, V) \to 0$ as $i \to \infty$;
(ii) If V is piecewise $C^{1,\beta}$ in the sense of Definition 4.2 then there exist constants $C, R > 0$ such that for all balls B with radius $r_B \in (0, R)$ centered on the support of $\|V\|$ one has

$$\Delta_B^{1,1}(V_i, V) \le C\left(\delta_i^\beta + \frac{\delta_i}{r_B + \delta_i}\right) \|V\|(B^{C\delta_i}) \tag{4.7}$$

and

$$\Delta^{1,1}(V_i, V) \le C\left(\delta_i^\beta + \frac{\delta_i}{R}\right) \|V\|(\mathbb{R}^n) \tag{4.8}$$

Proof. The proof is split into some steps.
Step 1. We show that for all i there exists $A^i : \mathbb{R}^n \to L(\mathbb{R}^n; \mathbb{R}^n)$ constant in each cell $K \in \mathcal{K}_i$, such that

$$\int_{\mathbb{R}^n\times G_{d,n}} \left\|A^i(y) - \Pi_T\right\| \, dV(y, T) = \int_{y\in\mathbb{R}^n} \left\|A^i(y) - \Pi_{T_yM}\right\| \, d\|V\|(y) \xrightarrow[i\to+\infty]{} 0 \,. \tag{4.9}$$

Indeed, let us fix $\varepsilon > 0$. Since $x \mapsto \Pi_{T_xM} \in \mathrm{L}^1(\mathbb{R}^n, L(\mathbb{R}^n; \mathbb{R}^n), \|V\|)$, there exists a Lipschitz map $A : \mathbb{R}^n \to L(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\int_{y\in\mathbb{R}^n} \left\|A(y) - \Pi_{T_yM}\right\| \, d\|V\|(y) < \varepsilon \,.$$

For all i and $K \in \mathcal{K}_i$, define for $x \in K$,

$$A^i(x) = A^i_K = \frac{1}{\|V\|(K)} \int_K A(y) \, d\|V\|(y) \,.$$

Then

$$\begin{aligned}\int_{y\in\mathbb{R}^n} \left\|A^i(y) - \Pi_{T_yM}\right\| \, d\|V\|(y) &\le \int_{y\in\mathbb{R}^n} \left\|A^i(y) - A(y)\right\| \, d\|V\|(y) + \int_{y\in\mathbb{R}^n} \left\|A(y) - \Pi_{T_yM}\right\| \, d\|V\|(y) \\ &\le \varepsilon + \sum_{K\in\mathcal{K}_i} \int_{y\in K} \left\| \frac{1}{\|V\|(K)} \int_K A(u) \, d\|V\|(u) - A(y) \right\| \, d\|V\|(y) \\ &\le \varepsilon + \sum_{K\in\mathcal{K}_i} \frac{1}{\|V\|(K)} \int_{y\in K} \int_{u\in K} \left\|A(u) - A(y)\right\| \, d\|V\|(u) \, d\|V\|(y) \\ &\le \varepsilon + \delta_i \operatorname{lip}(A) \|V\|(\mathbb{R}^n) \le 2\varepsilon \text{ for } i \text{ large enough,}\end{aligned}$$

which proves (4.9).
Step 2. Here we make the result of Step 1 more precise, i.e., for all i, we prove that there exists $T^i : \mathbb{R}^n \to G_{d,n}$ constant in each cell $K \in \mathcal{K}_i$ such that

$$\int_{\mathbb{R}^n\times G_{d,n}} \left\|T^i(y) - T\right\| \, dV(y, T) = \int_{y\in\mathbb{R}^n} \left\|T^i(y) - T_yM\right\| \, d\|V\|(y) \xrightarrow[i\to+\infty]{} 0 \,. \tag{4.10}$$

Indeed, let $\varepsilon > 0$ and, thanks to Step 1, take i large enough and $A^i : \mathbb{R}^n \to L(\mathbb{R}^n; \mathbb{R}^n)$ as in (4.9), such that

$$\sum_{K\in\mathcal{K}_i} \int_K \left\|A^i(y) - \Pi_{T_yM}\right\| \, d\|V\|(y) < \varepsilon \,.$$

As a consequence we find

$$\int_K \left\| A^i(y) - \Pi_{T_y M} \right\| \, d\|V\|(y) = \varepsilon_K^i \quad \text{with} \quad \sum_{K \in \mathcal{K}_i} \varepsilon_K^i < \varepsilon .$$

In particular, for all $K \in \mathcal{K}_i$, there exists $y_K \in K$ such that

$$\left\| A^i(y_K) - \Pi_{T_{y_K} M} \right\| \le \frac{\varepsilon_K^i}{\|V\|(K)} .$$

Define $T^i : \mathbb{R}^n \to G_{d,n}$ by $T^i(y) = T_{y_K} M$ for $K \in \mathcal{K}_i$ and $y \in K$, hence T^i is constant in each cell K and

$$\begin{aligned} \int_{\mathbb{R}^n \times G_{d,n}} \left\| T^i(y) - T \right\| dV(y, T) &= \sum_{K \in \mathcal{K}_i} \int_K \left\| \Pi_{T_{y_K} M} - \Pi_{T_y M} \right\| \, d\|V\|(y) \\ &\le \sum_{K \in \mathcal{K}_i} \int_K \| \Pi_{T_{y_K} M} - \underbrace{A^i(y)}_{=A^i(y_K)} \| \, d\|V\|(y) + \int_{\mathbb{R}^n \times G_{d,n}} \left\| A^i(y) - \Pi_T \right\| dV(y, T) \\ &\le \sum_{K \in \mathcal{K}_i} \int_K \frac{\varepsilon_K^i}{\|V\|(K)} d\|V\|(y) + \varepsilon \le 2\varepsilon , \end{aligned} \tag{4.11}$$

which implies (4.10).

Step 3*: proof of (i)*. We preliminarily show that

$$\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - T\| \, dV(y, T) \xrightarrow[i \to \infty]{} 0. \tag{4.12}$$

Indeed, thanks to Step 2, let $T^i : \mathbb{R}^n \to G_{d,n}$ be such that (4.10) holds. We have

$$\begin{aligned} \sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - T\| \, dV(y, T) &\le \sum_{K \in \mathcal{K}_i} \int_{K \times G_{d,n}} \left\| T_K^i - T \right\| dV(y, T) \\ &= \int_{\mathbb{R}^n \times G_{d,n}} \left\| T^i(y) - T \right\| dV(y, T) \\ &\xrightarrow[i \to +\infty]{} 0 , \end{aligned}$$

which proves (4.12). Then (i) follows by combining (4.12) with Lemma 4.1.

Step 4. Assume that V is piecewise $C^{1,\beta}$ and let $R, C > 0$ be as in Definition 4.2. We shall now prove that for any ball $B \subset \mathbb{R}^n$ centered on the support of $\|V\|$ with radius $r_B < R/2$ and for any infinitesimal sequence $\eta_i \ge C\delta_i$, assuming also i large enough so that $\delta_i \le (R - 2r_B)/(C + 1)$, there exists a decomposition $\mathcal{K}_i = \mathcal{K}_i^{reg} \sqcup \mathcal{K}_i^{sing}$ such that

$$\|T_x S - T_y S\| \le C|x - y|^\beta, \qquad \forall K \in \mathcal{K}_i^{reg}, \ \forall x, y \in K \cap S \tag{4.13}$$

and

$$\|V\| \left(\bigcup \mathcal{K}_i^{sing} \cap B \right) \le C' \frac{\delta_i}{r_B + \eta_i} \|V\|(B^{\eta_i}) . \tag{4.14}$$

