Gaussian lower bounds for non-homogeneous Kolmogorov equations with measurable coefficients

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Abstract

We prove Gaussian upper and lower bounds for the fundamental solutions of a class of degenerate parabolic equations satisfying a weak Hörmander condition. The bound is independent of the smoothness of the coefficients and generalizes classical results for uniformly parabolic equations.

Keywords: Kolmogorov equations, fundamental solution, linear stochastic equations, Harnack inequalities.

1 Introduction

We consider the Kolmogorov backward equation

$$\mathcal{L}u := \sum_{i,j=1}^{m_0} \partial_{x_i}(a_{ij}\partial_{x_j}u) + \sum_{i=1}^{m_0} (\partial_{x_i}(a_i u) + b_i\partial_{x_i}u) + cu + \sum_{i,j=1}^{d} b_{ij}x_j\partial_{x_i}u + \partial_t u = 0, \tag{1.1}$$

where $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, $m_0 \le d$ and $\mathcal{L}$ verifies the following two standing assumptions:

Assumption 1.1. *The coefficients $a_{ij} = a_{ji}, a_i, b_i, c$, for $1 \le i,j \le m_0$, are bounded, measurable functions of $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ and*

$$\mu^{-1}|\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(t,x)\xi_i\xi_j \le \mu|\xi|^2, \qquad \xi \in \mathbb{R}^{m_0},\ (t,x) \in \mathbb{R}^{d+1}, \tag{1.2}$$

for some positive constant μ.

Assumption 1.2. *The matrix $B := (b_{ij})_{1\le i,j\le d}$ has constant real entries and takes the block-form*

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & * \end{pmatrix} \tag{1.3}$$

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where each B_i is a $(m_i \times m_{i-1})$-matrix of rank m_i with

$$m_0 \geq m_1 \geq \cdots \geq m_\nu \geq 1, \qquad \sum_{i=0}^{\nu} m_i = d,$$

and the blocks denoted by "$*$" are arbitrary.

Our main result extends the bounds for the fundamental solution proved in [4] and [28, 29] for uniformly parabolic operators with measurable coefficients: we refer to [15] for a description of the development of this theory for non-degenerate parabolic operators, which includes the relevant contributions in [30] and [9]. In the following statement, Γ denotes the fundamental solution of the operator $\mathcal{L}$, as given in Definition 2.2: the existence of Γ is briefly discussed in Remark 2.3.

Theorem 1.3. *Let $\mathcal{L}$ be an operator in the form* (1.1)*, satisfying Assumptions 1.1 and 1.2. Assume in addition that $n = m_0 < d = 2n$ and*

$$B = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix} \tag{1.4}$$

where I_n and 0_n denote the $(n \times n)$-identity matrix and the $(n \times n)$-zero matrix, respectively. Let $I =]T_0, T_1[$ be a bounded interval. Then, there exist four positive constants $\lambda^+, \lambda^-, C^+, C^-$ such that

$$C_I^- \Gamma^{\lambda^-}(t, x; T, y) \leq \Gamma(t, x; T, y) \leq C_I^+ \Gamma^{\lambda^+}(t, x; T, y) \tag{1.5}$$

for every $(t, x), (T, y) \in \mathbb{R}^{d+1}$ with $T_0 < t < T < T_1$. The constants λ^-, λ^+ depend only on d and $\mathcal{L}$, while C^-, C^+ also depend on $T_1 - T_0$. In (1.5) *Γ^{λ^-} and Γ^{λ^+} denote the fundamental solutions of $\mathcal{L}^{\lambda^-}$ and $\mathcal{L}^{\lambda^+}$, respectively, where*

$$\mathcal{L}^\lambda := \tfrac{\lambda}{2} \sum_{i=1}^{m_0} \partial_{x_i x_i} + \sum_{i,j=1}^{d} b_{ij} x_j \partial_{x_i} + \partial_t. \tag{1.6}$$

The explicit expression of $\Gamma^{\lambda^\pm}$ is given in (2.8) *below.*

Remark 1.4. *Our proof of the lower bound in (1.5) is based on a local Harnack inequality for the operator $\mathcal{L}$. This inequality was recently proved by Golse, Imbert, Mouhot and Vasseur in [17] for the case where the matrix B is of the form* (1.4)*. This motivates the presence of that additional assumption in Theorem 1.3. However, since our method is not restricted to that particular case, we preferred to derive the bounds in (1.5) for those operators satisfying only Assumptions 1.1 and 1.2. This approach has the advantage of highlighting the geometric structure of the operator which is a cornerstone of our techniques. Moreover, the validity of Theorem 1.3 will be automatically extended to the family of operators $\mathcal{L}$ satisfying only Assumptions 1.1 and 1.2 once the corresponding local Harnack inequality will be proved.*

Degenerate equations of the form (1.1) naturally arise in the theory of stochastic processes, in physics and in mathematical finance. For instance, if W denotes a real Brownian motion, then the simplest non-trivial Kolmogorov operator

$$\tfrac{1}{2}\partial_{vv} + v\partial_x + \partial_t, \qquad t \geq 0,\ (v, x) \in \mathbb{R}^2,$$

is the backward Fokker-Planck operator of the classical Langevin stochastic equation

$$\begin{cases} dV_t = dW_t, \\ dX_t = V_t dt, \end{cases}$$

that describes the position X and velocity V of a particle in the phase space (cf. [25]). Notice that in this case we have $1 = m_0 < d = 2$.

Linear Fokker-Planck equations (cf. [11] and [37]), non-linear Boltzmann-Landau equations (cf. [26] and [7]) and non-linear equations for Lagrangian stochastic models commonly used in the simulation of turbulent flows (cf. [6]) can be written in the form

$$\sum_{i,j=1}^{n} \partial_{v_i}(a_{ij}\partial_{v_j} f) + \sum_{j=1}^{n} v_j \partial_{x_j} f = \partial_t f, \qquad t \geq 0,\, v \in \mathbb{R}^n,\, x \in \mathbb{R}^n, \tag{1.7}$$

with the coefficients $a_{ij} = a_{ij}(t, v, x, f)$ that may depend on the solution f through some integral expressions. Clearly the operator in (1.7) is the *forward* expression of a particular case of (1.1), namely, the one with $n = m_0 < d = 2n$ and

$$B = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}$$

where I_n and 0_n denote the $(n \times n)$-identity matrix and the $(n \times n)$-zero matrix, respectively. Of course, our main result does apply to forward degenerate parabolic equations, as the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ writes in this form (see equation (2.9) and Definition 2.2 below).

In mathematical finance, equations of the form (1.1) appear in various models for the pricing of path-dependent derivatives such as Asian options (cf., for instance, [31], [5]), stochastic volatility models (cf. [18], [34]) and in the theory of stochastic utility (cf. [2], [3]).

