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PhD Thesis

DEFECTIVENESS AND IDENTIFIABILITY: A GEOMETRIC POINT OF VIEW ON TENSOR ANALYSIS

XXXIII CICLO

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Abstract

Given a projective variety $X \subset \mathbb{P}^N$ we can define its $h-$secant variety $\mathbb{S}ec_h(X) \subset \mathbb{P}^N$, i.e. the Zariski closure of all points $p \in \mathbb{P}^N$ lying on a $\mathbb{P}^{h-1}$ which is $h-$secant to X. The variety X is said to be $h-$identifiable if the general point $p \in \mathbb{S}ec_h(X)$ can be expressed uniquely as a linear combination $p = \lambda_1 p_1 + \cdots + \lambda_h p_h$ with $p_1, \ldots, p_h$ points of X. Thanks to Terracini's lemma it is possible to rephrase the problem of secant dimensions and identifiability in the birational setting. This turns out in the study of the dimension and the singularities of linear systems of the form $|\mathcal{O}_X(1) \otimes \mathcal{I}_{p_1^2,\ldots,p_h^2}|$, i.e. hyperplane divisors of X singular at h general points.

In the area of tensor analysis these notions are related to the properties of tensor decomposition. For applications ranging from biology to Blind Signal Separation, data compression algorithms and analysis of mixture models, uniqueness of decompositions allows to solve the problem once a solution is determined.

The thesis studies the relation between defectiveness and identifiability. It is shown how to link the geometry of the tangential contact locus to the secant defect, proving that under mild numerical conditions the non $h-$secant defectiveness imply the $(h-1)-$identifiability, where h is less than the generic rank. With our techinques it is possible to give new bounds for the identifiability in the case of many important tensor varieties such as Veronese, Segre and Grassmannians.

In the case of generic identifiability it is studied the nested singularities of tangential linear system. With this, together with the classical Noether-Fano inequalities, it is proved a new statement on generic identifiability of many partially symmetric tensors.

Next it is studied the defectiveness for Flag varieties, i.e. special tensor varieties parametrizing chains of vector spaces $0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{P}^N$. We improve the osculating projection techinque from Araujo, Massarenti and Rischter, giving completely new bounds on secant defectiveness and identifiability.

The new notion of $(h,s)-$tangential weak defectiveness is introduced and studied for the case of Segre-Veronese varieties.

The study of Secant varieties of Veronese embedding allowed also to check Comon's conjecture under improved numerical bounds.

Keywords: secant defectiveness, weakly defectiveness, tangential weak defectiveness, identifiability, tensor varieties

Sintesi

Data una varietà proiettiva $X \subset \mathbb{P}^N$ possiamo definire la varietà $h-$secante $\mathbb{S}ec_h(X) \subset \mathbb{P}^N$ come la chiusura nella topologia di Zariski di tutti i punti $p \in \mathbb{P}^N$ contenuti in uno spazio lineare $\mathbb{P}^{h-1}$ $h-$secante rispetto a X. La varietà X si dice essere $h-$identificabile se il punto generale $p \in \mathbb{S}ec_h(X)$ può essere espresso in modo univoco come una combinazione lineare $p = \lambda_1 p_1 + \cdots + \lambda_h p_h$ con $p_1, \ldots, p_h$ punti di X. Grazie al lemma di Terracini è possibile riformulare il problema delle dimensioni secanti e dell'identificabilità nel contesto della geometria birazionale. Ciò si traduce nello studio della dimensione e delle singolarità dei sistemi lineari della forma $|\mathcal{O}_X(1) \otimes \mathcal{I}_{p_1^2,\ldots,p_h^2}|$, ovvero sistemi lineari di sezioni iperpiane di X singolari in h punti generali.

Nell'area dell'analisi tensoriale queste nozioni sono legate alle proprietà della decomposizione tensoriale. Per le applicazioni che vanno dalla biologia al Blind Signal Separation, algoritmi di compressione dei dati e analisi di mixture models l'unicità delle decomposizioni tensoriali consente di risolvere il problema una volta determinata la soluzione.

In questa tesi si studia la relazione tra difettività e identificabilità. Viene mostrato come collegare la geometria del luogo di contatto tangenziale al difetto secante, dimostrando che sotto opportune condizioni numeriche la non $h-$difettività implica la $(h-1)-$identificabilità, dove h è strettamente minore del rango generico. Grazie a queste tecniche è possibile migliorare i risultati di identificabilità nel caso di molte importanti varietà tensoriali quali Veronese, Segre e Grassmanniane.

Nel caso dell'identificabilità generica invece, vengono studiate le singolarità del sistema lineare tangenziale. Da questo, insieme alle classiche disuguaglianze di Noether-Fano, si ottiene un nuovo risultato sull'identificabilità generica di molti tensori parzialmente simmetrici.

Successivamente viene studiata la difettività per le varietà Flag, ovvero speciali varietà tensoriali che parametrizzano particolari configurazioni di spazi vettoriali

$$0 \subset V_1 \subset \cdots \subset V_k \subset \mathbb{P}^N$$

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Viene in seguito introdotta e studiata la nuova nozione di $(h, s)-$difettività tangenziale debole e applicata nel caso delle varietà di Segre-Veronese.

Infine lo studio delle varietà secanti nel caso delle varietà di Veronese ha permesso anche di verificare la congettura di Comon in alcuni casi non noti precedentemente.

Parole Chiave: Difettività secante, difettività debole, difettività tangenzialmente debole, identificabilità, geometria tensoriale.

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Introduction

In the last twenty years Algebraic Geometry has played a crucial role in many applications to the most revelant fields of sciences. Biology, Blind Signal Separation, data compression algorithms, analysis of mixture models psycho-metrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis, see for instance [DDL13a], [DDL13b], [DL15], [KADL11], [SB00], [KB09a], [CGLM08], [LO15], [MR13]. The mathematical background that links all of these fields together is Tensor Analysis.

In particular if $X \subset \mathbb{P}^N$ is a projective variety parametrizing special type of tensors (tensor varieties), where $\mathbb{P}^N = \mathbb{P}(\Gamma)$ for some submodule $\Gamma \subseteq V_1 \otimes \cdots \otimes V_s$, a crucial problem is to determine the dimension of the locus of tensors $T \in \mathbb{P}^N$ such that T can be expressed as a limit of linear combinations of "simpler" tensors, i.e. the ones belonging to the variety X.

More generally we can consider the $h-$secant variety $\mathbb{S}ec_h(X) \subset \mathbb{P}^N$ attached to a projective variety $X \subset \mathbb{P}^N$, i.e. the Zariski closure of the points $z \in \mathbb{P}^N$ lying in the linear span of h points of X:

$$\mathbb{S}ec_h(X) = \overline{\{z \in \mathbb{P}^N | z \in \langle x_1, \ldots, x_h \rangle \; s.t. \; x_1, \ldots, x_h \in X\}} \subset \mathbb{P}^N$$

By a straightforward dimensional count the expected dimension of $\mathbb{S}ec_h(X)$ is

$$\operatorname{expdim}(\mathbb{S}ec_h(X)) = \min\{h \dim(X) + h - 1, N\}$$

The actual dimension of $\mathbb{S}ec_h(X)$ however can be smaller than the expected one, i.e. $\dim(\mathbb{S}ec_h(X)) < \operatorname{expdim}(\mathbb{S}ec_h(X))$: in this case we say that X is $h-$defective.

The classification of $h-$defective tensor varieties is far from being complete. The only class completely classified is that of Veronese varieties, see [AH95]. For other types of tensor varieties bounds on h such that X is not $h-$defective are given, see for instance [AB09], [AB12], [AB13], [AOP09], [BBC12], [BCC11], [BDDG07], [LP13], [MR17],[AMR19] and many others.

Thanks to [CC02, Proposition 5.4] it is possible to control the $h-$ secant defect studying the $h-$ tangential projection

$$\tau_h : X \dashrightarrow \mathbb{P}^M$$

where τ_h is the rational map associated to the linear system $|\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\ldots,x_h^2}|$. In particular if τ_h is generically finite then X is not $(h+1)-$defective. This result will be improved with the technique of osculating projection, introduced in [MR17],[AMR19], see Chapter 5.

If $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ is a tensor variety another important question to ask is whenever a given tensor $T \in \mathbb{S}ec_h(X)$ can be written uniquely as

$$T = \lambda_1 T_1 + \cdots + \lambda_h T_h$$

where $T_1, \ldots, T_h$ are tensors in X. This leads to the notion of identifiability for a tensor variety, and more generally for every variety $X \subset \mathbb{P}^N$. An element p of a projective space $\mathbb{P}^N$ is h-identifiable via a variety X if there is a unique way to write p as linear combination of h elements of X. In the classical setting this very often translates into rationality problems and it is linked to the existence of birational parameterizations.

An important notion is that of weak defectiveness, introduced by Chiantini and Ciliberto in [CC02]: given a projective variety $X \subset \mathbb{P}^N$ and $x_1, \ldots, x_h$ general points of X we can consider $\Gamma_h(H)$ as the union of irreducible components of $Sing(H)$ passing through $x_1, \ldots, x_h$, where $H \in |\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2, \ldots, x_h^2}|$ is a general hyperplane section singular at the prescribed points. The variety X is said to be $h-$weakly defective if $\dim(\Gamma_h(H)) > 0$, see Section 1.3 for a specific treatment.

Weak defectiveness has been connected to the study of identifiability problems, see for instance [Mel06].

If a variety X is not $h-$weakly defective then it is $h-$identifiable. In general, however, controlling the $h-$weak defectiveness property is quite hard, for this reason in [CO12] the authors introduced the related notion of $h-$tangential weak defectiveness. Given a projective variety $X \subset \mathbb{P}^N$ and $x_1, \ldots, x_h$ general points of X we denote by Γ_h the union of the irreducible components of the singular locus of the scheme $\langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_h}X \rangle \cap X$ passing through the points $x_1, \ldots, x_h$. The variety X is said to be $h-$tangential weakly defective is $\dim(\Gamma_h) > 0$, and Γ_h is called the $h-$tangential contact locus, see Section 1.4. If X is not h-tangential weakly defective then it is identifiable [CO12, Proposition 2.4]. Even though the converse does not hold in general, the h-tangential contact locus of X gives the right information on the number of decompositions of the general element of $\mathbb{S}ec_h(X)$, in fact the h-secant degree of X is equal to the h-secant degree of the h-tangential contact locus of X [CC10].

A common problem in tensor analysis is to compute the rank of a tensor whenever it is known a decomposition of it lying in a suitable tensor subvariety. In particular if $T \in \langle Y \rangle$ and h is the minimum such that

$$T = \lambda_1 T_1 + \cdots + \lambda_h T_h$$

with $Y \subset X$ a tensor subvariety and $T_i \in Y$, it could exist tensors $T_1', \ldots, T_s'$ in X such that

$$T = \beta_1 T_1' + \cdots + \beta_s T_s'$$

and $s < h$.

Comon's conjecture [CGLM08] states that $h = s$ in the case where Y is a Veronese variety and X is a Segre variety.

More precisely, Comon's conjecture predicts that the rank of a homogeneous polynomial $F \in k[x_0, \ldots, x_n]_d$ with respect to the Veronese variety V_d^n is equal to its rank with respect to the Segre variety $SV_1^{\boldsymbol{n}} \cong (\mathbb{P}^n)^d$ into which V_d^n is diagonally embedded, that is

$$\mathrm{rank}_{V_d^n}(F) = \mathrm{rank}_{SV_1^n}(F)$$

Now, let Y, Z be tensor subvarieties of an irreducible projective tensor variety $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$, spanning two linear subspaces $\mathbb{P}^{N_Y} := \langle Y \rangle, \mathbb{P}^{N_Z} := \langle Z \rangle \subseteq \mathbb{P}^N$. Fix two tensors $T_Y \in \mathbb{P}^{N_Y}, T_Z \in \mathbb{P}^{N_Z}$, and consider a tensor $T \in \langle T_Y, T_Z \rangle$. There exist $h = \mathrm{rank}_Y(T_Y)$ tensors $T_1, \ldots, T_h$ and $s = \mathrm{rank}_Z(T_Z)$ tensors $T_1', \ldots, T_s'$ such that

$$T_Y = \lambda_1 T_1 + \cdots + \lambda_h T_h$$

and

$$T_Z = \alpha_1 T_1' + \cdots + \alpha_s T_s'$$

The line $\langle T_Y, T_Z \rangle$ is indeed contained in the linear span $\langle T_1, \ldots, T_h, T_1', \ldots, T_s' \rangle$ and so

$$\mathrm{rank}_X(T) \leqslant \mathrm{rank}_Y(T_Y) + \mathrm{rank}_Z(T_Z) \tag{0.0.1}$$

Strassen's conjecture predicts that the previous inequality is in fact an equality. Strassen's conjecture was originally stated for triple tensors and then generalized to a number of different context. For instance, for homogeneous polynomials it says that if $F \in k[x_0, \ldots, x_n]_d, G \in k[y_0, \ldots, y_m]_d$ are homogeneous polynomials in distinct sets of variables then

$$\mathrm{rank}_{V_d^{n+m}}(F + G) = \mathrm{rank}_{V_d^n}(F) + \mathrm{rank}_{V_d^m}(G)$$

In 2017 Shitov has exhibited examples of tensors T for which both Comon's and Strassen's conjecture do not hold, see [Shi18] and [Shi17]. However in the given counterexamples the rank of T is fairly bigger than its border rank. Comon's and Strassen's conjecture for general tensors, i.e. tensors T for which $\mathrm{rank}(T) = \underline{\mathrm{rank}}(T)$, are still open. Here with $\underline{\mathrm{rank}}(T)$ we mean the border rank of T, i.e. the minimum h such that T can be written as a limit of tensors of rank h. See Chapter 1 for the discussion.

This thesis is organized as follows. In Chapter 1 we recall all the basic results we need in the next chapters. We start introducing the main tensor varieties, i.e. Segre, Veronese, Grassmannians and Segre-Veronese, with its related notion of flattenings. Next we move to the construction of secant varieties, together with the notions of secant defectiveness, weak defectiveness, tangential weak defectiveness and identifiability. Finally we recall the strategy introduced in [MR17] which tackle the problem of secant defectiveness using osculating spaces and projections. This technique was successfully applied to study the problem of secant defectiveness for Grassmannians [MR17] and Segre-Veronese varieties [AMR19].

In Chapter 2, while surveying the state of the art on Comon's and Strassen's conjectures, we push a bit forward standard techniques, based on catalecticant matrices and more generally on flattenings, to extend some results on these conjectures, known in the setting of Veronese and Segre varieties, for Segre-Veronese and Segre-Grassmannian varieties that is to the context of mixed tensors.

Next we introduce a method to improve a classical result on Comon's conjecture. By standard arguments involving catalecticant matrices it is not hard to prove that Comon's conjecture holds for the general polynomial in $k[x_0,\dots,x_n]_d$ of symmetric rank h as soon as $h < \binom{n+\lfloor\frac{d}{2}\rfloor}{n}$, see Proposition 2.2.2. We manage to improve this bound looking for equations for the $(h-1)$-secant variety $\mathbb{S}ec_{h-1}(V_d^n)$, not coming from catalecticant matrices, that are restrictions to the space of symmetric tensors of equations of the $(h-1)$-secant variety $\mathbb{S}ec_{h-1}(SV_1^n)$. We will do so by embedding the space of degree d polynomials into the space of degree $d+1$ polynomials by mapping F to x_0F and then considering suitable catalecticant matrices of x_0F rather than those of F itself.

Implementing this method in Macaulay2 we are able to prove for instance that Comon's conjecture holds for the general cubic polynomial in $n+1$ variables of rank $h=n+1$ as long as $n\leqslant 30$. Note that for cubics the usual flattenings work for $h\leqslant n$.

The main result of this Chapter is the following:

Theorem. Assume $n\geqslant 2$ and set $h=\binom{n+\lfloor\frac{d}{2}\rfloor}{n}$. Then Comon's conjecture holds for the general degree d homogeneous polynomial in $n+1$ variables of rank h in the following cases:

- $d=3$ and $2\leqslant n\leqslant 30$;
- $d=5$ and $3\leqslant n\leqslant 8$;
- $d=7$ and $n=4$.

In Chapter 3 we develop an entirely new approach to study generic identifiability. Starting from the seminal paper [CC10], where the geometry of contact loci has been carefully studied, and the improvement presented in [BBC$^+$18], we derive identifiability statements for non secant defective varieties. Even if new this is not really surprising since weak defectiveness and tangential weak defectiveness, thanks to the Terracini Lemma 1.2.5, have secant defectiveness as a common ancestor. With this new approach we are able to translate all the literature on defective varieties into identifiability statements, providing in many cases sharp classification of h-identifiability.

The main result of this Chapter is the following theorem:

Theorem. Let $X\subset\mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety of dimension n. Assume that:

a) X is k-twd,

b) X is not $(k-1)$-twd

c) $k>n$ and $N\geqslant(k+1)(n+1)-1$.

Then π^X_{k+1} is of fiber type.

Thanks to this theorem we are able to classify completely the subgeneric identifiability for Segre varieties that are product of $\mathbb{P}^1$'s:

Theorem. The Segre embedding of n copies of $\mathbb{P}^1$, with $n\geqslant 5$ is h-identifiable for any $h\leqslant\lfloor\frac{2^n}{n+1}\rfloor-1$.

Recall that the generic rank of the Segre embedding of $(\mathbb{P}^1)^n$ is $\lceil \frac{2^n}{n+1} \rceil$, therefore our result shows generic identifiability of all sub-generic binary tensors, qbits in the quantum computing dictionary, in the perfect case, that is when $\frac{2^n}{n+1}$ is an integer, and all but the last one in general, as predicted by the conjecture posed in [BC13].

Finally we give many more applications of our results to several classes of tensor varieties.

In Chapter 4 we move to the study of generic identifiability. In [Mel06] generic identifiability of symmetric tensors has shown its close connection to modern birational projective geometry and especially to the maximal singularities methods. In a series of papers, [Mel06] [Mel09], [GM19], the generic identifiability problem for symmetric tensors has been completely solved.

In this chapter we extend this theory to arbitrary tensors. As for the symmetric case it is expected that identifiability is very rare and our result support this convincement.

The main tool in [Mel06] was the use, after [CC02], of non weakly defective varieties to study identifiability. Unfortunately it is very hard to determine the weak defectiveness of general tensors. Thanks to the main result in [CM19] (see Chapter 3) for the generic identifiability we may assume without loss of generality the non tangential weakly defectiveness under mild numerical assumptions.

Tangential weak defectiveness does not behave as weak defectiveness with respect to the maximal singularities method. Therefore we have to develop tools to plug in maximal singularities methods for non tangentially weakly defective varieties.

Let us denote with $\mathcal{H}(h)$ the linear system $|\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\dots,x_h^2}|$, where $x_1, \dots, x_h$ are general points of X.

The main technical result of this Chapter is the following:

Theorem. Let $X \subseteq \mathbb{P}^N$ be a projective irreducible,reduced and non-degenerate variety of general rank g. Let $\{x_1, ..., x_{g-1}\}$ be general points on X and $\mathcal{H} = \mathcal{H}(g-1)$. Assume that:

- X is perfect and non defective
- X is not $(g-1)$-twd

Then there is a variety Y and a birational map $\nu : Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor D, with $D \equiv \nu_*^{-1}\mathcal{H}$ such that for any point $y \in Y$

$$\operatorname{mult}_y D < 1 + \epsilon.$$

Thanks to the implementation of the so called Noether-Fano inequalities [Cor95] we were able to prove the non generic identifiability for many partially symmetric tensors:

Theorem. Fix two multiindexes $\boldsymbol{n} = (n_1, \dots, n_r)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$. Let $X = SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ the corresponding Segre-Veronese variety. Assume that $d_i > n_i + 1$, for $i = 1, \dots, r$, and

$$\lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i + 1} \rceil > 2(\sum n_i).$$

Then X is not generically identifiable.

In Chapter 5 we investigate secant defectiveness of flag varieties applying the machinery introduced in [MR17]. We would like to stress that its application to flag varieties involves much more difficult computations compared with the case of the Grassmannians, this is particularly reflected in Section 5.5 where we introduce submersions of flag varieties into product of Grassmannians in order to study the relation among their higher osculating spaces.

Furthermore, our results on secant defectiveness, combined with a recent result in [CM19] (content of Chapter 3), allow us to produce a bound for identifiability of flag varieties. Our main result can be summarized as follows:

Theorem. Consider a flag variety $\mathbb{F}(k_1,\dots,k_r;n)$. Assume that $n \geqslant 2k_j+1$ for some index j and let l be the maximum among these j. Then, for

$$h \leqslant \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1)\rfloor},$$

$\mathbb{F}(k_1,\dots,k_r;n)$ is not $(h+1)$-defective. Furthermore, if for such h we have

$$h > 2\dim(\mathbb{F}(k_1,\dots,k_r;n)$$

then the general point of the h-secant variety of $\mathbb{F}(k_1,\dots,k_r;n)$ is h-identifiable.

Finally in Chapter 6 we approach the problem of weak defectiveness using degenerations of tangent linear spaces to osculating linear spaces, and we apply this method to Segre-Veronese varieties. Furthermore, we introduce the concept of (h,s)-tangential weak defectiveness, where h,s are positive integers. A variety $X \subset \mathbb{P}^N$ is (h,s)-tangential weakly defective if a general linear subspace of dimension s, which is tangent to X at h general points $p_1,\dots,p_h \in X$, is tangent to X along a positive dimensional subvariety of X containing at least one of the p_i. In particular, when $s = \dim\langle T_{p_1}X,\dots,T_{p_h}X\rangle$ we recover the notion of h-tangential weak defectiveness while for $s = N-1$ we get the notion of h-weak defectiveness. We also classify the 1-weakly defective Segre-Veronese varieties. Our main result in this direction is the following:

Theorem. If $h \leqslant (n_1+1)^{\lfloor \log_2(d)\rfloor}$ then the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \mathbb{P}^N$ is not h-weakly defective. Furthermore, $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is 1-weakly defective if and only if $d_r = 1$ and $n_r > \sum_{i=1}^{r-1} n_i$.

Moreover, if $\boldsymbol{n} = (1,n)$ and $\boldsymbol{d} = (c,1)$ then $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not $(1,s)$-tangential weakly defective if and only if $s \leqslant c(n+1)$.

Lastly, if $n_r > \sum_{i=1}^{r-1} n_i$ and $s \leqslant \prod_{i=2}^{r}\binom{n_i+d_i}{n_i} - n_r\sum_{i=1}^{r-1} n_i$, then $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not $(1,s)$-tangential weakly defective.

Chapter 1

Preliminaries

In this chapter we introduce all the tools we need in the rest of the thesis. We start with the construction of tensor spaces together with the notion of flattening in section 1.1. Next we review the notion of secant variety in section 1.2, weak defectiveness in section 1.3, tangentially weak defectiveness and identifiability in section 1.4. In the last section 1.5 we introduce the technique of osculating projections, which will be central for Chapter 5 and Chapter 6. Finally we give some examples for our constructions, in order to make clearer for the reader the computations we will see in the next chapters.

Throughout the thesis we will always work over the field of complex numbers $\mathbb{C}$.

1.1 Tensor Spaces

Tensor product of vector spaces is a central construction in both abstract and applied algebraic geometry. Our attention will be focused on some special variety natural embedded in tensor spaces, parametrizing special type of tensors. For a complete treatment we invite the reader to look at [Lan12].

Let $V_1, \dots, V_s$ be complex vector spaces with $\dim(V_i) = a_i + 1$. We denote with

$$V_1 \otimes V_2 \otimes \cdots \otimes V_s$$

the tensor product of $V_1, \dots, V_s$. Let $\{e^i_j\}_{j=1,\dots,a_i+1}$ be a choice of a basis for every vector space V_i. Then every vector $T \in V_1 \otimes \cdots \otimes V_s$ can be expressed uniquely as

$$T = \sum \alpha_{j_1,\dots,j_s} e^1_{j_1} \otimes \cdots \otimes e^s_{j_s}$$

If $s = 2$ let us fix $\{e_i\}_{i=1,\dots,s_1}$ and $\{f_j\}_{j=1,\dots,s_2}$ bases of V_1 and V_2 respectively. Then an element $T \in V_1 \otimes V_2$ with

$$T = \sum a_{i,j} e_i \otimes f_j$$

can be viewed in three ways:

1) As a linear map $T : V_1^* \to V_2$ given by

$$T(e_k^*) = a_{k,1} f_1 + \cdots + a_{k,s_2} f_{s_2}$$

2) As a linear map $T : V_2^* \to V_1$ given by

$$T(f_k^*) = a_{1,k}e_1 + \cdots + a_{s_1,k}e_{s_1}$$

3) As a bilinear map $T : V_1^* \otimes V_2^* \to \mathbb{C}$ given by

$$T(e_k^* \otimes f_l^*) = a_{k,l}$$

Under the identification of T with a $s_1 \times s_2$ array of numbers the three maps as above corresponds to the three possible slices of the array. More generally for every susbet $A \subset \{1, \ldots, s\}$ we can view the tensor

$$T = \sum \alpha_{j_1,\ldots,j_s} e^1_{j_1} \otimes \cdots \otimes e^s_{j_s}$$

as a linear map

$$T : V_A^* \to V_{A^c}$$

with $V_A = V_{i_1} \otimes \cdots \otimes V_{i_k}$ where $A = \{i_1, \ldots, i_k\} \subset \{1, \ldots, s\}$.

Definition 1.1.1. The linear map $\widetilde{T} : V_A^* \to V_{A^c}$ associated to the tensor T is called an $(A, A^c)-$flattening of T.

A tensor $T \in V_1 \otimes \cdots \otimes V_s$ is said to be a rank-1 tensor if there exist $v_1 \in V_1, \ldots, v_s \in V_s$ vectors such that $T = v_1 \otimes \cdots \otimes v_s$.

Definition 1.1.2. The rank of the tensor T is the minimal $r = \mathrm{rank}(T) = R(T)$ such that there exists an expression

$$T = \sum_{j=1}^{r} a_j v_1^j \otimes \cdots \otimes v_s^j$$

with $\{v_i^j\}_{j=1,\ldots,r} \subset V_i$ and $a_j \in \mathbb{C}$.

Definition 1.1.3. The border rank of the tensor T is the minimal $r = \underline{\mathrm{rank}}(T) = \underline{R}(T)$ such that T can be written as a limit of tensors of rank r but not as a limit of tensors of rank s with $s < r$.

1.1.3 Segre Varieties

Let $\mathbb{P}^{a_i} = \mathbb{P}(V_i)$ be the projective space of lines in V_i and consider the variety

$$X = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_s}$$

If we denote with

$$\pi_i : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_s} \to \mathbb{P}^{a_i}$$

the natural projection on the $i-$th factor we have that the linear system

$$|\mathcal{O}_{\mathbb{P}^{a_1}\times\cdots\times\mathbb{P}^{a_s}}(1)| := |\pi_1^*(\mathcal{O}_{\mathbb{P}^{a_1}}(1)) \otimes \cdots \otimes \pi_s^*(\mathcal{O}_{\mathbb{P}^{a_s}}(1))|$$

is very ample and $\mathcal{O}_{\mathbb{P}^{a_1}\times\cdots\times\mathbb{P}^{a_s}}(1)_{|\mathbb{P}^{a_i}} = \mathcal{O}_{\mathbb{P}^{a_i}}(1)$. The associated map

$$\varphi_{|\mathcal{O}_X(1)|} : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_s} \to \mathbb{P}^{\Pi(a_i+1)-1}$$

is called the Segre embedding. We will use the notation $\nu_{(1,\ldots,1)}^{(a_1,\ldots,a_s)} := \varphi_{|\mathcal{O}_X(1)|}$ and we call the image

$$SV_{(1,\ldots,1)}^{(a_1,\ldots,a_s)} := \nu_{(1,\ldots,1)}^{(a_1,\ldots,a_s)}(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_s}) \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$$

a Segre variety. It is straightforward to see that

$$\varphi_{|\mathcal{O}_X(1)|}([v_1], \ldots, [v_s]) = [v_1 \otimes \cdots \otimes v_s]$$

and so the Segre variety parametrizes rank-1 tensors up to scalar equivalence.

The rank and the border rank can be different for a particular tensor. Let us show an example of a tensor T for which $R(T) > \underline{R}(T)$.