Define $\mathcal{K}_i^{sing}$ as the set of $K \in \mathcal{K}_i$ for which $\Sigma^{(C\delta_i)} \cap K$ is non-empty, and set $\mathcal{K}_i^{reg} = \mathcal{K}_i \setminus \mathcal{K}_i^{sing}$. It is immediate to check that (4.13) holds, thanks to (4.4). Let now B be a fixed ball of radius $0 < r_B < R/2$ centered at some point $x \in S$. Take $K \in \mathcal{K}_i^{sing}$ and assume without loss of generality that $K \cap B$ is not empty, hence there exists $p \in K \cap B$ and $z \in \Sigma$ such that $|p - z| < (C + 1)\delta_i$. Consequently, $B \subset B(z, 2r_B + (C + 1)\delta_i)$. Then $K \cap B \subset \Sigma^{(C+1)\delta_i} \cap B(z, 2r_B + (C + 1)\delta_i)$ and thus, assuming in addition that $\eta_i < R - r_B$ for i large enough, we obtain

$$\begin{aligned} \|V\| \left(\bigcup_{K \in \mathcal{K}_i^{sing}} K \cap B \right) &\le \|V\| \left(\Sigma^{(C+1)\delta_i} \cap B(z, 2r_B + (C + 1)\delta_i) \right) \\ &\le C\mathcal{H}^d \left(S \cap \Sigma^{(C+1)\delta_i} \cap B(z, 2r_B + (C + 1)\delta_i) \right) \\ &\le C^2 (C + 1)\delta_i \, \mathcal{H}^{d-1} \left(\Sigma \cap B(z, 2r_B + (C + 1)\delta_i) \right) , \end{aligned} \tag{4.15}$$

thanks to (4.5) and (4.6). On the other hand, since $C\delta_i \le \eta_i < R - r_B$ one has by (4.2), (4.3) and (4.6) that

$$\begin{aligned}\|V\|(B^{\eta_i}) &\ge C^{-1}\mathcal{H}^d(S\cap B^{\eta_i}) \ge C^{-2}(r_B+\eta_i)^d \ge C^{-2}(r_B+\eta_i)(r_B+C\delta_i)^{d-1}\\ &\ge \frac{1}{2^{d-1}C^2(C+1)}\frac{(r_B+\eta_i)}{\delta_i}(C+1)\delta_i\,(2r_B+(C+1)\delta_i)^{d-1}\\ &\ge \frac{1}{2^{d-1}C^3(C+1)}\frac{(r_B+\eta_i)}{\delta_i}C(C+1)\delta_i\mathcal{H}^{d-1}(\Sigma\cap B(z,2r_B+(C+1)\delta_i))\\ &\ge \frac{1}{2^{d-1}C^5(C+1)}\frac{(r_B+\eta_i)}{\delta_i}\|V\|\left(\bigcup_{K\in\mathcal{K}^{sing}} K\cap B\right),\end{aligned}$$

which by (4.15) gives (4.14) with $C' = 2^{d-1}C^5(C+1)$.
Step 5. Define $T^i_K = T_{y_K}M$ for each cell $K\in\mathcal{K}_i$ and for some $y_K\in K$. Set

$$A_i = \sum_{K\in\mathcal{K}_i}\min_{P\in G_{d,n}}\int_{(B\cap K)\times G_{d,n}}\|P-T\|\;dV(y,T)\,.$$

Then for every ball B of radius $r>0$, and choosing $\eta_i = C\delta_i$, we have

$$\begin{aligned}A_i &= \sum_{K\in\mathcal{K}_i}\int_{K\cap B}\|T_{y_K}M - T_yM\|\;d\|V\|(y)\\ &= \sum_{K\in\mathcal{K}_i^{reg}}\int_{K\cap B}\|T_{y_K}M - T_yM\|\;d\|V\|(y) + \sum_{K\in\mathcal{K}_i^{sing}}\int_{K\cap B}\|T_{y_K}M - T_yM\|\;d\|V\|(y)\\ &\le \sum_{K\in\mathcal{K}_i^{reg}}\int_{K\cap B}C|y_K-y|^\beta\,d\|V\|(y) + 2\|V\|\left(\bigcup\mathcal{K}_i^{sing}\cap B\right)\\ &\le C\delta_i^\beta\|V\|(B) + 2\|V\|\left(\bigcup\mathcal{K}_i^{sing}\cap B\right) \le C\left(\delta_i^\beta + \frac{\delta_i}{r_B+\eta_i}\right)\|V\|(B^{C\delta_i})\\ &\le C\left(\delta_i^\beta + \frac{\delta_i}{r_B+\delta_i}\right)\|V\|(B^{C\delta_i})\end{aligned}\tag{4.16}$$

(the constant C appearing in the various inequalities of (4.16) may change from line to line). Then, the local estimate (4.7) is a consequence of Lemma 4.1 combined with (4.16).
Step 6. For the proof of the global estimate (4.8) we set $r = R/2$ and apply Besicovitch Covering Theorem to the family of balls $\{B_r(x)\}_{x\in M}$, so that we globally obtain a subcovering $\{B_\alpha\}_{\alpha\in I}$ with overlapping bounded by a dimensional constant ζ_n. We notice that I is necessarily a finite set of indices, by the Ahlfors regularity of M. We now set $U = \mathbb{R}^n\setminus M$ and associate to the family $\{B_\alpha\}_{\alpha\in I}\cup\{U\}$ a partition of unity $\{\psi_\alpha\}_{\alpha\in I}\cup\{\psi_U\}$ of class C^∞, so that by finiteness of I there exists a constant $L\ge 1$ with the property that $\mathrm{lip}(\psi_U)\le L$ and $\mathrm{lip}(\psi_\alpha)\le L$ for all $\alpha\in I$. Moreover, the fact that the support of ψ_U is disjoint from the closure of M implies that there exists i_0 depending only on M, such that the support of $\|V_i\|$ is disjoint from that of ψ_U for every $i\ge i_0$. Then we fix a generic test function $\varphi\in C^0_c(\mathbb{R}^n\times G_{d,n})$ and define $\varphi_\alpha(x,S) = \varphi(x,S)\psi_\alpha(x)$ and $\varphi_U(x,S) = \varphi(x,S)\psi_U(x)$, so that $\varphi(x,S) = \varphi_U(x,S)+\sum_{\alpha\in I}\varphi_\alpha(x,S)$. By the fact that $\mathrm{lip}(\varphi_\alpha)\le\mathrm{lip}(\varphi)+\mathrm{lip}(\psi_\alpha)$ and $\mathrm{lip}(\varphi_U)\le\mathrm{lip}(\varphi)+\mathrm{lip}(\psi_U)$, by the Ahlfors regularity of M, by (4.7), and for $i\ge i_0$, we deduce that

$$\begin{aligned}|V_i(\varphi)-V(\varphi)| &\le \sum_{\alpha\in I}|V_i(\varphi_\alpha)-V(\varphi_\alpha)| \le (1+L)\sum_{\alpha\in I}\Delta^{1,1}_{B_\alpha}(V_i,V)\\ &\le C(1+L)\sum_{\alpha\in I}\left(\delta_i^\beta+\frac{\delta_i}{r}\right)\|V\|(B_\alpha^{C\delta_i}) \le C(1+L)\left(\delta_i^\beta+\frac{\delta_i}{r}\right)\sum_{\alpha\in I}\|V\|(B_\alpha)\\ &\le C(1+L)\zeta_n\left(\delta_i^\beta+\frac{\delta_i}{r}\right)\|V\|(\mathbb{R}^n) \le C\left(\delta_i^\beta+\frac{\delta_i}{R}\right)\|V\|(\mathbb{R}^n)\end{aligned}$$

where, as before, the constant C appearing in the above inequalities can change from one step to the other. This concludes the proof of (4.8) and thus of the theorem.

□

5 Weak Second Fundamental Form of a Varifold

The content of this section refers to the forthcoming paper [3].

Following Hutchinson [16], we begin by recalling a useful way of representing the second fundamental form of a d-dimensional manifold embedded in $\mathbb{R}^n$.

Let M be a smooth, d-dimensional submanifold of $\mathbb{R}^n$ with the standard metric. For $x \in M$, we denote by $P(x)$ the orthogonal projection onto the tangent space $T_x M$; such a projection is represented by the matrix $P_{ij}(x)$ with respect to the standard basis of $\mathbb{R}^n$. The usual covariant derivative in $\mathbb{R}^n$ is denoted by D. Assuming $x \in M$ fixed, and given a vector $v \in T_x\mathbb{R}^n = \mathbb{R}^n$, we let $v^T = P(x)\,v$ and $v^\perp = v - v^T$.

We denote by, respectively, TM and $(TM)^\perp$ the tangential and the normal bundle associated with M, so that we have the splitting $TM \oplus (TM)^\perp = T\mathbb{R}^n$. We also denote by $\Gamma(TM)$ the space of smooth sections of TM (the smooth tangential vector fields) and by $\Gamma(TM)^\perp$ the space of smooth sections of $(TM)^\perp$ (the smooth normal vector fields).