Besides its applicative interest, the operator $\mathcal{L}$ in (1.1) has been studied by several authors because of its challenging theoretical features. As in the study of uniformly parabolic operators, the theoretical results mainly depend on the assumptions on the coefficients. We summarize here the main results available in the literature and we focus in particular on those that are useful for the purpose of this work:

- *Constant coefficients.* If the a_{ij}'s, the a_i's and the b_i's are constant and $c = 0$, the operator $\mathcal{L}$ appears as the prototype of *hypoelliptic operators* in the seminal Hörmander's work [19]. In particular, Hörmander proves that a smooth fundamental solution for $\mathcal{L}$ exists if, and only if, Assumptions 1.1 and 1.2 are satisfied. We emphasize that this regularity property is not obvious for strongly degenerate operators of the form (1.1). Based on the explicit expression of the fundamental solution, mean value formulas and Harnack inequalities for the non-negative solutions of $\mathcal{L}u = 0$ have been proved in [21, 22, 16, 24]. In particular, [24] studies the invariance of the solutions of $\mathcal{L}u = 0$ with respect to suitable *non-Euclidean* translations and *non-homogeneous* dilations: it is then proved a Harnack inequality which is translation- and dilation-invariant. In Section 2 we give the precise statement of the above assertions.

- *Hölder continuous coefficients.* The existence of a fundamental solution for operators $\mathcal{L}$ with Hölder continuous coefficients has been proved by several authors using the parametrix method. We refer to the papers [40, 20, 38] where a restricted class of operators $\mathcal{L}$ are considered and [35, 12, 10] where the general family of operators satisfying Assumptions 1.1 and 1.2 is considered. In [35] it is also assumed that all the $*$-blocks of B in (1.3) are null. As we will see in Remark 2.4, this condition is related to an invariance property of the operator $\mathcal{L}$ with respect to the anisotropic dilation (2.7). An invariant Harnack inequality has been proved in [35, 14] and a lower bound for the fundamental solution of $\mathcal{L}$ is obtained in [36, 14]. Also in [36] it is assumed that all the $*$-blocks of B in (1.3) are null.

- *Measurable coefficients.* An upper bound for the fundamental solution of $\mathscr{L}$ is obtained in [32, 23] by adapting the Aronson's method [4]: the latter is based on a local L^∞-estimate of the solutions proved by a Moser's iterative procedure which in turn relies on the combination of a Caccioppoli inequality with a Sobolev estimate (see [33, 8, 23]). The authors of [39] prove a weak form of the Poincaré inequality which yields the C^α-regularity of the solutions of $\mathscr{L}u = 0$. More recently, an invariant Harnack inequality for the positive solutions of (1.7) is proved in [17]: this is a remarkable result which comes more that 60 years after the analogous results for uniformly parabolic equations; in fact the classical techniques do not apply to degenerate equation like (1.1) and the authors of [17] use a different approach based on the so-called "velocity averaging method". It is worth noting that the main Lemma in [39] is a strictly positive lower bound, which is a step in the proof of the Harnack inequality, in accordance with the axiomatic approach described in [27].

The starting point of this paper is the Harnack inequality proved in [17] for the prototype equation (1.7). Actually, since our techniques apply without substantial changes, we consider the general equation (1.1). Our main result is a lower bound for the fundamental solution Γ of $\mathscr{L}$ under the mere assumption of measurability and boundedness of its coefficients, in the spirit of the works [4] and [28, 29]. Its proof is based on the repeated application of the Harnack inequality on suitable sequences of points that are usually called *Harnack chains*.

2 Preliminaries

Hereafter the operator $\mathscr{L}$ in (1.1) will be written in the compact form

$$\mathscr{L}u = \operatorname{div}(ADu + au) + \langle b, Du\rangle + cu + Yu = 0,$$

where $D = (\partial_{x_1}, \dots, \partial_{x_d})$ denotes the gradient in $\mathbb{R}^d$, $A := (a_{ij})_{1\le i,j\le d}$, $a := (a_i)_{1\le i\le d}$, $b := (b_i)_{1\le i\le d}$ with $a_{ij} = a_i = b_i \equiv 0$ for $i > m_0$ or $j > m_0$, and

$$Y := \langle Bx, D\rangle + \partial_t.$$

The *constant-coefficient Kolmogorov operator*

$$\mathscr{L}^1 := \tfrac{1}{2}\sum_{i=1}^{m_0} \partial_{x_i x_i} + Y$$

will be referred to as the *principal part of* $\mathscr{L}$. It will be clear in the sequel that $\mathscr{L}^1$ plays in this setting the role played by the heat operator in the uniformly parabolic case. We focus here, in particular, on the regularity properties of $\mathscr{L}^1$ and on its invariance with respect to a family of non-Euclidean translations and non-homogeneous dilations. It is known that Assumption 1.2 is equivalent to the hypoellipticity of $\mathscr{L}^1$; in fact, Assumption 1.2 is also equivalent to the well-known Hörmander's condition, which in our setting reads:

$$\operatorname{rank} \operatorname{Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)(t,x) = d+1, \qquad \text{for all } (t,x) \in \mathbb{R}^{d+1}, \tag{2.1}$$

where $\operatorname{Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)$ denotes the Lie algebra generated by the vector fields $\partial_{x_1}, \dots, \partial_{x_{m_0}}$ and Y (see Proposition 2.1 in [24]). Thus operator $\mathscr{L}$ can be regarded as a perturbation of its principal part $\mathscr{L}^1$: roughly speaking, Assumption 1.1 ensures that the sub-elliptic structure of $\mathscr{L}^1$ is preserved under perturbation.

Constant-coefficient Kolmogorov operators are naturally associated with *linear* stochastic differential equations: indeed, $\mathcal{L}^1$ is the backward Fokker-Planck operator of the d-dimensional SDE

$$dX_t = BX_t dt + \sigma dW_t, \tag{2.2}$$

where W is a standard m_0-dimensional Brownian motion and σ is the $(d \times m_0)$-matrix

$$\sigma = \begin{pmatrix} I_{m_0} \\ 0 \end{pmatrix}. \tag{2.3}$$

The solution X of (2.2) is a Gaussian process with transition density

$$\Gamma^1(t, x; T, y) = \frac{1}{\sqrt{(2\pi)^d \det \mathcal{C}(T-t)}} \exp\left(-\frac{1}{2}\langle \mathcal{C}(T-t)^{-1}\big(y - e^{(T-t)B}x\big), \big(y - e^{(T-t)B}x\big)\rangle\right) \tag{2.4}$$

for $t < T$ and $x, y \in \mathbb{R}^d$, where

$$\mathcal{C}(t) = \int_0^t \left(e^{sB}\sigma\right)\left(e^{sB}\sigma\right)^* ds \tag{2.5}$$

is the covariance matrix of X_t. Assumption 1.2 ensures (actually, is equivalent to the fact) that $\mathcal{C}(t)$ is positive definite for any positive t. Moreover Γ^1 in (2.4) is the fundamental solution of $\mathcal{L}^1$ and the function

$$u(t, x) := E\left[\varphi\left(X_T\right) \mid X_t = x\right] = \int_{\mathbb{R}^d} \Gamma^1(t, x; T, y)\varphi(y)dy, \qquad t < T,\ x \in \mathbb{R}^d,$$

solves the backward Cauchy problem

$$\begin{cases} \mathcal{L}^1 u(t, x) = 0, & t < T,\ x \in \mathbb{R}^d, \\ u(T, x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases}$$

for any bounded and continuous function φ.