Example 1.1.4. Let $V_1 = V_2 = \mathbb{C}^2$ and $V_3 = \mathbb{C}^3$, with $(a_1, a_2), (b_1, b_2), (c_1, c_2, c_3)$ basis of V_1, V_2 and V_3 respectively. Consider the Segre variety $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^{12}$ parametrizing rank 1 tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. Consider the tensor

$$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$$

It can be proved that $R(T) = 3$ but $\underline{R}(T) = 2$. In fact T can be seen as a limit of tensors of rank 2. Indeed let

$$T(\epsilon) = (\epsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \epsilon a_2) \otimes (b_1 + \epsilon b_2) \otimes (c_1 + \epsilon c_2)$$

and note that

$$T = \lim_{\epsilon \to 0} \frac{T(\epsilon)}{\epsilon}$$

1.1.4 Veronese Varieties

Let us now consider the case where $V = V_1 = \cdots = V_d$, with $\dim(V) = n + 1$, and denote with

$$V^{\otimes d} := \underbrace{V \otimes \cdots \otimes V}_{d-\text{times}}$$

Let $S = S_d$ be the symmetric group on d elements. Consider the map:

$$\pi_{\text{Sym}} : V^{\otimes d} \to V^{\otimes d}$$

given on rank 1 tensors as

$$\pi_{\text{Sym}}(v_1 \otimes \cdots \otimes v_d) = \frac{1}{d!} \sum_{\sigma \in S} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

Definition 1.1.5. A tensor

$$T \in \text{Sym}^d(V) := \{X \in V^{\otimes d} | \pi_{\text{Sym}}(X) = X\}$$

is called a symmetric tensor.

Let

$$\mathbb{P}(V) \subset \mathbb{P}(V) \times \cdots \times \mathbb{P}(V)$$

be the natural diagonal inclusion given by associating to a point $[v] \in \mathbb{P}(V)$ the point

$$([v], \ldots, [v]) \in \mathbb{P}(V) \times \cdots \times \mathbb{P}(V)$$

If we restrict the Segre embedding

$$\nu_{(1,\ldots,1)}^{(n,\ldots,n)} : \underbrace{\mathbb{P}(V) \times \cdots \times \mathbb{P}(V)}_{d-\text{times}} \to \mathbb{P}(V^{\otimes d})$$

to $\mathbb{P}(V)$ we have an embedding

$$\nu_d = \varphi_{|\mathbb{P}(V)} : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^d(V))$$

called the $d-$th Veronese embedding of $\mathbb{P}(V)$. The variety

$$V_d^n := \nu_d(\mathbb{P}(V)) \subset \mathbb{P}(\mathrm{Sym}^d(V))$$

is called a Veronese variety.

Remark 1.1.6. Note that elements $F \in \mathrm{Sym}^d(V^*)$ can be identified with homogeneous polynomials on V. Under this identification the Veronese embedding

$$\nu_d : \mathbb{P}(V^*) \to \mathbb{P}(\mathrm{Sym}^d(V^*))$$

is the map that associates to every linear form $[L]$ its $d-$th power $[L^d]$. In particular $\nu_d = \varphi_{|\mathcal{O}_{\mathbb{P}(V)}(d)|}$ and V_d^n parametrizes homogeneous form $F \in \mathbb{C}[x_0, \ldots, x_n]$ such that $F = L^d$.

Given a symmetric tensor $F \in \mathrm{Sym}^d(V)$ for any $k \leqslant d$ we can restrict the $(k, d-k)-$flattening

$$\widetilde{F} : (V^*)^{\otimes k} \to V^{\otimes d-k}$$

to the subspace $\mathrm{Sym}^k(V)^* \subset (V^*)^{\otimes k}$ to obtain a symmetric flattening

$$\widetilde{F} : \mathrm{Sym}^k(V)^* \to \mathrm{Sym}^{d-k}(V)$$

Definition 1.1.7. The matrix $A_F^{(k,d-k)} = Cat_F^{(k,d-k)}$ representing the $(k, d-k)$ symmetric flattening $\widetilde{F}$ is called the $(k, d-k)-$catalecticant matrix of F.

It is straightforward to see that the colums of $A_F^{(k,d-k)}$ are the coefficients of the partial derivatives of order k of F.

Definition 1.1.8. The symmetric rank of a symmetric tensor $F \in \mathrm{Sym}^d(V)$ is the minimum $r = \mathrm{srk}(F) = sR(F)$ such that there exists an expression

$$F = \sum_{j=1}^{r} L_j^d$$

with L_j^d symmetric rank 1 tensors. The definition of symmetric border rank $\underline{sR}(F) = \underline{\mathrm{srk}}(F)$ is analogous to the previous case for general tensors.

1.1.8 Grassmannian Varieties

Consider as before the case where $V = V_1 = \cdots = V_k$, with $\dim(V) = n+1$.

Consider the map:

$$\pi_{\bigwedge} : V^{\otimes k} \to V^{\otimes k}$$

given on rank 1 tensors as

$$\pi_{\bigwedge}(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

Definition 1.1.9. A tensor

$$T \in \bigwedge^k(V) := \{X \in V^{\otimes k} | \pi_{\bigwedge}(X) = X\}$$

is called an alternate (or antisymmetric, skew symmetric) tensor.

Let $\mathbb{G}(k,n)$ be the Grassmannian parametrizing $k-$linear spaces of $\mathbb{P}^n$. Any point $[\Pi] \in \mathbb{G}(k,n)$ represents a unique linear space $\mathbb{P}^k \subset \mathbb{P}^n$.

Given a point $[\Pi] \in \mathbb{G}(k,n)$ we can choose a basis $\{v_1 \ldots, v_{k+1}\}$ of Π and construct the $(n+1) \times (k+1)$ matrix A_Π where

$$A_\Pi = \begin{bmatrix} v_1 \\ \cdots \\ \cdots \\ v_{k+1} \end{bmatrix}$$

Since $\dim(\Pi) = k$ we have that not all the determinants of the $(k+1)-$minors of A_Π vanish simultaneously. Finally note that the determinants of the $(k+1)-$minors are exactly the coordinates of the vector

$$v_1 \wedge \cdots \wedge v_{k+1} \in \bigwedge^{k+1}(V)$$

A different choice of a basis $\{w_1, \ldots, w_{k+1}\}$ for Π yields

$$w_1 \wedge \cdots \wedge w_{k+1} = det(M)(v_1 \wedge \cdots \wedge v_{k+1})$$

with $M \in GL(k+1)$ the matrix of change of basis. Thus we have a well defined map

$$p_k^n : \mathbb{G}(k,n) \to \mathbb{P}(\bigwedge^{k+1}(V))$$

$$p_k^n(\{v_1, \ldots, v_{k+1}\}) = [v_1 \wedge \cdots \wedge v_{k+1}]$$

It can be proved that p_k^n is in fact an embedding, called the Plücker embedding of $\mathbb{G}(k,n)$. The variety $\mathbb{G}(k,n) \subset \mathbb{P}(\bigwedge^{k+1} V)$ naturally parametrizes skew symmetric tensors $T \in \bigwedge^{k+1} V$ such that

$$T = v_1 \wedge \cdots \wedge v_{k+1}$$

with $v_i \in V$.

Definition 1.1.10. The skew symmetric rank $r = \operatorname{skrk}(T)$ of a skew symmetric tensor T is the minimum integer r such that there exists an expression

$$T = \sum_{j=1}^{r} v_1^j \wedge \cdots \wedge v_{k+1}^j$$

with $v_i^j \in V$. The definition of the skew symmetric border rank $\underline{\operatorname{skrk}}(T)$ is analogous to the case of general tensors.

As for the case of symmetric tensor given $T \in \bigwedge^{k+1}(V)$ we can consider the $(i, k+1-i)-$skew flattening given by

$$\widetilde{T} : \bigwedge^{i}(V^*) \to \bigwedge^{k+1-i}(V)$$

1.1.10 Mixed Tensors

We can mix the construction of Segre embedding with the Veronese embedding and the Plücker embedding in order to have a more general notion of Segre-Veronese embedding and Segre-Plücker embedding.

Given a multiindex $\boldsymbol{n} = (n_1, \ldots, n_s)$ consider the vector spaces $V_1, \ldots, V_s$ such that $\dim(V_i) = n_i + 1$. Consider moreover the degree multiindex $\boldsymbol{d} = (d_1, \ldots, d_s)$.

The Segre-Veronese embedding of $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_s)$ with degree $(d_1, \ldots, d_s)$ is the map:

$$\nu_{\boldsymbol{d}}^{\boldsymbol{n}} = \nu_{(d_1,\ldots,d_s)}^{(n_1,\ldots,n_s)} : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_s) \to \mathbb{P}(\operatorname{Sym}^{d_1}(V_1) \otimes \cdots \otimes \operatorname{Sym}^{d_s}(V_s))$$

$$\nu_{\boldsymbol{d}}^{\boldsymbol{n}}(([v_1], \ldots, [v_s])) = [v_1^{d_1}] \otimes \cdots \otimes [v_s^{d_s}]$$

Note that the Segre-Veronese embedding is the map associated with the linear system

$$|\mathcal{O}_{\mathbb{P}(V_1)\times\cdots\times\mathbb{P}(V_s)}(d_1, \ldots, d_s)| = |\pi_1^*(\mathcal{O}_{\mathbb{P}^{a_1}}(d_1)) \otimes \cdots \otimes \pi_s^*(\mathcal{O}_{\mathbb{P}^{a_s}}(d_s))|$$

We can generalize the notion of flattening for the case of Segre-Veronese varieties. Given a tensor $T \in \operatorname{Sym}^{d_1}(V_1) \otimes \cdots \otimes \operatorname{Sym}^{d_s}(V_s)$ we can consider the $(A, B)-$mixed symmetric flattening

$$\widetilde{T} : \operatorname{Sym}^{a_1}(V_1^*) \otimes \cdots \otimes \operatorname{Sym}^{a_s}(V_s^*) \to \operatorname{Sym}^{b_1}(V_1) \otimes \cdots \otimes \operatorname{Sym}^{b_s}(V_s)$$

with $A = \{a_i, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_s\}$, such that $d_i = a_i + b_i$, induced by the natural inclusion

$$\operatorname{Sym}^{a_i}(V_i) \otimes \operatorname{Sym}^{b_i}(V_i) \subset V^{\otimes d_i}$$

Similarly given vector spaces $V_1, \ldots, V_s$ with $\dim(V_i) = n_i + 1$ and multiindices $\boldsymbol{k} = (k_1, \ldots, k_s)$ $\boldsymbol{n} = (n_1, \ldots, n_s)$ such that $k_i + 1 \leqslant n_i + 1$ we can consider the Segre-Plücker embedding:

$$p_{\boldsymbol{k}}^{\boldsymbol{n}} : \mathbb{G}(k_1, n_1) \times \cdots \times \mathbb{G}(k_s, n_s) \to \mathbb{P}(\bigwedge^{k_1+1} V_1 \otimes \cdots \otimes \bigwedge^{k_s+1} V_s)$$

$$p^{\boldsymbol{n}}_{\boldsymbol{k}}(\{v^1_j\}_{j\leqslant k_1+1},\dots,\{v^s_j\}_{j\leqslant k_s+1}) = ([v^1_1\wedge\cdots\wedge v^1_{k_1+1}],\dots,[v^1_s\wedge\cdots\wedge v^s_{k_s+1}])$$

where $\{v^i_1,\dots,v^i_{k_i+1}\}$ is a choice of a basis for the k_i-linear space $[\Pi_i]\in\mathbb{G}(k_i,n_i)$.

Finally for a tensor $T\in\bigwedge^{k_1+1}(V_1)\otimes\cdots\otimes\bigwedge^{k_s+1}(V_s)$ we can consider the $(A,B)-$mixed skew flattening

$$\widetilde{T}:\bigwedge^{a_1}(V_1^*)\otimes\cdots\otimes\bigwedge^{a_s}(V_s^*)\to\bigwedge^{b_1}(V_1)\otimes\cdots\otimes\bigwedge^{b_s}(V_s)$$

with $A=\{a_1,\dots,a_s\}$ and $B=\{b_1,\dots,b_s\}$ such that $a_i+b_i=k_i+1$ induced by the natural inclusion

$$\bigwedge^{a_i}(V_i)\otimes\bigwedge^{b_i}(V_i)\subset V^{\otimes(k_i+1)}$$

1.1.10 Flattenings and Border rank

Flattenings are a useful tool to find equations for tensors of a given border rank. Let us start recalling the following theorem:

Theorem 1.1.11. *Let $T\in V_1\otimes\cdots\otimes V_s$. Then $R(T)$ equals the number of rank one matrices needed to span a space containing $\tilde{T}(V_{A^*})\subset V_{A^c}$ for every $(A,A^c)-$flattening of T.*

Proof. See [Lan12, Theorem 3.1.1.1]. □

The same result holds if we take

- $T\in\mathrm{Sym}^d(V)$ with its corresponding $(k,d-k)-$flattening.
- $T\in\bigwedge^{k+1}V$ with its corresponding $(i,k+1-i)-$flattening.
- $T\in\mathrm{Sym}^{d_1}(V_1)\otimes\cdots\otimes\mathrm{Sym}^{d_s}(V_s)$ with its corresponding $(A,B)-$mixed symmetric flattening.
- $T\in\bigwedge^{k_1+1}V_1\otimes\cdots\otimes\bigwedge^{k_s+1}V_s$ with its corresponding $(A,B)-$mixed skew flattening.

Theorem 1.1.11 shows that whenever a tensor $T\in V_1\otimes\cdots\otimes V_s$ has rank $R(T)=r$ then the rank of every $(A,A^c)-$flattening $\tilde{T}$ as a linear map is less or equal to r. Since the locus of linear maps between two vector spaces with rank less or equal to a given rank r is closed, we see that whenever the border rank $\underline{R}(T)$ is equal to r then every flattening $\tilde{T}$ has rank less or equal to r.

Let $T\in V_1\otimes\cdots\otimes V_s$ be a tensor, with $V_i=\mathrm{Sym}^{d_i}(V)$ or $V_i=\bigwedge^{k_i+1}V$ at the occurence.

Definition 1.1.12. The variety

$$\hat{\sigma}_{r,V_1\otimes\cdots\otimes V_s}=\overline{\{T\in V_1\otimes\cdots\otimes V_s|\underline{R}(T)=r\}}\subset V_1\otimes\cdots\otimes V_s$$

is called the $r-$border rank variety.

When there is no ambiguity we will write $\hat{\sigma}_r$ for $\hat{\sigma}_{r,V_1\otimes\cdots\otimes V_s}$. As we will see in the next section the variety $\hat{\sigma}_r$ is an algebraic variety. We can thus test the membership $T \in \hat{\sigma}_r$ using polynomials. More precisely, if a polynomial vanishes on all tensors T of rank r then it will also vanish on a tensor that is a limit of tensors of rank r, i.e. a tensor of border rank r.

Let us now consider the sub-space variety

$$Sub_{b_1,\dots,b_s}(V_1 \otimes \cdots \otimes V_s) = \{T \in V_1 \otimes \cdots \otimes V_s | \dim(\tilde{T}(V_i^*)) \leqslant b_i, \forall 1 \leqslant i \leqslant s\}$$

Thanks to theorem 1.1.11 we have that

$$\hat{\sigma}_r \subset Sub_r(V_1 \otimes \cdots \otimes V_s) := Sub_{r,\dots,r}(V_1 \otimes \cdots \otimes V_s)$$

and so polynomials in the ideal of $Sub_r(V_1 \otimes \cdots \otimes V_s)$ furnish tests for membership in $\hat{\sigma}_r$.

Since

$$\tilde{T} : V_i^* \to V_1 \otimes \cdots \otimes \hat{V}_i \otimes \cdots \otimes V_s$$

is a linear map, imposing the condition $\dim(\tilde{T}(V_i^*)) \leqslant r$ is equivalent to impose $\operatorname{rank}(A(T)) \leqslant r$ where $A(T)$ is the matrix representing the flattening $\tilde{T}$ in a fixed basis. The entries

$$A(T)_{i,j} \in \operatorname{Sym}^1(V_1 \otimes \cdots \otimes V_s)^*$$

are linear polynomials in the coordinates of T and so the $(r+1) \times (r+1)-$minors of $A(T)$ give for every $1 \leqslant i \leqslant s$ a vector space of polynomials

$$\bigwedge^{r+1} V_i^* \otimes \bigwedge^{r+1} (V_1 \otimes \dots V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_s)^* := \bigwedge^{r+1} V_i^* \otimes \bigwedge^{r+1} V_{\hat{i}}^*$$

vanishing in $\hat{\sigma}_r$ for every $1 \leqslant i \leqslant s$. In particular these are equations for $\hat{\sigma}_r$ with homogeneous degree $(r+1)$ in $\operatorname{Sym}^d(V_1 \otimes \cdots \otimes V_s)^*$.

This construction can easily be generalized to arbitrary $(A, A^c)-$flattening:

the module

$$\bigwedge^{r+1} V_A^* \otimes \bigwedge^{r+1} V_{A^c}^* \subset \operatorname{Sym}^{r+1}(V_1 \otimes \cdots \otimes V_s)^*$$

furnish equations for $\hat{\sigma}_r$. These equations are called equations of flattenings.

Example 1.1.13. Let $V_1 = \mathbb{C}^3$ and $V_2 = \mathbb{C}^4$, with $\hat{\sigma}_{2,\mathbb{C}^3,\mathbb{C}^4}$ the variety parametrizing tensors $T \in \mathbb{C}^3 \otimes \mathbb{C}^4$ with border rank 2. There is only one possible non trivial partition $(\{1\},\{2\})$ of $\{1,2\}$. Consider the $(\{1\},\{2\})-$flattening

$$\tilde{T} : (\mathbb{C}^3)^* \to \mathbb{C}^4$$

If

$$T = \sum_{1\leqslant i\leqslant 3, 1\leqslant j\leqslant 4} a_{i,j} e_i \otimes f_j$$

then the matrix representing $\tilde{T}$ is

$$A(T) = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$$

Considering the determinant of the 3×3 minors of $A(T)$ gives 4 equations F_1, F_2, F_3, F_4 of degree 3 for $\hat{\sigma}_{2,\mathbb{C}^3\mathbb{C}^4}$.

Remark. *Note that equations of flattenings, as we will see in the next section, in general do not generate the ideal of* $\hat{\sigma}_r$*. Moreover if* $r > max\{\dim(V_A) | A \subset \{1, \ldots, s\}\}$ *then there are no equations for* $\hat{\sigma}_r$ *coming from flattenings.*

The same construction work for symmetric tensors, skew tensors and mixed tensors.

For the reader convenience we briefly explain the construction in the symmetric case.

For every symmetric tensor $F \in \mathrm{Sym}^d(V)$ and for every unordered partition $(k, d-k)$ of d consider the symmetric flattening

$$\tilde{F} : \mathrm{Sym}^k(V)^* \to \mathrm{Sym}^{d-k}(V)$$

Viewing $\tilde{F}$ as a tensor in $\mathrm{Sym}^k(V) \otimes \mathrm{Sym}^{d-k}(V)$ we have that if

$$F = x_1^d + \cdots + x_r^d$$

with $x_i \in V$ then

$$\tilde{F} = x_1^k \otimes x_1^{d-k} \otimes \cdots \otimes x_r^k \otimes x_r^{d-k}$$

Now by Theorem 1.1.11 we have that the module

$$\bigwedge^{r+1}(\mathrm{Sym}^k(V))^* \otimes \bigwedge^{r+1}(\mathrm{Sym}^{d-k}(V)^*) \subset \mathrm{Sym}^{r+1}(\mathrm{Sym}^d(V))^*$$

gives equations for $\hat{\sigma}_{r,\mathrm{Sym}^d(V)}$.

Such equations are called symmetric flattenings and corresponds to minors of catalecticant matrices.

1.2 Secant varieties

The notion of Secant variety is quite classical in algebraic geometry. Here we review the definition and the main properties that are useful for the results stated in the thesis. For a more complete treatment see for instance [Rus16, Chapter 1].

Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non degenerate projective variety of dimension $\dim(X) = n$. Let $k \geqslant 1$ be a positive integer and let

$$(sec_k(X))^0 := \{((x_1, \ldots, x_k), z) | z \in \langle x_1, \ldots, x_k \rangle, \dim(\langle x_1, \ldots, x_k \rangle) = k-1\} \subset (X \times \cdots \times X) \times \mathbb{P}^N$$

Definition 1.2.1. The abstract $k-$secant variety of X is the Zariski closure

$$sec_k(X) = \overline{(sec_k(X))^0} \subset (X \times \cdots \times X) \times \mathbb{P}^N$$

The closed set $sec_k(X)$ has a natural structure of $\mathbb{P}^{k-1}-$bundle over the open dense subset $(X \times \cdots \times X) \backslash \Delta_X$ of X^k. Thus $sec_k(X)$ is irreducible of dimension $\dim(sec_k(X)) = k(\dim(X) + 1) - 1$. Consider the two canonical projections of $sec_k(X)$ over the factors X^k and $\mathbb{P}^N$,

$$\begin{array}{ccc} sec_k(X) & \xrightarrow{\pi_2} & \mathbb{P}^N \\ \downarrow{\scriptstyle \pi_1} & & \\ X^k & & \end{array}$$

Definition 1.2.2. The $k-$secant variety $\mathbb{S}ec_k(X)$ is the scheme-theoretic image of $sec_k(X)$ in $\mathbb{P}^N$, i.e. $\mathbb{S}ec_k(X) = \pi_2(sec_k(X))$.

It is straightforward from the definition to see that

$$\mathbb{S}ec_k(X) = \overline{\bigcup_{x_i \in X, \dim(\langle x_1,\dots,x_k\rangle)=k-1} \langle x_1,\dots,x_k\rangle} \subseteq \mathbb{P}^N$$

is the Zariski closure of the union of all $k-$secant linear space to X in $\mathbb{P}^N$. It is an irreducible projective variety of dimension $\dim(\mathbb{S}ec_k(X)) \leqslant \min\{N, k(\dim(X)+1)-1\}$. The value $\min\{N, k(\dim(X)+1)-1\}$ is called the expected dimension and it is often denoted with $\mathrm{expdim}(\mathbb{S}ec_k(X))$.

Definition 1.2.3. X is said to be $k-$secant defective if $\dim(\mathbb{S}ec_k(X)) < \mathrm{expdim}(\mathbb{S}ec_k(X))$. The $k-$defect of X is $\delta_k(X)$ with $\delta_k(X) = \mathrm{expdim}(\mathbb{S}ec_k(X)) - \dim(\mathbb{S}ec_k(X))$.

If $\mathbb{S}ec_k(X) \subsetneq \mathbb{P}^N$ then X is not $k-$defective if the projection map $\pi_2 = \pi_k^X : sec_k(X) \to \mathbb{P}^N$ is generically finite.

We now review the construction of the secant varieties from another perspective, which nevertheless produces an equivalent definition of $\mathbb{S}ec_k(X)$. This alternative point of view will be useful in Chapter 3.

Let $\mathbb{G}(k-1, N)$ be the Grassmannian parametrizing $(k-1)-$linear spaces in $\mathbb{P}^N$. Let

$$\Gamma_k(X) \subseteq X^k \times \mathbb{G}(k-1, N)$$

be the closure of the rational map

$$\alpha : X^k \dashrightarrow \mathbb{G}(k-1, N)$$

$$\alpha((x_1,\dots,x_k)) = [\langle x_1,\dots,x_k\rangle]$$

Let $\mathcal{S}_k(X)$ be the projection of $\Gamma_k(X)$ into $\mathbb{G}(k-1, N)$ and let

$$\mathcal{I}_k = \{(z, [\Pi]) | z \in \Pi\} \subseteq \mathbb{P}^N \times \mathbb{G}(k-1, N)$$

Observe that since $\mathcal{I}_k$ with the canonical projection $\psi_k : \mathcal{I}_k \to \mathbb{G}(k-1, N)$ has a structure of $\mathbb{P}^{k-1}-$bundle over $\mathbb{G}(k-1, N)$, it is irreducible.

Definition 1.2.4. The $k-$secant variety $\mathbb{S}ec_k(X)$ is the scheme theoretic image $\mathbb{S}ec_k(X) = \pi_k((\psi_k)^{-1}(\mathcal{S}_k(X)))$, where $\pi_k : \mathcal{I}_k \to \mathbb{P}^N$ is the canonical projection.

In general computing explicitely $\dim(\mathbb{S}ec_k(X))$ is a hard task. The idea is that $\mathbb{S}ec_k(X)$ is strictly related to the embedding of X in $\mathbb{P}^N$, so we expect that we can relate the extrinsic geometry of X to the study of $\mathbb{S}ec_k(X)$. The first and the main result in this direction is the celebrated Terracini's Lemma:

Theorem 1.2.5 (Terracini's Lemma)**.** *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth and reduced projective variety. Then:*

- *For every* $x_1, \ldots, x_k \in X$ *and for every* $z \in \langle x_1, \ldots, x_k \rangle$,

$$\langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_k} X \rangle \subseteq \mathbb{T}_z \mathbb{S}ec_k(X)$$

- *There exists an open subset* $U \subset X^k$ *such that*

$$\langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_k} X \rangle = \mathbb{T}_z \mathbb{S}ec_k(X)$$

for every $(x_1, \ldots, x_k) \in U$ *and general* $z \in \langle x_1, \ldots, x_k \rangle$.

Let us briefly recall the notion of linear projection. Let $L \subset \mathbb{P}^N$ be a linear space with $\dim(L) = l$ and let $M \subset \mathbb{P}^N$ another linear space with $\dim(M) = N - l - 1$ such that $L \cap M = \varnothing$ and $\langle L, M \rangle = \mathbb{P}^N$. The linear projection

$$\pi_L : X \dashrightarrow \mathbb{P}^{N-l-1} = M$$

is the rational map defined in the open set $X \backslash (L \cap X)$ by $\pi_L(x) =< L, x > \cap M$.

The first application of Terracini's Lemma that we use is the so called Trisecant Lemma (see [Rus16, Proposition 1.3.3] for the proof).

Proposition 1.2.6 (Trisecant Lemma)**.** *Let* $X \subset \mathbb{P}^N$ *be a non degenerate, irreducible projective variety with* $\operatorname{codim}(X) > k$*. Then a general* $(k+1)-$*secant* $L = \mathbb{P}^k = \langle x_0, \ldots, x_k \rangle$ *is not* $(k+2)-$*secant, i.e.* $L \cap X = \{x_0, \ldots, x_k\}$ *scheme theoretically.*

Taking $L = \langle \mathbb{T}_{x_1}, \ldots, \mathbb{T}_{x_k} \rangle$ we can consider the projection $\tau_k : X \dashrightarrow \mathbb{P}^{N_k}$ where

$$N_k = N - \dim(\langle \mathbb{T}_{x_1}, \ldots, \mathbb{T}_{x_k} \rangle) - 1$$

By Theorem 1.2.5 for a general choice of $x_1, \ldots, x_k$ we have $N_k = \dim(\mathbb{S}ec_k(X))$. The rational map τ_k is called a general $k-$tangential projection.

It turns out that τ_k encodes many informations about the dimension of $\mathbb{S}ec_k(X)$. We have the following:

Proposition 1.2.7. *[CC02, Proposition 3.5] Let* $X \subset \mathbb{P}^N$ *be an irreducible, non degenerate projective variety. Assume that* $s_k = \dim(\langle \mathbb{T}_{x_1}, \ldots, \mathbb{T}_{x_k} \rangle) < N - n - 1$ *where* $n = \dim(X)$*. If* $X_k = \tau_k(X) \subset \mathbb{P}^{N-s_k-1}$ *then* $\dim(X_k) = n - \delta_{k+1}(X)$.

We give an example of an application of the previous proposition for the case of defectiveness of quadratic Veronese varieties:

Example 1.2.8. Let $\nu_2^n : \mathbb{P}^n \longrightarrow \mathbb{P}^{N_n}$ be the 2-Veronese embedding of $\mathbb{P}^n$, with $N_n = \frac{1}{2}(n+2)(n+1)-1$, and $V_2^n = \nu_2^n(\mathbb{P}^n)$ the corresponding Veronese variety. Fix $p_1, \ldots, p_h \in V_2^n$ general points with $h \leqslant n-1$. Then, the linear system of hyperplanes in $\mathbb{P}^{N_n}$ containing $\langle T_{p_1} V_2^n, \ldots, T_{p_h} V_2^n \rangle$ corresponds to the linear system of quadrics in $\mathbb{P}^n$ whose vertex contains $\Lambda = \langle \nu_2^{n-1}(p_1), \ldots, \nu_2^{n-1}(p_h) \rangle$. Therefore, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_2^n} & V_2^n \subset \mathbb{P}^{N_n} \\ {\scriptstyle \pi_\Lambda} \downarrow & & \downarrow {\scriptstyle \tau_{X,h}} \\ \mathbb{P}^{n-h} & \xrightarrow{\nu_2^{n-h}} & V_2^{n-h} \subset \mathbb{P}^{N_{n-h}} \end{array}$$

where, π_Λ is the projection of $\mathbb{P}^n$ from Λ. Since $\dim(\pi_\Lambda^{-1}(p)) = h-1$ for a general point $p \in \mathbb{P}^{n-h}$ we conclude that $\tau_{X,h}$ has positive relative dimension, then by Proposition 1.2.7 V_2^n is h-defective for $h \leqslant n$.

