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Uniqueness and representation theorems for solutions of Kolmogorov-Fokker-Planck equations

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RIASSUNTO: *Si considerano operatori parabolici degeneri del tipo seguente*

$$L = div(A(x,t)D) + \langle x, BD\rangle - \partial_t,$$

ove B è una matrice costante, $A(z) = A^T(z) \geq 0$. Adattando un metodo noto per gli operatori parabolici classici, basato sostanzialmente su precise stime puntuali della solizione fondamentale di L, si dimostra che se u è soluzione di $Lu = 0$ in $\mathbb{R}^N \times]0,T[$ e $u(x,0) = 0$, allora una qualunque delle seguenti condizioni: $|u(x,t)|$ si maggiora con una funzione del tipo $e^{c|x|^2}$, oppure $u \geq 0$, implica $u \equiv 0$.

Vengono inoltre forniti un risultato di rappresentazione ed un teorema di tipo Fatou per le soluzioni non negative di $Lu = 0$ in $\mathbb{R}^N \times]0,T[$.

ABSTRACT: *We consider a class of ultraparabolic operators of the following type*

$$L = div(A(x,t)D) + \langle x, BD\rangle - \partial_t,$$

where B is a constant matrix, $A(z) = A^T(z) \geq 0$. We show that if u is a solution of $Lu = 0$ on $\mathbb{R}^N \times]0,T[$ and $u(x,0) = 0$, then each of the following conditions: $|u(x,t)|$ can be bounded (in some sense) by $e^{c|x|^2}$, or $u \geq 0$, implies $u \equiv 0$. We use a technique which is well known in the classic parabolic case and which relies on some pointwise estimates of the fundamental solution of L.

Next, we prove a representation theorem and a Fatou type theorem for non-negative solutions of $Lu = 0$ in $\mathbb{R}^N \times]0,T[$.

KEY WORDS AND PHRASES: *Ultraparabolic equation – Cauchy problem – Fatou theorem*
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1 – Introduction and main results

We consider a class of degenerate second order parabolic operators of the following type

$$L = \mathrm{div}\,(A(z)D) + \langle x, BD\rangle - \partial_t, \tag{1.1}$$

where $z = (x,t) \in \mathbb{R}^{N+1}$, $\mathrm{div}(\cdot)$, $D = (\partial_{x_1}, ..., \partial_{x_N})$ and $\langle\cdot\,,\cdot\rangle$ denote the divergence, the gradient and the inner product in $\mathbb{R}^N$, respectively. In (1.1), $B = (b_{i,j})$ denotes an $N \times N$ matrix with constant real entries and $A(z) = \big(a_{i,j}(z)\big)_{i,j=1,\dots,N}$ is a non-negative symmetric matrix for any $z \in \mathbb{R}^{N+1}$.

The interest for the equation (1.1) comes from the fact that it arises in studying diffusion processes from a probabilistic point of view. Equations like (1.1) also appear in describing the brownian motion of a particle in a fluid (see [12]).

In this note we shall always assume that

HYPOTHESIS H.1. *For some basis in* $\mathbb{R}^N$, $A(z)$ *and* B *can be written as*

$$A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0, \end{pmatrix} \qquad B = \begin{pmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0, \end{pmatrix} \tag{1.2}$$

where each B_j *is a* $p_{j-1} \times p_j$ *block matrix of rank* $p_j, j = 1, 2, ..., r$, *with* $p_0 \geq p_1 \geq ... \geq p_r \geq 1$ *and* $p_0 + p_1 + ... + p_r = N$. *Moreover, there exists* $\mu > 0$ *such that*

$$\mu^{-1}|\xi|^2 \leq \langle A_0(z)\xi, \xi\rangle \leq \mu|\xi|^2$$

for every $\xi \in \mathbb{R}^{p_0}$ *and for every* $z \in \mathbb{R}^{N+1}$.

An important consequence of Hypothesis H.1 is that it ensures that the "frozen" operator

$$L_\zeta u = \mathrm{div}(A(\zeta)Du) + \langle x, BDu\rangle - \partial_t u, \tag{1.3}$$

is hypoelliptic, for every $\zeta \in \mathbb{R}^{N+1}$ (as we will clarify in Section 2, Theorem 2.1). Later on we will introduce further hypotheses on the operator L.

In this paper we prove some uniqueness results for the Cauchy problem related to L, that extend the classic parabolic ones (see [4], [1], [2], [14]) and rely on some pointwise estimates of the fundamental solution Γ of L, recently proved in [8] and [9], under a few regularity conditions on the coefficients of $A(z)$.

We recall that our method was already used by Scornazzani in '82 [11] for the Kolmogorov operator in $\mathbb{R}^3$: $L = \partial_x (a(x,y,t)\partial_x) + x\partial_y - \partial_t$, which is a prototype of the operators (1.1).

We emphasize that our uniqueness classes are the same classes Tychonov found for the heat equation $\Delta u = u_t$ (see [14]). Indeed, in Section 3 we prove that if u is a solution of $Lu = 0$ in $S_I = \mathbb{R}^N \times]0,T[$ and $u(x,0) = 0$, then each of the following conditions:

$$\int_{S_I} |u(x,t)| e^{-c|x|^2} dxdt < \infty$$

(Theorem 3.1), or $u \geq 0$ (Theorem 3.2) implies $u \equiv 0$.

Since our results apply to parabolic operators, the Tychonov example [14] shows that the growth condition allowed in Theorem 3.1 cannot be improved.

In Section 4 we show that, if u is a non-negative solution of (1.1) in S_I, then there exists a Borel measure $\rho \geq 0$ such that

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0) d\rho(\xi), \tag{1.4}$$

where Γ denotes the fundamental solution of L. We denote by $\varphi(x)dx$ the density of the absolutely continuous part of ρ with respect the Lebesgue measure. Then, using a method introduced by Kato [4], we prove that

$$\lim_{t \to 0+} u(x,t) = \varphi(x), \tag{1.5}$$

for almost every $x \in \mathbb{R}^N$. More precisely, we first prove a covering result of Vitali type (Lemma 4.1), corresponding to a family of "boxes" suitably related to the natural geometry of the operator L in (1.1). Then we show that (1.5) holds for every point in $\mathbb{R}^N$ which is a Lebesgue point for the function φ with respect to the above family of boxes (see Theorem 4.1).

Finally, we obtain a further uniqueness result for non-negative solutions of $Lu = 0$ in S_I such that $\lim_{t\to 0+} u(\cdot, t) = \rho$ in the measure sense (Corollary 4.1).

Now, we complete the list of the hypotheses on the operator L.