We can now introduce the second fundamental form of M, as the bilinear and symmetric map $II : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)^\perp$ defined as

$$II(u, v) = (D_u v)^\perp .$$

For our purposes it is convenient to extend the second fundamental form in such a way that it can take any pair of (tangent) vectors of $\mathbb{R}^n$ as input. To this end we define the *extended second fundamental form* of M as

$$\mathbf{B}(u, v) = II(u^T, v^T)$$

for all smooth vector fields u, v defined on M with values in $T\mathbb{R}^n$. We set

$$B_{ij}^k = \langle \mathbf{B}(e_i, e_j), e_k \rangle , \tag{5.1}$$

where $\{e_i : i = 1, \dots, n\}$ is the canonical basis of $\mathbb{R}^n$. By tensoriality of the covariant derivative one infers that the coefficient set $\{B_{ij}^k : i, j, k = 1, \dots, n\}$ uniquely identifies $\mathbf{B}$.

An equivalent way of defining the extended second fundamental form is by computing tangential derivatives of the orthogonal projection $P(x)$ on the tangent space $T_x M$. More precisely, let us set

$$A_{ijk}(x) = \langle \nabla^M P_{jk}(x), e_i \rangle \tag{5.2}$$

whenever $x \in M$ and $i, j, k = 1, \dots, n$. It is not difficult to check that

$$A_{ijk} = B_{ij}^k + B_{ik}^j$$

and, reciprocally,

$$B_{ij}^k = \frac{1}{2}\left(A_{ijk} + A_{jik} - A_{kij}\right)$$

at every point of M.

We note for future reference the symmetry properties $A_{ijk} = A_{ikj}$ and $B_{ij}^k = B_{ji}^k$. The symmetry of A_{ijk} follows from $P_{jk} = P_{kj}$. The symmetry of B_{ij}^k relies upon the identity $(D_u v)^\perp = (D_v u)^\perp$, a consequence of the fact that the Levi-Civita connection is torsion-free and that the commutator $[u, v]$ is a tangent vector field whenever u and v are tangent vector fields.

5.1 Generalized Second Fundamental Form of a varifold

Here we recall the definition of *generalized curvature* proposed by Hutchinson [16] (see also the recent reformulation due to Menne [20]).

First of all we consider the easier case of a smooth manifold M with constant multiplicity. We fix a test function $\varphi(x, S)$ defined on $\mathbb{R}^n \times G_{d,n}$ and a d-dimensional manifold M without boundary, then let $P(x)$ be orthogonal projection onto $T_x M$, as before. We define the tangent vector field

$$Y_i(x) = \varphi(x, P(x))\, P(x)(e_i) .$$

By the divergence theorem on M, we get for $i = 1, \dots, n$

$$0 = \int_M \nabla_i^M \varphi + D_{jk}^\star \varphi \underbrace{\nabla_i^M P_{jk}}_{A_{ijk}} + \varphi \underbrace{\nabla_q^M P_{iq}}_{A_{qiq}} .$$

Here we have used Einstein's summation notation and denoted by ∇_i^M the i-th component of the tangential gradient operator, while $D_{jk}^\star$ denotes differentiation with respect to S_{jk}. This identity can be used as a definition of *generalized curvature* for a varifold. Following Hutchinson, we thus say that an integral varifold V is a *curvature varifold* if there exists a family of functions $\{A_{ijk}(x, S)\}$ in $L^1_{loc}(V)$, called *generalized curvature*, such that for all $\varphi \in C_c^1(\mathbb{R}^n \times R^{n^2})$ one has

$$0 = \int \left(\nabla_i^S \varphi + D_{jk}^\star \varphi \, A_{ijk} + \varphi \, A_{qiq} \right) \, dV . \tag{5.3}$$

The notion of curvature varifold has been later extended by Mantegazza [19] to that of *curvature varifold with boundary*. Quite interestingly, it turns out that the boundary measure of a curvature varifold with boundary is $(d-1)$–rectifiable and has an integral multiplicity (this follows from a very nice argument showing first the local orientability of the varifold, and then applying Federer-Fleming's Integrality Theorem for currents). Moreover, the notion of curvature varifold has been shown to be equivalent to the so-called V-weak differentiability of the approximate tangent map (see the recent work by Menne [20]).

However, some important facts concerning Hutchinson's definition should be pointed out in order to explain the obstacles that we encountered while trying to adapt such a notion to the general (and, in particular, discrete) varifold setting.

First, the existence of the generalized curvature is not always guaranteed, and it is not clear from the definition what kind of alternative object (measure, distribution) should be considered as its natural replacement in more general cases.

Second, the uniqueness result proved by Hutchinson (see [16, Proposition 5.2.2]) is based on suitably testing (5.3) with functions of the form $\psi(x, S) = S_{ij}\varphi(x)$, which gives for every $i, j, k = 1, \dots n$

$$0 = \int \left(\nabla_i^S \varphi(x) + A_{ijk}(x, S) + \varphi(x) \, A_{qiq}(x, S) \right) \, dV . \tag{5.4}$$

Moreover, as a byproduct, the proof shows that the curvature functions $A_{ijk}(x, S)$ depend $\|V\|$-almost everywhere only on x and not on S. This simply follows from the fact that, recalling the decomposition $V = \nu_x \otimes \|V\|$, one has $\nu_x = \delta_{P(x)}$ for $\|V\|$-almost all x thanks to the rectifiability of V. Therefore, $A_{ijk}(x, P(x))$ is a curvature function for V not depending upon the variable S.

Third, another comment about uniqueness and existence. In linear algebra it is well-known that uniqueness implies existence. In this sense, Hutchinson's definition is overdetermined in that it mixes conditions for existence of the generalized curvature with constraints on the class of admissible varifolds. Indeed, all curvature varifolds satisfy a very peculiar blow-up property, that is, every tangent varifold of a curvature varifold V, obtained by blowing-up at each points of the support of $\|V\|$, consists of a finite sum of d-planes with integral multiplicities. This means that only a certain kind of singularities for a curvature varifold are admitted, i.e. those of *crossing type*. The following regularity result due to Hutchinson reflects the above tangential property (see [15, Theorem 3.7]).

Theorem 5.1. *Let V be a curvature varifold such that the functions A_{ijk} belong to $L^p_{loc}(\|V\|)$ for $p > d$. Then V is locally a finite sum of graphs of multiple–valued functions of class $C^{1,1-d/p}$.*

For the reasons explained above, Hutchinson's definition of curvature varifold cannot be easily extended to more general varifolds, in the spirit of the regularization technique that we have proposed for the first variation and the mean curvature. Thus, in view of the applications we have in mind, it seems unavoidable to further weaken the original definition in order to guarantee existence of the curvature functions in a distributional sense.

Definition 5.2 (Variations of V). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω. We define for $i, j, k = 1 \dots n$ the following distributions of order 1:*

$$\begin{array}{rcccl} \delta_{jk} V & : & \mathrm{C}^1_c(\Omega, \mathbb{R}^n) & \to & \mathbb{R} \\ & & X & \mapsto & \int_{\Omega \times G_{d,n}} S_{jk} \operatorname{div}_S X(y) \, dV(y, S) \end{array} \tag{5.5}$$

or equivalently,

$$\begin{array}{rcccl} \delta_{ijk} V & : & \mathrm{C}^1_c(\Omega, \mathbb{R}) & \to & \mathbb{R} \\ & & \varphi & \mapsto & \int_{\Omega \times G_{d,n}} S_{jk} \nabla_i^S \varphi \, dV(y, S) \, . \end{array} \tag{5.6}$$

Since for any $S \in G_{d,n}$ we have trace$(S) = d$, we obtain

$$\operatorname{trace} \left(\delta_{jk} V \right)_{jk} = d \, \delta V \, .$$

For a d–varifold V associated with a C^2 compact d–sub-manifold M without boundary, we have for every $\varphi \in \mathrm{C}^1(\Omega)$

$$\begin{aligned} \delta_{ijk} V(\varphi) &= - \int_\Omega \Big(A_{ijk}(x) + P_{jk}(x) \sum_q A_{qiq}(x) \Big) \varphi(x) \, d\|V\|(x) \\ &= - \int_\Omega \Big(A_{ijk}(x) + \int_{G_{d,n}} S_{jk} \, d\nu_x(S) \sum_q A_{qiq}(x) \Big) \varphi(x) \, d\|V\|(x) \end{aligned}$$

Moreover, if M has a boundary ∂M and $\eta = (\eta_1, \dots, \eta_n)$ denotes the inner normal to ∂M, it follows from the divergence theorem that

$$\delta_{ijk} V(\varphi) \;=\; - \int_\Omega \Big(A_{ijk}(x) \;+\; \int_{G_{d,n}} S_{jk} \, d\nu_x(S) \sum_q A_{qiq}(x) \Big) \varphi(x) \, d\|V\|(x) \;-\; \int_{\partial M} \varphi(x) S_{jk} \eta_i(x) \, d\mathcal{H}^{d-1}(x) \;\; .$$

In particular, $\delta_{ijk} V$ is a Radon measure and the second fundamental form is contained in the part absolutely continuous with respect to $\|V\|$ while the boundary term is singular with respect to $\|V\|$. This motivates the following definitions.