Operator $\mathcal{L}^1$ has some remarkable invariance properties that were first studied in [24]. Denote by $\ell_{(\tau,\xi)}$, for $(\tau, \xi) \in \mathbb{R}^{d+1}$, the left-translations in $\mathbb{R}^{d+1}$ defined as

$$\ell_{(\tau,\xi)}(t, x) := (\tau, \xi) \circ (t, x) := \left(t + \tau, x + e^{tB}\xi\right), \tag{2.6}$$

Then, $\mathcal{L}^1$ is invariant with respect to ℓ_ζ in the sense that

$$\mathcal{L}^1\left(u \circ \ell_\zeta\right) = \left(\mathcal{L}^1 u\right) \circ \ell_\zeta, \qquad \zeta \in \mathbb{R}^{d+1}.$$

Moreover, let $\mathcal{D}(r)$ be defined as

$$\mathcal{D}(r) := \operatorname{diag}(rI_{m_0}, r^3 I_{m_1}, \ldots, r^{2\nu+1} I_{m_\nu}), \qquad r \geq 0, \tag{2.7}$$

where I_{m_i} denotes the $(m_i \times m_i)$-identity matrix. Then, $\mathcal{L}^1$ is homogeneous with respect to the dilations in $\mathbb{R}^{d+1}$ defined as

$$\delta_r(t, x) := \left(r^2 t, \mathcal{D}(r)x\right),$$

if and only if all the $*$-blocks of B in (1.3) are null ([24], Proposition 2.2). In this case, we have

$$\mathcal{L}^1(u \circ \delta_r) = r^2(\mathcal{L}^1 u) \circ \delta_r.$$

The natural number

$$Q := m_0 + 3m_1 + \cdots + (2\nu+1)m_\nu.$$

is usually called the *homogeneous dimension* of $\mathbb{R}^d$ with respect to $(\mathcal{D}(r))_{r>0}$, because the Jacobian of $\mathcal{D}(r)$ is equal to r^Q. We also note that $Q/2$ is the rate of the diagonal decay of the fundamental solution of $\mathcal{L}^1$ (see inequalities (4.7) in Remark 4.5 below).

In accordance with (2.4), the fundamental solution of the operator $\mathcal{L}^\lambda$ defined in (1.6) is

$$\Gamma^\lambda(t,x;T,y) = \frac{1}{\sqrt{(2\pi\lambda)^d \det \mathcal{C}(T-t)}} \exp\left(-\frac{1}{2\lambda}\langle \mathcal{C}(T-t)^{-1}\big(y-e^{(T-t)B}x\big)\big), \big(y-e^{(T-t)B}x\rangle\right) \tag{2.8}$$

for $t<T$ and $x,y\in\mathbb{R}^d$.

We end this section with the definitions of weak and fundamental solutions utilized in the sequel.

Definition 2.1. *A* weak solution *of* (1.1) *in a domain* Ω *of* $\mathbb{R}^{d+1}$ *is a function* u *such that*

$$u, \partial_{x_1}u, \dots, \partial_{x_{m_0}}u, Yu \in L^2_{\text{loc}}(\Omega)$$

and

$$\int_\Omega -\langle ADu, D\psi\rangle - u\langle a, D\psi\rangle + \psi\langle b, Du\rangle + u\psi + \psi Yu = 0,$$

for any $\psi \in C_0^\infty(\Omega)$.

We recall that the formal adjoint operator of $\mathcal{L}$ is defined as

$$\mathcal{L}^* v := \sum_{i,j=1}^{m_0} \partial_{y_i}(a_{ij}\partial_{y_j}v) - \sum_{i=1}^{m_0}\left(a_i\partial_{y_i}v + \partial_{y_i}(b_i v)\right) + (c - \operatorname{tr}(B))\,v - \sum_{i,j=1}^{d} b_{ij}y_j\partial_{y_i}v - \partial_T v. \tag{2.9}$$

Definition 2.2. *A fundamental solution for* $\mathcal{L}$ *is a continuous and positive function* $\Gamma = \Gamma(t,x;T,y)$, *defined for* $t<T$ *and* $x,y\in\mathbb{R}^d$, *such that:*

i) $\Gamma(\cdot,\cdot;T,y)$ *is a weak solution of* $\mathcal{L}u=0$ *in* $]-\infty,T[\times\mathbb{R}^d$ *and* $\Gamma(t,x;\cdot,\cdot)$ *is a weak solution of* $\mathcal{L}^*u=0$ *in* $]t,+\infty[\times\mathbb{R}^d$;

ii) for any bounded function $\varphi\in C(\mathbb{R}^d)$ *and* $x,y\in\mathbb{R}^d$, *we have*

$$\begin{cases} \mathcal{L}u(t,x) = 0, & (t,x)\in\,]-\infty,T[\times\mathbb{R}^d, \\ \lim\limits_{\substack{(t,x)\to(T,y)\\ t<T}} u(t,x) = \varphi(y), & y\in\mathbb{R}^d, \end{cases}$$

$$\begin{cases} \mathcal{L}^*v(T,y) = 0, & (T,y)\in\,]t,+\infty[\times\mathbb{R}^d, \\ \lim\limits_{\substack{(T,y)\to(t,x)\\ T>t}} v(T,y) = \varphi(x), & x\in\mathbb{R}^d, \end{cases}$$

where

$$u(t,x) := \int_{\mathbb{R}^d}\Gamma(t,x;T,y)\varphi(y)dy, \qquad v(T,y) := \int_{\mathbb{R}^d}\Gamma(t,x;T,y)\varphi(x)dx.$$

Remark 2.3. *A Harnack inequality and the existence of a fundamental solution for* $\mathcal{L}$ *were proven under the additional assumption that the coefficients are Hölder continuous and* $a_i = 0$ *for* $i=1,\dots,m_0$ *(see [14] and [12]). To our knowledge, the existence of a fundamental solution for* $\mathcal{L}$ *with discontinuous coefficients has not been proven yet. Actually, the a priori bounds for* Γ *provided in this note and the Hölder continuity of the weak solutions seem can be used to prove its existence.*

Remark 2.4. *Let u be a weak solution of* (1.1) *and $r > 0$. Then $v := u \circ \delta_r$ solves $\mathcal{L}^{(r)} v = 0$ where*

$$\mathcal{L}^{(r)} v := \operatorname{div}(A^{(r)} Dv) + \operatorname{div}(a^{(r)} v) + \langle b^{(r)}, Dv\rangle + c^{(r)} v + \langle B^{(r)} x, Dv\rangle + \partial_t v, \tag{2.10}$$

with $A^{(r)} = A \circ \delta_r$, $a^{(r)} = r(a \circ \delta_r)$, $b^{(r)} = r(b \circ \delta_r)$, $c^{(r)} = r^2(c \circ \delta_r)$ and $B^{(r)} = r^2 \mathcal{D}_r B \mathcal{D}_{\frac{1}{r}}$, that is

$$B^{(r)} = \begin{pmatrix} r^2 B_{1,1} & r^4 B_{1,2} & \cdots & r^{2\nu} B_{1,\nu} & r^{2\nu+2} B_{1,\nu+1} \\ B_1 & r^2 B_{2,2} & \cdots & r^{2\nu-2} B_{2,\nu} & r^{2\nu} B_{2,\nu+1} \\ 0 & B_2 & \cdots & r^{2\nu-4} B_{3,\nu} & r^{2\nu-2} B_{3,\nu+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & r^2 B_{\nu+1,\nu+1} \end{pmatrix},$$

where $B_{i,j}$ denotes the $*$-block in the (i,j)-th position in (1.3).