HYPOTHESIS H.2. *There exists the fundamental solution Γ of L, with the following usual properties:*

i) *For every function $\varphi \in C_0(\mathbb{R}^N)$ we have*

$$\lim_{t\to s+} \int_{\mathbb{R}^N} \Gamma(x,t;y,s)\varphi(y)dy = \varphi(x) \qquad \forall x \in \mathbb{R}^N, \tag{1.6}$$

ii) *if we denote by Γ^* the fundamental solution of the adjoint operator L^*, then*

$$\Gamma^*(z;w) = \Gamma(w;z). \tag{1.7}$$

HYPOTHESIS H.3. *There exists a positive constant $\lambda > 0$ such that, if we denote by Γ^+ and Γ^-, respectively, the fundamental solutions of*

$$L^+ = \lambda\Delta_{p_0} + \langle x, BD\rangle - \partial_t,$$

and

$$L^- = \lambda^{-1}\Delta_{p_0} + \langle x, BD\rangle - \partial_t,$$

then for every $T > 0$ there exist two positive constants c^+, c^- such that:

$$\begin{aligned} &c^-\Gamma^-(z,\zeta) \le \Gamma(z,\zeta) \le c^+\Gamma^+(z,\zeta) \\ &\left|\partial_{x_i}\Gamma(z,\zeta)\right| \le \frac{c^+}{\sqrt{t-\tau}}\,\Gamma^+(z,\zeta) \end{aligned} \tag{1.8}$$

for every $z, \zeta \in \mathbb{R}^{N+1}, 0 < t - \tau < T$ and for any $i = 1, ..., p_0$.

REMARK 1.1. Hypotheses H.2 and H.3 are fulfilled if, for every bounded interval $I \subset \mathbb{R}$, the coefficients $a_{i,j}(z)$ and their first derivatives $\partial_{x_i} a_{i,j}, 1 \le i,j \le p_0$ satisfies on $S_I = \mathbb{R}^N \times I$ a uniformly Hölder continuity condition related to the geometry associated to the operator L. For the precise statement of these results see [8], Theorem 1.1, Corollary 2.5, Proposition 4.1, Lemma 2.2 and Main Theorem of [9].

REMARK 1.2. Here and in the sequel, when we say that "a function u is a solution of $Lu = 0$", we implicitly mean that each derivative $\partial_{x_i} u, \partial^2_{x_i,x_j} u (1 \le i,j \le p_0), Yu = \langle x, BDu \rangle - \partial_t u$ exists and is a continuous function.

Estimates (1.8) are meaningful, since we can write explicitly the fundamental solutions Γ^+ and Γ^-, as next section shows.

2 – Some known results

We first recall some results proved in [5] regarding operators (1.1) with constant $A = A(z)$.

THEOREM 2.1. *Let $A = A(z)$ in (1.1) be a constant matrix, and set*

$$E(t) = \exp(-tB^T) \qquad C(t) = \int_0^t E(s)AE^T(s)ds. \tag{2.1}$$

Then:

i) *the operator L is invariant with respect to the left translations of the group $\left(\mathbb{R}^{N+1}, \circ\right)$, where "$\circ$" is defined as*

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \quad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{2.2}$$

In particular we have

$$\Gamma(z;\zeta) = \Gamma(\zeta^{-1} \circ z; 0) = \Gamma(\zeta^{-1} \circ z) \quad \forall z, \zeta \in \mathbb{R}^{N+1},\ z \neq \zeta; \tag{2.3}$$

ii) *Hypothesis H.1 implies the hypoellipticity of L, and the hypoellipticity of L is equivalent to the following condition*

$$C(t) > 0 \qquad \textit{for every} \quad t > 0; \tag{2.4}$$

iii) *Hypothesis H.1 implies that the operator* L *commutes with the dilation group* $(D(\lambda), \lambda^2))_{\lambda>0}$, *where*

$$D(\lambda) = \mathrm{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \dots, \lambda^{2r+1} I_{p_r}). \tag{2.5}$$

The integers p_j *are the same as in Hypothesis H.1 (whence* $p_0 \geq p_1 \geq \dots \geq p_r \geq 1$ *and* $p_0 + p_1 + \dots + p_r = N$*);*

iv) *the following identities hold*

$$E(\lambda^2 t) = D(\lambda) E(t) D\left(\frac{1}{\lambda}\right), \qquad C(\lambda^2 t) = D(\lambda) C(t) D(\lambda), \tag{2.6}$$

for every $t, \lambda > 0$*;*

v) *the fundamental solution* Γ *of* L *is*

$$\Gamma(x,t) = \frac{c}{t^{\frac{Q}{2}}} \exp\left(-\frac{1}{4} < C^{-1}(1) D\left(\frac{1}{\sqrt{t}}\right) x, D\left(\frac{1}{\sqrt{t}}\right) x >\right), \tag{2.7}$$

where $Q = p_0 + 3p_1 + \dots + (2r+1)p_r$ *is the homogeneous dimension of* $\mathbb{R}^N$ *with respect to* $D(\lambda)$.

We end this section with some results which we only state, since the proof is similar to the one in the classic parabolic case.

MAXIMUM PRINCIPLE. *If* $Lu \geq 0$ *on* $S_I = \mathbb{R}^N \times]0, T[$ *and if*

$$\limsup_{(x,t)\to(x_0,0)} u(x,t) \leq 0 \qquad \forall x_0 \in \mathbb{R}^N,$$

$$\limsup_{|x|\to\infty} \left(\sup_{t\in I} u(x,t)\right) \leq 0,$$

then $u \leq 0$ *on* S_I.

A PRIORI INTERIOR ESTIMATES. *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$. *Denote by* ∂ *one of the following derivatives:* $\partial_{x_i}, \partial^2_{x_i,x_j}, Y = \langle x, BD\rangle - \partial_t, 1 \leq i,j \leq p_0$. *Then, for every compact subset* K *of* Ω *there exists a positive constant* c *such that*

$$\sup_K |\partial u| \leq c \sup_K |u| \tag{2.8}$$

for every solution u *of* $Lu = 0$ *in* Ω.

3 – Uniqueness results

The main purpose of this section is the proof of the following results

THEOREM 3.1. *Let $u \in C(\overline{S_I})$ be a solution of the Cauchy problem*

$$(3.1) \qquad \begin{cases} Lu = 0 & in \quad S_I, \\ u(\cdot, 0) = 0. \end{cases}$$

If there exists a positive constant c such that

$$(3.2) \qquad \int_{S_I} e^{-c|x|^2} |u(x,t)| dx\, dt < \infty,$$

then $u \equiv 0$ in S_I.

THEOREM 3.2. *Let $u \in C(\overline{S_I})$ be a solution of the Cauchy problem*

$$(3.1) \qquad \begin{cases} Lu = 0 & in \quad S_I, \\ u(\cdot, 0) = 0. \end{cases}$$

If $u \geq 0$, then $u \equiv 0$ in S_I.