Definition 5.3 (Bounded variations). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω. We say that V has* bounded variations *if and only if for $i, j, k = 1 \dots n$, $\delta_{ijk} V$ is a Radon measure. In this case there exist $\beta_{ijk} \in \mathrm{L}^1(\|V\|)$ and $\left(\delta_{ijk} V\right)_s$ Radon measures singular w.r.t. $\|V\|$ such that*

$$\delta_{ijk} V = -\beta_{ijk} \, \|V\| + \left(\delta_{ijk} V \right)_s \;\; .$$

It follows from our calculations above that, in the regular case, the A_{ijk} are connected to the Radon-Nikodym derivative of $\delta_{ijk} V$ w.r.t. $\|V\|$ through the linear equations

$$\beta_{ijk}(x) = A_{ijk}(x) + \int_{G_{d,n}} S_{jk} \, d\nu_x(S) \sum_q A_{qiq}(x) \, . \tag{5.7}$$

Lemma 5.4. *Let c be a $n \times n$ nonnegative definite and symmetric matrix, and let $b = (b_{ijk}) \in \mathbb{R}^{n^3}$. Let us consider the set of n^3 equations of unknowns $(a_{ijk})_{i,j,k=1\dots n}$*

$$a_{ijk} + c_{jk} \sum_q a_{qiq} = b_{ijk} \, , \quad \textit{for } i, j, k = 1 \dots n \, . \tag{5.8}$$

Then the unique solution of the system is

$$a_{ijk} = b_{ijk} - c_{jk} [(I + c)^{-1} H]_i \, , \tag{5.9}$$

where $H = (H_1, \dots, H_n)$ is defined by $H_i := \sum_q b_{qiq}$.

Proof sketch. The proof is split in two steps. First, one has to show that the matrix $I + c$ is invertible (this follows from the fact that $c = \int_{G_{d,n}} S\, dv(S)$ and by an application of Jensen's inequality). Second, by plugging (5.9) into (5.8) and by using the matricial identity

$$(I + c)^{-1} - I + c(I + c)^{-1} = 0\,.$$

□

An immediate consequence of Lemma 5.4 is the following proposition.

Proposition 5.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω with bounded variations $\beta_{ijk} \in \mathrm{L}^1(\|V\|)$. Then the linear system* (5.7) *admits a unique solution $\{A^V_{ijk}\} \subset \mathrm{L}^1(\|V\|)$ that additionally satisfies the symmetry property $A^V_{ijk} = A^V_{ikj}$ for all $i, j, k = 1, \dots, n$. The collection of functions A^V_{ijk} is called weak curvature of V.*

From the definition of weak curvature, that is incorporated in Proposition 5.5, we derive the definition of weak second fundamental form

$$B^V_{ijk} = \frac{1}{2}\left(A_{ijk} + A_{jik} - A_{kij}\right)$$

Similarly as before, we fix three non-negative kernel profiles $\rho, \xi, \eta \in \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1, with the properties listed below:

- for all $t \geq 1$, $\rho(t) = \xi(t) = \eta(t) = 0$;
- ρ is decreasing, $\rho'(0) = 0$, and $\int_{\mathbb{R}^n} \rho(|x|)\, dx = 1$;
- $\xi(t) > 0$ for all $0 < t < 1$, and $\int_{\mathbb{R}^n} \xi(|x|)\, dx = 1$.

Then we set $\rho_\varepsilon, \xi_\varepsilon, \eta_\varepsilon$ as usual. Given any d–varifold $V = \|V\| \otimes \nu_x$ and $\varepsilon > 0$, the following quantities are defined for $\|V\|$–almost every x:

$$\beta^{V,\varepsilon}_{ijk} = -\frac{C_\xi}{C_\rho}\frac{\delta_{ijk} V \star \rho_\varepsilon}{\|V\| \star \xi_\varepsilon}\,, \tag{5.10}$$

$$c^{V,\varepsilon}_{jk} = \frac{\left(\int_{G_{d,n}} S_{jk}\, dv.(S)\|V\|\right) \star \eta_\varepsilon}{\|V\| \star \eta_\varepsilon}\,, \tag{5.11}$$

$$A^{V,\varepsilon}_{ijk} = \beta^{V,\varepsilon}_{ijk} - c^{V,\varepsilon}_{jk}\left[(I + c^{V,\varepsilon})^{-1}\beta^{V,\varepsilon}_{qiq}\right], \tag{5.12}$$

$$B^{V,\varepsilon}_{ijk} = \frac{1}{2}\left(A^{V,\varepsilon}_{ijk} + A^{V,\varepsilon}_{jik} - A^{V,\varepsilon}_{kij}\right)\,. \tag{5.13}$$

We stress that the regularized weak fundamental form can be defined for any varifold V, even when V does not have bounded variations! We state here one of the results proved in the forthcoming paper [3], as an example showing that we can essentially recover similar convergence results as those proved for the approximate mean curvature.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a rectifiable d–varifold with bounded variations. Then, for $\|V\|$–almost any $x \in \Omega$ the quantities $\beta^{V,\varepsilon}(x), c^{V,\varepsilon}(x), M^{V,\varepsilon}(x), A^{V,\varepsilon}(x)$ respectively converge to $\beta^V(x), c^V(x), M^V(x), A^V(x)$ as $\varepsilon \to 0$.*

6 Natural Kernel Pairs and Numerical Tests

6.1 Natural Kernel Pairs

Up to now we have considered generic pairs (ρ, ξ) of kernel profiles, with various regularity assumptions (see Hypothesis 1). One might ask if some special choice of kernel pairs could lead to better convergence rates than those proved in Theorems 3.4 and 3.6. Although the pairs (ρ, ρ) seem quite natural, as they allow for instance some algebraic simplifications in the formula for the ε-mean curvature for a point cloud varifold, from the point of view of numerical convergence rates there are more appropriate choices.

We propose a criterion for selecting the pair (ρ, ξ), that is related to what we define as the *natural kernel pair* property, or shortly (NKP).

Definition 6.1 (Natural Kernel Pair). *We say that (ρ, ξ) is a natural kernel pair, or equivalently that it satisfies the (NKP) property, if it satisfies Hypothesis 1 and*

$$\xi(s) = -\frac{s\rho'(s)}{n} \qquad \text{for all } s \in (0,1)\,. \tag{6.1}$$

Even though it is not clear whether the (NKP) property might produce better convergence rates in the previously mentioned theorems, we have an experimental validation of its effectiveness. Indeed, all the tests that we have performed have shown increased convergence rates, even in presence of noise. We now sketch the heuristic argument leading to Definition 6.1.

Given $1 \le d < n$ and ρ, ξ as in Hypothesis 1 we set

$$C_{\rho,\xi} = \frac{\int_0^1 \rho(t)\, t^{d-1}\, dt}{\int_0^1 \xi(t)\, t^{d-1}\, dt} = \frac{C_\rho}{C_\xi}\,.$$

We fix a d–dimensional submanifold $M \subset \mathbb{R}^n$ of class C^3 and define the associated varifold $V = v(M, 1)$. Then we perform a Taylor expansion of the difference $H^V_{\rho,\xi,\varepsilon}(x) - H(x)$ at a point $x \in M$ (here $H(x)$ denotes the classical mean curvature of M at x). By focusing on the expression of the constant term of this expansion, which must be 0 because of Theorem 3.3, one can see after some computations that it is proportional to

$$\int_0^1 \left(s\rho'(s) + d\, C_{\rho,\xi}\, \xi(s)\right) s^{d-1}\, ds\,,$$

see [4]. On one hand, this integral is 0 for any kernel pair (ρ, ξ), as shown through an integration by parts coupled with the definition of the constant $C_{\rho,\xi}$. On the other hand one might want to strengthen the nullity of the integral by additionally requiring the nullity of the integrand. This precisely amounts to require (6.1) and thus leads to Definition 6.1.