Notation 2.5. *Let $M > 0$ and $B := (b_{ij})_{1 \le i,j \le d}$ a matrix that satisfies Assumption 1.2. We denote by $\mathcal{K}_{M,B}$ the class of Kolmogorov operators of the form* (2.10) *with $r \in [0,1]$ and the coefficients a_{ij}, a_i, b_i, c, for $1 \le i,j \le m_0$, that satisfy Assumption 1.1 with the non-degeneracy constant μ in* (1.2) *and the norms $\|a_i\|_\infty$, $\|b_i\|_\infty$, $\|c\|_\infty$ smaller than M.*

Remark 2.6. *Let $\mathcal{L} \in \mathcal{K}_{M,B}$. If u is a solution of $\mathcal{L}u = 0$ then, for any $\zeta \in \mathbb{R}^{d+1}$, $v := u \circ \ell_\zeta$ solves $(\mathcal{L} \circ \ell_\zeta) v = 0$ where $(\mathcal{L} \circ \ell_\zeta)$ is the operator obtained from $\mathcal{L}$ by ℓ_ζ-translating its coefficients. Moreover, operator $(\mathcal{L} \circ \ell_\zeta)$ still belongs to $\mathcal{K}_{M,B}$.*

3 Harnack inequalities

Let B be a matrix that satisfies Assumption 1.2. We associate to B the cylinders

$$Q_1^+ = \{(t,x) \in \mathbb{R} \times \mathbb{R}^d \mid 0 \le t < 1, \, |x| < 1\},$$

and

$$Q_r^+(z_0) := z_0 \circ \delta_r \left(Q_1^+\right) = \{z \in \mathbb{R}^{d+1} \mid z = z_0 \circ \delta_r(\zeta), \, \zeta \in Q_1^+\},$$

for $z_0 \in \mathbb{R}^{d+1}$ and $r > 0$. The first step in deriving the lower bound in (1.5) is based on the following remarkable result proven in [17].

Theorem 3.1 (**Local Harnack inequality).** *Let $\mathcal{L} \in \mathcal{K}_{M,B}$ and assume B to be of the form* (1.4). *If u is a non-negative weak solution of* (1.1) *in Q_1^+ then*

$$\sup_{Q_r^+(\beta,0)} u \le C \inf_{Q_r^+(0,0)} u, \tag{3.1}$$

where the constants $C \ge 1$ and $\beta, r \in]0,1[$ depend only on M and B.

Remark 3.2. *It seems that the method introduced in [17] can be readily extended to the entire class of operators satisfying Assumptions 1.1 and 1.2, in order to guarantee the validity of Theorem 3.1 in this general setting.*

Remark 3.3. *The constants β, r in* Theorem 3.1 *are small so that the cylinders $Q_r^+(0,0)$ and $Q_r^+(\beta,0)$ are disjoint subsets of Q_1^+, as in the usual statement of the parabolic Harnack inequality. The article [1] contains a geometric statement of the Harnack inequality that explains how to choose those constants.*

Remark 3.4. *By Remark 2.6, the Harnack inequality* (3.1) *is valid for cylinders centered at an arbitrary point* $z_0 \in \mathbb{R}^d$ *with the same constants* C, β, r, *dependent only on* M *and* B.

Next we prove a global version of the Harnack inequality based on a classical argument which makes use of the so-called Harnack chains. We first prove a preliminary result. For $\beta, r, R > 0$ and $z_0 \in \mathbb{R}^{d+1}$, we define the cones

$$P_{\beta,r,R} = \left\{ z \in \mathbb{R}^{d+1} \mid z = \delta_\varrho(\beta, \xi),\ |\xi| < r,\ 0 < \varrho \le R \right\},$$

and $P_{\beta,r,R}(z_0) := z_0 \circ P_{\beta,r,R}$. Here $|\xi|$ denotes the Euclidean norm of the vector $\xi \in \mathbb{R}^d$. Theorem 3.1 combined with Remark 2.4 gives the following

Lemma 3.5. *Let* $z \in \mathbb{R}^{d+1}$, $R \in]0, 1]$ *and assume* B *to be of the form* (1.4). *Let* u *be a continuous and non-negative weak solution of* (1.1) *in* $Q_R^+(z)$. *Then we have*

$$\sup_{P_{\beta,r,R}(z)} u \le C u(z),$$

where the constants C, β *and* r *are the same as in Theorem 3.1 and depend only on* M *and* B.

Proof. Let u be a continuous and non-negative weak solution of (1.1) in $Q_R^+(z)$ and let $w \in P_{\beta,r,R}(z)$. Then $w = z \circ \delta_\varrho(\beta, \xi)$ for some $\varrho \in]0, R]$ and $|\xi| < r$. By using the notation introduced in (2.6), we obtain from Remark 2.4 that the function $u_{z,\varrho} := u \circ \ell_z \circ \delta_\varrho$ is a continuous and non-negative weak solution in $Q^+_{\frac{R}{\varrho}}(0,0) \supseteq Q_1^+(0,0)$ of $\mathcal{L}^{(\varrho)} u_\varrho = 0$, where $\mathcal{L}^{(\varrho)}$ is the operator defined in (2.10). Since $\mathcal{L}^{(\varrho)} \in \mathcal{K}_{M,B}$, by the Harnack inequality (3.1) for $\mathcal{L}^{(\varrho)}$, we have

$$u(w) = u_{z,\varrho}(\beta, \xi) \le \sup_{Q_r^+(\beta,0)} u_{z,\varrho} \le C \inf_{Q_r^+(0,0)} u_{z,\varrho} \le C u_{z,\varrho}(0,0) = C u(z).$$

□

Theorem 3.6 (Global Harnack inequality). *Let* $\mathcal{L} \in \mathcal{K}_{M,B}$, $T \in \mathbb{R}$, $\tau \in]0, 1]$ *and assume* B *to be of the form* (1.4). *If* u *is a continuous and non-negative weak solution of* (1.1) *in* $]T - \tau, T + \tau[\times \mathbb{R}^d$, *then we have*

$$u(T, y) \le c_0 e^{c_0 \langle \mathcal{C}^{-1}(T-t)(y - e^{(T-t)B}x), y - e^{(T-t)B}x \rangle} u(t, x), \qquad t \in]T - \tau, T[,\ x, y \in \mathbb{R}^d,$$

where $\mathcal{C}$ *is the covariance matrix in* (2.5) *and* c_0 *is a positive constant that depends only on* M *and* B.

Before proving Theorem 3.6, we recall (see, for instance, Sect.9.5 in [31]) that the Hörmander condition (2.1) is equivalent to the fact that the pair of matrices (B, σ), with σ as in (2.3), is controllable in the following sense: for any $(t, x), (T, y) \in \mathbb{R}^{d+1}$ with $t < T$, there exists $v \in L^2([t, T]; \mathbb{R}^{m_0})$ such that the system

$$\begin{cases} \gamma'(s) = B\gamma(s) + \sigma v(s), \\ \gamma(t) = x, \quad \gamma(T) = y, \end{cases} \tag{3.2}$$

has solution. The function v is called a *control for* (B, σ) *on* $[t, T]$. In the proof of Theorem 3.6 we will use the following

Lemma 3.7. *Let* γ *be the solution of the linear problem*

$$\begin{cases} \gamma'(s) = B\gamma(s) + \sigma v(s), \qquad s \in [t, T], \\ \gamma(t) = x, \end{cases} \tag{3.3}$$

with $T-t\le 1$, initial datum $x\in\mathbb{R}^d$ and control function $v\in L^2([t,T];\mathbb{R}^{m_0})$. Then we have

$$(s,\gamma(s))\in P_{1,\kappa\|v\|_{L^2([t,T])},\sqrt{T-t}}(t,x),\qquad s\in[t,T],$$

where κ is a positive constant which depends only on B.