This result enable us to control the defectiveness of X looking carefully at its general tangential projection. Indeed we have the following:

Corollary 1.2.9. *The general tangential projection τ_k is generically finite if and only if X is not $(k+1)-$defective.*

1.2.9 Secant varieties for tensor spaces

Let $X = SV^{(a_1,\dots,a_s)}_{(1,\dots,1)} = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_s} \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ be the Segre variety parametrizing rank one tensors, i.e. tensors of the form

$$v_1 \otimes \cdots \otimes v_s \in V_1 \otimes \cdots \otimes V_s$$

By definition a tensor T with $R(T) = r$ is such that $T \in \langle x_1, \dots, x_r \rangle$ with $x_1, \dots, x_r \in X$.

It is now straightforward from 1.1.10 that the following characterization holds

$$\mathbb{S}ec_r(X) = \hat{\sigma}_{r, V_1 \otimes \cdots \otimes V_s}$$

Similarly if we consider $X = V_d^n \subset \mathbb{P}(\operatorname{Sym}^d V)$ the $d-$th Veronese embedding of V we have that

$$\mathbb{S}ec_r(X) = \hat{\sigma}_{r, \operatorname{Sym}^d V}$$

Question. *What is the expected dimension for $\mathbb{S}ec_r(X)$ where X is either the Segre, Veronese, Grassmannian or a Segre-Veronese variety?*

A tensor $T \in X$ with $R(T) = r$ is expected to depends on

$$r \dim(X) + r - 1$$

parameters, i.e.:

- If $X = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_s}$ then $\operatorname{expdim}(\mathbb{S}ec_r(X)) = r(\sum_{1 \leqslant i \leqslant s} a_i) + r - 1$.
- If $X = V_d^n$ with $\dim(V) = n+1$ then $\operatorname{expdim}(\mathbb{S}ec_r(X)) = r(n+1) - 1$.

- If $X = \mathbb{G}(k, n)$ then $\operatorname{expdim}(\mathbb{S}ec_r(X)) = r((n-k)(k+1)+1) - 1$.

- If $X = SV^{(a_1,\dots,a_s)}_{d_1,\dots,d_s}$ then $\operatorname{expdim}(\mathbb{S}ec_r(X)) = r(\sum_{1\leqslant i\leqslant s} a_i) + r - 1$.

Furthermore for a tensor variety we can give a definition of typical rank (general rank):

Definition 1.2.10. Given a tensor variety $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ we say that X has typical rank r if the set of tensors $T \in \langle X \rangle \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ with $\operatorname{rank}_X(T) = r$ (rank with respect to the variety X) has positive measure.

The main drawback with flattenings is that they give equations for $\mathbb{S}ec_r(X)$ only up to a certain value of r, which in general is less than the typical rank.

Example 1.2.11. Let $X = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^{54}$ be the Segre variety with 4 factors. We have that $\operatorname{expdim}(\mathbb{S}ec_6(X)) = 47 < 54$ and so we expect many equations vanishing on $\mathbb{S}ec_4(X)$. However for any choice of a $(A, A^c)-$flattening of $T \in \mathbb{P}^{54}$ we have that the induced linear map

$$\tilde{T} : (\mathbb{C}^A)^* \to \mathbb{C}^{A^c}$$

has rank at most 6 and so there are no equations for $\mathbb{S}ec_6(X)$ coming from minors of flattenings.

The problem of finding the dimension of the secant varieties for tensor spaces is far from being solved, even if many results in these directions are proved, see for instance [AB09], [AB12], [AB13], [AOP09], [BBC12], [BCC11], [BDDG07], [LP13], [MR17],[AMR19] and many others..

The only complete classification is given for Veronese varieties V^n_d:

Theorem 1.2.12. *[AH95] Let $V^n_d = \nu^n_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ the Veronese variety of degree d and dimension n. Then $\mathbb{S}ec_h(V^n_d)$ has the expected dimension for all triples (d, n, k) except for:*

- $d = 2,\ 2 \leqslant h \leqslant n$
- $d = 4,\ n = 2,\ h = 5$
- $d = 4,\ n = 3,\ h = 9$
- $d = 3,\ n = 4,\ h = 7$
- $d = 4,\ n = 4,\ h = 14$

As we will see in the next chapter we must consider a slightly different realizations of flattenings in order to ensure new equations for the secant varieties.

1.3 Weak defectiveness

In this section we introduce the notion of weakly defective varieties and we study their main properties. For a complete treatment see [CC02].

Given $X \subset \mathbb{P}^N$ a irreducible, reduced projective variety by Theorem 1.2.5 for a general choice of $x_1, \ldots, x_k \in X$ and $z \in \langle x_1, \ldots, x_k \rangle$ we have

$$\dim(\mathbb{S}ec_k(X)) = \dim(\langle \mathbb{T}_{x_1}, \ldots, \mathbb{T}_{x_k} \rangle)$$

Let $H \in |\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{I}_{\langle \mathbb{T}_{x_1},\ldots,\mathbb{T}_{x_k} \rangle}|$ be a hyperplane in $\mathbb{P}^N$ containing $\mathbb{T}_z\mathbb{S}ec_k(X)$. Then the subscheme $H \cap X \subset X$ is singular at $x_1, \ldots, x_k$. Conversely if $x_1, \ldots, x_k$ are general points on X, with X linearly normal, then a hyperplane divisor $H \in |\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\ldots,x_k^2}|$ is the restriction of an hyperplane $\overline{H} \subset \mathbb{P}^N$ such that $H \supset \langle \mathbb{T}_{x_1}, \ldots, \mathbb{T}_{x_k} \rangle$. With this identification in mind given $H \in |\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\ldots,x_k^2}|$ as above we set

$$\Gamma_k(H) := \Gamma_{x_1,\ldots,x_k}(H)$$

the union of the irreducible components of $Sing(H)$ passing through the points $x_1, \ldots, x_k$. The subvariety $\Gamma_k(H)$ is called the contact locus of H. By monodromy if $\Gamma_k(H)_i \subset \Gamma_k(H)$ is an irreducible component containing the point x_i then $\dim(\Gamma_k(H)_i)$ is constant for every $1 \leqslant i \leqslant k$. So it makes sense to define $\nu_k(X) := \dim(\Gamma_k(H))$ for a general H.

Definition 1.3.1. The variety X is called $k-$weakly defective if for general points $x_1, \ldots, x_k$ and general $H \in |\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\ldots,x_k^2}|$ the contact locus is positive dimensional, i.e. $\nu_k(X) > 0$.

Finally we set

$$h(\Gamma_k(H)) = N - \dim(|\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{I}_{\Gamma_k(H)}|)$$

i.e. $h(\Gamma_k(H))$ is the number of conditions that $\Gamma_k(H)$ imposes to a general hyperplane in order to be contained in it. We have the following theorem due to Terracini:

Theorem 1.3.2. *[CC02, Theorem 1.1] Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non degenerate projective variety. If $x_1, \ldots, x_k \in X$ are general and if $H \in |\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{I}_{\langle \mathbb{T}_{x_1},\ldots,\mathbb{T}_{x_k} \rangle}|$ is a general hyperplane section singular in $x_1, \ldots, x_k$ with contact locus $\Gamma(H)$ then*

$$k \leqslant h(\Gamma_k(H)) \leqslant k(1 + \nu_k) - \delta_k$$

In particular if X is $k-$defective then it is also $k-$weakly defective.

In general the converse of Theorem 1.3.2 is false: there are varieties which are $k-$weakly defective but not $k-$defective.

Example 1.3.3. The Segre varieties $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2(n+1)-1}$ embedded with $\mathcal{O}_X(1)$ are $1-$weakly defective but not $1-$defective for $n \geqslant 2$. Let $[x_0 : x_1]$ and $[y_0 : \cdots : y_n]$ be the homogeneous coordinates of $\mathbb{P}^1$ and $\mathbb{P}^n$ respectively. Let $Z_{i,j} = x_i y_j$ be the homogeneous coordinates of $\mathbb{P}^{2(n+1)-1}$ where $0 \leqslant i \leqslant 1$ and $0 \leqslant j \leqslant n$. In the open affine $U \subset \mathbb{P}^1 \times \mathbb{P}^n$ defined by $x_0 \neq 0, y_0 \neq 0$ the Segre embedding is

$$\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^{2(n+1)-1}$$

$$\varphi(x_1, y_1, \dots, y_n) = (y_1, \dots, y_n, x_1, x_1y_1, \dots, x_1y_n)$$

Evaluating the Jacobian matrix of φ at the origin yields that $\mathbb{T}_{[1:0],[1:0\cdots:0]}(\mathbb{P}^1 \times \mathbb{P}^n)$ is the projective completion of the vector subspace of $\mathbb{C}^{2(n+1)-1}$ spanned by $e_1, \dots, e_{n+1}$. Thus a general hyperplane H containing $\mathbb{T}_{[1:0],[1:0\cdots:0]}(\mathbb{P}^1 \times \mathbb{P}^n)$ is of the form

$$\alpha_1 Z_{1,1} + \cdots + \alpha_n Z_{1,n} = 0$$

Now the hyperplane section $H \cap (\mathbb{P}^1 \times \mathbb{P}^n)$ is the locus of points $(x_1, y_1, \dots, y_n)$ such that

$$\alpha_1 x_1 y_1 + \cdots + \alpha_n x_1 y_n = 0$$

Since x_1 is a common factor we see that in U the divisor $H \cap (\mathbb{P}^1 \times \mathbb{P}^n)$ is the union of two hyperplanes passing through the origin. This proves that $\Gamma(H)$ is a codimension 2 linear subspace.

In general $\Gamma_k(H) \subsetneq Sing(H)$, i.e. for a general $H \in |\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\dots,x_k^2}|$ we could have an irreducible component $\Delta(H) \subset Sing(H)$ such that $x_1, \dots, x_k \notin \Delta(H)$.

However if $\dim(\Gamma_k(H)) = 0$ then $\Delta(H) = \varnothing$. In fact we have the following:

Theorem 1.3.4. *[CC02, Theorem 1.4] Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non-degenerate projective variety. Then the following properties hold:*

- *If X is $k-$weakly defective then the general point of every irreducible component of $\Gamma_k(H)$ is a double point for a general H.*
- *If X is not $k-$weakly defective and H is general then $\Gamma_k(H) = \{x_1, \dots, x_k\}$ scheme theoretically. Moreover $\Delta(H) = \varnothing$ and $x_1, \dots, x_k$ are ordinary double points.*

Let now $X \subset \mathbb{P}^N$ be a smooth projective variety together with a family of Cartier divisors $\{H_y\}_{y \in Y}$ parametrized by the reduced, irreducible variety Y. We have the following:

Theorem 1.3.5 (Infinitesimal Bertini's Theorem)**.** *Let $y \in Y$ be a general point and $S = S_y = Sing(H_y)$. Let $v \in T_yY$ be a tangent vector with the corresponding section $s \in H^0(H, \mathcal{N}_{H|X})$ given by the first order deformation σ determined by v. Then $S \subset Z(s)$ with $Z(s)$ the zero subscheme of the section s.*

Proof. See [CC02, Theorem 2.2]. □

Theorem 1.3.5 enable us to describe explicitly the infinitesimal deformations of singular divisors $H_y \subset X$ whenever Y is contained in a suitable linear system.

Corollary 1.3.6. *If $Y \subset |\mathcal{H}|$ is contained in a linear system then $\mathbb{T}_yY \subset |\mathcal{H} \otimes \mathcal{I}_{S_y}|$ for a general $y \in Y$.*

1.4 Tangential weak defectiveness and Identifiability

In this section we introduce the notions of tangentially weakly defective varieties and identifiability. In general verifying when a variety is $k-$weakly defective is quite hard. To overcome this problem in [CO12] the authors introduce the notion of tangentially weakly defective variety. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non-degenerate projective variety. For a subset of general points $A = \{x_1, \ldots, x_k\}$ we set

$$M_A = \langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_k}X \rangle$$

The $k-$tangential contact locus $\Gamma_k(A)$ is the closure in X of the union of all the irreducible components which contain at least one point of A, of the locus of point in X where M_A is tangent to X. Sometimes, where the set A is assumed, we will write simply Γ_k for $\Gamma_k(A)$.

Definition 1.4.1. The variety X is said to be $k-$tangentially weakly defective if for a general $A = \{x_1, \ldots, x_k\}$ we have $\gamma_k(X) := \dim(\Gamma_k(A)) > 0$.

As for the case of weakly defectiveness it makes sense to define $\gamma_k(X)$ because by monodromy the dimension of the irreducible component fo $\Gamma_k(A)$ passing through x_i is independent of the point.

Miming what we did for the case of tensors we give a more general notion of identifiability and we relate it to weakly defectiveness and tangential weak defectiveness. Let us introduce first the notion of rank of a point $p \in \mathbb{P}^N$ with respect to a variety X. First let $X^{(k)}$ be the $k-$symmetric product of X with $U_k^X \subset X^{(k)}$ the locus parametrizing distinct points.

Definition 1.4.2. Let $X \subset \mathbb{P}^N$ be a non-degenerate subvariety. We say that a point $p \in \mathbb{P}^N$ has rank h with respect to X if $p \in \langle z \rangle$ for some $z \in U_h^X$ and $p \notin \langle z' \rangle$ for every $z' \in U_{h'}^X$ with $h' < h$.

Remark 1.4.3. Note that in the case where $X \subset \mathbb{P}^N = \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ is a tensor variety the definition agree with the previous notion of tensor rank.

Definition 1.4.4. A point $p \in \mathbb{P}^N$ is $h-$identifiable with respect to X if $\mathrm{rank}_X(p) = h$ and $(\pi_h^X)^{-1}(p)$ is a single point. The variety is said to be $h-$identifiable if the map

$$\pi_h^X : sec_h(X) \to \mathbb{S}ec_h(X)$$

is birational.

In the next proposition we resume the relations between the four properties we introduced, namely secant defectiveness, weakly defectiveness, tangential weak defectiveness and identifiability.

Proposition 1.4.5. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non-degenerate projective variety.*

1) If X is $h-$defective then it is $h-$weakly defective.

2) If X is $h-$defective then it is $h-$tangentially weakly defective.

3) If X is $h-$tangentially weakly defective then it is $h-$weakly defective.

4) If X is not $h-$tangentially weakly defective then it is $h-$identifiable.

Proof. 1) Is straightforward from Theorem 1.3.4.

2) If X is $h-$defective by Proposition 1.2.7 the general tangential projection

$$\tau_h = \pi_{M_A} : X \dashrightarrow \mathbb{P}^{N_h}$$

is not generically finite. By [CC10, Remark 3.6] the general fiber of τ_h is contained in $\Gamma_h(A)$ yielding $\gamma_h(X) > 0$.

3) If $Y \subset M_A \cap X$ is an irreducible component of $\Gamma_h(A)$ passing through x_i then for every $H \in |\mathcal{O}_X(1) \otimes \mathcal{I}_{x_1^2,\dots,x_k^2}|$ we have $Y \subset \Gamma_h(H)_i$.

4) Let $z \in \langle x_1, \dots, x_k\rangle$ be a general point in $\mathbb{S}ec_k(X)$. By [CO12, Proposition 3.9] we have that $\Gamma_h(A) = x_1, \dots, x_h$ scheme theoretically. If X is not $h-$identifiable then $z \in \langle y_1, \dots, y_k\rangle$ with say $y_1 \neq x_i$ for any i. Then by Terracini's Lemma

$$\mathbb{T}_{y_1}X \subset \langle \mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_k}X\rangle$$

and so $y_1 \in \Gamma_h(A)$, contradiction.

□

Note that all the implications of the previous proposition are sharp. In the previous section we saw an example of a variety that is weakly defective but not defective, namely the Segre variety $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2(n+1)-1}$. Let us analyze other examples in which the converse of Proposition 1.4.5 does not hold.

Example 1.4.6 (**A variety which is $h-$tangentially weakly defective but not $h-$defective**)**.** Let us consider the Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^5 \subset \mathbb{P}^{53}$ and let $A = \{x_1, \dots, x_5\} \subset X$ be general points. Then by [AOP09, Theorem 4.12] $\mathbb{S}ec_5(X)$ has the expected dimension. On the other hand by [BBC⁺18] we have that $\Gamma_5(A) = \Gamma^1 \cup \dots \cup \Gamma^5$ where each $\Gamma^i \cong \mathbb{P}^4$ is a linear space.

Example 1.4.7 (**A variety which is $h-$ weakly defective but not $h-$tangentially weakly defective**)**.** In the example of the Segre variety $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2(n+1)-1}$ we proved that X is $1-$weakly defective. Since X is smooth it can not be $1-$tangentially weakly defective.

Example 1.4.8. [**A variety which is $h-$identifiable but is $h-$tangentially weakly defective**] By [BV18] we have that the Grassmannian $\mathbb{G}(2, 7)$ is $3-$tangentially weakly defective. Let $A = \{x_1, x_2, x_3\} \subset X$ be general points. By explicit computation it is showed that $\Gamma_3(A) = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ where each $\Gamma^i \cong \mathbb{P}^3$ is a linear space passing through x_i. Moreover the linear span $\Pi = \langle \Gamma_3(A) \rangle$ is also tangent to X along a linear space $W \cong \mathbb{P}^5$ such that $x_i \notin W$ and $L_i = W \cap \Gamma^i \cong \mathbb{P}^1$ for every i. One again compute that $\langle L_i, L_j, W \rangle \subsetneq \Pi$. As we will see by [BBC+18, Corollary 4.3] the 3−secant degree of X is equal to the 3−secant degree of $\Gamma_3(A)$ and so we have that $\mathbb{G}(2,7)$ is 3−identifiable.

1.5 Osculating Projections

In this section we introduce a new method to study the defectiveness of a variety, following [MR17] and [AMR19]. From the idea of Proposition 1.2.7 we see that if we can control the dimension of the fiber of a general k−tangential projection then we are able to control the dimension of the k−th Secant variety $\mathbb{S}ec_k(X)$. In general describe explicitly the map given by the tangential projection, when k is big, is quite hard. To overcome this problem we degenerate several tangent spaces $\mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_i}X$ to a single osculating space $\mathbb{T}^a_y X$ where $a \geqslant 2$ and $x_1, \dots, x_i$ degenerate to y. In many cases, as we will see in Chapter 5 and Chapter 6, we can bound explicitely the dimension of the general fiber of a osculating projection. This finally allows us to conclude the non secant defectiveness for many interesting tensor varieties [AMR19] [MR17] [FCM20].

Let $X \subset \mathbb{P}^N$ be an integral projective variety of dimension n, $p \in X$ a smooth point, and

$$\begin{array}{rccc} \phi : & \mathcal{U} \subseteq \mathbb{C}^n & \longrightarrow & \mathbb{C}^N \\ & (t_1, \dots, t_n) & \mapsto & \phi(t_1, \dots, t_n) \end{array}$$

with $\phi(0) = p$, be a local parametrization of X in a neighborhood of $p \in X$.

For any $s \geqslant 0$ let $O^s_p X$ be the affine subspace of $\mathbb{C}^N$ passing through $p \in X$, and whose direction is given by the subspace generated by the vectors $\phi_I(0)$, where $I = (i_1, \dots, i_r)$ is a multi-index such that $|I| \leqslant s$ and

$$\phi_I = \frac{\partial^{|I|}\phi}{\partial t_1^{i_1} \dots \partial t_r^{i_r}}$$

Definition 1.5.1. The *s-osculating space* $\mathbb{T}^s_p X$ of X at p is the projective closure in $\mathbb{P}^N$ of the affine subspace $O^s_p X \subseteq \mathbb{C}^N$.

For instance, $\mathbb{T}^0_p X = \{p\}$, and $\mathbb{T}^1_p X$ is the usual tangent space of X at p. When no confusion arises we will write $\mathbb{T}^s_p$ instead of $\mathbb{T}^s_p X$.

Note that while the dimension of the tangent space at a smooth point is always equal to the dimension of the variety, higher order osculating spaces can be strictly smaller than expected even at a general point.

In general, we have

$$\dim(\mathbb{T}^s_p X) = \min\left\{ \binom{n+s}{n} - 1 - \delta_{s,p}, N \right\}$$

where $\delta_{s,p}$ is the number of linear relations satisfied by partial derivatives of ϕ of order $\leqslant s$ at the point $p \in X$.

For $p_1, \dots, p_l \in X \subset \mathbb{P}^N$ be general points and $k_1, \dots, k_l$ non-negative integers we will call the linear projection

$$\Pi_{\mathbb{T}^{k_1,\dots,k_l}_{p_1,\dots,p_l}} : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_{k_1,\dots,k_l}}$$

with center $\langle \mathbb{T}^{k_1}_{p_1} X, \dots, \mathbb{T}^{k_l}_{p_l} X \rangle$, a general $(k_1, \dots, k_l)$-osculating projection of X.

In order to count the number of tangent spaces we can degenerate into a higher order osculating space we will need the following notion.

Definition 1.5.2. Let $X \subset \mathbb{P}^N$ be a projective variety. We say that X has *m-osculating regularity* if the following property holds: given general points $p_1, \dots, p_m \in X$ and an integer $s \geqslant 0$, there exists a smooth curve C and morphisms $\gamma_j : C \to X$, $j = 2, \dots, m$, such that $\gamma_j(t_0) = p_1$, $\gamma_j(t_\infty) = p_j$, and the flat limit $\mathbb{T}_0$ in the Grassmannian of the family of linear spaces

$$\mathbb{T}_t = \left\langle \mathbb{T}^s_{p_1}, \mathbb{T}^s_{\gamma_2(t)}, \dots, \mathbb{T}^s_{\gamma_m(t)} \right\rangle, \ t \in C \backslash \{t_0\}$$

is contained in $\mathbb{T}^{2s+1}_{p_1}$. We say that $\gamma_2, \dots, \gamma_m$ realize the m-osculating regularity of X for $p_1, \dots, p_m$.

We say that X has *strong* 2*-osculating regularity* if the following property holds: given general points $p, q \in X$ and integers $s_1, s_2 \geqslant 0$, there exists a smooth curve $\gamma : C \to X$ such that $\gamma(t_0) = p$, $\gamma(t_\infty) = q$ and the flat limit $\mathbb{T}_0$ in the Grassmannian of the family of linear spaces

$$\mathbb{T}_t = \left\langle \mathbb{T}^{s_1}_p, \mathbb{T}^{s_2}_{\gamma(t)} \right\rangle, \ t \in C \backslash \{t_0\}$$

is contained in $\mathbb{T}^{s_1+s_2+1}_p$.

The notions of m-osculating regularity and strong 2-osculating regularity were introduced in [MR17, Section 5] and [AMR19, Section 4].

Now if $X \subseteq \mathbb{P}^N$ is a variety having m-osculating regularity, one is able to degenerate the join of m tangent spaces $\langle \mathbb{T}^1_{p_1} \dots \mathbb{T}^1_{p_m} \rangle$ into the single osculating space $\mathbb{T}^3_p X$. Then one further degenerates a general span $\langle \mathbb{T}^3_{p_1} \dots \mathbb{T}^3_{p_m} \rangle$ into $\mathbb{T}^7_p X$ and so on. Thus inductively we degenerate a general h-tangential projection into a linear projection with center contained in a suitable linear span of osculating spaces, and then check whether this projection is generically finite. Building on an argument based on semi continuity we are able to control the secant dimension of X. In fact:

Theorem 1.5.3. *Let $X \subseteq \mathbb{P}^N$ be a projective variety having m-osculating regularity. Let $k_1, \dots, k_l$ positive integers such that the general osculating projection $\Pi_{\mathbb{T}^{k_1,\dots,k_l}_{p_1,\dots,p_l}}$ is generically finite. Then X is not $(h+1)$-defective for*

$$h \leqslant \sum_{j=1}^{l} m^{\lfloor \log_2(k_j+1) \rfloor - 1}$$

Proof. See [MR17, Theorem 5.3]. □

Furthermore, if X in addition has strong 2-osculating regularity, this method can be made more effective. We start introducing an important numerical function, which will be useful in order to control the arithmetic of the degeneration technique.

Definition 1.5.4. Given an integer $m \geqslant 0$ we define a function

$$h_m : \mathbb{N}_{\geqslant 0} \longrightarrow \mathbb{N}_{\geqslant 0}$$

as follows: $h_m(0) = 0$, and for any $k > 0$ write

$$k + 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geqslant 1$ and $\varepsilon \in \{0, 1\}$, then

$$h_m(k) = m^{\lambda_1 - 1} + m^{\lambda_2 - 1} + \cdots + m^{\lambda_l - 1}$$

Theorem 1.5.5. *[MR17, Theorem 5.3] Let $X \subset \mathbb{P}^N$ be a projective variety having m-osculating regularity and strong 2-osculating regularity. Let $s_1, \ldots, s_l \geqslant 1$ integers such that the general osculating projection $\Pi^{s_1,\ldots,s_l}_{p_1,\ldots,p_l}$ is generically finite. Then X is not $(h+1)$-defective for*

$$h \leqslant \sum_{j=1}^{l} h_m(s_j).$$

The power of this new approach was shown in [MR17] and in [AMR19], where the authors obtained new results on secant defectiveness of Grassmannians and Segre-Veronese varieties respectively. The main results are:

Theorem 1.5.6. *[MR17, Theorem 5.4] Let $\mathbb{G}(r, n)$ be the Grassmaniann parametrizing all $(r+1)$-linear subspaces of $\mathbb{C}^{n+1}$. Assume that $r \geqslant 2$ and $n \geqslant 2r + 1$. Set,*

$$\alpha := \left\lfloor \frac{n+1}{r+1} \right\rfloor$$

and write $r = 2^{\lambda_1} + \cdots + 2^{\lambda_s} + \epsilon$ with $\lambda_1 > \lambda_2 > \cdots \geqslant \lambda_s$ and $\epsilon \in \{0, 1\}$. If either

- *$h \leqslant (\alpha - 1)(\alpha^{\lambda_1 - 1} + \cdots + \alpha^{\lambda_s - 1}) + 1$ or*
- *$n \geqslant r^2 + 3r + 1$ and $h \leqslant \alpha^{\lambda_1} + \cdots + \alpha^{\lambda_s} + 1$.*

Then, $\mathbb{G}(r, n)$ is not h-defective. Asymptotically, $\mathbb{G}(r, n)$ is not h-defective for

$$h \leqslant \left(\frac{n+1}{r+1}\right)^{\lfloor \log_2(r) \rfloor}$$

Theorem 1.5.7. *[AMR19, Theorem 4.8] Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers with, $n_1 \leqslant \cdots \leqslant n_r$ and $d = d_1 + \cdots + d_r \geqslant 3$. Let $SV^{\boldsymbol{d}}_{\boldsymbol{n}}$ be the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ embedded by the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}}(d_1, \ldots, d_r)|$. Write*

$$d - 1 = 2^{\lambda_1} + \cdots + 2^{\lambda_s} + \epsilon$$

with $\lambda_1 > \lambda_2 > \cdots \geqslant \lambda_s$ and $\epsilon \in \{0, 1\}$. Then $SV^{\boldsymbol{d}}_{\boldsymbol{n}}$ is not h-defective for

$$h \leqslant n_1((n_1 + 1)^{\lambda_1 - 1} + \cdots (n_1 + 1)^{\lambda_s - 1}) + 1$$

Asymptotically, $SV^{\boldsymbol{d}}_{\boldsymbol{n}}$ is not h-defective for

$$h \leqslant n_1^{\lfloor \log_2(d-1) \rfloor}$$

1.5.7 Osculating spaces of linear sections

Here we review the notion of osculating well-behaved subvariety Y with respect to a variety $X \subset \mathbb{P}^N$, where Y can be thought as a linear section $Y = \mathbb{P}^k \cap X$. We follow the treatment of [FMR20].

In the case of a tensor variety $Y \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_s)$ we have seen that in general $\langle Y \rangle$ is not equal to the whole $\mathbb{P}(V_1 \otimes \cdots \otimes V_s)$. In particular, as for the case of the Veronese V_d^n and the Segre $SV_{(1,\ldots,1)}^{(n,\ldots,n)}$, the variety Y can be equal to the intersection of a bigger variety X with a linear space, i.e. $Y = \Pi \cap X$. In fact, for example, we have

$$V_d^n = SV_{(1,\ldots,1)}^{(n,\ldots,n)} \cap \mathrm{Sym}^d(V)$$

Thus we are naturally interested to obtain a bound for the non-secant defectiveness of Y knowing a bound for the non-secant defectiveness of X.