We start by proving the following

LEMMA 3.1. *Let C be a $N \times N$, symmetric, positive, constant matrix, and let $z = (x,t)$ be a point of $\mathbb{R}^{N+1}$. Then there exist a positive constant c_0, depending only on the matrices B and C, and a positive constant $R = R(x)$ such that*

$$(3.3) \qquad \begin{gathered} \langle C\eta, \eta \rangle \geq \frac{c_0}{s} |\xi|^2, \\ \eta = D\left(\frac{1}{\sqrt{s}}\right)(x - E(s)\xi) \end{gathered}$$

for every $(\xi, s) \in B'_R(x) \times]0,1[= (\mathbb{R}^N \backslash B_R(x)) \times]0,1[$, where $B_R(x)$ is the (Euclidean) sphere of center x and radius R.

PROOF. Put

$$\begin{aligned}\widetilde{C} &= E^T(1)CE(1),\\ \tilde{\eta} &= D\left(\frac{1}{\sqrt{s}}\right)(\xi - E(-s)x).\end{aligned}$$

It follows from the first identity in (2.6) that

$$\langle C\eta, \eta\rangle = \langle \widetilde{C}\tilde{\eta}, \tilde{\eta}\rangle,$$

while (2.5) gives

$$\langle \widetilde{C}\tilde{\eta}, \tilde{\eta}\rangle \geq c|\tilde{\eta}|^2 \geq \frac{c}{s}|\xi - E(-s)x|^2 \geq \frac{c}{s}(|\xi| - |E(-s)x|)^2.$$

This proves Lemma 3.1, if we set $c_0 = c/4$ and

$$R \geq 2 \sup_{0<s<1} |E(-s)x|.$$

PROOF OF THEOREM 3.1. Let φ a nonincreasing $C^2(\mathbb{R})$ function such that $\varphi(t) = 0$ for $t \geq 2$, $\varphi(t) = 1$ for $t \leq 1$. Fix $\bar{x} \in \mathbb{R}^N$ and set, for every $R > 0$

$$h_R(x) = \varphi\left(\frac{|x - \bar{x}|}{R}\right). \tag{3.4}$$

Note that

$$\text{supp}(\partial_{x_i} h_R) \subset B_{2R}(\bar{x}) \backslash B_R(\bar{x}), \qquad \left|\partial_{x_i} h_R(x)\right| \leq \frac{c_1}{R},$$

for every $i = 1, ..., N$, thus $|Yh_R| \leq c_2$ for every $R > 0$. As a consequence $L^* h_R$ is a *bounded* function, uniformly with respect to $R \geq 1$.

Next fix $\delta \in]0, T], \delta \leq 1$ and denote by S_I' the set $\mathbb{R}^N \times]0, \delta[$. Starting from the Green's identity, we obtain with standard arguments that

$$u(\bar{z}) = \int_0^{\bar{t}} \left(\int_{B'_R(\bar{x})} \Big(\Gamma L^* h_R + 2\langle A(\xi, \tau) D\Gamma, Dh_R\rangle\Big) u(\xi, \tau) d\xi \right) d\tau, \tag{3.5}$$

for every $\bar{z} = (\bar{x}, \bar{t}) \in S'_I$, where $B'_R(\bar{x}) = \mathbb{R}^N \backslash B_R(\bar{x})$. Using the upper bounds in (1.8) and applying Lemma 3.1 to the matrix $C^+(1)$ associated to the operator L^+, and defined in (2.1), we deduce from the identity (3.5) that

$$|u(\bar{z})| \leq c_3 \int_0^{\bar{t}} \Big(\int_{B'_R(\bar{x})} \Big(1 + \frac{1}{\sqrt{\bar{t} - \tau}}\Big)\Gamma^+(\bar{z}; \xi, \tau)|u(\xi, \tau)| d\xi \Big) d\tau \leq$$

$$\leq c'_3 \int_0^{\bar{t}} \Big(\int_{B'_R(\bar{x})} (\bar{t} - \tau)^{-\frac{Q+1}{2}} \exp\Big(- c_0 \frac{|\xi|^2}{4(\bar{t} - \tau)}\Big) |u(\xi, \tau)| d\xi \Big) d\tau$$

for every $R \geq R(\bar{x})$. Now put $\delta = \min\left\{\frac{c_0}{8c}, 1\right\}$ (where c is the constant appearing in (3.2)). Since the function $(\xi, \tau) \mapsto (\bar{t} - \tau)^{-\frac{Q+1}{2}} \exp\Big(-c_0 \frac{|\xi|^2}{8(\bar{t} - \tau)}\Big)$ is bounded on $B'_R(\bar{x}) \times I$, uniformly in $R \geq R(\bar{x})$, we have

$$|u(\bar{z})| \leq c_4 \int_0^{\bar{t}} \Big(\int_{B'_R(\bar{x})} e^{-c|\xi|^2} |u(\xi, \tau)| d\xi \Big) d\tau. \tag{3.6}$$

On the other hand condition (3.2) implies

$$\lim_{R \to \infty} \int_0^{\bar{t}} \Big(\int_{B'_R(\bar{x})} e^{-c|\xi|^2} |u(\xi, \tau)| d\xi \Big) d\tau = 0 \tag{3.7}$$

then $u(\bar{z}) = 0$ for every $\bar{z} \in S'_I$.

Theorem 3.1 follows by iterating the previous argument, since δ depends only on the constant c in (3.2) and on the operator L.

As a simple consequence of Hypothesis H.3 and Theorem 3.1 we derive the following

COROLLARY 3.1 - Reproduction property. *If Γ denotes the fundamental solution of L, then*

$$\Gamma(x, t; \xi, \tau) = \int_{\mathbb{R}^N} \Gamma(x, t; y, s)\Gamma(y, s; \xi, \tau) dy$$

for every $\tau < s < t$ and for every $x, \xi \in \mathbb{R}^N$.

LEMMA 3.2. *Let $u \in C(\overline{S_I})$ be a nonnegative solution of $Lu = 0$. Then*

$$u(z) \geq \int_{\mathbb{R}^N} \Gamma(z;\xi,\tau) u(\xi,\tau) d\xi, \tag{3.8}$$

for every $z = (x,t) \in S_I$ and for every $\tau \in I, \tau < t$.