Proof. The explicit solution of (3.3) is

$$\gamma(s)=e^{(s-t)B}x+\int_t^s e^{(s-\tau)B}\sigma v(\tau)d\tau,\qquad s\in[t,T].$$

Thus, setting $\varrho=\sqrt{s-t}$, we have that $(s,\gamma(s))\in P_{1,r,\sqrt{T-t}}(t,x)$ if and only if

$$\int_t^{t+\varrho^2} e^{(t+\varrho^2-\tau)B}\sigma v(\tau)d\tau=\mathcal{D}(\varrho)\,\xi\quad\text{ with }\quad|\xi|\le r. \tag{3.4}$$

To check this, we first notice that, according to (2.7), the space $\mathbb{R}^d$ admits a natural decomposition as a direct sum

$$\mathbb{R}^d=\bigoplus_{j=0}^{\nu}V_j,\qquad \dim V_j=m_j.$$

Then, for $x\in\mathbb{R}^d$, with obvious notation we have $x=x^{(0)}\oplus\cdots\oplus x^{(\nu)}$ where

$$\mathcal{D}(r)x^{(j)}=r^{2j+1}x^{(j)},\qquad j=0,\dots,\nu.$$

We also write a $(d\times d)$-matrix E in block form as in (1.3), that is $E=\left(E^{(ij)}\right)_{i,j=0\dots,\nu}$ where $E^{(ij)}$ is a block of dimension $m_i\times m_j$. In particular, given the definition of exponential $E(t):=e^{tB}$ as the sum of a power series, a direct computation shows that

$$\begin{aligned}
E^{(00)}(t)&=I_{m_0}+t\mathrm{O}(t),\\
E^{(0j)}(t)&=\frac{t^j}{j!}\left(I_{m_j}+t\mathrm{O}(t)\right)B_j\cdots B_1,\qquad j=1,\dots,\nu,
\end{aligned} \tag{3.5}$$

as $t\to 0$, where I_{m_j} denotes the $(m_j\times m_j)$-identity matrix. Now, $\sigma v\in V_0$ and therefore, by (3.5), we have

$$\left|\left(e^{(t+\varrho^2-\tau)B}\sigma v(\tau)\right)^{(j)}\right|\le\kappa(t+\varrho^2-\tau)^j|v(\tau)|,\qquad \tau\in[t,T],$$

with the constant κ dependent only on B. Thus we have

$$\left|\int_t^{t+\varrho^2}\left(e^{(t+\varrho^2-\tau)B}\sigma v(\tau)\right)^{(j)}\right|d\tau\le\kappa\int_t^{t+\varrho^2}(t+\varrho^2-\tau)^j|v(\tau)|d\tau\le$$

(by Hölder's inequality)

$$\le\kappa\|v\|_{L^2([t,T])}\varrho^{2j+1},$$

and recalling the properties of the dilation operators, see again (2.7), this proves (3.4). □

Let us consider the control problem (3.2) one more time. Among the paths γ satisfying (3.2), one is often interested in one minimizing the *total cost*

$$\|v\|^2_{L^2([t,T])}=\int_t^T|v(s)|^2ds.$$

Classical control theory provides the explicit expression of an optimal control and of its cost (see, for instance, [31], Theor. 9.55).

Lemma 3.8. *The optimal control for problem* (3.2) *is given by*

$$\bar{v}(s) = \left(e^{(T-s)B}\sigma\right)^* \mathcal{C}^{-1}(T-t)\left(y - e^{(T-t)B}x\right), \qquad s \in [t,T].$$

The corresponding minimal cost will be denoted by

$$V(t,x;T,y) := \|\bar{v}\|^2_{L^2([t,T])}$$

and is equal to

$$V(t,x;T,y) = \langle \mathcal{C}^{-1}(T-t)(y - e^{(T-t)B}x), y - e^{(T-t)B}x\rangle.$$

Proof of Theorem 3.6. In order to use the previous versions of the Harnack inequality, we first notice that by assumption, for every $z \in]T-\tau, T[\times\mathbb{R}^d$, u is a continuous and non-negative weak solution of (1.1) in $Q^+_{\sqrt{\tau}}(z)$. Next we fix $x,y \in \mathbb{R}^d$, $t \in]T-\tau,T[$ and consider the solution γ of the control problem (3.2) corresponding to the optimal control $\bar{v}$ given in Lemma 3.8. Moreover, we set $c_2 = \left(\frac{r}{\kappa}\right)^2$ where r and κ are the constants in Theorem 3.1 and Lemma 3.7 respectively.

Now, if $T \le t + \tau\beta$ and $\|\bar{v}\|^2_{L^2([t,T])} \le c_2$, then by Lemma 3.7 we have

$$(T,y) \in P_{1,r,\sqrt{\tau}}(t,x) \cap \left(]t, t+\tau\beta] \times \mathbb{R}^d\right) \subseteq P_{\beta,r,\sqrt{\tau}}(t,x)$$

and therefore by Lemma 3.5 we get

$$u(T,y) \le Cu(t,x)$$

where C is the constant in Theorem 3.1, which depends only on M and B.

Viceversa, setting $t_0 = t$ and

$$t_{j+1} = (t_j + \tau\beta) \wedge \inf\{s \in [t_j, T] \mid \|\bar{v}\|^2_{L^2([t_j,s])} \ge c_2\},$$

we have that $t_j = T$ for $j \ge \frac{1}{\beta} + \frac{\|\bar{v}\|^2_{L^2([t,T])}}{c_2}$ and

$$(t_{j+1}, \gamma(t_{j+1})) \in P_{1,r,\sqrt{\tau}}(t_j, \gamma(t_j)) \cap \left(]t_j, t_j + \tau\beta] \times \mathbb{R}^d\right) \subseteq P_{\beta,r,\sqrt{\tau}}(t_j, \gamma(t_j))$$

if $t_j < T$. By Lemma 3.5 we have

$$u(t_j, \gamma(t_j)) \le Cu(t_{j-1}, \gamma(t_{j-1})),$$

which yields

$$u(T,y) \le C^{\frac{1}{\beta} + \frac{1}{c_2}V(t,x;T,y)} u(t,x).$$

The thesis follows by using the expression of the optimal cost given in Lemma 3.8. □

4 Lower bounds for fundamental solutions

In the proof of the lower bound for the fundamental solution we will make use of the following upper bound.

Theorem 4.1 (Gaussian upper bound). *Let $\mathcal{L} \in \mathcal{K}_{M,B}$. There exists a positive constant c_3, only dependent on M and B, such that*

$$\Gamma(t,x;T,y) \le \frac{c_3}{(T-t)^{\frac{Q}{2}}} \exp\left(-\frac{1}{c_3}\left|\mathcal{D}\left((T-t)^{-\frac{1}{2}}\right)\left(y - e^{(T-t)B}x\right)\right|^2\right), \tag{4.1}$$

for $0 < T-t \le 1$ and $x,y \in \mathbb{R}^d$.