In particular we want to relate the osculating spaces of a projective variety to those of its linear sections. It is well-known that the tangent space $\mathbb{T}_pY$ of $Y = \Pi \cap X$ at a smooth point $p \in Y$ is equal to $\mathbb{T}_pX \cap \Pi$. This is not always the case for higher order osculating spaces.

Definition 1.5.8. Let $X \subset \mathbb{P}^N$ be an irreducible variety and $Y = \mathbb{P}^k \cap X$ be a linear section of X. We say that Y is *osculating well-behaved* if for each smooth point $p \in Y$ we have

$$\mathbb{T}_p^sY = \mathbb{P}^k \cap \mathbb{T}_p^sX$$

for every $s \geqslant 0$.

Example 1.5.9 (A variety not osculating well-behaved). In the projective space $\mathbb{P}^{k+2}$ consider two complementary subspaces $\mathbb{P}^1, \mathbb{P}^k$, and let $C \subset \mathbb{P}^k$ be a degree k rational normal curve. Fixed an isomorphism $\psi : \mathbb{P}^1 \to C$ we consider the rational normal scroll

$$S_{(1,k)} = \bigcup_{p \in \mathbb{P}^1} \langle p, \psi(p) \rangle \subset \mathbb{P}^{k+2}$$

where $\langle p, \psi(p) \rangle$ is the line through p and $\psi(p)$. Then $S_{(1,k)}$ can be locally parametrized by the map

$$\begin{array}{cccc} \phi : & \mathbb{A}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^{k+2} \\ & (u, [\alpha_0 : \alpha_1]) & \mapsto & [\alpha_0 u : \alpha_0 : \alpha_1 u^k : \alpha_1 u^{k-1} : \cdots : \alpha_1 u : \alpha_1]. \end{array}$$

Now, consider the Segre embedding

$$\begin{array}{cccc} \sigma : & \mathbb{P}^1 \times \mathbb{P}^k & \longrightarrow & \mathbb{P}^{2k+1} \\ & ([u : v], [\alpha_0 : \cdots : \alpha_k]) & \mapsto & [\alpha_0 u : \cdots : \alpha_k u : \alpha_0 v : \cdots : \alpha_k v]. \end{array}$$

and let $\Sigma_{(1,k)}$ be its image. Note that $\Sigma_{(1,k)}$ is locally parametrized by

$$\begin{array}{cccc} \widetilde{\sigma} : & \mathbb{A}^1 \times \mathbb{P}^k & \longrightarrow & \mathbb{P}^{2k+1} \\ & ([u : 1], [\alpha_0 : \cdots : \alpha_k]) & \mapsto & [\alpha_0 u : \cdots : \alpha_k u : \alpha_0 : \cdots : \alpha_k]. \end{array}$$

and that $\deg(\Sigma_{(1,k)}) = \deg(S_{(1,k)}) = k+1$. Now, take $\alpha_i = \alpha_1 u^{i-1}$ for $i = 2, \dots, k$. Then

$$\widetilde{\sigma}(u, [\alpha_0 : \alpha_1 : \dots : \alpha_1 u^{k-1}]) = [\alpha_0 u : \alpha_1 u : \dots : \alpha_1 u^{k-1} : \alpha_1 u^k : \alpha_0 : \alpha_1 : \dots \alpha_1 u^{k-1}]$$

and the coordinate functions of this last map are exactly the ones appearing in the expression of ϕ. Therefore, if $[Z_0 : \dots : Z_{2k+1}]$ are the homogeneous coordinates on $\mathbb{P}^{2k+1}$ and

$$H^{k+2} = \{Z_j - Z_{k+j+2} = 0,\ j = 1, \dots, k-1\} \cong \mathbb{P}^{k+2}$$

then we have

$$S_{(1,k)} = \Sigma_{(1,k)} \cap H^{k+2} \subset \mathbb{P}^{2k+1}$$

By [AMR19, Example 4.12] we have that if $p \in S_{(1,k)}$ is a general point and $k \geqslant 2$ then $\dim(\mathbb{T}^2_p S_{(1,k)}) = 4$. On the other hand, [AMR19, Corollary 2.6] yields $\mathbb{T}^2_p \Sigma_{(1,k)} = \mathbb{P}^{2k+1}$. We conclude that

$$\mathbb{T}^2_p S_{(1,k)} \subsetneqq \mathbb{T}^2_p \Sigma_{(1,k)} \cap H^{k+2} = H^{k+2}$$

for all $k \geqslant 3$.

In order to keep our program working we have to study how osculating regularity behaves under linear sections.

Lemma 1.5.10. *Let $H \subset \mathbb{P}^n$ be a linear subspace, H_t a family of linear subspaces parametrized by $\mathbb{P}^1 \backslash \{0\}$, and H_0 its flat limit. Then*

$$\lim_{t \mapsto 0} \{H_t \cap H\} \subseteq H_0 \cap H$$

Proof. We may assume that $H = V(x_0, \dots, x_r) \subset \mathbb{P}^n$, where $0 \leqslant r \leqslant n-1$. Write

$$H_t = \left\{ \sum_{i=0}^n \alpha_i^1(t) x_i = \dots = \sum_{i=0}^n \alpha_i^k(t) x_i = 0 \right\}$$

Therefore,

$$H_t \cap H = \{x_0 = \dots = x_r = \sum_{i=r+1}^n \alpha_i^1(t) x_i = \dots = \sum_{i=r+1}^n \alpha_i^k(t) x_i = 0\}$$

and

$$\lim_{t \mapsto 0} \{H_t \cap H\} = \left\{ x_0 = \dots = x_r = \sum_{i=r+1}^n \frac{\alpha_i^1}{t^{s_1}}(0) x_i = \dots = \sum_{i=r+1}^n \frac{\alpha_i^k}{t^{s_k}}(0) x_i = 0 \right\}$$

where s_j is the biggest power of t that divides simultaneously $\alpha_{r+1}^j, \dots, \alpha_n^j$. On the other hand

$$\lim_{t \mapsto 0} \{H_t\} \cap H = \left\{ x_0 = \dots = x_r = \sum_{i=r+1}^n \frac{\alpha_i^1}{t^{u_1}}(0) x_i = \dots = \sum_{i=r+1}^n \frac{\alpha_i^k}{t^{u_k}}(0) x_i = 0 \right\}$$

where u_j is the biggest power of t that divides simultaneously $\alpha_0^j, \dots, \alpha_n^j$. Note that $u_j \leqslant s_j$ for $j = 1, \dots, k$, and thus we conclude that $\lim_{t \mapsto 0}\{H_t \cap H\} \subseteq \lim_{t \mapsto 0}\{H_t\} \cap H$. □

As a consequence of Lemma 1.5.10 and Definition 1.5.8 we have the following.

Proposition 1.5.11. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and $Y = \mathbb{P}^k \cap X$ a linear section of X that is osculating well-behaved. Assume that given general points $p_1, \dots, p_m \in Y$ one can find smooth curves $\gamma_j : C \to X, j = 2, \dots, m$, realizing the m-osculating regularity of X for $p_1, \dots, p_m$ such that $\gamma_j(C) \subset Y$. Then Y has m-osculating regularity as well. Furthermore, the analogous statement for strong 2-osculating regularity holds as well.*

Proof. By hypothesis given general points $p_1, \dots, p_m \in Y$ and an integer $s \geqslant 0$ there exist smooth curves $\gamma_j : C \to X$ with $\gamma_j(t_0) = p_1$ and $\gamma_j(\infty) = p_j$ for $j = 2, \dots, m$ such that $\gamma_j(C) \subset Y$. Consider the family of linear spaces

$$\mathbb{T}_t = \langle \mathbb{T}^s_{p_1} Y, \mathbb{T}^s_{\gamma_2(t)} Y, \dots, \mathbb{T}^s_{\gamma_m(t)} Y \rangle$$

parametrized by $C \backslash \{t_0\}$. Since Y is osculating well-behaved we can write $\mathbb{T}_t$ as follows

$$\begin{aligned} \mathbb{T}_t &= \langle \mathbb{T}^s_{p_1} Y, \mathbb{T}^s_{\gamma_2(t)} Y, \dots, \mathbb{T}^s_{\gamma_m(t)} Y \rangle = \langle \mathbb{T}^s_{p_1} X \cap \mathbb{P}^s, \mathbb{T}^s_{\gamma_2(t)} X \cap \mathbb{P}^s, \dots, \mathbb{T}^s_{\gamma_m(t)} X \cap \mathbb{P}^s \rangle \\ &\subseteq \langle \mathbb{T}^s_{p_1} X, \mathbb{T}^s_{\gamma_2(t)} X, \dots, \mathbb{T}^s_{\gamma_m(t)} X \rangle \cap \mathbb{P}^s \end{aligned}$$

Therefore

$$\lim_{t \mapsto 0} \{\mathbb{T}_t\} \subseteq \lim_{t \mapsto 0} \{\langle \mathbb{T}^s_{p_1} X, \mathbb{T}^s_{\gamma_2(t)} X, \dots, \mathbb{T}^s_{\gamma_m(t)} X \rangle\} \cap \mathbb{P}^s = \mathbb{T}^{2s+1}_p Y$$

where the last inclusion comes from Lemma 1.5.10. This argument, with the obvious changes, proves that strong 2-osculating regularity passes from X to Y as well. □

As a consequence of Proposition 5.7.1 we have the following:

Corollary 1.5.12. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and $Y = \mathbb{P}^k \cap X$ a linear section of X that is osculating well-behaved. Assume that given general points $p_1, \dots, p_m \in Y$ one can find smooth curves $\gamma_j : C \to X, j = 2, \dots, m$, realizing the m-osculating regularity of X for $p_1, \dots, p_m$ such that $\gamma_j(C) \subset Y$. Let $s_1, \dots, s_l \geqslant 1$ integers such that the general osculating projection $\Pi_{\mathbb{T}^{s_1, \dots, s_l}_{p_1, \dots, p_l}}$ of X restricted to Y is generically finite. Then Y is not $(h+1)$-defective for*

$$h \leqslant \sum_{j=1}^{l} h_m(s_j).$$

Proof. Since Y is osculating well-behaved, we have just to observe that the general osculating projection $\Pi_{\mathbb{T}^{s_1, \dots, s_l}_{p_1, \dots, p_l} X}$ of X factors trough the general osculating projection $\Pi_{\mathbb{T}^{s_1, \dots, s_l}_{p_1, \dots, p_l} Y}$ of Y. Now, the statement follows immediately from Proposition 5.7.1 and Theorem 5.4.2. □

Chapter 2

Comon's and Strassen's Conjectures

Tensor decomposition problems come out naturally in many areas of mathematics and applied sciences. For instance, in signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience, graph analysis, control theory and electrical networks [KB09a], [CM96], [CGLM08], [LO15], [MR13], [MR14], [BFFX17]. In pure mathematics tensor decomposition issues arise while studying the additive decompositions of a general tensor [Dol04], [DK93], [MM13], [Mas16], [RS00], [TZ11], [MMS18].

Comon's conjecture [CGLM08], which states the equality of the rank and symmetric rank of a symmetric tensor, and Strassen's conjecture on the additivity of the rank of tensors [Str73] are two of the most important and guiding problems in the area of tensor decomposition.

More precisely, Comon's conjecture predicts that the rank of a homogeneous polynomial $F \in k[x_0, \ldots, x_n]_d$ with respect to the Veronese variety V_d^n is equal to its rank with respect to the Segre variety $SV_1^{\boldsymbol{n}} \cong (\mathbb{P}^n)^d$ into which V_d^n is diagonally embedded, that is $\operatorname{rank}_{V_d^n}(F) = \operatorname{rank}_{SV_1^{\boldsymbol{n}}}(F)$.

Strassen's conjecture was originally stated for triple tensors and then generalized to a number of different contexts. For instance, for homogeneous polynomials it says that if $F \in k[x_0, \ldots, x_n]_d$ and $G \in k[y_0, \ldots, y_m]_d$ are homogeneous polynomials in distinct sets of variables then $\operatorname{rank}_{V_d^{n+m}}(F + G) = \operatorname{rank}_{V_d^n}(F) + \operatorname{rank}_{V_d^m}(G)$.

In Sections 2.2 and 2.3, while surveying the state of the art on Comon's and Strassen's conjectures, we push a bit forward some standard techniques, based on catalecticant matrices and more generally on flattenings, to extend some results on these conjectures, known in the setting of Veronese and Segre varieties, for Segre-Veronese and Segre-Grassmann varieties that is to the context of mixed tensors.

In Section 2.4 we introduce a method to improve a classical result on Comon's conjecture. By standard arguments involving catalecticant matrices it is not hard to prove that Comon's conjecture holds for the general polynomial in $k[x_0, \ldots, x_n]_d$ of symmetric rank h as soon as $h < \binom{n+\lfloor \frac{d}{2} \rfloor}{n}$, see Proposition 2.2.2. We manage to improve this bound looking for equations for the $(h-1)$-secant variety $\mathbb{S}ec_{h-1}(V_d^n)$, not coming from catalecticant matrices, that are restrictions to the space of symmetric tensors of equations of the $(h-1)$-secant variety $\mathbb{S}ec_{h-1}(SV_1^{\boldsymbol{n}})$. We will do so by embedding the space of degree d polynomials into the space of degree $d+1$ polynomials by mapping F to x_0F and then considering suitable catalecticant matrices of x_0F rather than those of F itself.

Implementing this method in Macaulay2 we are able to prove for instance that Comon's conjecture holds for the general cubic polynomial in $n+1$ variables of rank $h = n+1$ as long as $n \leqslant 30$. Note that for cubics the usual flattenings work for $h \leqslant n$. Note that a similar problem was considered by Friedman in [Fri16]. In his work he focused on a sufficient condition ensuring the equality of the symmetric and ordinary tensor rank. In Section 2.4 we use a different technique to prove Friedman's result for $F \in \mathrm{Sym}^d(\mathbb{C}^n)$ in the case $(d, n) = \{(3, 2), (4, 2), (3, 3)\}$.

2.1 Notation

Let us briefly recall the main notations we will use throughout this chapter. Let $\boldsymbol{n} = (n_1, \dots, n_p)$ and $\boldsymbol{d} = (d_1, \dots, d_p)$ be two p-uples of positive integers. Set

$$d = d_1 + \cdots + d_p, \ n = n_1 + \cdots + n_p, \text{ and } N(\boldsymbol{n}, \boldsymbol{d}) = \prod_{i=1}^{p} \binom{n_i + d_i}{n_i}$$

Let $V_1, \dots, V_p$ be vector spaces of dimensions $n_1 + 1 \leqslant n_2 + 1 \leqslant \cdots \leqslant n_p + 1$, and consider the product

$$\mathbb{P}^{\boldsymbol{n}} = \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*).$$

The line bundle

$$\mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(d_1, \dots, d_p) = \pi_1^* \mathcal{O}_{\mathbb{P}(V_1^*)}(d_1) \otimes \cdots \otimes \pi_p^* \mathcal{O}_{\mathbb{P}(V_1^*)}(d_p)$$

where $\pi_i : \mathbb{P}^{\boldsymbol{n}} \to \mathbb{P}(V_i^*)$ is the natural projection induces an embedding

$$\begin{array}{rccc} \nu_{\boldsymbol{d}}^{\boldsymbol{n}} : & \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*) & \longrightarrow & \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_p} V_p^*) = \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-1}, \\ & ([v_1], \dots, [v_p]) & \longmapsto & [v_1^{d_1} \otimes \cdots \otimes v_p^{d_p}] \end{array}$$

where $v_i \in V_i$. We call the image

$$SV_{\boldsymbol{d}}^{\boldsymbol{n}} = \nu_{\boldsymbol{d}}^{\boldsymbol{n}}(\mathbb{P}^{\boldsymbol{n}}) \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-1}$$

a *Segre-Veronese variety.*

When $p = 1$, SV_d^n is a Veronese variety. In this case we write V_d^n for SV_d^n, and ν_d^n for the Veronese embedding. When $d_1 = \cdots = d_p = 1$, $SV_{1,\dots,1}^{\boldsymbol{n}}$ is a Segre variety. In this case we write $SV_1^{\boldsymbol{n}}$ for $SV_{(1,\dots,1)}^{\boldsymbol{n}}$, and $\sigma^{\boldsymbol{n}} = \nu_{(1,\dots,1)}^{\boldsymbol{n}}$ for the Segre embedding. Note that

$$\sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}} = \sigma^{\boldsymbol{n}'} \circ \left(\nu_{d_1}^{n_1} \times \cdots \times \nu_{d_p}^{n_p}\right),$$

where $\boldsymbol{n}' = (N(n_1, d_1) - 1, \dots, N(n_p, d_p) - 1)$.

Similarly, given a p-uple of positive integers $\boldsymbol{k} = (k_1+1,...,k_p+1)$ we may consider the Segre-Plücker embedding

$$\begin{array}{rccc} \sigma p^{\boldsymbol{n}}_{\boldsymbol{k}}: & \mathbb{G}(k_1,n_1)\times\cdots\times\mathbb{G}(k_p,n_p) & \longrightarrow & \mathbb{P}(\bigwedge^{k_1+1}V_1\otimes\cdots\otimes\bigwedge^{k_p+1}V_p) = \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{k})-1}, \\ & ([H_1],\dots,[H_p]) & \longmapsto & [p^{n_1}_{k_1}(H_1)\otimes\cdots\otimes p^{n_p}_{k_p}(H_p)] \end{array}$$

where $N(\boldsymbol{n},\boldsymbol{k}) = \prod_{i=1}^{p}\binom{n_i+1}{k_i+1}$ and $p^{n_i}_{k_i}$ is the standard Plücker embedding. We call the image

$$SG^{\boldsymbol{n}}_{\boldsymbol{d}} = \sigma p^{\boldsymbol{n}}_{\boldsymbol{k}}(\mathbb{G}(k_1,n_1)\times\cdots\times\mathbb{G}(k_p,n_p)) \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{k})-1}$$

a *Segre-Grassmann variety.*

For every $A \subset \{1,\dots,p\}$ and $B = A^c$ we may interpret a tensor

$$T \in V_1\otimes\dots\otimes V_p = V_A\otimes V_B$$

as a linear map $\widetilde{T}: V_A^* \to V_{A^c}$. In Section 1.1 we have seen that if the rank of T is at most r then the rank of $\widetilde{T}$ is at most r as well. The matrix associated to the linear map $\widetilde{T}$ is called an (A,B)*-flattening* of T.

In the case of mixed tensors, as we saw in Section 1.1, we can consider the embedding

$$\mathrm{Sym}^{d_1}V_1\otimes\dots\otimes\mathrm{Sym}^{d_p}V_p \hookrightarrow V_A\otimes V_B$$

where $V_A = \mathrm{Sym}^{a_1}V_1\otimes\dots\otimes\mathrm{Sym}^{a_p}V_p$, $V_B = \mathrm{Sym}^{b_1}V_1\otimes\dots\otimes\mathrm{Sym}^{b_p}V_p$, with $d_i = a_i+b_i$ for any $i = 1,...,p$. In particular, if $n=1$ we may interpret a tensor $F \in \mathrm{Sym}^{d_1}V_1$ as a degree d_1 homogeneous polynomial on $\mathbb{P}(V_1^*)$. In this case the matrix associated to the linear map $\widetilde{F}: V_A^* \to V_B$ is nothing but the a_1-th *catalecticant matrix* of F, that is the matrix whose rows are the coefficient of the partial derivatives of order a_1 of F.

Similarly, by considering the inclusion

$$\bigwedge^{k_1+1}V_1\otimes\dots\otimes\bigwedge^{k_p+1}V_p \hookrightarrow V_A\otimes V_B$$

where $V_A = \bigwedge^{a_1}V_1\otimes\dots\otimes\bigwedge^{a_p}V_p$, $V_B = \bigwedge^{b_1}V_1\otimes\dots\otimes\bigwedge^{b_p}V_p$, with $k_i+1 = a_i+b_i$ for any $i = 1,...,p$, we get the so called *skew-flattenings.*

Remark 2.1.2. The partial derivatives of an homogeneous polynomials are particular flattenings. The partial derivatives of a polynomial $F \in k[x_0,...,x_n]_d$ are $\binom{n+s}{n}$ homogeneous polynomials of degree $d-s$ spanning a linear space $H_{\partial^s F} \subseteq \mathbb{P}(k[x_0,...,x_n]_{d-s})$.

If $F \in k[x_0,...,x_n]_d$ admits a decomposition

$$F = \lambda_1 L_1^d + \cdots + \lambda_h L_h^d$$

then $F \in \mathbb{S}ec_h(V_d^n)$, and conversely a general $F \in \mathbb{S}ec_h(V_d^n)$ can be written as a linear combination of $L_1^d,\dots,L_h^d$ with L_i suitable linear forms. If $F = \lambda_1 L_1^d + \dots + \lambda_h L_h^d$ is a decomposition then the partial derivatives of order s of F can be decomposed as linear combinations of $L_1^{d-s},...,L_h^{d-s}$ as well. In fact we have

$$\frac{\partial}{\partial x_i}F = \lambda_1\frac{\partial}{\partial x_i}L_1^d + \cdots + \lambda_h\frac{\partial}{\partial x_i}L_h^d$$

and if $L_j^d = (a_{0,j}x_0 + \cdots + a_{n,j}x_n)^d$ then

$$\frac{\partial}{\partial x_i} L_j^d = da_{i,j} L_j^{d-1}$$

By induction on the order of differentiation we see that the partial derivatives of order s of F can be decomposed as linear combinations of $L_1^{d-s}, ..., L_h^{d-s}$.

Therefore, the linear space $\left\langle L_1^{d-s}, \ldots, L_h^{d-s} \right\rangle$ contains $H_{\partial^s F}$.

Let us briefly recall the notions of rank and border rank for general varieties.

Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate variety. We define the rank $\mathrm{rank}_X(p)$ with respect to X of a point $p \in \mathbb{P}^N$ as the minimal integer h such that there exist h points $x_1, \ldots, x_h \in X$ with $p \in \langle x_1, \ldots, x_h \rangle$. Clearly, if $Y \subseteq X$ we have that

$$\mathrm{rank}_X(p) \leqslant \mathrm{rank}_Y(p) \tag{2.1.4}$$

The border rank $\underline{\mathrm{rank}}_X(p)$ of $p \in \mathbb{P}^N$ with respect to X is the smallest integer $r > 0$ such that p is in the Zariski closure of the set of points $q \in \mathbb{P}^N$ such that $\mathrm{rank}_X(q) = r$. In particular $\underline{\mathrm{rank}}_X(p) \leqslant \mathrm{rank}_X(p)$.

Recall that given an irreducible and reduced non-degenerate variety $X \subset \mathbb{P}^N$, and a positive integer $h \leqslant N$ the *h-secant variety* $\mathbb{S}ec_h(X)$ of X is the subvariety of $\mathbb{P}^N$ obtained as the Zariski closure of the union of all $(h-1)$-planes spanned by h general points of X.

In other words $\underline{\mathrm{rank}}_X(p)$ is computed by the smallest secant variety $\mathbb{S}ec_h(X)$ containing $p \in \mathbb{P}^N$.

Now, let Y, Z be subvarieties of an irreducible projective variety $X \subset \mathbb{P}^N$, spanning two linear subspaces $\mathbb{P}^{N_Y} := \langle Y \rangle, \mathbb{P}^{N_Z} := \langle Z \rangle \subseteq \mathbb{P}^N$. Fix two points $p_Y \in \mathbb{P}^{N_Y}, p_Z \in \mathbb{P}^{N_Z}$, and consider a point $p \in \langle p_Y, p_Z \rangle$. There exist $a = \mathrm{rank}_Y(p_Y)$ points $y_1, \ldots, y_a$ and $b = \mathrm{rank}_Z(p_Z)$ points $z_1, \ldots, z_b$ such that

$$p_Y = \lambda_1 y_1 + \cdots + \lambda_a y_a$$

and

$$p_Z = \alpha_1 z_1 + \cdots + \alpha_b z_b$$

The line $\langle p_Y, p_Z \rangle$ is indeed contained in the linear span $\langle y_1, \ldots, y_a, z_1, \ldots, z_b \rangle$ and so

$$\mathrm{rank}_X(p) \leqslant \mathrm{rank}_Y(p_Y) + \mathrm{rank}_Z(p_Z) \tag{2.1.5}$$

2.2 Comon's conjecture

It is natural to ask under which assumptions (2.1.4) is indeed an equality. Consider the Segre-Veronese embedding

$$\sigma\nu_{\boldsymbol{d}}^{\boldsymbol{n}} : \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*) \to \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_p} V_p^*) = \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-1}$$

with $V_1 \cong \cdots \cong V_p \cong V$ k-vector spaces of dimension $n+1$. Its composition with the diagonal embedding

$$i : \mathbb{P}(V^*) \to \mathbb{P}(V_1^*) \times \cdots \times \mathbb{P}(V_p^*)$$

is the Veronese embedding of ν_d^n of degree $d = d_1 + \cdots + d_p$. Let $V_d^n \subseteq SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ be the corresponding Veronese variety. We will denote by $\Pi_{n,d}$ the linear span of V_d^n in $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-1}$.

In the notations of Section 2.1 set $X = SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ and $Y = V_d^n$. For any symmetric tensor $T \in \Pi_{n,d}$ we may consider its symmetric rank $\mathrm{srk}(T) := \mathrm{rank}_{V_d^n}(T)$ and its rank $\mathrm{rank}(T) := \mathrm{rank}_{SV_d^n}(T)$ as a mixed tensor. Comon's conjecture predicts that in this particular setting the inequality (2.1.4) is indeed an equality [CGLM08].

Conjecture 1 (Comon's)**.** *Let T be a symmetric tensor. Then* $\mathrm{rank}(T) = \mathrm{srk}(T)$.

Conjecture 1 has been generalized in a number of directions for complex border rank, real rank and real border rank, see [Lan12, Section 5.7.2] for a full overview.

Note that when $d = 2$ and $p = 2$ Comon's conjecture is true. Indeed, $\mathbb{S}ec_h(SV_1^{\boldsymbol{n}})$ is cut out by the size $(h+1) \times (h+1)$ minors of a general square matrix and $\mathbb{S}ec_h(V_2^n)$ is cut out by the size $(h+1) \times (h+1)$ minors of a general symmetric matrix, that is

$$\mathbb{S}ec_h(V_2^n) = \mathbb{S}ec_h(SV_1^{\boldsymbol{n}}) \cap \Pi_{n,2}$$

Conjecture 1 has been proved in several special cases. For instance, when the symmetric rank is at most two [CGLM08], when the rank is less than or equal to the order [ZHQ16], for tensors belonging to tangential varieties to Veronese varieties [BB13], for tensors in $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ [BL13], when the rank is at most the flattening rank plus one [Fri16], for the so called Coppersmith–Winograd tensors [LM17], for symmetric tensors in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ and also for symmetric tensors of symmetric rank at most seven in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ [Sei18].

On the other hand, a counter-example to Comon's conjecture has recently been found by Y. Shitov [Shi18]. The counter-example consists of a symmetric tensor T in $\mathbb{C}^{800} \times \mathbb{C}^{800} \times \mathbb{C}^{800}$ which can be written as a sum of 903 rank one tensors but not as a sum of 903 symmetric rank one tensors. It is important to stress that for this tensor T rank and border rank are quite different. Comon's conjecture for border ranks is still completely open [Shi18, Problem 25].

Even though it has been recently proven false in full generality, we believe that Comon's conjecture is true for a general symmetric tensor, perhaps it is even true for those tensor for which $\mathrm{rank}\, T = \underline{\mathrm{rank}}\, T$.

In what follows we use simple arguments based on flattenings to give sufficient conditions for Comon's conjecture, recovering a known result, and its skew-symmetric analogue.

Lemma 2.2.1. *The tensors $T \in \mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ such that $\dim(\widetilde{T}(V_A^*)) \leqslant h-1$ for a given flattening $\widetilde{T}$ associated to the partition (A, A^c) form a proper closed subset of $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$. Furthermore, the same result holds if we replace the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ with the Segre-Grassmann variety $SG_{\boldsymbol{k}}^{\boldsymbol{n}}$.*

Proof. Let $T \in \mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ be a general point, with

$$T = \lambda_1[(v_1^1)^{d_1} \otimes \cdots \otimes (v_p^1)^{d_p}] + \cdots + \lambda_h[(v_1^h)^{d_1} \otimes \cdots \otimes (v_p^h)^{d_p}]$$

Assume that $\dim(\widetilde{T}(V_A^*)) \leqslant h-1$. This condition forces the (A,B)-flattening matrix to have rank at most $h-1$ and gives a module of equations

$$\bigwedge^h (V_A)^* \otimes \bigwedge^h (V_B)^* \subset Sym^h(Sym^{d_1}(V_1) \otimes \cdots \otimes Sym^{d_p}(V_p))$$

On the other hand, by [SU00, Proposition 4.1] these minors do not vanish on $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$, and therefore define a proper closed subset of $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$. In the Segre-Grassmann setting we argue in the same way by using skew-flattenings. □

Proposition 2.2.2. *[IK99] For any integer $h < \binom{n+\lfloor \frac{d}{2}\rfloor}{n}$ there exists an open subset $\mathcal{U}_h \subseteq \mathbb{S}ec(V_n^d)$ such that for any $T \in \mathcal{U}_h$ the rank and the symmetric rank of T coincide, that is*

$$\operatorname{rank}(T) = \operatorname{srk}(T)$$

Proof. First of all, note that we always have $\operatorname{rank}(T) \leqslant \operatorname{srk}(T)$. Furthermore for any (A, B)-flattening $\widetilde{T} : V_A^* \to V_B$ the inequality $\operatorname{rank}(T) \geqslant \dim(\widetilde{T}(V_A^*))$ holds. Since T is symmetric and its catalecticant matrices are particular flattenings we get that

$$\operatorname{rank}(T) \geqslant \dim(H_{\partial^s T})$$

for any $s \geqslant 0$.