PROOF. Let $\tau \in I, \tau < t$ and define, for every $n \in \mathbb{N}, h_n(\xi) = \varphi\left(\frac{|\xi|}{n}\right)$ (see (3.4)),

$$f_n(\xi,\tau) = h_n(\xi) u(\xi,\tau)$$

and

$$U_n(z;\tau) = \int_{\mathbb{R}^N} \Gamma(z;\xi,\tau) f_n(\xi,\tau) d\xi. \tag{3.9}$$

Then $LU_n(\cdot\,;\tau) = 0$ in $\mathbb{R}^N \times]\tau, T[$ and

$$\lim_{(x,t)\to(y,\tau)} U_n(x,t;\tau) = f_n(y,\tau) \leq u(y,\tau)$$

for every $y \in \mathbb{R}^N$. On the other hand, using the upper bound (1.8) of Γ, we get

$$0 \leq U_n(z;\tau) \leq c^+ \int_{\mathbb{R}^N} \Gamma^+(z;\xi,\tau) f_n(\xi,\tau) d\xi.$$

Then, since f_n is a bounded function with compact support, we easily obtain from the explicit expression of Γ^+ that

$$\lim_{|x|\to\infty} \left(\sup_{\tau \leq t \leq T} U_n(x,t;\tau) \right) = 0.$$

Hence, the maximum principle, gives

$$0 \leq U_n(z,\tau) \leq u(z) \tag{3.10}$$

for every $z \in \mathbb{R}^N \times]\tau, T[$. Moreover, being the sequence (f_n) increasing, the function

$$U(z,\tau) = \lim_{n\to\infty} U_n(z,\tau), \tag{3.11}$$

is well defined and, by the monotone convergence theorem, $U(z,\tau) \leq u(z)$. This completes the proof of Lemma 3.2.

COROLLARY 3.2. *Let $u \in C(\overline{S_I})$ be a nonnegative solution of $Lu=0$. Then the following integral*

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)u(\xi,0)d\xi$$

converges for every $(x,t) \in S_I$. Moreover

$$\lim_{t\to 0+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)u(\xi,0)d\xi = u(x,0)$$

for every $x \in \mathbb{R}^N$.

PROOF. The first assertion follows from (3.8). To study the second one, we note that, for every $x \in \mathbb{R}^N$, and for every $n > |x|+1$, we have

$$\lim_{t\to 0+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)f_n(\xi,0)d\xi = f_n(x,0) = u(x,0),$$

where (f_n) is the sequence introduced in the proof of the previous Lemma. On the other hand

$$\begin{aligned}
0 \leq &\lim_{t\to 0+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\left[u(\xi,0) - f_n(\xi,0)\right]d\xi \leq \\
\leq &\lim_{t\to 0+} \int_{\mathbb{R}^N\setminus B(0,n)} \Gamma(x,t;\xi,0)u(\xi,0)d\xi \leq \\
\leq &c^+ \lim_{t\to 0+} \int_{\mathbb{R}^N\setminus B(0,n)} \Gamma^+(x,t;\xi,0)u(\xi,0)d\xi = 0,
\end{aligned}$$

since, for every $k > 0$, there exists $\delta = \delta(k) > 0$ such that the integrating function in the last term is bounded by $e^{-k|\xi|^2}u(\xi,0)$ uniformly in $t \in]0,\delta[$

and such a function is integrable on $\mathbb{R}^N$ for k big enough. Indeed, for every $t_0 \in]0,T[$ there exist two positive constants k_1, k_2 such that

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-k_1|\xi|^2} u(\xi,0)d\xi &\leq k_2 c^- \int_{\mathbb{R}^N} \Gamma^-(0,t_0;\xi,0)u(\xi,0)d\xi \leq \\ \text{(by (1.8))} \quad &\leq k_2 \int_{\mathbb{R}^N} \Gamma(0,t_0;\xi,0)u(\xi,0)d\xi \leq k_2 u(0,t_0), \end{aligned} \tag{3.12}$$

where last inequality follows from Lemma 3.2.

PROOF OF THEOREM 3.2. Let u be a nonnegative solution of (3.1). Fix $s, t \in I$ such that $t > s$ and note that constants k_1 and k_2 appearing in (3.12) depend continuously on $t_0 \in]0,T[$. Hence, integrating with respect to $\tau \in]0,s[$, we obtain

$$u(0,t) \geq \frac{c_0}{s} \int_0^s \Big(\int_{\mathbb{R}^N} e^{-c|\xi|^2} u(\xi,\tau)d\xi \Big) d\tau$$

for some positive constants c_0, c depending only on t and s. Then the proof of Theorem 3.2 follows from Theorem 3.1.

We will obtain a first representation result from the interior a priori estimates stated in the introduction. First we prove a preliminary result

PROPOSITION 3.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a monotone increasing and locally bounded sequence such that $Lu_n = 0$ in S_I, for every $n \in \mathbb{N}$. Then the function $u := \sup u_n$ is a solution of $Lu = 0$ in S_I.*

PROOF. We start by showing that the sequence u_n is locally equicontinuous. Let K be a compact subset of S_I. Being the Lie algebra generated by $\partial_{x_1}, ..., \partial_{x_{p_0}}, Y = \langle x, BD\rangle - \partial_t$ equal to $\mathbb{R}^{N+1}$, we can find a bounded open set Ω such that $K \subset \Omega \subset \overline{\Omega} \subset S_I$, two constants $\overline{c}(K) > 0$ and $\alpha \in]0,1[$ such that the following condition holds:

for every pair of points $z, \zeta \in K$, there exist a positive constant $\delta \leq \overline{c}(K)|z-\zeta|^\alpha$ and a continuous and piecewise differentiable path

$$\gamma : [0,\delta] \to \Omega, \quad \gamma(0) = z, \quad \gamma(\delta) = \zeta$$

such that

$$\gamma'(s) = \sum_{j=1}^{p_0} c_j(s)e_j + c_0(s)(B^T\gamma(s) - e_{N+1}),$$

for any $s \in [0,\delta]$ at which γ' is defined. Here $|c_j(s)| \leq 1$ for $0 \leq j \leq p_0$ and $e_j = (0,...,\underset{j}{1},...,0), 1 \leq j \leq N+1$ (for the above result see [6]).

Then, denoting by $\widetilde{D}$ the gradient in $\mathbb{R}^{N+1}$ and using the interior a priori estimates (2.8), we get

$$\begin{aligned}|u(z) - u(\zeta)| =& \Big|\int_0^\delta \frac{d}{ds}u(\gamma(s))ds\Big| \leq \\ \leq& \int_0^\delta \Big|\langle\widetilde{D}u(\gamma(s)), \gamma'(s)\rangle\Big| ds \leq \\ \leq& \int_0^\delta \Big(\sum_{j=1}^{p_0} |\partial_{x_j}u(\gamma(s))| + |Yu(\gamma(s))|\Big) ds \leq \\ \leq& \delta c(\overline{\Omega}) \sup_\Omega |u| \leq \sup_\Omega |u| c(\overline{\Omega})\overline{c}(K)|z-\zeta|^\alpha,\end{aligned}$$

for every $z,\zeta \in K$. Hence the sequence u_n is equicontinuous, and it converges uniformly on compact subsets of S_I.