Proof. The Gaussian upper bound (4.1) has been proven by the first two authors in [23] under the assumption that the low order terms $b_1, \ldots, b_{m_0}$ are null. The general case where $b_1, \ldots, b_{m_0}$ are bounded measurable functions can be treated in a very similar way: here we limit ourselves to sketch the few adjustments needed in the proof given in [23].

The first modification is in the proof of the Caccioppoli inequality [23, Theorem 2.3]. We set

$$Q_1 = \{(t, x) \in \mathbb{R} \times \mathbb{R}^d \mid |t| < 1, |x| < 1\}$$

and, for any $z_0 \in \mathbb{R}^{d+1}$ and $r > 0$,

$$Q_r(z_0) := z_0 \circ \delta_r (Q_1) = \{z \in \mathbb{R}^{d+1} \mid z = z_0 \circ \delta_r(\zeta), \, \zeta \in Q_1\}.$$

With this notation, we consider a weak sub-solution u of (1.1) in $Q_r(z_0)$, that is u such that

$$\int_{Q_r(z_0)} -\langle ADu, D\varphi\rangle - \langle a, D\varphi\rangle u + \langle b, Du\rangle\varphi + \varphi c u + \varphi Y u \geq 0, \tag{4.2}$$

and we use $\varphi := 2qu^{2q-1}\psi^2$ as a test function in (4.2), where $\psi \in C_0^\infty(Q_r(z_0))$. Focusing on the new term $\langle b, Du\rangle\varphi = 2\langle b, Du^q\rangle u^q \psi^2$, we find that the following inequality holds for every positive δ:

$$\begin{aligned}\left| \int_{Q_r(z_0)} \langle b, Du\rangle\varphi \right| &\leq 2 \left(\int_{Q_r(z_0)} |D_{m_0} u^q|^2 \psi^2 \right)^{1/2} \left(\int_{Q_r(z_0)} |b|^2 u^{2q} \psi^2 \right)^{1/2} \\ &\leq \delta(2q-1) \int_{Q_r(z_0)} |D_{m_0} u^q|^2 \psi^2 + \frac{\|b\|_{L^\infty(Q_r(z_0))}}{\delta(2q-1)} \int_{Q_r(z_0)} u^{2q}\psi^2.\end{aligned}$$

From this point we get the Caccioppoli inequality by following the proof of [23, Theorem 2.3].

The second modification is in the proof of the Sobolev inequality. Referring to the proof of [23, Theorem 2.5], we find an extra term in the representation formula of the sub-solution u: according to the notations in [23, Theorem 2.5], we denote it by

$$I_5(z) := \int_{Q_r(z_0)} \left(\Gamma_0(z; \cdot)\langle b, Du\rangle\psi\right)(\zeta) d\zeta,$$

for which the following estimate holds:

$$\|I_5\|_{L^{2\kappa}(Q_\rho(z_0))} \leq C_3 \|b\|_{L^\infty(Q_r(z_0))} \|D_{m_0} u\|_{L^2(Q_r(z_0))}.$$

Again, we infer the Sobolev inequality by following the rest of the proof of [23, Theorem 2.5].

The last adjustment is in the proof of the inequality (3.2) in [23, Theorem 3.3]. In the identity (A.1) in Appendix A of [23] we have to add the term

$$\mathbf{I}_5 := \iint_{[\tau,\eta]\times\mathbb{R}^d} u\gamma_R^2 e^{2h}\langle b, Du\rangle,$$

where $\gamma_R \in C_0^\infty(\mathbb{R}^d, [0, 1])$ is such that $\gamma_R(x) = 1$ whenever $|x| < R$, and $|D\gamma_R|$ bounded by a constant independent of R. We easily see that, for every positive δ, we have

$$|\mathbf{I}_5| \leq \frac{\delta}{2} \iint_{[\tau,\eta]\times\mathbb{R}^d} |D_{m_0} u|^2 \gamma_R^2 e^{2h} + \frac{1}{2\delta} \iint_{[\tau,\eta]\times\mathbb{R}^d} |b|^2 u^2 \gamma_R^2 e^{2h}.$$

The rest of the proof follows the same lines of the proof of [23, Theorem 3.3]. ∎

Lemma 4.2. *Let $\mathcal{L} \in \mathcal{K}_{M,B}$. There exist two positive constants R and c_4, which depend only on M and B, such that*

$$\int_{\left|\mathcal{D}(\sqrt{T-t})\left(y-e^{(T-t)B}x\right)\right|\le R} \Gamma(t,x;T,y)dx \ge c_4, \qquad 0 < T-t \le 1,\ y \in \mathbb{R}^d. \tag{4.3}$$

Proof. First notice that, for a suitably large constant c_5 dependent only on M and B, the function

$$v(T,y) := \int_{\mathbb{R}^d} \Gamma(t,x;T,y)dx - e^{-c_5(T-t)}, \qquad T > t,\ y \in \mathbb{R}^d,$$

is a weak super-solution of the forward Cauchy problem

$$\begin{cases} \mathcal{L}^* v(T,y) = -e^{-c_5(T-t)}\left(c - \mathrm{tr}(B) + c_5\right) \le 0, & T > t,\ y \in \mathbb{R}^d, \\ v(t,y) = 0 & y \in \mathbb{R}^d, \end{cases}$$

for the adjoint operator $\mathcal{L}^*$ in (2.9). In order to apply the maximum principle as in [13] (cf. Proposition 3.4), we note that, for any positive c, the super-level set $\{(T,y) \mid \Gamma(t,x;T,y) \ge c\}$ is a (possibly empty) compact subset of $]t,t+1] \times \mathbb{R}^d$ as a consequence of the upper bound (4.1). Thus we have $v \ge 0$, that is

$$\int_{\mathbb{R}^d} \Gamma(t,x;T,y)dx \ge e^{-c_5(T-t)}, \qquad T > t,\ y \in \mathbb{R}^d,$$

and (4.3) follows from the following estimate:

$$\int_{\left|\mathcal{D}(\sqrt{T-t})\left(y-e^{(T-t)B}x\right)\right|\ge R} \Gamma(t,x;T,y)dx \le$$

(by the upper bound (4.1))

$$\le \frac{c_3}{(T-t)^{\frac{Q}{2}}} \int_{\left|\mathcal{D}(\sqrt{T-t})\left(y-e^{(T-t)B}x\right)\right|\ge R} \exp\left(-\frac{1}{c_3}\left|\mathcal{D}\left((T-t)^{-\frac{1}{2}}\right)\left(y-e^{(T-t)B}x\right)\right|^2\right) dx =$$

(by the change of variable $z = \mathcal{D}\left((T-t)^{-\frac{1}{2}}\right)\left(y - e^{(T-t)B}x\right)$)

$$= c_3 \int_{|z|\ge R} \exp\left(-\frac{1}{c_3}|z|^2\right) dz$$

which gives the thesis. □

We are now ready to state and prove the main result of the present paper.