6.2 Numerical Tests

In this section we provide numerical computations of the approximate mean curvature of various $2D$ and $3D$ point clouds. In particular, we illustrate numerically its dependence on the regularization kernel, the regularization parameter ε, and the sampling resolution. Our purpose is not a thorough comparison with the many numerical approaches for computing the mean curvature of point clouds, triangulated meshes, or digital objects, this will be done in a subsequent paper for obvious length reasons.

Given a point cloud varifold $V_N = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$, its orthogonal approximate mean curvature is given by

$$\begin{aligned} H^{V_N,\perp}_{\rho,\xi,\varepsilon}(x_{j_0}) &= \int_{P \in G_{d,n}} \Pi_{P^\perp} H^{V_N}_{\rho,\xi,\varepsilon}(x_{j_0})\, dv_{x_{j_0}}(P) \\ &= -\frac{C_\xi}{C_\rho} \cdot \frac{\sum_{j=1}^N \mathbb{1}_{\{|x_j - x_{j_0}| < \varepsilon\}} m_j \rho'\left(\frac{|x_j - x|}{\varepsilon}\right) \Pi_{P_{j_0}^\perp}\left(\frac{\Pi_{P_j}(x_j - x_{j_0})}{|x_j - x_{j_0}|}\right)}{\sum_{j=1}^N \mathbb{1}_{\{|x_j - x_{j_0}| < \varepsilon\}} m_j \varepsilon \xi\left(\frac{|x_j - x_{j_0}|}{\varepsilon}\right)}\,. \end{aligned} \tag{6.2}$$

We focus on the orthogonal approximate mean curvature, for it is at a given resolution more robust with respect to inhomogeneous local distribution of points than the approximate mean curvature, and as it can even be seen directly on simple examples. Take indeed a sampling $\{x_j\}_1^N$ of the planar line segment $[-1,1]\times\{0\}$ with more points having a negative first coordinate, and let $P_j = P = \{y=0\}$. Assume that there exists j_0 such that $x_{j_0} = (0,0)$. Then the sum of all vectors $\frac{\Pi_{P_j}(x_j - x_{j_0})}{|x_j - x_{j_0}|}$ is nonzero, whereas its projection onto $P^\perp$ is zero, which is consistent with the (mean) curvature of the continuous segment at the origin.

The formula above involves densities m_j, the computation of which for a given point cloud being a question we have not focused on up to now, despite it is an important issue. Nevertheless, if we assume that $m_j = m(1+o(1))$ whenever x_j belongs to the ball B_ε and for some constant m possibly depending on B_ε, then we can cancel m_j from formula (6.2) up to a small error. This justifies the following formula approximating the value of $H^{V_N,\perp}_{\rho,\xi,\varepsilon}(x_{j_0})$:

$$H^{V_N,\perp}_{\rho,\xi,\varepsilon}(x_{j_0}) \simeq \frac{C_\xi}{C_\rho} \cdot \frac{-\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x_{j_0}|<\varepsilon\}} \rho'\left(\frac{|x_j - x|}{\varepsilon}\right) \Pi_{P_{j_0}^\perp}\left(\frac{\Pi_{P_j}(x_j - x_{j_0})}{|x_j - x_{j_0}|}\right)}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x_{j_0}|<\varepsilon\}} \varepsilon\xi\left(\frac{|x_j - x_{j_0}|}{\varepsilon}\right)} . \tag{6.3}$$

The advantages of Formula (6.3) are numerous: it is very easy to compute, it does not require a prior approximation of local length or area, it does *not depend on any orientation* of the point cloud (because the formula is grounded on varifolds which have no orientation) and as we shall see right now, it behaves well from a numerical perspective.

In the next subsection, we study how this formula behaves on $2D$ point cloud varifolds built from parametric curves, for different choices of radial kernels and various sampling resolutions. The last subsection is devoted to $3D$ point clouds.

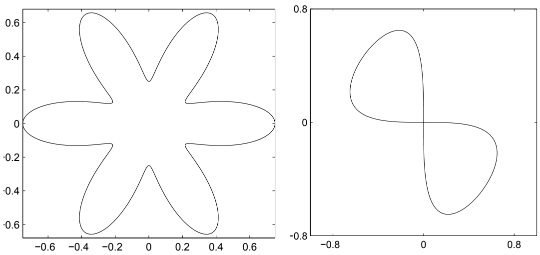
6.2.1 Test shapes, sample point cloud varifolds, and kernel profiles

In [4] we have tested the numerical behavior of formula (6.3) for different choices of 2D parametric shapes, kernel profiles ρ, ξ, number N of points in the cloud, and values of the parameter ε used to define the kernels ρ_ε and ξ_ε. Here we only present a selection of those tests. We denote as N_{neigh} the average number of points in a ball of radius ε centered at a point of the cloud. The chosen, 2D parametric test shapes are (see Figure 1):

(a) A "flower" parametrized by $r(\theta) = 0.5(1 + 0.5\sin(6\theta + \frac{\pi}{2}))$;
(b) An "eight" parametrized by $x(t) = 0.5\ \sin(t)(\cos t + 1)$, $y(t) = 0.5\ \sin(t)(\cos t - 1)$, $t \in (0, 2\pi)$.

We test formula (6.3) with some profiles ρ, ξ defined on $[0, 1]$:

- the “tent” kernel pair $(\rho_{tent}, \rho_{tent})$, with $\rho_{tent}(r) = (1 - r)$;
- the “natural tent” pair $(\rho_{tent}, \xi_{tent})$, with $\xi_{tent}(r) = -\frac{1}{n} r\rho'_{tent}(r) = r$;
- the “exp” kernel pair $(\rho_{\exp}, \rho_{exp})$, with $\rho_{exp}(r) = \exp\left(-\frac{1}{1-r^2}\right)$;
- the “natural exp” pair (ρ_{exp}, ξ_{exp}), with $\xi_{exp}(r) = -\frac{1}{n} r\rho'_{exp}(r)$.



(a) (b)

Figure 1: some 2D parametric test shapes

Notice that ρ_{exp}, ξ_{exp} satisfy Hypothesis 1; on the contrary, ρ_{tent} is only in $\mathrm{W}^{1,\infty}$ and ξ_{tent} is not even continuous.

To define point clouds from samples of these parametric test shapes, we use two approaches:

- either we compute the exact tangent line $T(t) \in G_{1,2}$ at the N points $\{0, h, 2h, \ldots, (N-1)h\}$ for $h = \frac{2\pi}{N}$, and we set

$$V_N = \sum_{j=1}^{N} m_j \delta_{(x(jh),y(jh))} \otimes \delta_{T(jh)} \,, \tag{6.4}$$

- or we compute by linear regression a tangent line $T^{app} \in G_{1,2}$ at each sample point and we set

$$V_N = \sum_{j=1}^{N} m_j \delta_{(x(jh),y(jh))} \otimes \delta_{T^{app}(jh)} \,. \tag{6.5}$$

For all shapes under study, the exact vector curvature $H(t)$ can be computed explicitly and evaluated at jh, $j = 0 \ldots N-1$. To quantify the accuracy of approximation (6.3), we use the following relative average error

$$E^{rel} = \frac{1}{N} \sum_{j=1}^{N} \frac{|H^{V_N}_{\rho,\xi,\varepsilon}(x_j) - H(jh)|}{\|H\|_\infty} \,, \tag{6.6}$$

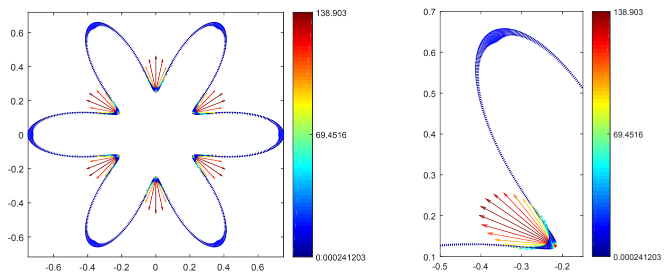
where $x_j = (x(jh), y(jh))$.

6.2.2 Numerical illustration of orthogonal approximate mean curvature

We first test formula (6.3) on the flower with exact normals. We represent in Figure 2 the curvature vectors computed for $N = 10^5$ points and $\varepsilon = 0.001$ with the natural kernel pair $(\rho_{exp}, \xi_{\exp})$. Arrows indicate the vectors and colors indicate their norms. Remark that the sample points are obtained from a uniform sampling in parameter space (polar angle), therefore sample points are not regularly spaced on the flower. Still, these spatial variations are negligible and (6.3) provides a good approximation of the continuous mean curvature, as we already know from Theorem 3.6, and as it will be illustrated numerically in the next section.