Next, again using the interior a priori estimates (2.8), it is clear that $Lu = 0$, and Proposition 3.1 is proved.

PROPOSITION 3.2. *Let $u \in C(\overline{S_I})$ be a nonnegative solution of $Lu = 0$ in S_I. Then*

$$u(z) = \int_{\mathbb{R}^N} \Gamma(z;\xi,0)u(\xi,0)d\xi. \tag{3.13}$$

PROOF. The sequence U_n defined in (3.9) satisfies the hypotheses of Proposition 3.1, then the function U, defined in (3.11), solves $Lu = 0$ in S_I. Moreover, from (3.10) it follows that the function $u(z) - U(z;0)$ is nonnegative and from Corollary 3.2 we get $u(x,0) = U(x,0;0)$ for every $x \in \mathbb{R}^N$. Hence Theorem 3.2 provides the claim.

4 – Representation and Fatou type results

In this section we shall prove the following

THEOREM 4.1.

(i) *Let u be a nonnegative solution of $Lu = 0$ on S_I ($I =]0, T[$). Then there exists a nonnegative Borel measure ρ such that*

$$\int_{\mathbb{R}^N} e^{-c|x|^2} d\rho(x) < \infty, \tag{4.1}$$

for some positive constant c, and

$$u(z) = \int_{\mathbb{R}^N} \Gamma(z; \xi, 0) d\rho(\xi), \tag{4.2}$$

for every $z \in S_I$.

(ii) *For every nonnegative Borel measure ρ verifying (4.1), there exists an interval $I =]0, T[$ such that the function u defined in (4.2) is a solution of $Lu = 0$ on S_I.*

(iii) *Denote by φ the density of the absolutely continuous part of ρ (with respect to the Lebesgue measure). Then*

$$\lim_{t \to 0+} u(x, t) = \varphi(x) \tag{4.3}$$

for almost every $x \in \mathbb{R}^N$.

(iv)

$$\lim_{t \to 0+} u(\cdot, t) = \rho \tag{4.4}$$

in the measure sense.

REMARK 4.1. Here and in the sequel, the expression "almost every" will always understood "with respect to the Lebesgue measure", denoted by m.

PROOF. (i) Let u be a nonnegative solution of $Lu = 0$ in S_I. Then Proposition 3.2 gives

$$u\left(x, t+\frac{1}{n}\right) = \int_{\mathbb{R}^N} \Gamma\left(x, t+\frac{1}{n}; \xi, \frac{1}{n}\right) u\left(\xi, \frac{1}{n}\right) d\xi$$

for every $n \in \mathbb{N}$. As a consequence, using the lower bound in (1.8) it is clear that, for every $t_0 \in I$, there exist two positive constants c, c' which do not depend on n, such that

$$\begin{aligned} u\left(0, t_0+\frac{1}{n}\right) \geq & c^- \int_{\mathbb{R}^N} \Gamma^-\left(0, t_0+\frac{1}{n}; \xi, \frac{1}{n}\right) u\left(\xi, \frac{1}{n}\right) d\xi \geq \\ \geq & c' \int_{\mathbb{R}^N} e^{-c|\xi|^2} u\left(\xi, \frac{1}{n}\right) d\xi \geq 0 \end{aligned}$$

for every $n > \frac{1}{T-t_0}$. Therefore the sequence of measures

$$d\mu_n = e^{-c|\xi|^2} u\left(\xi, \frac{1}{n}\right) d\xi,$$

is bounded, thus, by the Frostman's Selection Theorem, there exists a nonnegative Borel measure μ such that, up to a subsequence,

$$\mu_n \xrightarrow[n\to\infty]{w} \mu.$$

In order to prove (4.2) we first observe that, as a consequence of Lemma 3.1 and of the upper bound in (1.8), there exist two constants $M > c$ and $T_1 \in I$ such that

$$\Gamma\left(x, t+\frac{1}{n}; \xi, \frac{1}{n}\right) \leq c(x,t) e^{-M|\xi|^2} \tag{4.5}$$

for every $(x,t) \in \mathbb{R}^N \times]0, T_1[$ and $\xi \in \mathbb{R}^N, n \in \mathbb{N}$; hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} e^{c|\xi|^2} \Gamma(x, t; \xi, 0) \left(d\mu_n(\xi) - d\mu(\xi)\right) = 0.$$

Moreover, the sequence μ_n is bounded and

$$e^{c|\xi|^2}\Gamma\left(x,t+\frac{1}{n};\xi,\frac{1}{n}\right) \underset{n\to\infty}{\overset{\xi\in\mathbb{R}^N}{\rightrightarrows}} e^{c|\xi|^2}\Gamma\left(x,t;\xi,0\right),$$

then we conclude that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} e^{c|\xi|^2}\Gamma\left(x,t+\frac{1}{n};\xi,\frac{1}{n}\right)d\mu_n(\xi)=\int_{\mathbb{R}^N} e^{c|\xi|^2}\Gamma\left(x,t;\xi,0\right)d\mu(\xi) \tag{4.6}$$

for every $(x,t)\in\mathbb{R}^N\times[0,T_1[$. This proves (4.2) in the strip $\mathbb{R}^N\times[0,T_1[$, with $\rho=e^{c|\xi|^2}\mu$. Note that ρ satisfies (4.1).

Consider now a point $(x,t)\in S_I$, with $t\geq T_1$. For every $\tau\in]0,T_1[$ we obtain from Proposition 3.2 and Corollary 3.1 that

$$\begin{aligned}u(x,t)&=\int_{\mathbb{R}^N}\Gamma(x,t;\xi,\tau)u(\xi,\tau)d\xi=\\&=\int_{\mathbb{R}^N}\Gamma(x,t;\xi,\tau)\Big(\int_{\mathbb{R}^N}\Gamma(\xi,\tau;\eta,0)d\rho(\eta)\Big)d\xi=\int_{\mathbb{R}^N}\Gamma(x,t;\eta,0)d\rho(\eta),\end{aligned}$$

then the first assertion of Theorem 4.1 is proved.