Theorem 4.3 (Gaussian lower bound). *Let $\mathcal{L} \in \mathcal{K}_{M,B}$ and assume B to be of the form (1.4). There exists a positive constant C, dependent only on M and B, such that*

$$\Gamma(t,x;T,y) \ge \frac{C}{(T-t)^{\frac{Q}{2}}} e^{-\frac{1}{C}\langle \mathcal{C}^{-1}(T-t)(y-e^{(T-t)}x),\, y-e^{(T-t)}x\rangle}, \qquad 0 < T-t \le 1,\ x,y \in \mathbb{R}^d. \tag{4.4}$$

Remark 4.4. *In general, estimate (4.4) is valid for any $T-t > 0$, with C dependent also on $1 \vee (T-t)$.*

Proof. We prove a preliminary diagonal estimate. Let $\tau = \frac{T-t}{2}$: by the global Harnack inequality stated in Theorem 3.6, for any $\xi, y \in \mathbb{R}^d$ we have

$$\Gamma(t,y;T,y) \ge c_0 e^{-c_0\langle \mathcal{C}^{-1}(\tau)(\xi-e^{\tau B}y),\, \xi-e^{\tau B}y\rangle}\Gamma(t+\tau,\xi;T,y). \tag{4.5}$$

For any $y \in \mathbb{R}^d$ we set

$$D_R = \{\xi \in \mathbb{R}^d \mid \left|\mathcal{D}(\sqrt{\tau})\left(y - e^{\tau B}\xi\right)\right| \le R\}, \qquad R > 0,$$

and notice that, up to a constant dependent only on M and B, the Lebesgue measure of D_R equals τ^Q. We also note that, by Lemma 3.3 in [24], $\langle \mathcal{C}^{-1}(\tau)(\xi - e^{\tau B}y), \xi - e^{\tau B}y\rangle$ is bounded on D_R. Therefore, integrating (4.5) over D_R, we get

$$\Gamma(t,y;T,y) \geq \frac{c_8}{\tau^Q}\int_{\left|\mathcal{D}(\sqrt{\tau})(y-e^{\tau B}\xi)\right|\leq R} \Gamma(t+\tau,\xi;T,y)d\xi \geq \frac{c_9}{(T-t)^{\frac{Q}{2}}}, \tag{4.6}$$

where the last inequality follows from Lemma 4.2 and the constant c_9 depends only on M and B. Hence, by applying again the global Harnack inequality we get

$$\Gamma(t,0;T,y) \geq c_0 e^{-c_0\langle \mathcal{C}^{-1}(\tau)y,y\rangle}\Gamma(t+\tau,y;T,y) \geq$$

(by (4.6))

$$\geq \frac{c_{10}}{(T-t)^{\frac{Q}{2}}} e^{-c_0\langle \mathcal{C}^{-1}(\tau)y,y\rangle} \geq \frac{c_{11}}{(T-t)^{\frac{Q}{2}}} e^{-c_{11}\langle \mathcal{C}^{-1}(T-t)y,y\rangle},$$

where the last inequality is a consequence of (4.8) from Remark 4.5 below. This proves (4.4) for $x = 0$; the general statement follows by the translation-invariance property of the operator $\mathcal{L}$. □

Remark 4.5. *If we denote by $\mathcal{C}_0$ the covariance matrix appearing in the fundamental solution of the homogeneous principal part of $\mathcal{L}$, then there exist $\alpha_1,...,\alpha_4,\beta_1,...,\beta_4 > 0$ such that for any $\tau \in]0,1]$ and $z \in \mathbb{R}^d$*

$$\alpha_1\tau^Q \leq \alpha_2\det(\mathcal{C}_0(\tau)) \leq \det(\mathcal{C}(\tau)) \leq \alpha_3\det(\mathcal{C}_0(\tau)) \leq \alpha_4\tau^Q \tag{4.7}$$

and

$$\beta_1\left|\mathcal{D}\left((\tau)^{-\frac{1}{2}}\right)z\right|^2 \leq \beta_2\langle \mathcal{C}_0^{-1}(\tau)z,z\rangle \leq \langle \mathcal{C}^{-1}(\tau)z,z\rangle \leq \beta_3\langle \mathcal{C}_0^{-1}(\tau)z,z\rangle \leq \beta_4\left|\mathcal{D}\left((\tau)^{-\frac{1}{2}}\right)z\right|^2. \tag{4.8}$$

In fact, we recall (see Proposition 2.3 in [24]) that for any $\tau > 0$ one has

$$\mathcal{C}_0(\tau) = \mathcal{D}\left(\sqrt{\tau}\right)\mathcal{C}_0(1)\mathcal{D}\left(\sqrt{\tau}\right)$$

and

$$\mathcal{C}_0^{-1}(\tau) = \mathcal{D}\left(\tau^{-\frac{1}{2}}\right)\mathcal{C}_0^{-1}(1)\mathcal{D}\left(\tau^{-\frac{1}{2}}\right).$$

These identities imply that

$$\begin{aligned}\det\mathcal{C}_0(\tau) &= \det\left(\mathcal{D}\left(\sqrt{\tau}\right)\mathcal{C}_0(1)\mathcal{D}\left(\sqrt{\tau}\right)\right)\\ &= (\tau)^Q\det\mathcal{C}_0(1);\end{aligned}$$

moreover, if k_1 and k_2 denote, respectively, the least and the greatest eigenvalue of $\mathcal{C}_0^{-1}(1)$, we have that $k_1 > 0$ and that

$$k_1\left|\mathcal{D}\left((\tau)^{-\frac{1}{2}}\right)z\right|^2 \leq \langle \mathcal{C}_0^{-1}(\tau)z,z\rangle \leq k_2\left|\mathcal{D}\left((\tau)^{-\frac{1}{2}}\right)z\right|^2.$$

for all $z \in \mathbb{R}^d$ and $\tau > 0$. This proves the first and last inequalities in (4.7) and (4.8). To prove the equivalence between the matrices $\mathcal{C}_0^{-1}$ and $\mathcal{C}^{-1}$, we recall that, according to formula (3.14) in [24], we have

$$\frac{\det\mathcal{C}(\tau)}{\det\mathcal{C}_0(\tau)} = 1 + \tau O(1), \qquad \text{as } \tau \to 0^+.$$

Hence, if we set $\frac{\det \mathcal{C}(0)}{\det \mathcal{C}_0(0)} := 1$*, then* $\frac{\det \mathcal{C}(\tau)}{\det \mathcal{C}_0(\tau)}$ *is a strictly positive continuous function of* $\tau \geq 0$*. In particular, there exist two positive constants* k_3 *and* k_4 *such that*

$$k_3 \det \mathcal{C}_0(\tau) \leq \det \mathcal{C}(\tau) \leq k_4 \det \mathcal{C}_0(\tau), \qquad \text{for all } \tau \in]0, 1].$$

By the same argument we can prove that there exist two positive constant k_5 *and* k_6 *such that*

$$k_5 \langle \mathcal{C}_0^{-1}(\tau) z, z \rangle \leq \langle \mathcal{C}^{-1}(\tau) z, z \rangle \leq k_6 \langle \mathcal{C}_0^{-1}(\tau) z, z \rangle$$

for every $z \in \mathbb{R}^d$ *and* $\tau \in]0, 1]$ *(see inequality (2.12) in [14]). To this aim we recall that, for every* $z \in \mathbb{R}^d$*,*

$$\langle \mathcal{C}^{-1}(\tau) z, z \rangle = \langle \mathcal{C}^{-1}(\tau) z, z \rangle = 1 + \tau O(1), \qquad \text{as } \tau \to 0^+.$$

(see Lemma 3.3 in [24].) Then, the function $(z, \tau) \mapsto \frac{\langle \mathcal{C}^{-1}(\tau) z, z \rangle}{\langle \mathcal{C}^{-1}(\tau) z, z \rangle}$ *extends to a strictly positive continuous function defined in the compact set*

$$\big\{ (z, \tau) \in \mathbb{R}^{d+1} \mid |z| = 1, 0 \leq \tau \leq 1 \big\}.$$

Then, we conclude as above.