6.2.3 Convergence rate

In this section, we compute and represent the evolution with respect to the number of points N of the relative average error $E^{rel} = \frac{1}{N}\sum_{j=1}^{N} \frac{|H^N_\varepsilon(x_j) - H(t_j)|}{\|H\|_\infty}$ for the orthogonal approximate mean curvature vector (6.3) of point cloud varifolds sampled from the parametric flower. We compare the convergence rate of this error for the



(a) (b)

Figure 2: Orthogonal approximate curvature vectors along the discretized flower. Arrows indicate the curvature vectors and colors indicate their norms.

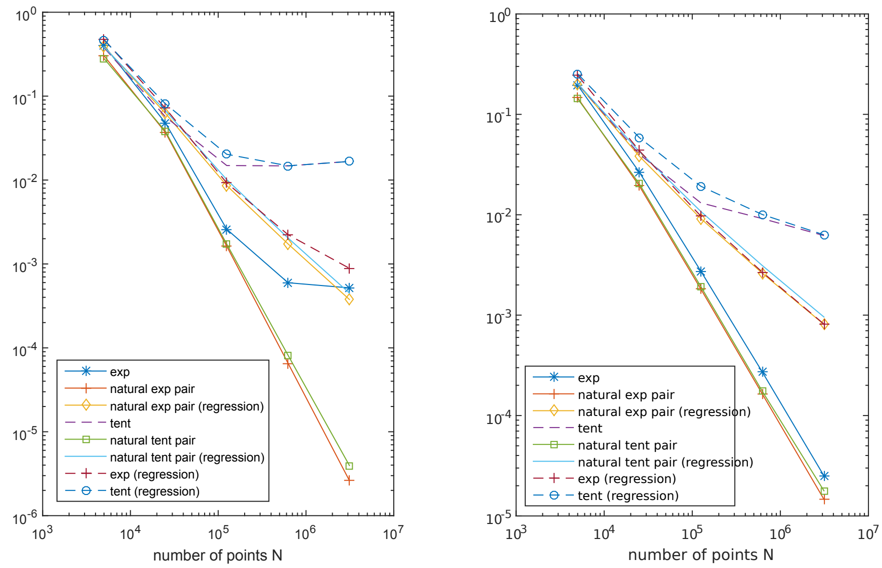
above choices of kernels pairs; more specifically we compute the convergence error for the varifold defined in (6.4) both in the case where $T(jh)$ is the exact tangent and in the case where T^{app} is computed by regression in an R–neighbourhood, with $R = \varepsilon/2$ (this situation is labelled as "regression" in all figures).

Theorem 3.6 guarantees the convergence under suitable assumptions of the orthogonal approximate mean curvature $H^{V_i,\perp}_{\rho,\xi,\varepsilon_i}$, and even provides a convergence rate. First, it is not very difficult to check that in the case where the point clouds are uniform samplings of a smooth curve, then the parameters $d_{i,1}$ and η_i of (3.10) are of order $\frac{1}{N}$. As we already pointed out, our sampling is not globally uniform, but locally almost uniform and we expect the same order for $d_{i,1}$ and η_i. As for $d_{i,2}$ in (3.11), if the tangents are exact, then $d_{i,2}$ is also of order $\frac{1}{N}$, otherwise, it depends essentially on the radius of the ball used to perform the regression. Here we set $R = \varepsilon/2$, which is not a priori optimal. If we want to estimate the mean curvature at some point x of the curve, then we will apply formula (6.3) to the closest point in the point cloud, which is at distance of order $\frac{1}{N}$ to x (this corresponds to what is denoted $|z_i - x|$ in Theorem 3.6). To summarize, according to these considerations together with Theorem 3.6, we expect to observe convergence under the assumption

$$\frac{1}{N\varepsilon} \to 0\,,$$

with a convergence rate of order $\dfrac{1}{N\varepsilon} + \varepsilon$, at least in the case where the tangents are exact. We start with studying two different cases: first with $\frac{1}{N\varepsilon} = N^{-1/4}$, where we expect convergence with rate at least $N^{-1/4}$, and then with $\frac{1}{N\varepsilon} = 0.01$, for which Theorem 3.6 is not sufficient to guarantee that convergence holds. In both cases, we focus on $\frac{1}{N\varepsilon}$ which is the leading term.

We use a log-log scale to represent the resulting relative average error (6.6) as a function of the number of sample points N for $\varepsilon = \frac{100}{N}$ (Figure 3(a)) and $\varepsilon = \left(\frac{10}{N}\right)^{3/4}$ (Figure 3(b)). We remark that the number N_{neigh} of points in a neighborhood $B_\varepsilon(x)$ is proportional to εN, which takes the values 100 and $10^{3/4}N^{1/4}$, respectively, for the above choices of ε. Interestingly, the experiments show a good convergence rate when choosing a natural kernel pair, even in the cases when $\frac{1}{N\varepsilon}$ is constant (thus when it does not converge to 0!). Furthermore, the convergence using natural kernel pairs and approximate tangents computed by regression is even faster than when using exact tangents and the tent kernel. We recall that the tent kernel does not satisfy Hypothesis 1 since it is only Lipschitz, nevertheless the corresponding natural pair $(\rho_{tent}, \xi_{tent})$ shows the same convergence properties as the smooth natural pair (ρ_{exp}, ξ_{exp}). This suggests that the (NKP) property is even more effective than the smoothness of the kernel profiles. Finally, when the tangents are not exact the convergence is slower. This is consistent with the fact that parameter $d_{i,2}$ in (3.11) depends on the radius R of the ball used to compute the regression tangent line (we recall that $R = \varepsilon/2$) which represents an additional parameter to be possibly optimized.



(a) (b)

Figure 3: Average error (log-log scale) for the orthogonal approximate mean curvature of the subsampled parametric flower, for increasing values of N, and with either $\varepsilon = \frac{100}{N}$ (left) or $\varepsilon = \left(\frac{10}{N}\right)^{3/4}$ (right). The number of points in the neighborhood used for estimating the curvature is constant for the left experiment, and scales as $10N^{1/4}$ for the right experiment.

6.2.4 The approximate mean curvature near singularities

Here we illustrate the specific features of the approximate mean curvatures $H^V_{\varepsilon,\rho,\xi}$ and $H^{V,\perp}_{\varepsilon,\rho,\xi}$ near singularities. Consistently with the properties of the classical generalized mean curvature of varifolds, $H^V_{\varepsilon,\rho,\xi}$ and $H^{V,\perp}_{\varepsilon,\rho,\xi}$ both preserve the zero mean curvature of straight crossings, as confirmed by the experiment on the "eight" (see Figure 4). In this case using $H^{V,\perp}_{\varepsilon,\rho,\xi}$ does not affect the reconstruction of the zero curvature at the crossing point, while it has the advantage of being more consistent at regular points.

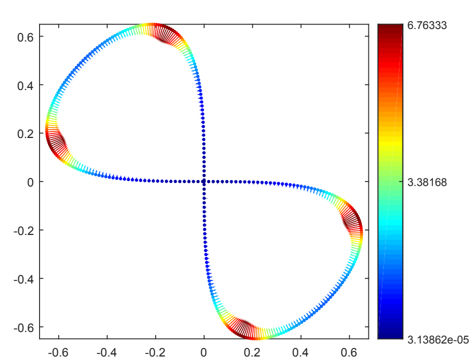


Figure 4: Curvature vector and intensity computed with the natural kernel pair (ρ_{exp}, ξ_{exp}) on the eight sampled with $N = 10000$ points and with $\varepsilon = 100/N = 0.01$, with exact tangents. For visualization purposes we only show 5% of the points in the cloud.

More generally, our model is able to deal correctly with singular configurations whose canonically associated varifold has a first variation δV which is absolutely continuous with respect to $\|V\|$. To illustrate this, we show the results of some tests performed on a union of two circles with equal radius and on a standard double bubble in the plane.