(ii). First note that, as a consequence of (4.5), there exists $T>0$ such that the function

$$u(x,t)=\int_{\mathbb{R}^N}\Gamma(x,t;\xi,0)d\rho(\xi)$$

is defined and continuous on $S_I=\mathbb{R}^N\times]0,T[$. In order to show that such a function is a solution of $Lu=0$ we consider, for any $n\in\mathbb{N}$ and for every $\varepsilon\in]0,T[$, the Cauchy problem

$$\begin{cases}Lv=0 & \text{in}\quad \mathbb{R}^N\times]\varepsilon,T[,\\ v(x,\varepsilon)=h_n(x)u(x,\varepsilon),\end{cases} \tag{4.7}$$

where h_n is the function defined in (3.4). Since the initial condition is a continuous function with compact support, the solution $v_{n,\varepsilon}$ of problem (4.7) can be written as

$$v_{n,\varepsilon}(x,t)=\int_{\mathbb{R}^N}\Gamma(x,t;\xi,\varepsilon)h_n(\xi)u(\xi,\varepsilon)d\xi.$$

On the other hand Corollary 3.1 yields

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\varepsilon)h_n(\xi)u(\xi,\varepsilon)d\xi \leq$$
$$\leq \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\varepsilon)u(\xi,\varepsilon)d\xi =$$
$$= \int_{\mathbb{R}^N} \Gamma(x,t;\eta,0)d\rho(\eta) = u(x,t),$$

whence, for every $\varepsilon > 0\, (v_{n,\varepsilon})_{n\in\mathbb{N}}$ is a monotone increasing and locally bounded sequence. Then Proposition 3.1 implies that $v_\varepsilon = \lim_{n\to\infty} v_{n,\varepsilon}$ is a solution of $Lu = 0$ in $\mathbb{R}^N \times]\varepsilon, T[$. Moreover, from the monotone convergence theorem, it follows that $v_\varepsilon = u$ in $\mathbb{R}^N \times]\varepsilon, T[$, hence, being ε arbitrary, we obtain assertion (ii).

In order to prove (4.3) and (4.4) we shall need some results from the Real Analysis, that are classic for the Euclidean geometry (see e.g. [3]). Here, we shall suitably adapt these results to the natural geometry related to every operator L, and we will give, for reader convenience, a self contained presentation of these in our setting.

For every $v \in \mathbb{R}^N$, we define

$$p(v) = \max\left\{|v_j|; 1 \leq j \leq N\right\} \tag{4.8}$$

and, for every $n \in \mathbb{N} \cup \{0\}, (x,t) \in \mathbb{R}^{N+1}, t > 0$, we set

$$C_n(x,t) = \left\{\xi \in \mathbb{R}^N : p\left(D\left(\frac{1}{\sqrt{t}}\right)\left(\xi - E(-t4^{-n})x\right)\right) < 1\right\}, \tag{4.9}$$

Then the following Vitali Covering Lemma holds

LEMMA 4.1. *For every finite family $\{C_1, ..., C_h\}$ of "boxes" of the type (4.9) there exists a disjoint subfamily $\{C_{i_1}, ..., C_{i_k}\}$ such that*

$$m\left(\bigcup_{j=1}^{h} C_j\right) \leq 3^N \sum_{l=1}^{k} m\left(C_{i_l}\right). \tag{4.10}$$

The proof of Lemma 4.1 is standard and it will be omitted.

LEMMA 4.2. *Let s be a nonnegative Borel measure on $\mathbb{R}^N$ and let $A \subset \mathbb{R}^N$ be a Borel set such that $s(A) = 0$. Then*

$$\lim_{t\to 0+}\left(\sup_{n\in\mathbb{N}\cup\{0\}} \frac{s\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)}\right) = 0 \tag{4.11}$$

for almost every $x \in A$.

PROOF. Let a be a fixed positive constant, and put

$$B_a = \left\{x \in A : \limsup_{t\to 0+}\left(\sup_{n\in\mathbb{N}\cup\{0\}} \frac{s\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)}\right) > a\right\}.$$

Note that, being the function $(x,t,n) \mapsto s\left(C_n(x,t)\right)$ lower semicontinuous, B_a is a Borel set. Choose $r \in]0,1[$ and let K be a compact subset of B_a. Then, for every $x \in K$ there exist $t \in]0,r[$ and $n \in \mathbb{N}$ such that

$$s\left(C_n(x,t)\right) > a\, m\left(C_n(x,t)\right). \tag{4.12}$$

Note that, for every $t > 0$ and $n \in \mathbb{N}\cup\{0\}$, each point $x \in \mathbb{R}^N$ belongs to the open set $x - E(-t4^{-n})x + C_n(x,t)$. By compactness, K is then covered by a finite family $\left\{x_j - E(-t_j 4^{-n_j})x_j + C_{n_j}(x_j,t_j), 1 \le j \le h : x_j, t_j, n_j \text{ verifying } (4.12)\right\}$, so that Lemma 4.1 gives

$$\begin{aligned} m(K) &\le m\left(\bigcup_{j=1}^{h}\left(x_j - E(-t_j 4^{-n_j})x_j + C_{n_j}(x_j,t_j)\right)\right) \le \\ &\le 3^N \sum_{l=1}^{k} m\left(x_{i_l} - E(-t_{i_l}4^{-n_{i_l}})x_{i_l} + C_{n_{i_l}}(x_{i_l},t_{i_l})\right) = \\ &= 3^N \sum_{l=1}^{k} m\left(C_{n_{i_l}}(x_{i_l},t_{i_l})\right). \end{aligned}$$

Using relation (4.12), we thus obtain

$$m(K) < \frac{3^N}{a}\sum_{l=1}^{k} s\left(C_{n_{i_l}}(x_{i_l},t_{i_l})\right) \le \frac{3^N}{a} s(K_\rho), \tag{4.13}$$

where $K_\rho = \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) < \rho\}$ and $\rho(r) = O(\sqrt{r})$. Indeed, if $x = x_{i_l} \in K, t = t_{i_l} \in]0, r[, n = n_{i_l} \in \mathbb{N} \cup \{0\}$, then

$$\begin{aligned} |\xi - x| \leq |\xi - E(-t4^{-n})x| + |(I - E(-t4^{-n}))x| \leq \\ \leq k\sqrt{Nt} + \|I - E(-t4^{-n})\| |x| \end{aligned}$$

for every $\xi \in C_n(x,t)$. Moreover, from the definition of $E(s)$ it directly follows that there exists a positive constant c such that $\|I - E(-t4^{-n})\| \leq c\,t$ for every $t \in]0, r[$ and for every $n \in \mathbb{N} \cup \{0\}$. As a consequence $|\xi - x| \leq c'\sqrt{t}$ for every $\xi \in C_n(x,t)$, for some positive constant c' depending only on K and c.

Therefore, since $\lim_{\rho\to 0} s(K_\rho) = s(K) = 0$, from (4.13), it follows that $m(K) = 0$ for every compact subset K of B_a. Hence $m(B_a) = 0$, and Lemma 4.2 is proved.