Now, for a general $T \in \mathbb{S}ec_h(V_d^n)$ we have $\operatorname{srk}(T) = h$, and if $h < \binom{n+\overline{s}}{n}$, where $\overline{s} = \lfloor \frac{d}{2} \rfloor$, then Lemma 2.2.1 yields $\dim(H_{\partial^{\overline{s}} T}) = h$. Therefore, under these conditions we have the following chain of inequalities

$$\dim(H_{\partial^{\overline{s}} T}) \leqslant \operatorname{rank}(T) \leqslant \operatorname{srk}(T) = \dim(H_{\partial^{\overline{s}} T})$$

and hence $\operatorname{rank}(T) = \operatorname{srk}(T)$. □

Now fix a vector space v of dimensione $n + 1$ and consider the Plücker embedding

$$p_k^n : \mathbb{G}(k, n) \to \mathbb{P}(\bigwedge^{k+1} V)$$

For any skew-symmetric tensor $T \in \Pi_{n,k} = \mathbb{P}(\bigwedge^{k+1} V) \subset V^{\otimes k}$ we may consider its skew rank $\operatorname{skrk}(T)$ that is its rank with respect to the Grassmannian $\mathbb{G}(k, n) \subseteq \Pi_{n,k}$, and its rank $\operatorname{rank}(T)$ as a mixed tensor. Playing the same game as in Proposition 2.2.2 we have the following.

Proposition 2.2.3. *For any integer $h < \binom{n}{\lfloor \frac{k}{2} \rfloor}$ there exists an open subset $\mathcal{U}_h \subseteq \mathbb{S}ec_h(\mathbb{G}(k, n))$ such that for any $T \in \mathcal{U}_h$ the rank and the skew rank of T coincide, that is*

$$\operatorname{rank}(T) = \operatorname{skrk}(T)$$

Proof. As before for any tensor T we have $\operatorname{rank}(T) \leqslant \operatorname{skrk}(T)$. For any (A, B)-skew-flattening $\widetilde{T} : V_A^* \to V_B$ we have $\operatorname{skrk}(T) \geqslant \dim(\widetilde{T}(V_A^*))$. Furthermore, since $\widetilde{T}$ is in particular a flattening also the inequality $\operatorname{rank}(T) \geqslant \dim(\widetilde{T}(V_A^*))$ holds.

Now, for a general $T \in \mathbb{S}ec_h(\mathbb{G}(k, n))$ we have $\operatorname{skrk}(T) = h$, and if $h < \binom{n}{\overline{s}}$, where $\overline{s} = \lfloor \frac{k}{2} \rfloor$, Lemma 2.2.1 yields $\operatorname{skrk}(T) = \dim(\widetilde{T_{\overline{s}}}(V_A^*))$, where $\widetilde{T_{\overline{s}}}$ is the skew-flattening corresponding to the partition $(\overline{s}, d - \overline{s})$ of d. Therefore, we deduce that

$$\dim(\widetilde{T_{\overline{s}}}(V_A^*)) \leqslant \operatorname{rank}(T) \leqslant \operatorname{skrk}(T) = \dim(\widetilde{T_{\overline{s}}}(V_A^*))$$

and hence $\operatorname{rank}(T) = \operatorname{skrk}(T)$. □

2.3 Strassen's conjecture

Another natural problem consists in giving hypotheses under which in (2.1.5) equality holds. Consider the triple Segre embedding

$$\sigma^{\boldsymbol{n}} : \mathbb{P}(V_1^*) \times \mathbb{P}(V_2^*) \times \mathbb{P}(V_3^*) = \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c \to \mathbb{P}(V_1^* \otimes V_2^* \otimes V_1^*) = \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-1}$$

and let $SV_1^{\boldsymbol{n}}$ be the corresponding Segre variety. Now, take complementary subspaces $\mathbb{P}^{a_1}, \mathbb{P}^{a_2} \subset \mathbb{P}^a$, $\mathbb{P}^{b_1}, \mathbb{P}^{b_2} \subset \mathbb{P}^b$, $\mathbb{P}^{c_1}, \mathbb{P}^{c_2} \subset \mathbb{P}^c$, and let $S^{(a_1,b_1,c_1)}, S^{(a_2,b_2,c_2)}$ be the Segre varieties associated respectively to $\mathbb{P}^{a_1} \times \mathbb{P}^{b_1} \times \mathbb{P}^{c_1}$ and $\mathbb{P}^{a_2} \times \mathbb{P}^{b_2} \times \mathbb{P}^{c_2}$.

In the notations of Section 2.1 set $X = SV_1^{\boldsymbol{n}}$, $Y = S^{(a_1,b_1,c_1)}$ and $Z = S^{(a_1,b_1,c_1)}$. Strassen's conjecture states that the additivity of the rank holds for triple tensors, or in onther words that in this setting the inequality (2.1.5) is indeed an equality [Str73].

Conjecture 2 (Strassen's)**.** *In the above notation let* $T_1 \in \left\langle S^{(a_1,b_1,c_1)} \right\rangle, T_2 \in \left\langle S^{(a_2,b_2,c_2)} \right\rangle$ *be two tensors. Then* $\operatorname{rank}(T_1 \oplus T_2) = \operatorname{rank}(T_1) + \operatorname{rank}(T_2)$.

Even though Conjecture 2 was originally stated in the context of triple tensors that is bilinear forms, with particular attention to the complexity of matrix multiplication, a number of generalizations are immediate. For instance, we could ask the same question for higher order tensors, symmetric tensors, mixed tensors and skew-symmetric tensors. It is also natural to ask for the analogue of Conjecture 2 for border rank. This has been answered negatively [Sch81].

In a more general setting, given $X \subset \mathbb{P}^N$ and $Y, Z \subset X$ subvarieties such that

$$\langle Y \rangle \cap \langle Z \rangle = \varnothing$$

and

$$\langle Y \rangle + \langle Z \rangle = \mathbb{P}^N$$

then:

Conjecture 3 (Strassen's General)**.** *In the above notation let* $y \in \langle Y \rangle$ *and* $z \in \langle Z \rangle$*. Then* $\operatorname{rank}(y \oplus z) = \operatorname{rank}(y) + \operatorname{rank}(z)$.

Conjecture 2 and its analogues have been proven when either T_1 or T_2 has dimension at most two, when $\operatorname{rank}(T_1)$ can be determined by the so called substitution method [LM17], when $\dim(V_1) = 2$ both for the rank and the border rank [BGL13], when T_1, T_2 are symmetric that is homogeneous polynomials in disjoint sets of variables, either T_1, T_2 is a power, or both T_1 and T_2 have two variables, or either T_1 or T_2 has small rank [CCC15], and also for other classes of homogeneous polynomials [CCO17], [Tei15].

As for Comon's conjecture a counterexample to Strassen's conjecture has recentely been given by Y. Shitov [Shi17]. In this case Y. Shitov proved that over any infinite field there exist tensors T_1, T_2 such that the inequality in Conjecture 2 is strict.

In what follows we give sufficient conditions for Strassen's conjecture, recovering a known result, and for its mixed and skew-symmetric analogues.

Proposition 2.3.1. *[IK99] Let V_1, V_2 be k-vector spaces of dimensions $n+1, m+1$, and consider $V = V_1 \oplus V_2$. Let $F \in \mathrm{Sym}^d(V_1) \subset \mathrm{Sym}^d(V)$ and $G \in \mathrm{Sym}^d(V_2) \subset \mathrm{Sym}^d(V)$ be two homogeneous polynomials. If there exists an integer $s > 0$ such that*

$$\dim(H_{\partial^s F}) = \mathrm{srk}(F), \quad \dim(H_{\partial^s G}) = \mathrm{srk}(G)$$

then $\mathrm{srk}(F+G) = \mathrm{srk}(F) + \mathrm{srk}(G)$.

Proof. Clearly, $\mathrm{srk}(F+G) \leqslant \mathrm{srk}(F) + \mathrm{srk}(G)$ holds in general. On the other hand, our hypothesis yields

$$\mathrm{srk}(F) + \mathrm{srk}(G) = \dim(H_{\partial^s F}) + \dim(H_{\partial^s F}) = \dim(H_{\partial^s F+G}) \leqslant \mathrm{srk}(F+G)$$

where the last inequality follows from Remark 2.1.2. □

Remark 2.3.2. The argument used in the proof of Proposition 2.3.1 works for $F \in \mathbb{P}^{N(n,d)}$ general only if for the generic rank we have $\lfloor \frac{\binom{n+d}{d}}{n+1} \rfloor \leqslant \binom{n+\lfloor \frac{d}{2} \rfloor}{n}$. For instance, when $n = 3, d = 6$ the generic rank is 21 while the maximal dimension of the spaces spanned by partial derivatives is 20.

Proposition 2.3.3. *Let $V_1, \ldots, V_p$ and $W_1, \ldots, W_p$ be k-vector spaces of dimension $n_1 + 1, \ldots, n_p + 1$ and $m_1 + 1, \ldots, m_p + 1$ respectively. Consider $U_i = V_i \oplus W_i$ for every $1 \leqslant i \leqslant p$. Let*

$$T_1 \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_p} V_p \subset \mathrm{Sym}^{d_1} U_1 \otimes \cdots \otimes \mathrm{Sym}^{d_p} U_p$$

and

$$T_2 \in \mathrm{Sym}^{d_1} W_1 \otimes \cdots \otimes \mathrm{Sym}^{d_p} W_p \subset \mathrm{Sym}^{d_1} U_1 \otimes \cdots \otimes \mathrm{Sym}^{d_p} U_p$$

be two mixed tensors.

If for any $i \in \{1, .., p\}$ there exists a pair (a_i, b_i) with $a_i + b_i = d_i$ and (A, B)-flattenings $\widetilde{T}_1 : V_A^* \to V_B$, $\widetilde{T}_2 : V_A^* \to V_B$ such that

$$\dim(\widetilde{T}_1(V_A^*)) = \mathrm{rank}(T_1), \quad \dim(\widetilde{T}_2(V_A^*)) = \mathrm{rank}(T_2)$$

then $\mathrm{rank}(T_1 + T_2) = \mathrm{rank}(T_1) + \mathrm{rank}(T_2)$.

Proof. Clearly, $\mathrm{rank}(T_1 + T_2) \leqslant \mathrm{rank}(T_1) + \mathrm{rank}(T_2)$. On the other hand, our hypothesis yields

$$\mathrm{rank}(T_1) + \mathrm{rank}(T_2) = \dim(\widetilde{T}_1(V_A^*)) + \dim(\widetilde{T}_2(V_A^*)) = \dim(\widetilde{T_1 + T_2}(V_A^*)) \leqslant \mathrm{rank}(T_1 + T_2)$$

where $\widetilde{T_1 + T_2}$ denotes the (A, B)-flattening of the mixed tensor $T_1 + T_2$. □

Arguing as in the proof of Proposition 2.3.3 with skew-symmetric flattenings we have an analogous statement in the Segre-Grassmann setting.

Proposition 2.3.4. *Let $V_1, \ldots, V_p$ and $W_1, \ldots, W_p$ be k-vector spaces of dimension $n_1 + 1, \ldots, n_p + 1$ and $m_1 + 1, \ldots, m_p + 1$ respectively. Consider $U_i = V_i \oplus W_i$ for every $1 \leqslant i \leqslant p$, and let*

$$T_1 \in \bigwedge^{k_1+1} V_1 \otimes \cdots \otimes \bigwedge^{k_p+1} V_p \subset \bigwedge^{k_1+1} U_1 \otimes \cdots \otimes \bigwedge^{k_p+1} U_p$$

and

$$T_2 \in \bigwedge^{k_1+1} W_1 \otimes \cdots \otimes \bigwedge^{k_p+1} W_p \subset \bigwedge^{k_1+1} U_1 \otimes \cdots \otimes \bigwedge^{k_p+1} U_p$$

be two skew-symmetric tensors with $k_i + 1 \leqslant \min\{n_i + 1, m_i + 1\}$.

If for any $i \in \{1, \ldots, p\}$ there exists a pair (a_i, b_i) with $a_i + b_i = k_i + 1$ and (A, B)-skew-flattenings $\widetilde{T}_1 : V_A^* \to V_B$, $\widetilde{T}_2 : V_A^* \to V_B$ such that

$$\dim(\widetilde{T}_1(V_A^*)) = \operatorname{rank}(T_1), \quad \dim(\widetilde{T}_2(V_A^*)) = \operatorname{rank}(T_2)$$

then $\operatorname{rank}(T_1 + T_2) = \operatorname{rank}(T_1) + \operatorname{rank}(T_2)$.

2.4 On the rank of x_0F: an application to Comon's conjecture

Propositions 2.2.2, 2.2.3 suggest that whenever we are able to write determinantal equations for secant varieties we are able to verify Comon's conjecture. This suggests a possible way to improve the range where the general Comon's conjecture holds giving a conjectural way to produce determinantal equations for some secant varieties.

Remark 2.4.1. Set $\boldsymbol{n} = (n, \ldots, n)$, $(d+1)$-times, $\boldsymbol{n}_1 = (n, \ldots, n)$, d-times, and consider the corresponding Segre varieties $X := SV_1^{\boldsymbol{n}}$, $X_1 := SV_1^{\boldsymbol{n}_1}$ and Veronese varieties $Y = V_{d+1}^n$, $Y_1 := V_d^n$. Fix the polynomial $x_0^{d+1} \in Y$ and let Π be the linear space spanned by the polynomials of the form x_0F, where F is a polynomial of degree d. This allows us to see $Y_1 \subseteq \Pi$. Note that polynomials of the form $x_0L_1^d$ lie in the tangent space of Y at L_1^{d+1}, and therefore $\underline{\operatorname{rank}}_Y(x_0L^{\otimes d}) = 2$.

Hence for a polynomial F of degree d we have $\operatorname{rank}_Y(x_0F) \leqslant 2\operatorname{rank}_{Y_1}(F)$. Our aim is to understand when the equality holds.

We may mimic the same construction for the Segre varieties X and X_1, and use determinantal equations for the secant varieties of X_1 to give determinantal equations of the secant varieties of X and henceforth conclude Comon's conjecture. In particular, as soon as d is odd and $d < n$, this produces new determinantal equations for $\mathbb{S}ec_h(X_1)$ and $\mathbb{S}ec_h(Y_1)$ with $2h < \binom{n+\frac{d+1}{2}}{n}$. Therefore, this would give new cases in which the general Comon's conjecture holds. Unfortunately, we are only able to successfully implement this procedure in very special cases.

Let us briefly recall, for the reader convenience, that for a smooth point $x \in X$, the *a-osculating space* $\mathbb{T}_x^aX$ of X at x is roughly the smaller linear subspace locally approximating X up to order a at x, i.e. the linear subspace spanned by all derivatives of order $\leqslant a$ of a parametrization of X centered in x. The *a-osculating variety* T^aX of X is defined as the closure of the union of all the osculating spaces

$$T^aX = \overline{\bigcup_{x\in X} \mathbb{T}^a_x X}$$

For the Veronese variety V^n_d we have the parametrization:

$$\nu^n_d : \mathbb{P}^n \to \mathbb{P}(Sym^d(V))$$

$$\nu^n_d(L) = [L^d]$$

with $V = k[x_0, \ldots, x_n]_1$ a vector space of dimension $n+1$. A curve passing through $[L^d]$ can be given locally as the image $\nu^n_d(L + tM)$ with $M \in k[x_0, \ldots, x_n]_1$ a linear form. For every $1 \leqslant a \leqslant d-1$ we have

$$\frac{d}{dt}(L + tM)^d_{|t=0} = d(d-1)\cdots(d-s+1)M^s(L+tM)^{d-s}$$

Since $k[x_0, \ldots, x_n]_s$ is spanned by $s-$th powers of linear forms the osculating space $\mathbb{T}^a_{[L^d]}V^n_d$ of order a at the point $[L^d] \in V_d$ can be written as

$$\mathbb{T}^a_{[L^d]}V^n_d = \left\langle L^{d-a}F \mid F \in k[x_0, \ldots, x_n]_a \right\rangle \subseteq \mathbb{P}^N$$

Equivalently, $\mathbb{T}^a_{[L^d]}V^n_d$ is the space of homogeneous polynomials whose derivatives of order less than or equal to a in the direction given by the linear form L vanish. Note that $\dim(\mathbb{T}^a_{[L^d]}V^n_d) = \binom{n+a}{n} - 1$ and $\mathbb{T}^b_{[L^d]}V^n_d \subseteq \mathbb{T}^a_{[L^d]}V^n_d$ for any $b \leqslant a$. Moreover, for any $1 \leqslant a \leqslant d$ and $[L^d] \in V^n_d$ we can embed a copy of V^n_a into the osculating space $\mathbb{T}^a_{[L^d]}V^n_d$ by considering

$$V^n_a = \{L^{d-a}M^a \mid M \in k[x_0, \ldots, x_n]_1\} \subseteq \mathbb{T}^a_{[L^d]}V^n_d$$

In particular we can embed

$$V^n_d = \{x_0L^d \mid L \in k[x_0, \ldots, x_n]_1\} \subseteq \mathbb{T}^d_{[x_0^d]}V^n_{d+1}$$

and Remark 2.4.1 yields that

$$\mathbb{S}ec_h(V^n_d) \subseteq \mathbb{S}ec_{2h}(V^n_{d+1}) \cap \mathbb{T}^d_{[L^{d+1}]}V^n_{d+1} \tag{2.4.2}$$

This embedding extends to an embedding at the level of Segre varieties, and, in the notation of Remark 2.4.1, we have that $\mathbb{S}ec_h(SV^{n_1}_1) \subseteq \mathbb{S}ec_{2h}(SV^n_1)$.

Assume that for a polynomial $F \in \mathbb{S}ec_h(V^n_d)$ we have $F \in \mathbb{S}ec_{h-1}(SV^{n_1}_1)$. Then $x_0F \in \mathbb{S}ec_{2h-2}(SV^n_1)$. Now, if we find a determinantal equation of $\mathbb{S}ec_{2h-2}(V^n_{d+1})$ coming as the restriction to Π, the space of symmetric tensors, of a determinantal equation of $\mathbb{S}ec_{2h-2}(SV^n_1)$, and not vanishing at x_0F then $x_0F \notin \mathbb{S}ec_{2h-2}(SV^n_1)$ and hence $F \notin \mathbb{S}ec_{h-1}(SV^{n_1}_1)$ proving Comon's conjecture for F.

This will be the leading idea to keep in mind in what follows. The determinantal equations involved will always come from minors of suitable catalecticant matrices, that can be therefore seen as the restriction to Π of determinantal equations for the secants of the Segre coming from non symmetric flattenings.

It is easy to give examples where the inequality (2.4.2) is strict. When $n = 1$ the generic rank is $g_d = \lceil \frac{d+1}{2} \rceil$. Then for d odd we have $g_d = g_{d-1}$ while for d even we have $g_d = g_{d-1} + 1$. Hence $\operatorname{rank}_{V_d} x_0F < 2\operatorname{rank}_{V_{d-1}} F$ if $2\operatorname{rank}_{V_{d-1}} F > \frac{g_d}{2}$, where $V_d := V_d^1$ is the rational normal curve. It is natural to ask if the inequality is indeed an equality as long as the rank is subgeneric. In the case $n = 1$ we have the following result.

Proposition 2.4.3. *Let $V_d := V_d^1$ be the degree d rational normal curve. If $2h < g_{d+1}$ then there are no $k_h > 0$ such that $\mathbb{S}ec_h(V_d) \subseteq \mathbb{S}ec_{2h-k_h}(V_{d+1}) \cap \mathbb{T}^d_{[x^{d+1}]}V_{d+1}$.*

Proof. Clearly, it is enough to prove the statement for $k_h = 1$. Let $p \in \mathbb{S}ec_h(V_d)$ be a general point. Then $p \in \langle [x_0L_1^d], \dots, [x_0L_h^d] \rangle$ with L_i general linear forms. In particular

$$p \in H := \left\langle \mathbb{T}_{[L_1^{d+1}]}V_{d+1}, \dots, \mathbb{T}_{[L_h^{d+1}]}V_{d+1} \right\rangle$$

Note that $\dim(H) = 2h - 1$. Now, assume that p is contained also in $\mathbb{S}ec_{2h-1}(V_{d+1})$. Then there exists a linear subspace $H' \subset \mathbb{P}^{d+1}$ of dimension $2h-2$ passing through p intersecting V_{d+1} at $2h - 1$ points $q_1, \dots, q_r$ counted with multiplicity. Let $q_{i_1}, \dots, q_{i_r}$ be the points among the q_i coinciding with some of the $[L_i^{d+1}]$ and such that the intersection multiplicity of H' and V_{d+1} at q_{i_j} is one, and $q_{j_1}, \dots, q_{j_r}$ be the points among the q_i coinciding with some of the $[L_i^{d+1}]$ and such that the intersection multiplicity of H' and V_d at q_{j_k} is greater that or equal to two.

Set $\Pi := \langle H, H' \rangle$, then

$$\dim(\Pi) = 2h - 1 + 2h - 2 - i_r - 2j_r$$

and Π intersects V_{d+1} at $2h + (2h - 1 - i_r - 2j_r)$ points counted with multiplicity. Consider general points $b_1, \dots, b_s \in V_{d+1}$ with $s = i_r + 2j_r$, and the linear space $\Pi' = \langle \Pi, b_1, \dots, b_s \rangle$. Therefore, $\dim(\Pi') = 4h-3$ and Π' intersects V_{d+1} at $4h-1$ points counted with multiplicity. Since $2h \leqslant \frac{d+3}{2}$ adding enough general points to Π' we may construct a hyperplane in $\mathbb{P}^{d+1}$ intersecting V_{d+1} at $d + 2$ points counted with multiplicity, a contradiction. □

2.4.3 Applications to Hankel Matrices

Proposition 2.4.3 can be applied to get results on the rank of a special class of matrices called Hankel matrices.

Definition 2.4.4. A matrix $A = (A_{i,j}) \in M(a, b)$ such that $A_{i,j} = A_{h,k}$ whenever $i + j = h + k$ is called a *Hankel matrix.*

Let

$$F = \binom{d}{0}Z_0x_0^d + \dots + \binom{d}{d-i}Z_ix_0^{d-i}x_1^i + \dots + \binom{d}{d}Z_dx_1^d$$

be a binary form and consider $[Z_0, \ldots, Z_d]$ as homogeneous coordinates on $\mathbb{P}(k[x_0, x_1]_d)$. Furthermore, for a given positive integer n, consider the matrices

$$M_{2n} = \begin{pmatrix} Z_0 & Z_1 & \ldots & Z_{n-1} & Z_n \\ Z_1 & \ldots & \ldots & \ldots & Z_{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ Z_{n-1} & \ldots & \ldots & \ldots & Z_{d-1} \\ Z_n & Z_{n+1} & \ldots & Z_{d-1} & Z_d \end{pmatrix} \quad M_{2n+1} = \begin{pmatrix} Z_0 & Z_1 & \ldots & Z_{n-1} & Z_n \\ Z_1 & \ldots & \ldots & \ldots & Z_{n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ Z_{n-1} & \ldots & \ldots & \ldots & Z_{d-2} \\ Z_n & \ldots & \ldots & \ldots & Z_{d-1} \\ Z_{n+1} & Z_{n+2} & \ldots & Z_{d-1} & Z_d \end{pmatrix}$$

It is well known that the ideal of $\mathbb{S}ec_h(V_d)$ is cut out by the minors of M_d of size $(h+1) \times (h+1)$ [LO15].

Now, consider a polynomial $F \in k[x_0, x_1]_d$ with homogeneous coordinates $[Z_0, \ldots, Z_d]$. Then $F' := x_0 F \in k[x_0, x_1]_{d+1}$ has homogeneous coordinates $[Z'_0, \ldots, Z'_{d+1}]$ with

$$Z'_i = \frac{d+1-i}{d+1} Z_i$$

In order to determine the rank of F' we have to relate the rank of the matrices

$$N_{2n} = \begin{pmatrix} Z_0 & \frac{d}{d+1} Z_1 & \ldots & \frac{d+1-n}{d+1} Z_n \\ \frac{d}{d+1} Z_1 & \ldots & \ldots & \frac{d-n}{d+1} Z_{n+1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{d-n+2}{d+1} Z_{n-1} & \ldots & \ldots & \frac{1}{d+1} Z_d \\ \frac{d-n+1}{d+1} Z_n & \ldots & \frac{1}{d+1} Z_d & 0 \end{pmatrix}$$

$$N_{2n+1} = \begin{pmatrix} Z_0 & \frac{d}{d+1} Z_1 & \ldots & \frac{d-n}{d+1} Z_n \\ \frac{d}{d+1} Z_1 & \ldots & \ldots & \frac{d-n-1}{d+1} Z_{n+1} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{d-n+2}{d+1} Z_n & \ldots & \ldots & \frac{1}{d+1} Z_d \\ \frac{d-n+1}{d+1} Z_{n+1} & \ldots & \frac{1}{d+1} Z_d & 0 \end{pmatrix}$$

with the rank of M_d.

In particular all the matrices of the form M_d and N_d considered above are Hankel matrices.

Let $M(a, b)$ be the vector space of $a \times b$ matrices with coefficients in the base field k. For any $h \leqslant \min\{a, b\}$ let $Rank_r(M(a, b)) \subseteq M(a, b)$ be the subvariety consisting of all matrices of rank at most h.

Now, consider the map

$$\beta : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$$

given by

$$\beta(2n) = (n+1, n+1)$$

$$\beta(2n+1) = (n+2, n+1)$$

For any $d \geqslant 1$ we can view the subspace $H_d \subseteq M(\beta(d))$ formed by matrices of the form M_d as the subspace of Hankel matrices. Now, given any linear morphism $f : M(a,b) \to M(c,d)$ we can ask if for some $s \leqslant \min\{c,d\}$ we have $f(Rank_h(M(a,b))) \subseteq Rank_s(M(c,d))$.

Corollary 2.4.5. *Consider the linear morphism*

$$\begin{array}{rccc} \alpha_d : & M(\beta(d)) & \longrightarrow & M(\beta(d+1)) \\ & (A_{i,j}) & \longmapsto & \left(\frac{d-(i+j-3)}{d+1} A_{i,j}\right) \end{array}$$

Then $\alpha_d(H_d) \subseteq H_{d+1}$ and in particular

$$\alpha_d(Rank_h(M(\beta(d)) \cap M_d)) \subseteq Rank_{2h}(M(\beta(d+1))) \cap M_{d+1}$$

Proof. Since $\alpha_d(A_{i,j}) = \alpha_d(A_{h,k})$ when $i+j = h+k$ we have that $\alpha_d(H_d) \subseteq H_{d+1}$. By Proposition 2.4.3

$$Rank_h(M(\beta(d)) \cap H_d = \mathbb{S}ec_h(V_d)$$

and by construction $\alpha_d(M_d)$ is the linear change of coordinates mapping a binary form $F \in k[x_0,x_1]_d$ to $F' = x_0F \in k[x_0,x_1]_{d+1}$.

Since

$$\mathbb{S}ec_h(V_d) \subseteq \mathbb{S}ec_{2h}(V_{d+1}) \cap \mathbb{T}^d_{[x^{d+1}]}V_{d+1}$$

if an $h \times h$ minor of a general matrix B in $M(\beta(d))$ does not vanish, under the assumption that all the $(h+1) \times (h+1)$ minors of B vanish, then there is a $2h \times 2h$ minor of $\alpha_d(B)$ that does not vanish. □

When $n \geqslant 2$ we are able to determine, via Macaulay2 [Mac92] aided methods, the rank of x_0F in some special cases.