First, we compare the behavior of $H^V_{\varepsilon,\rho,\xi}$ and $H^{V\perp}_{\varepsilon,\rho,\xi}$ in a neighborhood of an intersection point of the two circles (see Figure 5). From the point of view of pointwise almost everywhere convergence, both approximate curvatures behave equivalently well, since the error in the reconstruction of the curvature is localized in an ε-neighborhood of the crossing point. On one hand, due to the linearity of the first variation δV, the expected curvature H of the union $\mathcal{C}_1 \cup \mathcal{C}_2$ of the two circles at the crossing point p is the average of the curvatures H_1 and H_2 of, respectively, $\mathcal{C}_1$ and $\mathcal{C}_2$ at p. Indeed $\delta V = H_1\, d\mathcal{H}^1_{|\mathcal{C}_1} + H_2\, d\mathcal{H}^1_{|\mathcal{C}_2}$, whence one deduces that $H(p) = \frac{H_1(p)+H_2(p)}{2}$ and if p is an intersection point of the two circles, $|H(p)| = \sqrt{3} \approx 1.73$ which is consistent with the numerical value obtained at p (see Figure 5 (b)). On the other hand, the crossing point is negligible with respect to $\|V\|$ and therefore the pointwise value of $H(p)$ is not relevant in the continuous setting. Nevertheless, in the discrete setting there is a significant difference between the two proposed definitions of approximate mean curvature. More precisely, the one provided by $H^V_{\varepsilon,\rho,\xi}$ enforces a continuous mean curvature even at the crossing point, where one obtains the expected average value $H(p) = \frac{H_1(p)+H_2(p)}{2}$, see Figure 5 (b), whereas continuity cannot hold for $H^{V,\perp}_{\varepsilon,\rho,\xi}$, as one can see in Figure 5 (c).

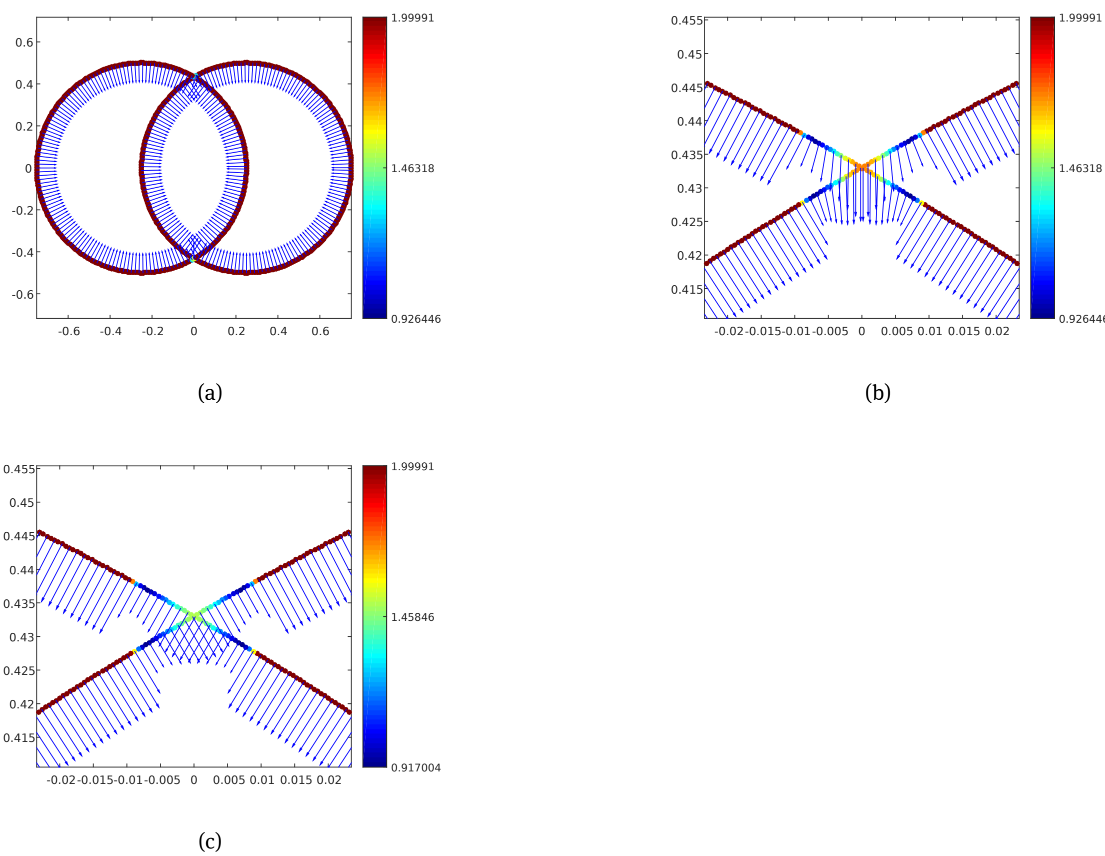


Figure 5: Curvature vector and intensity computed with the natural kernel pair (ρ_{exp}, ξ_{exp}) on two intersecting circles sampled with $N = 10000$ points and with $\varepsilon = 100/N = 0.01$, without projection onto the normal in (a) and (b) and with projection on (c). Tangents are exact.

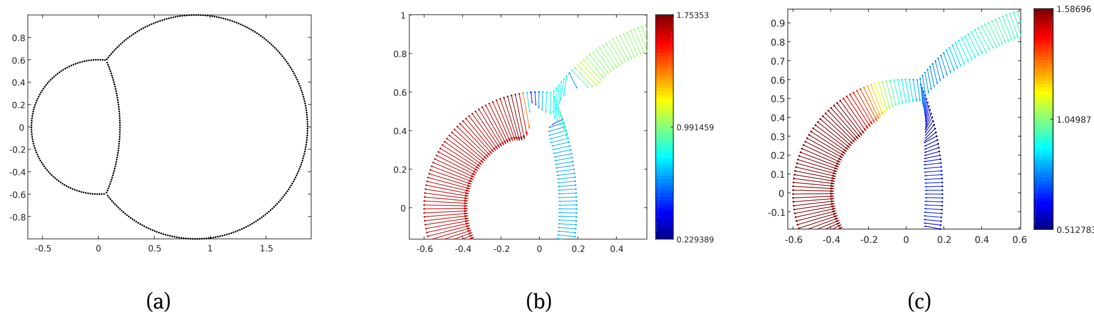


Figure 6: Curvature vectors and intensities computed with the natural kernel pair (ρ_{exp}, ξ_{exp}) on a standard double bubble (radii 1 and 0.6) sampled with $N = 800$ points and with $\varepsilon = 0.15$, without projection onto the normal in (b) and with additional averaging of the curvature at scale 2ε in (c). Tangents are computed by regression.