Consider the Lebesgue decomposition of ρ:

$$\begin{aligned} \rho =& \mu + s, \\ \mu \ll m & \quad s \perp m, \end{aligned} \tag{4.14}$$

and let φ be the density of μ:

$$\varphi \in L^1(\mathbb{R}^N), \qquad d\mu = \varphi dm.$$

PROPOSITION 4.1.

$$\lim_{t\to 0+} \left(\sup_{n\in\mathbb{N}\cup\{0\}} \frac{1}{m(C_n(x,t))} \int_{C_n(x,t)} \Big(|\varphi(\xi) - \varphi(x)| d\xi + ds(\xi) \Big) \right) = 0 \tag{4.15}$$

for almost every $x \in \mathbb{R}^N$.

PROOF. We start by proving that

$$\lim_{t\to 0+} \left(\sup_{n\in\mathbb{N}\cup\{0\}} \frac{1}{m(C_n(x,t))} \int_{C_n(x,t)} \Big((\varphi(\xi) - \varphi(x)) d\xi + ds(\xi) \Big) \right) = 0 \tag{4.16}$$

for almost every $x \in \mathbb{R}^N$.

First consider the measure s. Being $s \perp m$, there exists a set $B \subset \mathbb{R}^N$ such that $s(B) = 0$ and $m(B') = 0$. Then Lemma 3.2 yields

$$\lim_{t \to 0+} \left(\sup_{n \in \mathbb{N} \cup \{0\}} \frac{s\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)} \right) = 0 \tag{4.17}$$

for almost every $x \in B$, and so, since $m(B') = 0$, for almost every $x \in \mathbb{R}^N$.

Now consider the Lebesgue integral in (4.16). For every $a \in \mathbb{R}$ and for every Borel set $E \subset \mathbb{R}^N$, we define a nonnegative measure λ by setting

$$\lambda(E) = \int_{E \cap \{\varphi \geq a\}} (\varphi - a) dm.$$

Applying Lemma 3.2 to $A = \{\varphi < a\}$ we then get

$$\lim_{t \to 0+} \left(\sup_{n \in \mathbb{N} \cup \{0\}} \frac{\lambda\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)} \right) = 0$$

for almost every $x \in A$. Hence, noting that

$$\frac{\mu\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)} \leq a + \frac{\lambda\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)},$$

we have $\limsup_{t \to 0+} \left(\sup_{n \in \mathbb{N} \cup \{0\}} \frac{\mu(C_n(x,t))}{m(C_n(x,t))} \right) \leq a$ for almost every $x \in A$. Then, by setting

$$E_a = \left\{ x \in \mathbb{R}^N : \varphi(x) < a < \limsup_{t \to 0+} \left(\sup_{n \in \mathbb{N} \cup \{0\}} \frac{\mu\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)} \right) \right\},$$

we get $m(E_a) = 0$, thus

$$\limsup_{t \to 0+} \left(\sup_{n \in \mathbb{N} \cup \{0\}} \frac{\mu\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)} \right) \leq \varphi(x)$$

for almost every $x \in \mathbb{R}^N$.

Using $-\mu$ instead of μ and proceeding as above, we finally obtain

$$\liminf_{t\to 0+}\left(\sup_{n\in\mathbb{N}\cup\{0\}}\frac{\mu\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)}\right)\geq\varphi(x)$$

for almost every $x\in\mathbb{R}^N$. Hence

$$\lim_{t\to 0+}\left(\sup_{n\in\mathbb{N}\cup\{0\}}\frac{\mu\left(C_n(x,t)\right)}{m\left(C_n(x,t)\right)}\right)=\varphi(x) \tag{4.18}$$

for almost every $x\in\mathbb{R}^N$. Equations (4.17) and (4.18) implie (4.16), from which, with a standard argument, Lemma 4.2 follows (see Stein [13], page 11).

Proof of Theorem 4.1 - (iii) Let x be a point of $\mathbb{R}^N$ satisfying Proposition 4.1. Then, for every $\varepsilon>0$ there exists $\delta>0$ such that

$$0\leq\frac{1}{m\left(C_n(x,t)\right)}\int\limits_{C_n(x,t)}d\nu(\xi)<\varepsilon \tag{4.19}$$

for every $t\in]0,\delta[$ and for every $n\in\mathbb{N}\cup\{0\}$, where

$$d\nu(\xi)=\left|\varphi(\xi)-\varphi(x)\right|d\xi+ds(\xi).$$

Equation (4.2) gives immediately

$$u(x,t)-\varphi(x)=\int\limits_{\mathbb{R}^N}\Gamma(x,t;\xi,0)\Big(\left(\varphi(\xi)-\varphi(x)\right)d\xi+ds(\xi)\Big).$$

Using estimates (1.8) and the explicit expression of Γ^+ we can show that there exist two positive constants c_1,k such that

$$\Gamma(x,t;\xi,0)\leq c^+\Gamma^+(x,t;\xi,0)\leq c_1t^{-\frac{Q}{2}}e^{-kp^2\left(D\left(\frac{1}{\sqrt{t}}\right)\left(\xi-E(-t)x\right)\right)},$$

therefore

$$\left|u(x,t)-\varphi(x)\right|\leq\frac{c_1}{t^{\frac{Q}{2}}}\int\limits_{\mathbb{R}^N}e^{-kp^2\left(D\left(\frac{1}{\sqrt{t}}\right)\left(\xi-E(-t)x\right)\right)}d\nu(\xi). \tag{4.20}$$

Fix now $t \in]0,\delta[$ and put $k_t = \max\left\{n \in \mathbb{N} \cup \{0\} : 4^n t < \delta\right\}$. Then

(4.21)
$$\int_{\mathbb{R}^N} \frac{c_1}{t^{\frac{Q}{2}}} e^{-kp^2} d\nu(\xi) \leq \left(\int_{\{p<1\}} + \sum_{j=1}^{k_t} \int_{\{2^{j-1} \leq p < 2^j\}} + \int_{\left\{p \geq \sqrt{\frac{\delta}{4t}}\right\}} \right) \frac{c_1}{t^{\frac{Q}{2}}} e^{-kp^2} d\nu(\xi) =$$
$$= I + II + III.$$

Since $\{p < 1\} = C_0(x,t)$ and $m(C_0(x,t)) = 2^N t^{\frac{Q}{2}}$, from (4.19) it follows that

$$I \leq 2^N c_1 \varepsilon \tag{4.22}$$

for any $t \in]0,\delta[$.