- $(n,d) = (2,2)$

  By Theorem 1.2.12 the variety $\mathbb{S}ec_3(V_3^2)$ has the expected dimension, i.e. $\mathbb{S}ec_3(V_3^2) \subset \mathbb{P}^9$ is a hypersurface. Since the only possible flattening for a symmetric tensor $F \in Sym^3(\mathbb{C}^3)$ is given by a linear map

  $$\tilde{F} : (\mathbb{C}^3)^* \to \mathbb{C}^3$$

  there are no equations coming from $\tilde{F}$. In order to find equations for $\mathbb{S}ec_3(V_3^2)$ first embed

  $$Sym^3(\mathbb{C}^3) \to (\mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^3) \otimes (\mathbb{C}^3 \otimes (\mathbb{C}^3)^*)$$

  in the natural way. Now a polynomial $F \in Sym^3(\mathbb{C}^3)$ gives rise to a linear map $\bar{F} : (\mathbb{C}^9)^* \to \mathbb{C}^9$. In bases if

  $$F = \sum_{a+b+c=3} \binom{3}{a,b,c} Z_{a,b,c} x_0^a x_1^b x_2^c$$

  then

  $$\tilde{F} = \begin{pmatrix} 0 & H(\partial x_2) & -H(\partial x_1) \\ -H(\partial x_2) & 0 & H(\partial x_0) \\ H(\partial x_1) & -H(\partial x_0) & 0 \end{pmatrix}$$

where $H(\partial x_i)$ denotes the Hessian of the polynomial $\frac{\partial}{\partial x_i}F$. Finally we have that $\mathbb{S}ec_3(V_3^2)$ is the zero locus of the principal Pfaffian of size 8 of the matrix $\bar{F}$, called the Aronhold invariant. (see [LO15, Section 1.1] for more details).

With a Macaulay2 computation we prove that if $F \in \mathbb{S}ec_2(V_2^2)$ is general then the Aronhold invariant does not vanish at x_0F, hence $\operatorname{rank} x_0F = 2\operatorname{rank} F$.

- $(n, d) = (2, 3)$

  The varieties $\mathbb{S}ec_5(V_4^2)$ and $\mathbb{S}ec_3(V_3^2)$ are both hypersurfaces, given respectively by the determinant of the catalecticant matrix of second partial derivatives and the Aronhold invariant [LO15, Section 1.1]. With Macaulay2 we prove that the determinant of the second catalecticant matrix does not vanish at x_0F for $F \in \mathbb{S}ec_3(V_3^2)$ general, hence $\operatorname{rank} x_0F = 2\operatorname{rank} F$.

- $(n, d) = (3, 3)$

  The secant variety $\mathbb{S}ec_9(V_4^3)$ is the hypersurface cut out by the second catalecticant matrix [LO15, Section 1.1] while $\mathbb{S}ec_5(V_3^3)$ is the entire osculating space. A Macaulay2 computation shows that $\mathbb{T}^3_{[x_0^4]}V_4^3 \subseteq \mathbb{S}ec_9(V_4^3)$. This proves that $\operatorname{rank} x_0F < 2\operatorname{rank} F$, for F general.

- $(n, d) = (4, 3)$

  In this case $\mathbb{S}ec_8(V_3^4) = \mathbb{T}^3_{[x_0^4]}V_4^4$ and $\mathbb{S}ec_{14}(V_4^4)$ is given by the determinant of the second catalecticant matrix [LO15, Section 1.1]. Again using Macaulay2 we show that $\mathbb{T}^3_{[x_0^4]}V_4^4 \subseteq \mathbb{S}ec_{14}(V_4^4)$. This proves that $\operatorname{rank} x_0F < 2\operatorname{rank} F$, for F general.

Corollary 2.4.6. *For the osculating varieties $\mathbb{T}^3V_4^3$ and $\mathbb{T}^3V_4^4$ we have*

$$\mathbb{T}^3V_4^3 \subseteq \mathbb{S}ec_9(V_4^3), \quad \mathbb{T}^3V_4^4 \subseteq \mathbb{S}ec_{14}(V_4^4)$$

Proof. The action of $PGL(n+1)$ on $\mathbb{P}^n$ extends naturally to an action on $\mathbb{P}^{N(n,d)}$ stabilizing V_d^n and more generally the secant varieties $\mathbb{S}ec_h(V_d^n)$. Since this action is transitive on V_d^n we have $\mathbb{T}^a_{[x_0^d]}V_d^n \subseteq \mathbb{S}ec_h(V_d^n)$ if and only if $\mathbb{T}^a_{[L^d]}V_d^n \subseteq \mathbb{S}ec_hV_d^n$ for any point $[L^d] \in V_d^n$ that is $\mathbb{T}^aV_d^n \subseteq \mathbb{S}ec_hV_d^n$. Finally, we conclude by applying the previous results for $(n, d) = (3, 3)$ and $(n, d) = (4, 3)$. □

2.5 Macaulay2 implementation

In the Macaulay2 file `Comon-1.0.m2` we provide a function called `Comon` which operates as follows:

- `Comon` takes in input three natural numbers n, d, h;
- if $h < \binom{n+\lfloor \frac{d}{2} \rfloor}{n}$ then the function returns that Comon's conjecture holds for the general degree d polynomial in $n+1$ variables of rank h by the usual flattenings method in Proposition 2.2.2. If not, and d is even then it returns that the method does not apply;

- if d is odd and $\binom{n+k}{n} < 2\binom{n+k-1}{n}$, where $k = \lfloor \frac{d+1}{2} \rfloor$, then again it returns that the method does not apply;

- if d is odd, $\binom{n+k}{n} \geqslant 2\binom{n+k-1}{n}$ and $2h-1 > \binom{n+k}{n}$ then it returns that the method does not apply since $2h-2$ must be smaller than the number of order k partial derivatives;

- if d is odd, $\binom{n+k}{n} \geqslant 2\binom{n+k-1}{n}$ and $2h-1 \leqslant \binom{n+k}{n}$ then `Comon` produces a polynomial of the form
$$F = \sum_{i=1}^{h}(a_{i,0}x_0 + \cdots + a_{i,n}x_n)^d$$
then substitutes random rational values to the $a_{i,j}$, computes the polynomial $G = x_0F$, the catalecticant matrix D of order k partial derivatives of G, extracts the most up left $2h-1 \times 2h-1$ minor P of D, and compute the determinant $\det(P)$ of P;

- if $\det(P) = 0$ then `Comon` returns that the method does not apply, otherwise it returns that Comon's conjecture holds for the general degree d polynomial in $n+1$ variables of rank h.

Note that since the function `random` is involved `Comon` may return that the method does not apply even though it does. Clearly, this event is extremely unlikely. Thanks to this function we are able to prove that Comon's conjecture holds in some new cases that are not covered by Proposition 2.2.2. Since the case $n = 1$ is covered by Proposition 2.4.3 in the following we assume that $n \geqslant 2$.

Theorem 2.5.1. *Assume $n \geqslant 2$ and set $h = \binom{n+\lfloor \frac{d}{2} \rfloor}{n}$. Then Comon's conjecture holds for the general degree d homogeneous polynomial in $n+1$ variables of rank h in the following cases:*

- *$d = 3$ and $2 \leqslant n \leqslant 30$;*

- *$d = 5$ and $3 \leqslant n \leqslant 8$;*

- *$d = 7$ and $n = 4$.*

Proof. The proof is based on Macaulay2 computations using the function `Comon` exactly as shown in Example 2.5.2 below. □

Example 2.5.2. We apply the function `Comon` in a few interesting cases:

```
Macaulay2, version 1.12
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : loadPackage "Comon-1.0.m2";
i2 : Comon(5,3,4)
Lowest rank for which the usual flattenings method does not work = 6
o2 = Comon's conjecture holds for the general degree 3 homogeneous polynomial
     in 6 variables of rank 4 by the usual flattenings method
```

```
i3 : Comon(5,3,6)
Lowest rank for which the usual flattenings method does not work = 6
o3 = Comon's conjecture holds for the general degree 3 homogeneous polynomial
     in 6 variables of rank 6
i4 : Comon(5,3,7)
Lowest rank for which the usual flattenings method does not work = 6
o4 = The method does not apply --- The determinant vanishes
i5 : Comon(5,5,21)
Lowest rank for which the usual flattenings method does not work = 21
o5 = Comon's conjecture holds for the general degree 5 homogeneous polynomial
     in 6 variables of rank 21
i6 : Comon(4,7,35)
Lowest rank for which the usual flattenings method does not work = 35
o6 = Comon's conjecture holds for the general degree 7 homogeneous polynomial
     in 5 variables of rank 35
```

Here we present the Macaulay2 (pseudo)code that has been used for the previous computations:

```
Comon = method(TypicalValue => String);
Comon (ZZ,ZZ,ZZ) := (n,d,h) -> (
k = floor((d+1)/2);
<< "Lowest rank for which the usual flattenings method does not work = "
<< binomial(n+floor(d/2),n) << "";
if (h < binomial(n+floor(d/2),n)) then return
("Comon's conjecture holds for the general degree "|toString(d)|" homogeneous polynomial
in "|toString(n+1)|" variables of rank "|toString(h)|" by the usual flattenings method");
if (odd(d) == false) then return ("The method does not apply");
if (binomial(n+k,n) < 2*binomial(n+k-1,n)) then return ("The method does not apply");
if (2*h-1 > binomial(n+k,n)) then return
("The method does not apply, 2h-1 must be smaller
than or equal to the number of order "|toString(k)|" partial derivatives
--- For degree "|toString(d)|" homogeneous polynomial in
"|toString(n+1)|" variables usual flattenings work for
h strictly less than "|toString(binomial(n+k-1,n))|"
so the interesting cases are for h varying
between "|toString(binomial(n+k-1,n))|" and
"|toString(floor((1/2)*(binomial(n+k,n)+1)))|"");


R = QQ[x_0..x_n];
v = basis(k,R);
S = R[a_{0,1}..a_{n,h}];
for i from 1 to h do L_i = sum(0..n,j->a_{j,i}*x_j);
F = sum(1..h,i->L_i^d);
for i from 0 to n do for j from 1 to h do F=sub(F,a_{i,j}=>random(QQ));
G = x_0*F;
D = diff(v ** transpose v,G);
P = submatrix(D,{0..2*h-2},{0..2*h-2});
A = det P;
if A==0 then return
("The method does not apply --- The determinant vanishes ---
For degree "|toString(d)|" homogeneous polynomial
in "|toString(n+1)|" variables usual flattenings
work for h strictly less than "|toString(binomial(n+k-1,n))|"
so the interesting cases are for h varying between
"|toString(binomial(n+k-1,n))|" and "|toString(floor((1/2)*(binomial(n+k,n)+1)))|"");
return ("Comon's conjecture holds for the general degree "|toString(d)|"
homogeneous polynomial in "|toString(n+1)|"
variables of rank "|toString(h)|"
--- For degree "|toString(d)|" homogeneous polynomial
in "|toString(n+1)|" variables usual flattenings work
for h strictly less than "|toString(binomial(n+k-1,n))|"
so the interesting cases are for
h varying between "|toString(binomial(n+k-1,n))|"
and "|toString(floor((1/2)*(binomial(n+k,n)+1)))|"");
```

Chapter 3

Defectiveness and identifiability: subgeneric rank

Over a decade ago the notion of h-weakly defective varieties has been connected to identifiability of polynomials, [Mel06]. This provided the first systematic study of identifiability for Veronese varieties. More recently with the work of Luca Chiantini and Giorgio Ottaviani, [CO12], weakly defective varieties have been substituted by h-tangentially weakly defective varieties to study identifiability problems. In both approaches to provide identifiability one has to check the behavior of special linear systems and quite often this is done by an ad hoc degeneration argument. As a consequence identifiability has been proved in very few cases and quite often the results obtained are not expected to be sharp, [CO12] [BDDG07] [BC13] [BCO14] [Kru77].

In this chapter we want to develop an entirely new approach to study generic identifiability. Starting from the seminal paper [CC10], where the geometry of contact loci has been carefully studied, and the improvement presented in [BBC$^+$18], we derive identifiability statements for non secant defective varieties. Even if new this is not really surprising since weakly defectiveness and tangentially weakly defectiveness, thanks to the Terracini Lemma 1.2.5, have secant defectiveness as a common ancestor. With this new approach we are able to translate all the literature on defective varieties into identifiability statements, providing in many cases sharp classification of h-identifiability.

3.1 Relation between twd and defectiveness

Let us briefly recall the main notations that we will use in this chapter. For a more detailed explanation see Chapter 1.

A projective variety $X \subset \mathbb{P}^N$ is non degenerate if it is not contained in any hyperplane, i.e. if $|\mathcal{O}_X(1) \otimes \mathcal{I}_{\langle X \rangle}| = \emptyset$.

Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non degenerate variety.

For a subset $A = \{x_1, \ldots, x_h\} \subset X$ of general points we set

$$M_A := \langle \bigcup_i \mathbb{T}_{x_i} X \rangle.$$

By Terracini Lemma the space M_A is the tangent space to $\mathbb{S}ec_h(X)$ at a general point in $\langle A \rangle$.

The tangential h-contact locus $\Gamma_h :=: \Gamma(A)$ is the closure in X of the union of all the irreducible components which contain at least one point of A, of the locus of points of X where M_A is tangent to X. We will write $\gamma_h := \dim \Gamma(A)$. We say that X is h-twd (tangentially weakly defective) if $\gamma_h > 0$.

The h-tangential projection (from A) of X is

$$\tau_h : X \dashrightarrow \mathbb{P}^M$$

the linear projection from M_A. That is, by Terracini Lemma, the projection from the tangent space of a general point $z \in \langle A \rangle$ of $\mathbb{S}ec_h(X)$ restricted to X.

We start collecting properties of the tangential contact loci that will be useful for our purpose.

Theorem 3.1.1. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety. Let $A = \{x_1, \dots, x_h\} \subset X$ be a set of h general points and Γ the associated contact locus. Assume that $\mathbb{S}ec_{h-1}(X) \subsetneq \mathbb{P}^N$. Then we have:*

a) Γ is equidimensional and it is either irreducible (type I) or reduced (type II) with exactly h irreducible component, each of them containing a single point of A [CC10, Proposition 3.9],

b) $\langle \Gamma \rangle = \mathbb{S}ec_h(\Gamma)$ and $\mathbb{S}ec_i(\Gamma) \neq \langle \Gamma \rangle$ for $i < h$ [CC10, Proposition 3.9],

c) for $z \in \langle \Gamma \rangle$ general $\pi_h^X((\pi_h^X)^{-1}(z)) \subset \langle \Gamma \rangle$, [CC10, Proposition 3.9],

d) if we are in type I $\gamma_h > \gamma_{h-1}$, [BBC+18, Lemma 3.5]

e) if $\gamma_h = \gamma_{h+1}$ and $\mathbb{S}ec_{h+1}(X)$ is not defective and does not fill up $\mathbb{P}^N$ we are in type II, the irreducible components of both contact loci are linearly independent linear spaces, [BBC+18, Lemma 3.5],

f) if we are in type I and $\mathbb{S}ec_{h+1}(X)$ is not defective and does not fill up $\mathbb{P}^N$ then Γ_{h+1} is of type I.

Proof. a) Let $x_1, \dots, x_h$ be general points with $\Gamma(x_1, \dots, x_h)$ the tangential contact locus. Since for general $y_1, \dots, y_h \in \Gamma(x_1, \dots, x_h)$ one has

$$\mathbb{T}_{y_i} X \subset \mathbb{T}_x \mathbb{S}ec_h(X) = \langle \mathbb{T}_{x_1} X, \dots, \mathbb{T}_{x_h} X \rangle$$

it implies that

$$\langle \mathbb{T}_{x_1} X, \dots, \mathbb{T}_{x_h} X \rangle = \langle \mathbb{T}_{y_1} X, \dots, \mathbb{T}_{y_h} X \rangle$$

Now by monodromy on the general points $x_1, \dots, x_h$ we prove point (a).

b) We have

$$\mathbb{S}ec_h(\Gamma(x_1, \dots, x_k) \subset \langle x_1, \dots, x_h \rangle \subset \Gamma_{Sec_h(X)}(z)$$

with $z \in \langle x_1, \dots, x_h \rangle$ a general point. Let $y \in \Gamma_{\mathbb{S}ec_h(X)}(z)$ general, hence $y \in \langle y_1, \dots, y_h \rangle$ for suitable y_i. Since by Terracini's Lemma $\mathbb{T}_z\mathbb{S}ec_h(X) = \mathbb{T}_y\mathbb{S}ec_h(X)$ we have $y_1, \dots, y_h \in \Gamma(x_1, \dots, x_h)$, thus $y \in \mathbb{S}ec_h(\Gamma)$. If $\mathbb{S}ec_i(\Gamma) = \langle x_1, \dots, x_h \rangle$ for some $i \leqslant h$ then we would have $\mathbb{S}ec_i(X) = \mathbb{S}ec_h(X)$ contradicting the hypothesis.

c) Without loss of generality we can assume that Γ is irreducible. Let $z \in \langle x_1, \dots, x_h \rangle$ be general and $y \in \langle y_1, \dots, y_i \rangle$ general with $y_j \in \Gamma$ and $i \leqslant h$. By generality assumptions we have that $y_1, \dots, y_i$ are general points in X and thus y is general in $\mathbb{S}ec_i(X)$. By the choice of y_j we have

$$\mathbb{T}_y\mathbb{S}ec_i(X) = \langle \mathbb{T}_{y_1}X, \dots, \mathbb{T}_{y_i}X \rangle \subset \mathbb{T}_z\mathbb{S}ec_h(X)$$

$[\dim(\pi_h^X((\pi_h^X)^{-1}(z)))]-$dimensional family of $i-$secant to X passing through y. Let $\langle p_1, \dots, p_i \rangle$ be a general element of the family, then

$$\langle \mathbb{T}_{p_1}X, \dots, \mathbb{T}_{p_i}X \rangle = \mathbb{T}_y\mathbb{S}ec_i(X) \subset \mathbb{T}_z\mathbb{S}ec_h(X)$$

which shows that $p_1, \dots, p_i \in \Gamma$.

d) See [BBC+18][Lemma 3.5]

e) See [BBC+18][Lemma 3.5]

f) Let $A = \{x_1, \dots, x_h\}$ and $B = A \cup \{x_{h+1}\}$ be general sets in X. Assume that $\Gamma(B)$ is of type II. By definition $\Gamma(A) \subset \Gamma(B)$, on the other hand by point a) the irreducible component of $\Gamma(B)$ through x_1 does not contain x_2 and therefore it cannot contain $\Gamma(A)$. This contradiction proves the claim.

□

From the point of view of identifiability the notions of weakly defectiveness and twd behave the same. The following proposition is well known to the experts in the field but we were not able to found a written version of it.

Proposition 3.1.2. [CO12, Proposition 2.4] Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety. Assume that X is not h-twd, then X is h-identifiable.

Proof. Assume that X is not h-identifiable and let $z \in \mathbb{S}ec_h(X)$ be a general point. Let $z \in \langle x_1, \dots x_h \rangle$, for x_i general in X. The existence of a different decomposition yields a new set $\{y_1, \dots y_h\} \subset X$ such that $z \in \langle y_1, \dots, y_h \rangle$. Moving the point z in the linear space $\langle x_1, \dots, x_h \rangle$ yields a positive dimensional contact locus. □

Remark 3.1.3. We want to stress that h-identifiability is not equivalent to non h-twd. In [COV17] and [BV18] are described examples of Segre and Grassmannian varieties that are h-identifiable but h-twd (see for instance Example 1.4.8 in Chapter 1).

We aim to study the relation between twd and defectiveness. The next lemma is a first step in this direction.

Lemma 3.1.4. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety of dimension n,

$$\pi_k^X : sec_k(X) \to \mathbb{P}^N$$

the k-secant map, $\tau_{k-1}^X : X \dashrightarrow \mathbb{P}^M$ the $(k-1)$-tangential projection, and $\Gamma := \Gamma(x_1, \dots, x_k)$ the k-contact locus associated to the genral points $x_1, \dots, x_k$.

i) The map π_k^X is of fiber type if and only if τ_{k-1}^X is of fiber type.

ii) Let $\{x_1, \dots, x_k, y_1, y_2\}$ be general points. Then the irreducible component of

$$\Gamma(x_1, \dots, x_k, y_1) \cap \Gamma(x_1, \dots, x_k, y_2)$$

passing through the point x_i is positive dimensional only if either X is k-twd or π_{k+2}^X has positive dimensional fibers.

iii) The map $(\tau_{k-1}^X)_{|\Gamma} : \Gamma \dashrightarrow \mathbb{P}^{\gamma_k}$ is either of fiber type or dominant.

Proof. i) By Terracini Lemma π_k^X is of fiber type if and only if

$$\mathbb{T}_z\mathbb{S}ec_{k-1}(X) \cap \mathbb{T}_y X \neq \emptyset$$

for $y \in X$ general. This condition is clearly equivalent to have τ_{k-1}^X of fiber type.

ii) Assume that X is not k-twd and $\dim(\Gamma(x_1, \dots, x_k, y_1) \cap \Gamma(x_1, \dots, x_k, y_2)) > 0$ in a neighborhood of x_i. Set

$$M_{A_i} = \langle \mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_k}X, \mathbb{T}_{y_i}X \rangle,$$

the variety X is not k-twd therefore

$$M_{A_1} \cap M_{A_2} \supsetneq \langle \mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_k}X \rangle.$$

In particular we have

$$(M_{A_1} \cap M_{A_2}) \cap \mathbb{T}_{y_i}X \neq \emptyset,$$

and hence

$$\langle \mathbb{T}_{x_1}X, \dots, \mathbb{T}_{x_k}X, \mathbb{T}_{y_1}X \rangle \cap \mathbb{T}_{y_2}X \neq \emptyset.$$

This shows, by the generality of the points and point i) that π_{k+2}^X is of fiber type.

iii) Assume that $(\tau_{k-1}^X)_{|\Gamma}$ is not of fiber type. Then by point b) of Theorem 3.1.1 we have $\dim\langle\Gamma\rangle = k(\gamma_k + 1) - 1$. Hence $(\tau_{k-1}^X)_{|\Gamma} = \tau_{k-1}^\Gamma$ and both maps are dominant onto $\mathbb{P}^{\gamma_k}$. See also [CC06]. □

Next we prove a general statement for type II contact loci.

Lemma 3.1.5. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety. Assume that:

a) X is k-twd,

b) X is not $(k-1)$-twd

c) the k-contact locus is of type II.

Then π^X_{k+1} is of fiber type.

Proof. By point i) in Lemma 3.1.4 it is enough to prove that τ^X_k is of fiber type. Then by projection it is enough to prove the latter for $k = 2$. Let $\{x_1, x_2, y\} \subset X$ be a set of general points and $\Gamma = \Gamma(x_1, x_2, y)$ the contact locus associated to $\{x_1, x_2, y\}$. To conclude the proof it is enough to prove that $\langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle \cap \mathbb{T}_x X \neq \emptyset$, for $x \in \Gamma$ a general point.

For a general point $p \in \Gamma$ we set

$$\Gamma^i_p \subset \Gamma(x_i, p)$$

the irreducible component of the contact locus $\Gamma(x_i, p)$ through p. The contact locus is of type II, therefore $\Gamma^i_p \not\ni x_1, x_2$. Note that for a general point $x \in \Gamma^1_p$ we have $\mathbb{T}_x X \subset \langle \mathbb{T}_{x_1} X, \mathbb{T}_p X\rangle$. Then by semicontinuity for any point $w \in \Gamma^1_p$ there is a linear space of dimension n, say $A_w \subseteq \mathbb{T}_w X$, contained in the span.

Set

$$\mathbb{T}(\Gamma^1_p) = \langle \bigcup A_w\rangle_{w\in\Gamma^1_p}.$$

We may assume that X is not 2-defective, otherwise there is nothing to prove, that is

$$\mathbb{T}_{x_1} X \cap \mathbb{T}_{x_2} X = \emptyset, \tag{3.1.5}$$

and, since y is general,

$$\operatorname{codim}_{\mathbb{T}(\Gamma^1_y)}(\mathbb{T}(\Gamma^1_y) \cap \mathbb{T}_{x_1} X) = n + 1. \tag{3.1.5}$$

The variety X is not 1-twd, then there are points $z \in \Gamma^1_y$ with $A_z \cap \mathbb{T}_{x_1} X \neq \emptyset$.

Let $z \in \Gamma^1_y$ be a point with

$$A_z \cap \mathbb{T}_{x_1} X \neq \emptyset, \tag{3.1.5}$$

The contact locus is of type II, therefore $z \neq x_1$. We want to stress that this is the only point in the proof where we use the assumption that Γ is of type II.

If $A_z \cap \mathbb{T}_{x_2} X \neq \emptyset$, by Equations (3.1.5) and (3.1.5) we have

$$\operatorname{codim}_{\mathbb{T}(\Gamma^1_y)}(\mathbb{T}(\Gamma^1_y) \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle) \leqslant n$$

and we conclude $\mathbb{T}_y X \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle \neq \emptyset$, that is τ^X_2 is of fiber type.

Assume that $A_z \cap \mathbb{T}_{x_2} X = \emptyset$. Then we consider the span $\langle A_z, \mathbb{T}_{x_2}\rangle$. By semicontinuity to this linear space is associated a contact locus and we set Γ^2_z its irreducible component passing through z. As before we have

$$\operatorname{codim}_{\mathbb{T}(\Gamma^2_z)}(\mathbb{T}(\Gamma^2_z) \cap \mathbb{T}_{x_2}) = n + 1,$$

and by Equations (3.1.5) and (3.1.5) we conclude that

$$\operatorname{codim}_{\mathbb{T}(\Gamma^2_z)}(\mathbb{T}(\Gamma^2_z) \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2} \rangle) \leqslant n.$$

This yields

$$A_w \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2} \rangle \neq \emptyset, \tag{3.1.5}$$

for any point $w \in \Gamma^2_z$. We have $z \neq x_1$, then the general choice of the points x_i, and the assumption that X is not 2-defective ensure that

$$A_w \cap \mathbb{T}_{x_1} X = \emptyset \tag{3.1.5}$$

for general $w \in \Gamma^2_z$.

We set Γ^1_w the irreducible component through w of the contact locus associated to $\langle A_w, \mathbb{T}_{x_1} X \rangle$. Again $z \neq x_1$ and the general choice of the x_i ensure that $z \notin \Gamma^1_w$. In particular

$$\Gamma^1_w \neq \Gamma^2_z.$$

Set

$$S^2 := \bigcup_{v \in \Gamma^1_y \text{ general}} \Gamma^2_v.$$

Then Γ^2_z is in the closure of S^2 and, for $p \in \Gamma^1_y$ general, Γ^1_y is in the closure of

$$\bigcup_{w \in \Gamma^2_p \text{ general}} \Gamma^1_w.$$

Hence Γ^1_y is in the closure of

$$S^1 := \bigcup_{w \in \Gamma^2_z \text{ general}} \Gamma^1_w.$$

In particular the general point of S^1 is a general point of X. By construction we have

$$\operatorname{codim}_{\mathbb{T}(\Gamma^1_w)}(\mathbb{T}(\Gamma^1_w) \cap \mathbb{T}_{x_1} X) \leqslant n + 1.$$

Equations (3.1.5) and (3.1.5) then give

$$\operatorname{codim}_{\mathbb{T}(\Gamma^1_w)}(\mathbb{T}(\Gamma^1_w) \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2})\rangle \leqslant n$$

and this concludes the proof. □

We are ready to prove our main result that connects twd and defectiveness.

Theorem 3.1.6. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety of dimension n. Assume that:*

a) X is k-twd,

b) X is not $(k-1)$-twd

c) $k > n$ and $N \geqslant (k+1)(n+1) - 1$.

Then π^X_{k+1} is of fiber type.

Proof. Thanks to Lemma 3.1.5 we may assume that the contact locus is of type I. By hypothesis the variety X is k-twd. Let $A = \{x_1, \dots, x_k\} \subset X$ be a set of general points and $\Gamma := \Gamma(A)$ the associated contact locus of dimension $\gamma > 0$. Let $z \in \langle A \rangle$ be a general point, $\tau_k := \tau^X_k : X \dashrightarrow \mathbb{P}^M$ the k-tangential projection associated to A, and $y \in X$ a general point. For a general set $Y := \{y_1, \dots, y_{k-1}\} \subset \Gamma$ let $\Gamma(Y \cup \{y\})$ be the contact locus associated to $\{y_1, \dots, y_{k-1}, y\}$.