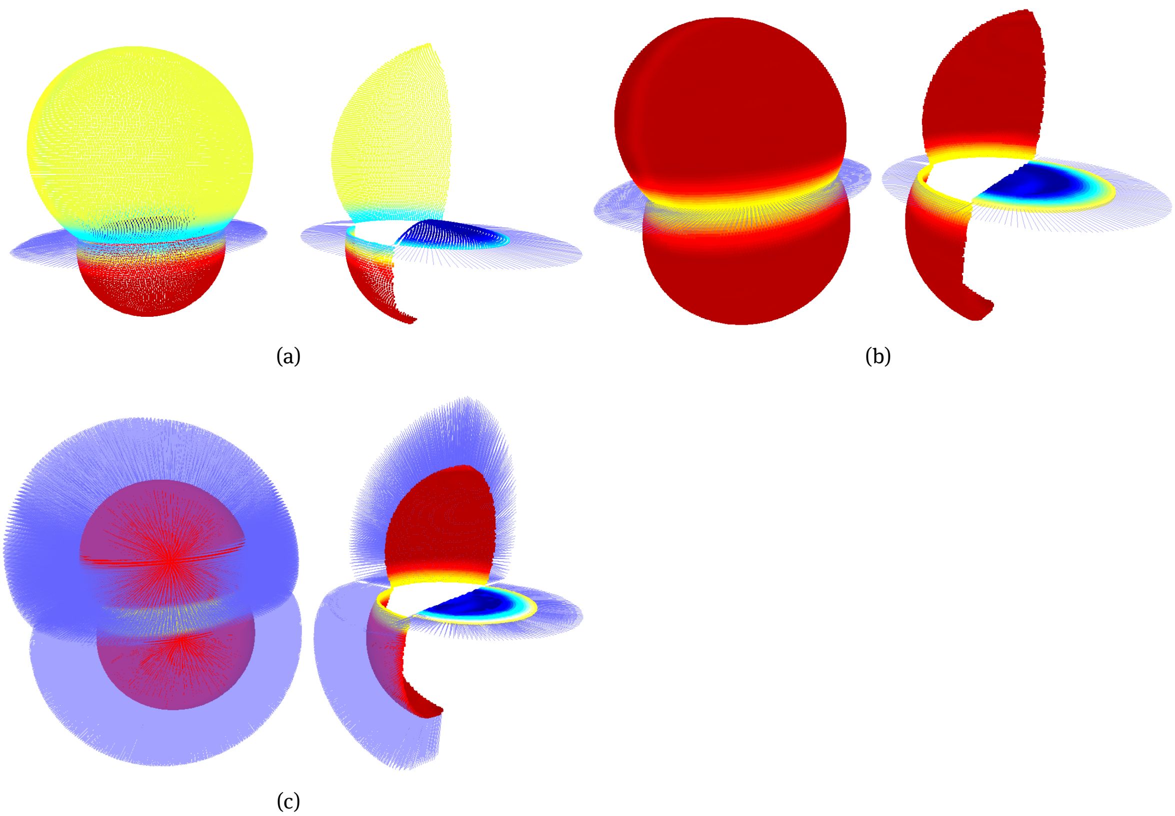
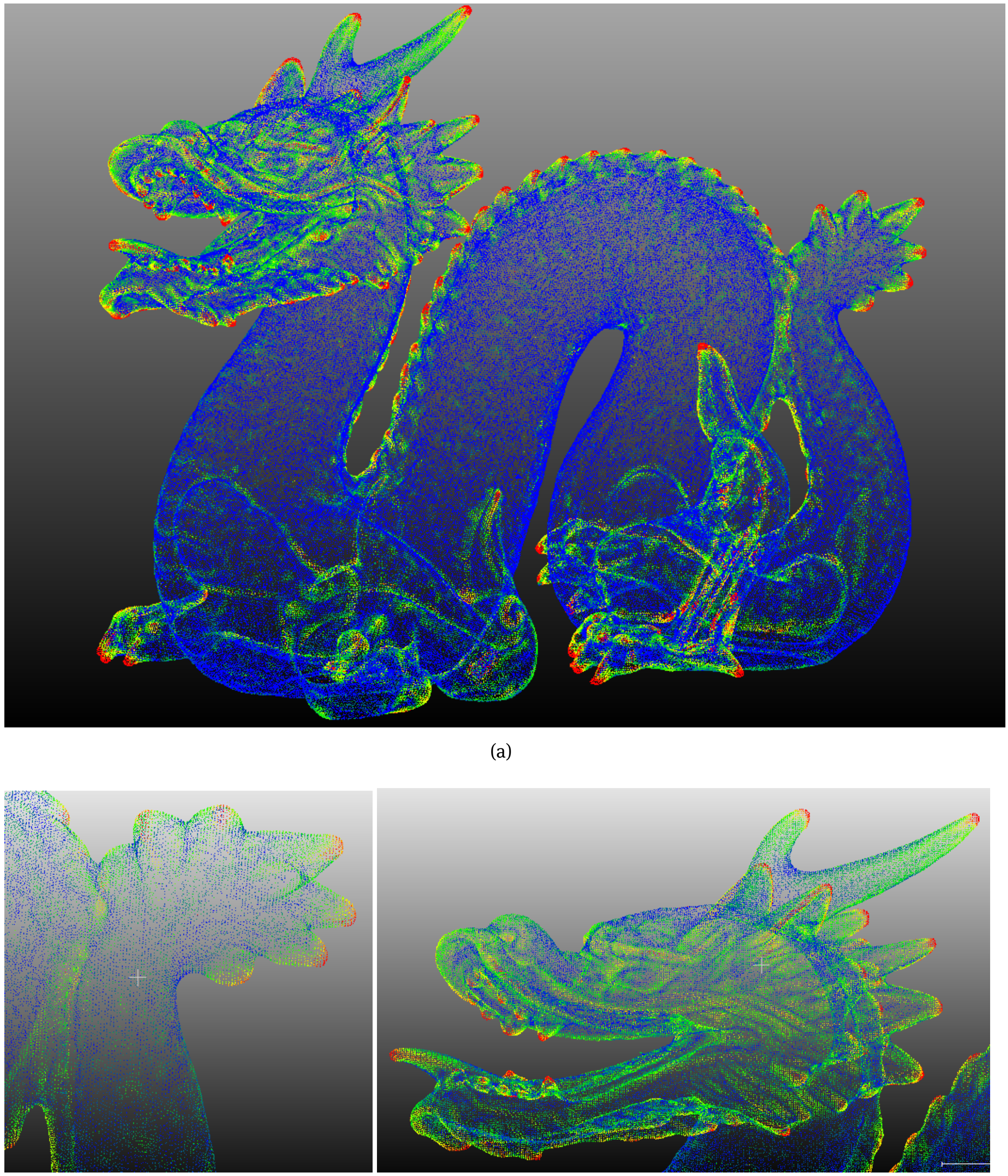


Figure 7: Curvature vectors and their intensities computed with the natural kernel pair (ρ_{exp}, ξ_{exp}) on sampled $3D$–double bubbles (show in full and partial views). In Figure a), the bubble has external caps with radii 0.7 and 1, is sampled with $N = 34378$ points, and the computations are made with $\varepsilon \approx 0.111$. The curvature vectors (with minus sign for the sake of readability) are shown only for the points which are closest to the singular circle. In b) and c), the double bubble has externals caps with same radius 1, is sampled with 33275 points, and $\varepsilon \approx 0.131$. All curvature vectors (with minus sign) are shown in c). To improve the visualization, points are shown with larger size in b) and c).

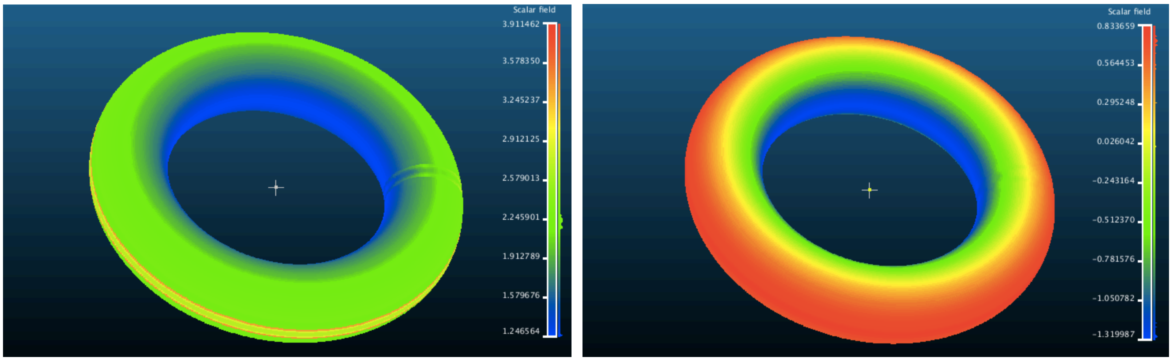
Second, we consider a standard double bubble in 2 dimensions (see Figure 6(a) and [11] for details on double bubbles), whose radii of the external boundary arcs are, respectively, 1 and 0.6. The corresponding point cloud varifold V is obtained by a uniform sampling of 800 points taken on the three arcs of the bubble, each endowed with a unit mass and tangent computed by regression. Again, we choose (ρ_{exp}, ξ_{exp}) as natural



(a)

(b) Zoom in on dragon's tail (c) Zoom in on dragon's head

Figure 8: Intensities of the approximate mean curvature of a dragon point cloud (435 545 points, diameter= 1) with $\varepsilon = 0.007$.



(a) approximate mean curvature (b) approximate Gaussian curvature

Figure 9

kernel pair, and $\varepsilon = 0.15$. Figure 6(b) shows the curvature vectors and intensities of $H^V_{\varepsilon,\rho,\xi}$ (up to a fixed renormalization that is applied for a better visualization). In order to get rid of the oscillation of the curvature near the singularities (as it occurred in the previous test, see again Figure 5) we have also applied a simple averaging of the reconstructed curvature at the scale 2ε, which gives the nicer result shown in Figure 6(c). We remark that the curvature vector defined on points that are very close to the theoretical singularity is consistent with the one obtained by direct computation on the (continuous) standard double bubble. More precisely, we obtain a numerical value of $(0.107, -0.809)$ for the mean curvature near the singularity shown in Figure 6, to be compared with the expected value $(0, -0.839)$, hence with a relative error of 13%. If we redo the same experiment but with twice the number of points, that is $N = 1600$ and $\varepsilon = 0.075$, we get a relative error of 7%.

Further pictures showing approximate mean curvatures of standard double bubbles in 3D and of a "classical" point-cloud dragon are presented in Figures 7 and 8. Then, we conclude with Figure 9 showing the approximate mean and Gaussian curvatures of a torus.

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