Consider now one of the terms of the sum II:

$$\int_{\{2^{j-1} \leq p < 2^j\}} \frac{c_1}{t^{\frac{Q}{2}}} e^{-kp^2} d\nu(\xi) \leq \frac{c_1}{t^{\frac{Q}{2}}} e^{-k4^{j-1}} \int_{\{p<2^j\}} d\nu(\xi) = \frac{c_2}{t^{\frac{Q}{2}}} e^{-k4^j} \int_{\{p<2^j\}} d\nu(\xi).$$

Note that, from the definition of $D(\lambda)$, we see that $p(D(n)v) \geq np(v)$ and $D(\lambda\mu) = D(\lambda)D(\mu)$ for every $\lambda, \mu > 0, n \in \mathbb{N}, v \in \mathbb{R}^N$. We can then write, setting $s = 4^j t$,

$$\left\{ p\left(D\left(\frac{1}{\sqrt{t}}\right)(\xi - E(-t)x) \right) < 2^j \right\} \subset C_j(x,s).$$

Moreover, since $j \leq k_t$ we have $s \in]0,\delta[$ and $m(C_j(x,s)) = 2^N s^{\frac{Q}{2}} = 2^{N+jQ} t^{\frac{Q}{2}}$. Then (4.19) yields

$$\int_{\{2^{j-1} \leq p < 2^j\}} \frac{c_1}{t^{\frac{Q}{2}}} e^{-kp^2} d\nu(\xi) \leq \frac{c_2 2^{N+jQ}}{2^{N+jQ} t^{\frac{Q}{2}}} e^{-k4^j} \int_{C_j(x,s)} d\nu(\xi) \leq \varepsilon c_3 2^{Qj} e^{-k4^j},$$

for every $t \in]0,\delta[$, therefore

$$II \leq \varepsilon c_3 \sum_{j=1}^{k_t} 2^{Qj} e^{-k4^j} \leq \varepsilon c_4, \tag{4.23}$$

uniformly in $t \in]0,\delta[$, being convergent the series $\sum_{j=1}^{\infty} 2^{Qj} e^{-k4^j}$.

In order to evaluate III we first note that

$$t^{-\frac{Q}{2}} e^{-\frac{k}{2}p^2} = \frac{p^Q e^{-\frac{k}{2}p^2}}{p^Q t^{\frac{Q}{2}}} \leq c_5 \left(\frac{4}{\delta}\right)^{\frac{Q}{2}}$$

on the set $\left\{p \geq \sqrt{\frac{\delta}{4t}}\right\}$, thus

$$(4.24)\qquad \begin{aligned} III \leq & \frac{c_6}{\delta^{\frac{Q}{2}}} \int\limits_{\left\{p \geq \sqrt{\frac{\delta}{4t}}\right\}} e^{-\frac{k}{2}p^2} d\nu(\xi) \leq \frac{c_6}{\delta^{\frac{Q}{2}}} \varphi(x) \int\limits_{\mathbb{R}^N} e^{-\frac{k}{2}p^2} d\xi + \\ & + \frac{c_6}{\delta^{\frac{Q}{2}}} e^{-\frac{\delta k}{8t}} \int\limits_{\mathbb{R}^N} \Big(\varphi(\xi) d\xi + ds(\xi)\Big) = III_1 + III_2. \end{aligned}$$

Then, since

$$\lim_{t\to 0} e^{-\frac{k}{2}p^2\left(D\left(\frac{1}{\sqrt{t}}\right)\left(\xi - E(-t)x\right)\right)} = 0$$

for almost every $\xi \in \mathbb{R}^N$, there exists $\delta_0 \in]0,\delta[$ such that

$$(4.25)\qquad 0 \leq III_1,\ III_2 \leq \varepsilon$$

for every $t \in]0,\delta_0[$.

Hence, substituting inequalities (4.22), (4.23) and (4.25) in (4.20), we obtain

$$\lim_{t\to 0+} u(x,t) = \varphi(x)$$

for every $x \in \mathbb{R}^N$ verifying Proposition 4.1, then for almost every $x \in \mathbb{R}^N$.

(iv). To prove equality (4.4) we need to show that

$$(4.26)\qquad \lim_{t\to 0+} \int\limits_{\mathbb{R}^N} u(x,t)\chi(x)dx = \int\limits_{\mathbb{R}^N} \chi(x)d\rho(x).$$

for every function $\chi \in C_0(\mathbb{R}^N)$. Relation (4.2) gives

$$(4.27)\qquad \int\limits_{\mathbb{R}^N} u(x,t)\chi(x)dx = \int\limits_{\mathbb{R}^N} \Big(\int\limits_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\chi(x)dx\Big) d\rho(\xi),$$

and, as a consequence of Hypothesis H.2,

$$\lim_{t\to 0+} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\chi(x)dx = \chi(\xi).$$

In order to prove (4.26) it is then sufficient to show that we can carry the limit under the integral sign in the right hand side of (4.27). To this end we first show that there exist $T_0 \in]0,T[$ and $M>0$ such that

$$\left| \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\chi(x)dx \right| \le Me^{-c|\xi|^2} \tag{4.28}$$

for every $t \in]0,T_0[$ and for every $\xi \in \mathbb{R}^N$, where c is the constant appearing in (4.1), then we apply the Lebesgue theorem.

It follows from estimates (1.8) and from Lemma 3.1 that there exist two positive constants: c_0, depending only on the operator L, and k_0, which may also depend on the (bounded) set supp(χ), such that

$$\begin{aligned}\Gamma(x,t;\xi,0) &\le \frac{c^+}{t^{\frac{Q}{2}}} \exp\left(-\frac{1}{4}\langle C^+\eta,\eta\rangle\right) \le \\ &\le \frac{k_0}{t^{\frac{Q}{2}}} \exp\left(-\frac{1}{8}\langle C^+\eta,\eta\rangle\right) \exp\left(-\frac{c_0}{8t}|\xi|^2\right); \\ \eta &= D\left(\frac{1}{\sqrt{t}}\right)(x-E(t)\xi)\end{aligned}$$

for every $x, \in$ supp(χ) and $t \in I$. Hence, setting $T_0 = \frac{c_0}{8c}$, we have

$$\left| \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0)\chi(x)dx \right| \le k_0 \sup|\chi| e^{-c|\xi|^2} \int_{\mathbb{R}^N} t^{-\frac{Q}{2}} \exp\left(-\frac{1}{8}\langle C^+\eta,\eta\rangle\right) dx,$$

then (4.28) holds, with

$$M = k_0 \sup|\chi| \int_{\mathbb{R}^N} e^{-\langle C^+y,y\rangle} dy.$$

This completes the proof of Theorem 4.1.

From (4.4) we immediately derive the following

COROLLARY 4.1. *Let u be a nonnegative solution of $Lu = 0$ in S_I. If*

$$u(\cdot, t) \xrightarrow[t \to 0+]{w} 0,$$

then $u \equiv 0$ in S_I.

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