Assume that π^X_{k+1} is not of fiber type. then, by i) in Lemma 3.1.4 τ_k is not of fiber type and by point iii) in Lemma 3.1.4, $\tau_k(\Gamma(Y \cup \{y\})$ is a linear space of dimension γ through $z := \tau_k(y)$. This gives a map

$$\chi : Hilb_{k-1}(\Gamma)_{red} \dashrightarrow \mathbb{G}(\gamma - 1, M - 1)$$

$$Y = \{y_1, \dots, y_{k-1}\} \longrightarrow \tau_k(\Gamma(y_1, \dots, y_{k-1}, y))$$

The point z is smooth hence all the linear spaces $\tau_k(\Gamma(Y \cup \{y\})$ sit in $\mathbb{T}_z\tau_k(X) \cong \mathbb{P}^n$. In other words we have a map

$$\chi : Hilb_{k-1}(\Gamma)_{red} \dashrightarrow \mathbb{G}(\gamma - 1, n - 1) \subset \mathbb{G}(\gamma - 1, M - 1).$$

Note that $\dim \mathbb{G}(\gamma - 1, n - 1) = \gamma(n - \gamma)$ and $\dim Hilb_{k-1}(\Gamma) \geqslant (k-1)\gamma$. By hypothesis $k > n$ and $\gamma > 0$ hence we have

$$(k-1)\gamma > \gamma(n - \gamma).$$

Then the map χ is of fiber type and fibers have, at least, dimension $\gamma(k - n + \gamma - 1)$.

Set $[Y_1], [Y_2] \in \chi^{-1}([\Lambda])$ general points, for $[\Lambda] \in \chi(Hilb_{k-1}(\Gamma)_{red}) \subset \mathbb{G}(\gamma - 1, n - 1)$ a general point. The variety X is not $(k-1)$-twd and we are assuming that π^X_{k+1} is not of fiber type therefore, by ii) in Lemma 3.1.4,

$$\dim(\Gamma \cap \Gamma(Y_i \cup \{y\})) = 0,$$

in a neighborhood of y_i. Since the fiber of χ is positive dimensional we have

$$\Gamma(Y_1 \cup \{y\}) \not\supset Y_2. \tag{3.1.6}$$

The contact loci are irreducible then, by Equation (3.1.6), we conclude that

$$\Gamma(Y_1 \cup \{y\}) \neq \Gamma(Y_2 \cup \{y\}).$$

Therefore, by point iii) in Lemma 3.1.4, the positive dimensional fiber of χ induces a positive dimensional fiber of τ_k and we derive, by point i) Lemma 3.1.4, the contradiction that that π^X_{k+1} is of fiber type. □

Remark 3.1.7. Both assumption b) and c) alone are reasonable and not over-demanding. Unfortunately the combination of them is quite restrictive and narrows the range of application we are aiming at.

We believe the statement is not optimal with respect to assumption c). But we are not sure it is true, in full generality, without any assumption of this kind. On the other hand we strongly believe that for many interesting varieties, like Segre, Grassmannian, Veronese and their combinations, twd can occur only one step before the secant map becomes of fiber type. This is not the case for weakly defectiveness as is shown in [BV18]. In [BV18, Theorem 1.1 a)] it is proven that $\mathbb{G}(2,7)$ is 2 and 3 weakly defective without being 3-defective. Note that this variety is 3-twd and it is not 2-twd.

The next result generalizes the main result in [BBC+18] and it allows to avoid the bottleneck introduced by conditions b) and c) of Theorem 3.1.6 in many interesting situations.

Lemma 3.1.8. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety. Assume that X is not 1-twd and π^X_{k+1} is generically finite, in particular X is not $(k+1)$-defective. If X is k-twd then $\gamma_k < \gamma_{k+1}$.

Proof. The variety X is not 1-twd, then we may assume, without loss of generality, that

$$\gamma_{k-1} < \gamma_k = \gamma_{k+1}.$$

Then $\gamma_{k+1} < n$ and $\mathbb{S}ec_{k+1}(X) \subsetneq \mathbb{P}^N$, hence by e) in Theorem 3.1.1, the contact loci are of type II and linearly independent linear spaces. Fix $\{x_1, \ldots, x_k, y\} \subset X$ a set of general points and let

$$\Gamma(x_1, \ldots, x_k, y) = \cup_1^k P_i \cup P_y$$

the contact locus. Moreover the assumption $\gamma_k = \gamma_{k+1}$ and point a) in Theorem 3.1.1 force

$$\Gamma(x_1, \ldots, x_{k-1}, y) = \cup_1^{k-1} P_i \cup P_y,$$

with the same P_i's. Then

$$\bigcap_{y \in X} \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X, \mathbb{T}_y X \rangle \supset \langle \mathbb{T}_z X \rangle_{z \in P_i, i=1,\ldots,k-1}$$

We are assuming that $\gamma_{k-1} < \gamma_k$ therefore

$$P_i \not\subset \Gamma(x_1, \ldots, x_{k-1}),$$

and we have a proper inclusion

$$\langle \mathbb{T}_z X \rangle_{z \in P_i, i=1,\ldots,k-1} \supsetneq \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X \rangle.$$

Set

$$M_{A_i} = \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X, \mathbb{T}_{y_i} X \rangle,$$

for general points $y_1, y_2 \in X$. Then we have

$$M_{A_1} \cap M_{A_2} \supset \langle \mathbb{T}_z X \rangle_{z \in P_i, i=1,\ldots,k-1} \supsetneq \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X \rangle.$$

and we conclude that

$$(M_{A_1} \cap M_{A_2}) \cap \mathbb{T}_{y_i} \neq \varnothing.$$

This shows that

$$\langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_{k-1}}X, \mathbb{T}_{y_1}X \rangle \cap \mathbb{T}_{y_2}X \neq \varnothing.$$

hence the k-tangential projection τ_k^X is of fiber type and by Lemma 3.1.4 we derive the contradiction that π_{k+1}^X is of fiber type. □

Remark 3.1.9. Let us recall that 1-twd varieties are classified in [GH79] and are essentially generalized developable varieties. In particular they are ruled by linear spaces and, with the unique exception of linear spaces, they are singular.

We are ready to apply the above results to get not tangentially weakly defectiveness and hence identifiability statements.

Corollary 3.1.10. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non degenerate variety that is not 1-twd, for instance a smooth variety or a variety that is not covered by linear spaces. Assume that π_k^X is generically finite and $k \geqslant \dim X$.

Then X is not $(k-\dim X)$-twd and it is not $(k-\dim X+1)$-twd if π_k^X is not dominant. If moreover either $k > 2\dim X$ or π_k^X is not dominant and $k \geqslant 2\dim X$ then X is not $(k-1)$-twd.

In all the above cases X is h-identifiable.

Proof. By hypothesis π_h is generically finite for any $h \leqslant k$. Then by Theorem 3.1.8 if it is j-twd

$$\gamma_j < \gamma_{j+1}.$$

The contact locus is a subvariety of X, hence $\gamma_{k-\dim X} = 0$. This proves the first statement.

If π_k^X is not dominant then the contact locus is a proper subvariety and we have $\gamma_{k-\dim X+1} = 0$.

Assume that $k \geqslant 2\dim X$ then by the first part X is not j-twd for some $j > \dim X$. Then we apply Theorem 3.1.6 recursively to conclude. We derive identifiability by Proposition 3.1.2. □

Remark 3.1.11. The first part of Corollary 3.1.10 extends the bounds in [BBC$^+$18] to non 1-twd varieties. The main novelty is the second part that allows to derive identifiability from non defectiveness for large enough secant varieties.

3.2 Application to tensor and structured tensor spaces

As we already mentioned identifiability is particularly interesting for tensor spaces. In this section we use our main result to explicitly state identifiability of a variety of tensor spaces. For this we will consider Segre, Segre-Veronese and Grassmannian varieties and their h-twd properties.

Let us recall some notation

Notation 3.2.1. Let $\boldsymbol{n} = (n_1, \dots, n_r)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$ two multiindices. The variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is the Segre-Veronese embedding

$$\nu_{\boldsymbol{d}}^{\boldsymbol{n}} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{\prod\binom{n_i+d_i}{n_i}-1}$$

via the complete linear system $|\mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^{n_r}}(d_r)|$.

When all d_i's are one we have the Segre embedding and we let $SV^{\boldsymbol{n}} := SV_{(1,\dots,1)}^{\boldsymbol{n}}$ and $SV^{n,r} := SV_{(1,\dots,1)}^{(n,\dots,n)} \cong (\mathbb{P}^n)^r$. The expected generic rank is

$$gr(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) = \lceil \frac{\prod\binom{n_i+d_i}{n_i}}{(\sum n_i) + 1} \rceil$$

Using the notations in [AOP09] we define

$$s(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) := \lfloor \frac{\prod\binom{n_i+d_i}{n_i}}{(\sum n_i) + 1} \rfloor.$$

For simplicity in the case $n_1 = \dots = n_r = n$ and $d_1 = \dots = d_r = 1$ we set

$$s_n^r := s(SV^{n,r})$$

The variety $\mathbb{G}(k, n)$ is the Grassmannian parameterizing $k-$planes in $\mathbb{P}^n$ embedded in $\mathbb{P}(\bigwedge^{k+1} V)$ via the Plücker embedding. The expected generic rank is

$$gr(\mathbb{G}(k, n)) = \lceil \frac{\binom{n+1}{k+1}}{(n-k)(k+1)+1} \rceil$$

Remark 3.2.2. Note that we always have

$$s(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) \geqslant gr(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) - 1$$

and equality occurs only when $\frac{\prod\binom{n_i+d_i}{n_i}}{(\sum n_i)+1}$ is not an integer. In particular for any $h < s(X)$ we have $\mathbb{S}ec_h(X) \subsetneq \mathbb{P}^N$.

The defectiveness of Segre and Segre-Veronese varieties is in general very far from being completely understood, [AOP09] [AB09] [AMR19], but it is in better shape than their identifiability. For the latter the best asymptotic bounds we are aware of is in [BBC$^+$18].

Before applying Theorem 3.1.6 to many examples of tensors spaces, let us recall the table of identifiability of tensors of bounded dimension given in [COV14, Theorem 1.1].

Theorem 3.2.3. *A generic tensor $T \in X = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$ of subgeneric rank $r < gr(X)$ is $r-$identifiable if $\prod_{1 \leqslant i \leqslant d} n_i \leqslant 15000$ except $(n_1, \dots, n_d)$ in the following list:*

$(n_1, \dots, n_d)$	r
$(4, 4, 3)$	5
$(4, 4, 4)$	6
$(6, 6, 3)$	8
$(n, n, 2, 2)$	$2n - 1$
$(2, 2, 2, 2, 2)$	5
unbalanced case	$r \geqslant \prod_{i \geqslant 2} n_i - \sum_{i \geqslant 2}(n_i - 1)$

Theorem 3.2.4. *Let $X = SV^{1,k} \cong (\mathbb{P}^1)^k$. Then X is not h-twd and hence h-identifiable in the following range:*

- $(k, h) = (2, 1), (3, 2), (4, 2), (5, 4), (6, 9)$,
- $k \geqslant 7$ *and* $h < s(X)$

Proof. For $k \leqslant 5$ this is well known, and can be easily checked also via a direct computation with commutative algebra software [Mac92]. For $k = 6$ this has been checked in [BC13] by a computer aided computation. Let us fix $k \geqslant 7$. By [CGG11, Theorem 4.1] X is never defective. In particular the morphism π_h^X is generically finite for $h \leqslant s_1^k$. When $k \geqslant 7$ we have

$$2 \dim X = 2k < \frac{2^k}{k+1} - 1 < s_1^k,$$

then we can apply Corollary 3.1.10. □

Remark 3.2.5. The Theorem answers positively Conjecture 1.2 in [BC13] when the generic rank is an integer, that is $\frac{2^k}{k+1} \in \mathbb{N}$. For $k \leqslant 6$ the one listed are the only identifiable cases.

For 3-factors Segre we plug [CO12] directly in Theorem 3.1.6 to get the following.

Theorem 3.2.6. *Let $X = SV^{n,3}$. Then X is h-identifiable for $h < s(X)$.*

Proof. For $n \leqslant 7$ the statement is proved in [CO12, Theorem 1.2]. For $n > 7$, by [Lic85], the variety X is not h-defective for $h \leqslant s_n^3$ and by the results in [CO12] X is not h-twd for $h = 3n$, confront the table in [CO12, Theorem 1.2] . Then we are in the condition to apply Theorem 3.1.6 recursively to prove that X is not h-twd, and hence identifiable, for $h < s_n^3$. □

For general diagonal Segre we have a similar statement using [AOP09].

Theorem 3.2.7. *Let $X = SV^{n,k}$, with $n \geqslant 2$ and $k \geqslant 4$. Let*

$$n \geqslant \delta(X) \equiv s_n^k \ mod(n+1)$$

Then X is not h-twd and hence $h-$identifiable for $h < s(X) - \delta(X)$. In particular when $\delta(X) = 0$ X is h-identifiable for all $h < s(X)$.

Proof. Using the notations in [CO12, Theorem 6.7] let α be the greatest integer such that $n + 1 \geqslant 2^\alpha$. First we prove the statement for all but finitely many cases.

Claim 1. If

$$(k, n) \notin \left\{ \begin{array}{ll} \text{(k,6) with } k \leqslant 6, & \text{(k,5) with } k \leqslant 5, \\ \text{(k,4) with } k \leqslant 5, & \text{(k,3) with } k \leqslant 4 \end{array} \right\}$$

then X is $h-$identifiable for $h < s(X) - \delta(X)$

Proof. By [AOP09, Theorem 5.2] we know that X is not h-defective as long as $h \leqslant s(X) - \delta(X)$. The variety $SV^{n,k}$ is not $h-$twd for

$$h \leqslant 2^{(k-1)\alpha-(k-1)} = 2^{(k-1)(\alpha-1)}$$

by [CO12, Theorem 6.7]. Let us assume that $n \neq 2$. A short hand computation shows that

$$2^{(k-1)(\alpha-1)} > dim(X) = kn$$

is satisfied for every (k,n) in the list. Then, using recursively Theorem 3.1.6, we conclude. □

For the case $n = 2$ it is easy to check that the inequality

$$s_2^k = \lfloor \frac{3^k}{2k+1} \rfloor - \delta(SV^{2,k}) > 4k = 2dim(SV^{2,k})$$

is satisfied for every $k \geqslant 5$ and so we can conclude using Corollary 3.1.10. When $(k,n) = (6,6),(5,6)$ we have the inequalities

$$s_6^6 - \delta(SV^{6,6}) > 2 \cdot 36 = 2 \dim SV^{6,6}$$

and

$$s_6^5 - \delta(SV^{5,6}) > 2 \cdot 30 = 2 \dim SV^{5,6}$$

Then we conclude by Corollary 3.1.10.

For all the remaining cases we have that $(n+1)^k \leqslant 15000$ and we conclude by Table 3.2.3. □

The next class of Segre varieties we treat in details is given by

$$SV[k,n] := \mathbb{P}^k \times (\mathbb{P}^n)^{k+1}.$$

For these varieties we have

$$gr(SV[k,n]) = \frac{(k+1)(n+1)^{k+1}}{(k+1)n+k+1} = (n+1)^k.$$

In particular $gr(SV[k,n]) = s(SV[k,n])$ is always an integer, that is $SV[k,n]$ is always perfect. Thanks to this special condition we have the following.

Theorem 3.2.8. *Let $X = SV[k,n]$ with n odd and $k > 1$. Then X is $h-$identifiable for $h < gr(X)$.*

Proof. The proof is entirely similar to that of Theorem 3.2.7. Indeed by [AOP09, Theorem 5.11] we know that all these Segre are non defective. If

$$(k,n) \neq (4,1),(3,1),(2,1),(2,3),(2,5)$$

the inequality

$$(n+1)^k > 2(k+kn+n) = 2dim(X)$$

is satisfied and we conclude using Corollary 3.1.10. For all the exceptional cases we have

$$(k+1)(n+1)^{k+1} \leqslant 15000$$

hence we may use Table 3.2.3. □

Remark 3.2.9. Defective Segre are expected to be quite rare, beside the unbalanced ones, see the conjecture in [AOP09]. This conjecture has been checked via a computer in many cases, [VVM15] [COV14]. For all these special values our argument gives identifiability confirming the numerical computation in [COV14].

Next we apply the same strategy to Segre–Veronese varieties. For this class of varieties the defectiveness results are much weaker and so are our bounds. Again the special case of binary forms is in better shape. We start recalling the notation of [LP13].

Definition 3.2.10. We say that $(d_1, ..., d_r; n)$ is special if

$$(d_1, ..., d_r; n) = (2, 2a; 2a+1), (1, 1, 2a; 2a+1), (2, 2, 2; 7), (1, 1, 1, 1; 3)$$

for $a \geqslant 1$. Otherwise $(d_1, ..., d_r; n)$ is called not special.

Theorem 3.2.11. *Let* $X = SV_{\boldsymbol{d}}^{(1,...,1)}$ *with* $r = \dim(X)$. *Assume that* $(d_1, ..., d_r; n)$ *is not special and* $r \geqslant 6$. *Then* X *is* $h-$*identifiable for* $h < s(X)$.

Proof. We are assuming that $(d_1, ..., d_r; n)$ is not special. Then, by [LP13, Theorem 2.1], the variety X is not h-defective for $h \leqslant gr(X)$. Thanks to Theorem 3.2.4 we may assume, without loss of generality that $d_1 > 1$ and we have

$$s(X) = \lfloor \frac{(d_1+1)\cdots(d_r+1)}{r+1} \rfloor \geqslant \frac{3\cdot 2^{r-1}}{r+1} - 1.$$

In particular

$$\frac{3\cdot 2^{r-1}}{r+1} - 1 > 2r = 2\dim(X)$$

holds for every $r \geqslant 6$.

The variety X is not 1-twd and so we conclude by Corollary 3.1.10. □

For general Segre-Veronese we have the following.

Theorem 3.2.12. *Let* $X := SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ *be the Segre–Veronese variety. Assume* $r \geqslant 2$,

$$n_1^{\lfloor log_2(d-1)\rfloor} \geqslant 2(n_1 + \ldots + n_r),$$

and set $d = d_1 + \ldots + d_r$. *Then* X *is* $h-$*identifiable for* $h \leqslant n_1^{\lfloor log_2(d-1)\rfloor} - 1$.

Proof. By [AMR19, Theorem 1.1] X is not h-defective for

$$h \leqslant n_1^{\lfloor log_2(d-1)\rfloor} - (n_1 + ... + n_r) + 1.$$

In our numerical assumptions $\mathbb{S}ec_h(X) \subsetneq \mathbb{P}^N$ and we may assume

$$h \geqslant 2 \dim X.$$

Then we conclude by Corollary 3.1.10. □

Remark 3.2.13. For the Veronese variety of $\mathbb{P}^n$, that is $SV_{d_1}^{n_1}$ it is easy, via Corollary 3.1.10 and [AH95], to reprove the identifiability results in [Mel06] and [COV17].

As in the Segre case, for special classes of Segre–Veronese there are better non defectiveness results. Here we recall the notation in [AB09]. Let $X := SV_{(1,2)}^{(m,n)}$ be the Segre-Veronese variety $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$ in $\mathbb{P}^N$ where

$$N = (m+1)\binom{n+2}{2} - 1$$

Let

$$r(m,n) = \begin{cases} m^3 - 2m & \text{if } m \text{ even and } n \text{ odd} \\ \frac{(m-2)(m+1)^2}{2} & \text{otherwise} \end{cases}$$

and

$$s(X) = \lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \rfloor$$

the meaningful numbers of X. With this in mind we have the following.

Corollary 3.2.14. Let $X = SV_{(1,2)}^{(m,n)}$. If $n > r(m,n)$ and

$$\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \rfloor \geqslant 2(m+n)$$

then X is not $h-$twd and hence $h-$identifiable for $h < s(X)$.

Proof. In our range X is not h-defective by [AB09, Theorem 1.1] and $\mathbb{S}ec_h(X) \subsetneq \mathbb{P}^N$. Moreover

$$s(X) = \lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \rfloor \geqslant 2(m+n) = 2 \dim X$$

and we may apply Corollary 3.1.10 to conclude. □

Let us consider now the case of $\mathbb{P}^m \times \mathbb{P}^n$ embedded with $\mathcal{O}(1,d)$ for $d \geqslant 3$.

Corollary 3.2.15. Let $X = SV_{(1,d)}^{(m,n)}$ with $d \geqslant 3$ and $m,n \geqslant 1$. Let

$$s(X) = max \left\{ s \in \mathbb{N} | \text{s is a multiple of } (m+1) \text{ and } s \leqslant \lfloor \frac{(m+1)\binom{n+d}{d}}{m+n+1} \rfloor \right\}$$

If $s(X) > 2(m+n)$ then X is not h-twd and hence h-identifiable for $h < s(X)$.

Proof. By [BCC11, Theorem 2.3] X is not h-defective for $h \leqslant s(X)$ and $\mathbb{S}ec_h(X) \subsetneq \mathbb{P}^{(m+1)\binom{n+d}{d}-1}$.

X is smooth, in particular it is not 1-twd. Since

$$s(X) > 2(m+n) = 2\dim(X)$$

we can apply Corollary 3.1.10 to conclude. □

Remark 3.2.16. Similar statements about subgeneric identifiability of $\mathbb{P}^n \times \mathbb{P}^1$ embedded with $\mathcal{O}(a,b)$ can be derived applying Corollary 3.1.10 using the non defectiveness results in [BBC12].

Finally we consider Grassmannian varieties. For this class of tensor spaces very few is known about identifiability. To the best of our knowledge the following is the first non computer aided result for them.

Theorem 3.2.17. *Let* $X = \mathbb{G}(k,n)$ *such that* $2k+1 \leqslant n$. *Assume that*

$$\lfloor \binom{n+1}{k+1}^{\lfloor log_2(k) \rfloor} \rfloor \geqslant 2(n-k)(k+1)$$

Then X *is* $h-$*identifiable for*

$$h \leqslant \binom{n+1}{k+1}^{\lfloor log_2(k) \rfloor} - 1$$

Proof. By [MR17, Theorem 5.4] in our numerical range X is not $h-$defective and $\mathbb{S}ec_h(X) \subsetneq \mathbb{P}^N$. Then we conclude by Corollary 3.1.10. □

The technique we developed can be applied to many other classes of varieties, once it is known their defectiveness behavior. As a sample we conclude with the following example.

Example 3.2.18. C. Améndola, J.-C. Faugère, K. Ranestad and B. Sturmfels in [AFS16] and [ARS18] studied the Gaussian moment variety

$$\mathcal{G}_{1,d} \subset \mathbb{P}^d$$

whose points are the vectors of all moments of degree $\leqslant d$ of a $1-$dimensional Gaussian distribution. They proved that $\mathcal{G}_{1,d}$ is a surface for every d and $\mathbb{S}ec_h(\mathcal{G}_{1,d})$ has always the expected dimension. In [BBC$^+$18, Example 5.8] it is shown that $\mathcal{G}_{1,d}$ is not uniruled by lines, in particular it is not 1-twd. As usual let

$$s(\mathcal{G}_{1,d}) = \lfloor \frac{d+1}{3} \rfloor \geqslant gr(\mathcal{G}_{1,d}) - 1$$

Then by Corollary 3.1.10 $\mathcal{G}_{1,d}$ is h-identifiable, for $h < s(\mathcal{G}_{1,d})$ when $d \geqslant 14$.

In Section 5 we will see another application of our technique to particular tensor spaces, called Flag varieties.

Chapter 4

Defectiveness and identifiability: generic rank

Why generic identifiability?

For statistical inference, it is meaningful to know if a probability distribution, arising from a model, uniquely determines the parameters that produced it. When this happens, the parameters are called *identifiable*. There are no useful models where all distributions are identifiable. Then the notion of *generic* identifiability for parametric models has been considered for instance in [AMR$^+$09] and in [RS12]. Conditions which guarantee the uniqueness of decomposition, for generic tensors in the model, are quite important in the applications. When generic identifiability holds, the set of non-identifiable parameters has measure zero, thus parameter inference is still meaningful. Notice that many decomposition algorithms converge to *one* decomposition, hence a uniqueness result guarantees that the decomposition found is the chased one. We refer to [KB09b] and its huge reference list, for more details.

From a purely theoretical point of view, the study of unique decompositions, or canonical forms in the early XX$^{\text{th}}$ century dictionary, has connection with both invariant theory, [Hil88], and projective geometry, [Pal03]. It is already over a decade, [Mel06], that generic identifiability of symmetric tensors has shown its close connection to modern birational projective geometry and especially to the maximal singularities methods. In a series of papers, [Mel06] [Mel09] [GM19], the generic identifiability problem for symmetric tensors has been completely solved.

This chapter is devoted to extend this theory to arbitrary tensors and can be considered as a first step, similar to [Mel06], in this direction. As for the symmetric case it is expected that identifiability is very rare and our result support this convincement.

The main tool in [Mel06] was the use, after [CC02], of non weakly defective varieties to study identifiability.

In recent years the notion of tangential weakly defectiveness, introduced in [CO12], has gradually substituted the weak defectiveness and proved valid to study generic identifiability of subgeneric tensors, [CO12] [BDDG07] [BC13] [BCO14] [Kru77] [CM19]. In particular thanks to the main result in [CM19] (see Chapter 3) for the generic identifiability we may assume without loss of generality the non tangential weakly defectiveness under mild numerical assumptions.

Tangential weak defectiveness does not behave as weak defectiveness with respect to the maximal singularities method. Therefore in this chapter we develop tools to plug in maximal singularities methods for non tangentially weakly defective varieties. In this way we are able to prove the non generic identifiability of many partially symmetric tensors.

4.1 Properties of contact locus for non twd varieties

Let us recall the main notation that we will use throughout this chapter. As always let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate variety with

$$\pi_h^X := \pi_{|sec_h(X)} : sec_h(X) \to \mathbb{P}^N$$

the h-secant map of X.

We call $g :=: g(X)$ the rank of a general point and we say that $X \subset \mathbb{P}^N$ is perfect if

$$\frac{N+1}{\dim X + 1} \in \mathbb{N}$$

Remark 4.1.1. Note that π_g^X is generically finite if and only if X is perfect and not defective. These are therefore necessary condition for identifiability.

Definition 4.1.2. Let $X \subset \mathbb{P}^N$ be a non-degenerate variety and $\{x_1, \dots, x_h\} \subset X$ general points. The variety X is said h-weakly defective if the general hyperplane singular along h general points is singular along a positive dimensional subvariety passing through the points. Let $H \in \mathcal{H}(h) := |\mathcal{I}(1)_{x_1^2,\dots,x_h^2}|$ be a general section, we call $\Gamma_h(H)$ its locus of tangency passing through $x_1, \dots, x_h$.

Definition 4.1.3. For a linear system $\mathcal{H}$ we set

$$\Gamma(\mathcal{H}) := \bigcap_{H \in \mathcal{H}} Sing(H)$$

the common singular locus.

Remark 4.1.4. We want to stress that, by [CC02], if $\Gamma_h(H)$ is zero dimensional in a neighborhood of $\{x_1, \dots, x_h\}$ then $\Gamma_h(H) = \{x_1, \dots, x_h\}$,

For a subset $A = \{x_1, \dots, x_h\} \subset X$ of general points we set

$$M_A := \langle \bigcup_i \mathbb{T}_{x_i} X \rangle.$$

and we denote with $\Gamma_h(A)$ its h-tangential contact locus.

Remark 4.1.5. Note that in general it is difficult to predict the behavior of $\Gamma(\mathcal{H}(h))$ for non h-twd varieties. By definition $\Gamma(\mathcal{H}(h))$ is zero dimensional in a neighborhood of the assigned singular points but not much is known about singular components away from these. Our Proposition 4.1.14 is a first attempt to study this problem, under strong hypothesis.

Remark 4.1.6. By definition τ_h is the rational map associated to the linear system $\mathcal{H}(h) = |\mathcal{I}_{x_1^2 \ldots, x_h^2}(1)|$.

In this section we study properties of the contact loci $\Gamma_{g-1}(H)$ (for a general $H \in \mathcal{H}(g-1)$) of projective varieties that are non defective and not $(g-1)$-twd. In particular in view of applications to Noether–Fano inequalities we are interested in studying the infinitesimally near singularities of $\mathcal{H}$.

We start recalling [CC02, Proposition 3.6] and its generalization to twd. This Proposition will be useful to reduce the study of $\Gamma_{g-1}(\mathcal{H})$ to the special case of $g = 2$.

Proposition 4.1.7. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non degenerate variety. Assume that X is not h-defective and $h(\dim X + 1) - 1 < N$. Let $X_s = \tau_s(X)$ be a general tangential projection.

i) X is $h-$weakly defective if and only if X_s is $(h-s)-$weakly defective.

ii) X is $h-$twd if and only if X_s is $(h-s)-$twd.

Proof. Point i) is [CC02, Proposition 3.6].

Point ii) is a simple adaptation of point i) substituting weakly defectiveness with twd. □

For future reference we observe the following fact.