The following corollary is a straightforward consequence of Theorem 4.1 and Remark 4.5

Corollary 4.6. *Let* $\mathscr{L} \in \mathcal{K}_{M,B}$*. There exists a positive constant* c_{12}*, only dependent on* M *and* B*, such that*

$$\Gamma(t, x; T, y) \leq \frac{c_{12}}{\sqrt{\det \mathcal{C}(T - t)}} \exp \left(-\frac{1}{c_{12}} \langle \mathcal{C}^{-1}(T - t) \left(y - e^{(T-t)B} x \right), \left(y - e^{(T-t)B} x \right) \rangle \right).$$

for $0 < T - t \leq 1$ *and* $x, y \in \mathbb{R}^d$*.*

References

[1] F. Anceschi, M. Eleuteri, and S. Polidoro, *A geometric statement of the Harnack inequality for a degenerate Kolmogorov equation with rough coefficients*, to appear in Commun. Contemp. Math., doi.org/10.1142/S0219199718500578, (2018).

[2] F. Antonelli, E. Barucci, and M. E. Mancino, *Asset pricing with a forward-backward stochastic differential utility*, Econom. Lett., 72 (2001), pp. 151–157.

[3] F. Antonelli and A. Pascucci, *On the viscosity solutions of a stochastic differential utility problem*, J. Differential Equations, 186 (2002), pp. 69–87.

[4] D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc., 73 (1967), pp. 890–896.

[5] E. Barucci, S. Polidoro, and V. Vespri, *Some results on partial differential equations and Asian options*, Math. Models Methods Appl. Sci., 11 (2001), pp. 475–497.

[6] M. Bossy, J.-F. Jabir, and D. Talay, *On conditional McKean Lagrangian stochastic models*, Probab. Theory Related Fields, 151 (2011), pp. 319–351.

[7] C. Cercignani, *The Boltzmann equation and its applications*, Springer-Verlag, New York, 1988.

[8] C. Cinti, A. Pascucci, and S. Polidoro, *Pointwise estimates for a class of non-homogeneous Kolmogorov equations*, Math. Ann., 340 (2008), pp. 237–264.

[9] E. B. Davies, *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math., 109 (1987), pp. 319–333.

[10] F. Delarue and S. Menozzi, *Density estimates for a random noise propagating through a chain of differential equations*, J. Funct. Anal., 259 (2010), pp. 1577–1630.

[11] L. Desvillettes and C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math., 54 (2001), pp. 1–42.

[12] M. Di Francesco and A. Pascucci, *On a class of degenerate parabolic equations of Kolmogorov type*, AMRX Appl. Math. Res. Express, 3 (2005), pp. 77–116.

[13] M. Di Francesco, A. Pascucci, and S. Polidoro, *The obstacle problem for a class of hypoelliptic ultraparabolic equations*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 155–176.

[14] M. Di Francesco and S. Polidoro, *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form*, Adv. Differential Equations, 11 (2006), pp. 1261–1320.

[15] E. B. Fabes, *Gaussian upper bounds on fundamental solutions of parabolic equations; the method of Nash*, in Dirichlet forms (Varenna, 1992), vol. 1563 of Lecture Notes in Math., Springer, Berlin, 1993, pp. 1–20.

[16] N. Garofalo and E. Lanconelli, *Level sets of the fundamental solution and Harnack inequality for degenerate equations of Kolmogorov type*, Trans. Amer. Math. Soc., 321 (1990), pp. 775–792.

[17] F. Golse, C. Imbert, C. Mouhot, and A. Vasseur, *Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation*, to appear in Ann. Scuola Norm. Sup. Pisa, (2017).

[18] D. G. Hobson and L. C. G. Rogers, *Complete models with stochastic volatility*, Math. Finance, 8 (1998), pp. 27–48.

[19] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), pp. 147–171.

[20] A. M. Il′in, *On a class of ultraparabolic equations*, Dokl. Akad. Nauk SSSR, 159 (1964), pp. 1214–1217.

[21] L. P. Kupcov, *Mean value theorem and a maximum principle for Kolmogorov's equation*, Mathematical notes of the Academy of Sciences of the USSR, 15 (1974), pp. 280–286.

[22] ——, *On parabolic means*, Dokl. Akad. Nauk SSSR, 252 (1980), pp. 296–301.

[23] A. Lanconelli and A. Pascucci, *Nash estimates and upper bounds for non-homogeneous Kolmogorov equations*, Potential Anal., 47 (2017), pp. 461–483.

[24] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), pp. 29–63.

[25] P. Langevin, *On the theory of Brownian motion*, C. R. Acad. Sci. (Paris), 146 (1908), pp. 530–533.

[26] P.-L. Lions, *On Boltzmann and Landau equations*, Philos. Trans. Roy. Soc. London Ser. A, 346 (1994), pp. 191–204.

[27] D. Maldonado, *On the elliptic Harnack inequality*, Proc. Amer. Math. Soc., 145 (2017), pp. 3981–3987.

[28] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.

[29] ——, *Correction to: "A Harnack inequality for parabolic differential equations"*, Comm. Pure Appl. Math., 20 (1967), pp. 231–236.

[30] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1958), pp. 931–954.

[31] A. Pascucci, *PDE and martingale methods in option pricing*, vol. 2 of Bocconi & Springer Series, Springer, Milan; Bocconi University Press, Milan, 2011.

[32] A. Pascucci and S. Polidoro, *A Gaussian upper bound for the fundamental solutions of a class of ultraparabolic equations*, J. Math. Anal. Appl., 282 (2003), pp. 396–409.

[33] ——, *The Moser's iterative method for a class of ultraparabolic equations*, Commun. Contemp. Math., 6 (2004), pp. 395–417.

[34] R. Peszek, *PDE models for pricing stocks and options with memory feedback*, Appl. Math. Finance, 2 (1995), pp. 211–223.

[35] S. Polidoro, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Matematiche (Catania), 49 (1994), pp. 53–105.

[36] ——, *A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations*, Arch. Rational Mech. Anal., 137 (1997), pp. 321–340.

[37] H. Risken, *The Fokker-Planck equation: Methods of solution and applications*, Springer-Verlag, Berlin, second ed., 1989.

[38] I. M. Sonin, *A class of degenerate diffusion processes*, Teor. Verojatnost. i Primenen, 12 (1967), pp. 540–547.

[39] W. Wang and L. Zhang, *The C^α regularity of weak solutions of ultraparabolic equations*, Discrete Contin. Dyn. Syst., 29 (2011), pp. 1261–1275.

[40] M. Weber, *The fundamental solution of a degenerate partial differential equation of parabolic type*, Trans. Amer. Math. Soc., 71 (1951), pp. 24–37.