Lemma 4.1.8. Let $Z \subset \mathbb{P}^n$ be a reduced projective variety of dimension $\dim(Z) = a$. Then $\operatorname{codim} |\mathcal{I}_Z(1)| \geqslant a + 1$ and equality is fulfilled only by linear spaces.

Proof. If Z is a linear space there is nothing to prove. Assume that Z is not a linear space, then $\dim\langle Z\rangle > \dim Z$. We have

$$\operatorname{codim} |\mathcal{I}_Z(1)| = \operatorname{codim} |\mathcal{I}_{\langle Z\rangle}(1)| = \dim\langle Z\rangle + 1 > \dim Z + 1.$$

□

Definition 4.1.9. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate and non h-defective variety. Let $\{x_1, \ldots, x_h\} \subset X$ be a set of general point and $\mathcal{H}(h) = |\mathcal{I}_{x_1^2,\ldots,x_h^2}(1)|$ the linear system of hyperplane sections singular in $\{x_1, \ldots, x_h\}$. Set

$$\mathcal{W}_h := \{(H, x) | H \in \mathcal{H}(h), x \in \Gamma_h(H)\} \in \mathcal{H} \times X$$

and $\pi_1^h : \mathcal{W}_h \to \mathcal{H}(h)$, $\pi_2^h : \mathcal{W}_h \to X$ the two canonical projections. We denote with $W_h := \pi_1^h(\mathcal{W}_h) \subset \mathcal{H}(h)$.

It is clear that $W_s \subset |\mathcal{I}_{x_1^2,\ldots,x_h^2}(1)|$ for any $h < s$. Then we may identify W_s as a subvariety of W_h for any $h \leqslant s$. Our next aim is to prove, in some cases, a more precise result.

Proposition 4.1.10. Assume that X is perfect and not defective with general rank g. Set $\mathcal{H} := \mathcal{H}(g-2) = |\mathcal{I}_{x_1^2,\dots,x_{g-2}^2}(1)|$ and assume

$$\dim(\Gamma_{g-1}(H)) = a,$$

for $H \in \mathcal{H}(g-1)$. Then we have $\operatorname{codim}_{\mathcal{H}(g-2)}(W_{g-1}) = a+1$.

Proof. The variety X is not defective, then $\dim(\mathcal{H}(g-2)) = 2n+1$. By a parameter count we have $\dim \mathcal{W}_{g-1} = 2n$.

By definition for a general $[H] \in W_{g-1}$ we have

$$\dim(\pi_1^{-1}(H)) = \dim\{x \in X | x \in \Gamma_{g-1}(H)\}) = \dim \Gamma_{g-1}(H)$$

therefore we conclude that

$$\dim(W_{g-1}) = \dim(\mathcal{W}_{g-1}) - \dim(\pi_1^{-1}(H)) = 2n - a$$

yielding $\operatorname{codim}_{\mathcal{H}(g-2)}(W_{g-1}) = a+1$. □

The following result is already implicitly used in [CC02] but we state it as a Proposition for the reader convenience.

Proposition 4.1.11. Let $X \subset \mathbb{P}^{2\dim X+1}$ be an irreducible, reduced non-degenerate variety. Assume that X is not defective and not 1−twd. Then for a general tangent hyperplane $H \in \mathcal{H}(1)$, the singular locus $\Gamma_1(H)$ is a linear space. In particular, under the hypothesis, also $\Gamma(\mathcal{H}(1))$ is a linear space.

Proof. If X is not 1-weakly defective, by Remark 4.1.4, $\Gamma_1(H)$ is a point. Assume that X is 1-weakly defective and $\dim \Gamma_1(H) = a$. Let $x \in X$ be a general point and $H \in \mathcal{H}(1)$ a general tangent section in x. Let us consider the variety

$$W_1 \subset |\mathcal{O}(1)| =: \mathcal{H}$$

parametrizing singular hyperplane sections. Proposition 4.1.10 yields $\operatorname{codim}_{\mathcal{H}}(W_1) = a+1$ and so $\operatorname{codim}(\mathbb{T}_{[H]}W_1) = a+1$. On the other hand, by the infinitesimal Bertini's theorem [CC02, Thm 2.2], we have

$$\mathbb{T}_{[H]}W_1 \subset \mathcal{H}(-Sing(H))$$

and so $\operatorname{codim}_{\mathcal{H}}(\mathcal{H}(-\Gamma_1(H))) \leqslant a+1$.

Hence we conclude by Proposition4.1.8 that $\Gamma_1(H)$ is a linear space. □

Lemma 4.1.12. Let $X \subset \mathbb{P}^N$ be an irreducible, reduced non-degenerate projective variety. Assume that X is 1−weakly defective with $\dim(\Gamma_1(H)) = a$, for $H \in \mathcal{H}(1)$ a general tangent hyperplane. Then a general hyperplane section X' of X satisfies $\dim(\Gamma_1(H')) = a-1$, for H' a general tangent hyperplane to X'.

Proof. Let $x \in X$ be a general point, $H \in |\mathcal{I}_{x^2}(1)|$ a general hyperplane section singular at x and $L \in |\mathcal{I}_x(1)|$ a general hyperplane section passing through x. The divisor L is smooth in a neighborhood of x and $\mathrm{Bs}\,|\mathcal{I}_x(1)| = \{x\}$. Hence, by Bertini's theorem,

$$\dim(Sing(H) \cap L)) = \dim \Gamma_1(H) - 1 = a - 1$$

To conclude observe that $H_{|L}$ is a general tangent section of L at x. □

Let $(z_1, \ldots, z_n)$ be a system of local coordinates at the point $(x \in X) \cong ((0, \ldots, 0) \in \mathbb{C}^n)$. Every divisor $H \in |\mathcal{I}_{x^2}(1)|$ can be expressed locally as

$$H = (Q_H(z_1, \ldots, z_n) + \sum_{d \geqslant 3} F_d(z_1, \ldots, z_n) = 0)$$

where $Q_H(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]_2$ is a Quadric and F_d are homogeneous polynomials of degree at least 3. The rank of the double point $x \in H$ is by definition the rank of the Quadric Q_H. The singular locus $\mathcal{A} = Sing(Q_H)$ is a linear space $\mathcal{A} \subset \mathbb{C}^n$ of dimension

$$dim(\mathcal{A}) = dim(X) - rank(Q_H)$$

It is called the asymptotic space of H at the point x. Let $\nu : X' \to X$ be the blow up of X at x with exceptional divisor E. Under the identification

$$E = \mathbb{P}((T_x X)^*) = \mathbb{P}^{n-1}$$

we have that

$$\nu_*^{-1}(H) \cap E = \mathbb{P}(Q_H)$$

and

$$Sing(\nu_*^{-1}(H)) \cap E \subseteq \mathbb{P}(\mathcal{A})$$

Note further that to every point $y \in E$ we can associate uniquely a line $l_y \in T_x X$ corresponding to the tangent direction represented by y.

With this notation in mind we are going to improve Proposition 4.1.11.

Proposition 4.1.13. Let $X \subset \mathbb{P}^{2\dim X + 1}$ be an irreducible, reduced non-degenerate projective variety. If X is not defective $\mathbb{P}(Sing(Q_H)) = \nu_*^{-1}(\Gamma_1(H)) \cap E$.

Proof. Let $H \in \mathcal{H}(1)$ be a generic hyperplane section singular at x. If $\dim(\Gamma_1(H)) = 0$, by [CC02, Theorem 1.4], x is an ordinary double point of H. Thus Q_H is a Quadric of maximal rank.

Assume $\dim(\Gamma_1(H)) = a > 0$. By Proposition 4.1.11 it is enough to prove that $rank(Q_H) = \dim X - a + 1$. Let $\nu : X' \to X$ be the blow up of X at the general point $x \in X$, with exceptional divisor E, and $H' = \nu_*^{-1}(H)$ the strict transform of H. We have

$$\nu_*^{-1}(\Gamma_1(H)) \cap E \subseteq Sing(H')$$

We already observed that $Sing(H') \cap E \subseteq \mathbb{P}(Sing(Q_H))$ hence

$$\nu_*^{-1}(\Gamma_1(H)) \cap E \subseteq \mathbb{P}(Sing(Q_H)).$$

This leads to

$$rank(Q_H) \leqslant \dim(X) - a + 1.$$

Let $H_1, \ldots, H_a \in \mathcal{H}(1)$ be general sections. Then Lemma 4.1.12 yields that $X^a := H_1 \cap \ldots \cap H_a$ is not 1−weakly defective. Hence, by the first part of the proof, we conclude

$$rank(Q_H) \geqslant \dim(X) - a + 1$$

and finish the proof. □

We take the opportunity to stress a property of $\Gamma(\mathcal{H}(g-1))$ for non twd varieties, recall Remark 4.1.5.

Proposition 4.1.14. Let $X \subset \mathbb{P}^N$ be a non defective, perfect, irreducible, reduced and non-degenerate variety with general rank g. Assume that X is not $(g-1)$-twd. Then $\langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_{g-1}}X \rangle$ is tangent only along a zero dimensional scheme.

Proof. Let

$$W \subset \langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_{g-1}}X \rangle \cap X$$

be an irreducible component of the locus where $\langle \mathbb{T}_{x_1}X, \ldots, \mathbb{T}_{x_{g-1}}X \rangle$ is tangent to X. By Proposition 4.1.7 we have that $X_{g-2} := \tau_{g-2}(X)$ is not 1-twd and not defective, where τ_{g-2} is the linear projection from $\langle \mathbb{T}_{x_2}X, \ldots, \mathbb{T}_{x_{g-1}}X \rangle$.

Claim 2. $\tau_{g-2}(W) = \tau_{g-2}(x_1)$

Proof. Let $y = \tau_{g-2}(x_1)$ and $H \in |\mathcal{I}_{y^2}(1)|$ a general tangent hyperplane section. By Proposition 4.1.7 X_{g-2} is not 1-twd and by Proposition 4.1.11 $\Gamma_1(H)$ is a linear space, therefore

$$\Gamma_1(H) \cap \mathbb{T}_y X_{g-2} = y.$$

On the other hand, by construction, we have

$$\tau_{g-2}(W) \subset \Gamma_1(H),$$

and this proves the claim. □

The variety X is not defective and $y = \tau_{g-2}(x_1)$ is a general point of X_{g-2}. Therefore $\tau_{g-2}^{-1}(y)$ is a finite scheme and we conclude by the Claim that W is 0-dimensional. □

Remark. *It would be very interesting to understand if the result in Proposition 4.1.14 is true for smaller values of the rank. Unfortunately our proof is based on Proposition 4.1.11 and cannot be extended in this direction.*

The following is the main result of this section.

Theorem 4.1.15. *Let $X \subseteq \mathbb{P}^N$ be a projective irreducible,reduced and non-degenerate variety of general rank g. Let $\{x_1, ..., x_{g-1}\}$ be general points on X and $\mathcal{H} = \mathcal{H}(g-1)$. Assume that:*

- *X is perfect and non defective*
- *X is not $(g-1)$-twd*

Then there is a variety Y and a birational map $\nu : Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor D, with $D \equiv \nu_*^{-1}\mathcal{H}$ such that for any point $y \in Y$

$$\operatorname{mult}_y D < 1 + \epsilon.$$

Proof. The variety X is non defective and not $(g-1)$-twd Therefore, by [CM19, Theorem 18] the tangential projection τ_{g-2}, from $\{x_2, \ldots, x_{g-1}\}$ is birational. Let $X_{g-2} = \tau_{g-2}(X)$ be the image of the tangential projection and define $\mathcal{H}' := (\tau_{g-2})_*\mathcal{H}$. Then, by Proposition 4.1.7 $X_{g-2} \in \mathbb{P}^{2\dim X+1}$ is not defective and not 1-twd.

Let $\sigma : Z \to X_{g-2}$ be the blow up of the point $\tau_{g-2}(x_1)$, with exceptional divisor E and $\mathcal{H}_Z = \sigma_*^{-1}\mathcal{H}'$.

Claim 3. $\Gamma(\mathcal{H}_Z)$ is empty.

Proof of the Claim. By Proposition 4.1.11 the singular locus $\Gamma_1(H)$ is a linear space. The variety X_{g-2} is not 1-twd therefore $\Gamma(\mathcal{H}') = \tau_{g-2}(x_1)$. This is enough to show that $\Gamma(\mathcal{H}_Z) \subset E$.

Assume that there is a point $z \in \Gamma(\mathcal{H}_Z) \cap E$ and denote by $l_z \subset \mathbb{P}^{2\dim X+1}$ the corresponding line in the projective space. By Proposition 4.1.13 this forces $l_z \subset \Gamma_1(H)$ and the contradiction $l_z \in \Gamma(\mathcal{H}')$. □

Let Y be the completion of the Cartesian square

$$\begin{array}{ccc} Y & \overset{\eta}{\dashrightarrow} & Z \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X & \underset{\tau_{g-2}}{\dashrightarrow} & X_{g-2} \end{array}$$

and $\mathcal{H}_Y = \nu_*^{-1}(\mathcal{H}) = \eta_*^{-1}(\mathcal{H}_Z)$ the strict transform linear system. By construction and Claim 3 the set $\Gamma(\mathcal{H}_Y)$ is contained in the locus where η is not an isomorphism. On the other hand, by monodromy, the same should be true for a different choice of $(g-2)$ points in $\{x_1, \ldots, x_{g-1}\}$. Thanks to the general choice of the points $\{x_1, \ldots, x_{g-1}\}$ this is enough to show that $\Gamma(\mathcal{H}_Y)$ is empty. In particular for any $y \in Y$ there are divisors $H \in \mathcal{H}_Y$ with $\operatorname{mult}_y H \leqslant 1$. To conclude we construct the divisor

$$D = \frac{1}{M}\sum_{1}^{M} H_i,$$

for $H_i \in \mathcal{H}_Y$ general. The locus $\Gamma(\mathcal{H}_Y)$ is empty therefore we may assume that for any $y \in Y$ there are at most $\dim \mathcal{H}_Y$ divisors in $\{H_i\}_{i=1,\ldots,M}$ singular in y. This is enough to conclude. □

4.2 Noether–Fano inequalities and generic identifiability

In this section we apply the previous results on the singular locus of linear system $\mathcal{H}(g-1)$ to produce non generic identifiability statements. We start recalling two results in this area.

Theorem 4.2.1 ([Mel06]). *Let $X \subseteq \mathbb{P}^N$ be a projective, irreducible non-degenerate variety. Suppose that X is generically identifiable. Then the $(g(X)-1)$-tangential projection $\tau_{g(X)-1} : X \dashrightarrow \mathbb{P}^{\dim(X)}$ is birational.*

Theorem 4.2.2 (Noether-Fano Inequalities [Cor95]). *Let $\pi : X \to X'$ and $\rho : Y \to Y'$ be two Mori fiber spaces and $\varphi : X \dashrightarrow Y$ a birational, not biregular, map*

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X' & & Y' \end{array}$$

Choose a very ample linear system $\mathcal{H}_Y$ in Y and let $\mathcal{H}_X = \varphi_*^{-1}(\mathcal{H}_Y)$. Let $a \in \mathbb{Q}$ such that $\mathcal{H}_X \equiv -aK_X + \pi^*(A)$ for some divisor $A \in Pic(S)$.

Then either $(X, \frac{1}{a}\mathcal{H}_X)$ has not canonical singularities or $K_X + \frac{1}{a}\mathcal{H}_X$ is not NEF.

We are ready to connect the contact loci properties and the Noether–Fano inequalities to produce a tool for non identifiability statements.

Theorem 4.2.3. *Let $X^n \subset \mathbb{P}^N$ be a projective smooth non-degenerate variety and $\tau_{g-1} : X \dashrightarrow \mathbb{P}^{\dim X}$ be a general tangential projection, associated to the linear system $\mathcal{H} := \mathcal{H}(g-1)$. Assume that*

- *$\pi : X \to S$ is a structure of a Mori fiber space such that*

$$\mathcal{H} \equiv_\pi -aK_X + \pi^*(A)$$

  with $a > 1$ a rational number and $A \in Pic(S)$

- *The $\mathbb{Q}$-divisor $K_X + \frac{1}{a}\mathcal{H}$ is NEF*

- *X is not $(g-1)-twd$*

Then τ_{g-1} is not birational, in particular X is not generically identifiable.

Proof. If $\pi_g^X : sec_g(X) \to \mathbb{P}^N$ is of fiber type then τ_{g-1} is of fiber type, see for instance [CM19, Lemma16 (i)], and we conclude, by Theorem 4.2.1, that X is not identifiable.

Then we may assume that X is perfect and not defective. In particular τ_{g-1} is a not biregular map onto $\mathbb{P}^{\dim X}$.

By Theorem 4.1.15 there is a variety Y and a birational map $\nu : Y \to X$ with the following property: for any $\epsilon > 0$ there is a $\mathbb{Q}$-divisor D, with $D \equiv \nu_*^{-1}\mathcal{H}(g-1)$ such that for any point $y \in Y$

$$\operatorname{mult}_y D < 1 + \epsilon.$$

In particular $(Y, \frac{1}{a}\nu_*^{-1}(\mathcal{H}(g-1)))$ and henceforth $(X, \frac{1}{a}\mathcal{H}(g-1))$ have canonical singularities. Then, by Theorem 4.2.2 applied to the diagram

$$\begin{array}{ccc} X & \overset{\tau_{g-1}}{\dashrightarrow} & \mathbb{P}^{\dim X} \\ \downarrow{\scriptstyle\pi} & & \downarrow \\ S & & Spec\mathbb{C} \end{array},$$

the map τ_{g-1} cannot be birational and therefore X is not generically identifiable by Theorem 4.2.1. □

We are ready to prove the non identifiability statement announced in the introduction.

Let $\mathbf{n} = (n_1, \dots, n_r)$ and $\mathbf{d} = (d_1, \dots, d_r)$ be two $r-$uples of positive integers and consider the Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$, i.e. the embedding

$$\nu_{\boldsymbol{d}}^{\boldsymbol{n}} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{\prod \binom{n_i+d_i}{n_i} - 1}$$

via the complete linear system $|\pi_1^*\mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes \dots \otimes \pi_r^*\mathcal{O}_{\mathbb{P}^{n_r}}(d_r)|$.

Theorem 4.2.4. *Fix two multiindexes $\boldsymbol{n} = (n_1, \dots, n_r)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$. Let $X = SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ the corresponding Segre-Veronese variety. Assume that $d_i > n_i + 1$, for $i = 1, \dots, r$, and*

$$\lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i + 1} \rceil > 2(\sum n_i).$$

Then X is not generically identifiable.

Proof. If X is defective or non perfect the statement is clear, recall Remark 4.1.1. Assume that X is not defective and perfect. Then $\tau_{g-1} : X \dashrightarrow \mathbb{P}^{\dim X}$ is generically finite. The numerical assumption reads

$$g(X) = \lceil \frac{\prod \binom{n_i+d_i}{n_i}}{\sum n_i + 1} \rceil > 2(\sum n_i) = 2\dim(X)$$

and, by [CM19, Corollary 22], the variety X is not $(g-1)-$twd.

After reordering the indexes we may assume that

$$\frac{n_1+1}{d_1} \geqslant \frac{n_i+1}{d_i}, \text{ for any } i. \tag{4.2.4}$$

Let $p : X \to Y$ be the canonical projection onto the Segre-Veronese $Y = SV_{(d_2,\dots,d_r)}^{(n_2,\dots,n_r)}$ and $a = \frac{d_1}{n_1+1} > 1$. Then p is a Mori fiber Space and

$$K_X + \frac{1}{a}\mathcal{H}(g-1) \equiv_p 0.$$

Further note that the cone of effective divisor of X is spanned by the lines in the factors $\mathbb{P}^{n_i}$ and, by Equation (4.2.4), we have

$$K_X + \frac{1}{a}\mathcal{H}(g-1)\cdot l_i = -(n_i+1) + \frac{n_1+1}{d_1}d_i \geqslant 0,$$

This shows that $K_X + \frac{1}{a}\mathcal{H}(g-1)$ is NEF and, by Theorem 4.2.3, we prove that X is not generically identifiable. □

Remark. *In recent years the Secant varieties of Segre-Veronese varieties have been studied intensively, see for instance [AB09],[AMR19],[BBC12],[BCC11]. However, to the best of our knowledge, this is the first result regarding non generic identifiability for infinite classes of Segre-Veronese varieties with $r \geqslant 2$.*

4.3 The example of $\mathbb{P}^1 \times \mathbb{P}^N$ embedded with $\mathcal{O}(d,1)$

In this section we study a special class of Segre-Veronese varieties. For this class we are able to explicitly compute the geometry of the tangential contact locus and confirm the validity of Theorem 4.1.15 3. We start recalling the notion of distance for Segre-Veronese varieties given in [AMR19, 2.4].

Let n and d be positive integers, and set

$$\Lambda_{n,d} = \{I = (i_1,\dots,i_d), 0 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant n\}.$$

For $I, J \in \Lambda_{n,d}$, we define their distance $d(I,J)$ as the number of different coordinates. More precisely, write $I = (i_1,\dots,i_d)$ and $J = (j_1,\dots,j_d)$. There are $r \geqslant 0$, distinct indexes $\lambda_1,\dots,\lambda_r \subset \{1,\dots,d\}$, and distinct indexes $\tau_1,\dots,\tau_r \subset \{1,\dots,d\}$ such that $i_{\lambda_k} = j_{\tau_k}$ for every $1 \leqslant k \leqslant r$, and

$$\{i_\lambda \mid \lambda \neq \lambda_1,\dots,\lambda_r\} \cap \{j_\tau \mid \tau \neq \tau_1,\dots,\tau_r\} = \varnothing$$

Then $d(I,J) = d - r$. Note that $\Lambda_{n,d}$ has diameter d and size $\binom{n+d}{n} = N(n,d)+1$.

Let $\boldsymbol{n} = (n_1,\dots,n_r)$ and $\boldsymbol{d} = (d_1,\dots,d_r)$ be two r-uples of positive integers, and set

$$\Lambda = \Lambda_{\boldsymbol{n},\boldsymbol{d}} = \Lambda_{n_1,d_1} \times \cdots \times \Lambda_{n_r,d_r}$$

For $I = (I^1,\dots,I^r), J = (J^1,\dots,J^r) \in \Lambda$, we define their distance as

$$d(I,J) = d(I^1,J^1) + \cdots + d(I^r,J^r)$$

Note that Λ has diameter d and size $\prod_{i=1}^r \binom{n_i+d_i}{n_i} = N(\boldsymbol{n},\boldsymbol{d})+1$. We will denote the homogeneous coordinates (respectively coordinates points) of $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ by X_J (respectively e_J), $J \in \Lambda$.

Proposition 4.3.1. Let $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{(d+1)(n+1)-1}$ be the Segre Veronese variety embedded with $\mathcal{O}_X(1,d)$. Let $x_1,\dots,x_{k(n+1)-1}$ be general points on X and let

$$X' = Bl_{x_2,\dots,x_{k(n+1)-1}}X \xrightarrow{\sigma} X$$

Assume that:

- $(d+1) = k(n+2)$
- X is not $[k(n+1)-1]$-twd

Then
$$\Gamma_1(\mathcal{H}')_{|E_i} = \varnothing$$
where $\mathcal{H}' = \sigma_*^{-1}(\mathcal{H}(k(n+1)-1))$.

Proof. We know that X is $[k(n+1)-1]$-weakly defective and its contact locus at a general point x is a hyperplane contained in a fiber of the canonical projection $\pi_1 : X \to \mathbb{P}^1$ (see [FCM20] and Chapter 6). Moreover X is not $k(n+1)$ defective, i.e. X is a perfect variety (see [MR17]). Up to a linear change of coordinates under the action of $PGL(2) \times PGL(n+1)$ we can assume that $x_1 = e_{\{0,\dots,0\},\{0\}}$ and $x_2 = e_{\{1,\dots,1\},\{1\}}$.

We have that
$$\mathcal{H}(k(n+1)-1) = |\mathcal{O}_X(d,1) \otimes \mathcal{I}_{x_1^2 \cup \dots \cup x_{k(n+1)-1}^2}|$$
is a sublinear system of
$$\mathcal{H}(2) = |\mathcal{O}_X(d,1) \otimes \mathcal{I}_{x_1^2 \cup x_2^2}|$$
with $dim(\mathcal{H}(k(n+1)-1) = dim(X)$.

We have that a divisor
$$H = (\sum \alpha_I Z_I = 0)$$
is in $\mathcal{H}(2)$ if and only if both
$$d((I^1, I^2), (\{0,\dots,0\},\{0\})) \geqslant 2$$
and
$$d((I^1, I^2), (\{1,\dots,1\},\{1\})) \geqslant 2$$
are satisfied.

Let
$$I_l^1 = \underbrace{\{1,\dots,1,0,\dots,0\}}_{\#1=l,\#0=d-l}, \{1\}$$
$$I_l^0 = \underbrace{\{1,\dots,1,0,\dots,0\}}_{\#1=l,\#0=d-l}, \{0\}$$

If we denote
$$\Lambda = \{I_1^1, \dots, I_{d-2}^1, I_2^0, \dots, I_d^0\}$$
we have that the set of hyperplane divisor defined by $H_I = (Z_I = 0)_{I \in \Lambda}$ is a basis of the linear system $\mathcal{H}(2)$.

By monodromy we can restrict ourselves to an affine neighborhood U_{x_1} of the point x_1. A basis of $\mathcal{H}(k(n+1)-1)$ is thus given by choosing $(n+1)-$tuples of numbers $(\alpha_i^I)_{i=1,\dots,n+1}$ for every $I \in \Lambda$.

In particular we have that

$$\mathcal{H}(k(n+1)-1) =< \sum_{I\in\Lambda} \alpha_1^I Z_I, \ldots, \sum_{I\in\Lambda} \alpha_{n+1}^I Z_I >$$

For simplicity let us denote $D_i = \sum_{I\in\Lambda} \alpha_i^I Z_I$ for $i = 1, \ldots, n+1$.

Locally in U_{x_1} we set $x_0 = y_0 = 1$ and so every D_i is of the form

$$D_i = (x_1(x_1 + F_i(y_1, \ldots, y_n) + G_i(x_1, y_1, \ldots, y_n)) = 0)$$

with $F_i(y_1, \ldots, y_n)$ linear forms and $G_i(x_1, y_1, \ldots, y_n)$ polynomials such that $deg(G_i) \geqslant 2$ and $deg_{y_j}(G_i) = 1$ for every $j = 1, \ldots, n$.

Let us denote

$$D_i' = (x_1 + F_i(y_1, \ldots, y_n) + G_i(x_1, y_1, \ldots, y_n) = 0)$$

Since every D_i is a reducible divisor we have that

$$Sing(D_i) = (x_1 = 0) \cap (D_i' = 0)$$

and so

$$Sing(D_j)_{|(x_1=0)} = (F_i(y_1, \ldots, y_n) = 0)$$

Since by assumptions we have that X is not $[k(n+1)-1]$-twd we have that

$$\bigcap_{1\leqslant i\leqslant n+1} (F_i(y_1, \ldots, y_n) = 0) = (0, \ldots, 0) \tag{4.3.1}$$

Let

$$\sigma : X' = Bl_{x_1, \ldots x_{k(n+1)-1}} X \to X$$

be the blow up with $E_1, \ldots, E_{k(n+1)-1}$ the exceptional divisors.

We have that

$$Sing(\sigma_*^{-1}(D_i))_{|E_1} = Sing(TC_{x_1} D_i)$$

with $TC_{x_1} D_i$ the tangent cone of D_i at the point x_i.

Now $TC_{x_1} D_i$ is the quadric Q_i given by $Q_i = x_1^2 + x_1 F_i(y_1, \ldots, y_n)$.

If $F_i = \gamma_i^1 y_1 + \ldots + \gamma_i^n y_n$ then Q_i can be represented by the symmetric matrix

$$M(Q_i) = \begin{pmatrix} 1 & \gamma_i^1 & \ldots & \gamma_i^n \\ \gamma_i^1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_i^n & 0 & \ldots & 0 \end{pmatrix}$$

Now $Sing(Q_i) = Ker(M(Q_i)) = \{x_1 = 0, F_i(y_1, \ldots, y_n) = 0\}$.

It follows from Equation (4.3.1) that $\bigcap_{1\leqslant i\leqslant n+1} Sing(\sigma_*^{-1}(D_i))_{|E_i} = \varnothing$, proving the result.

□

Chapter 5

Defectiveness and identifiability of Flag Varieties

In the most general context, a flag variety is a projective variety homogeneous under a complex linear algebraic group. Flag varieties play a central role in algebraic geometry, combinatorics, and representation theory.

Fix a vector space $V \cong \mathbb{C}^{n+1}$, over an algebraically closed field K of characteristic zero, and integers $k_1 \leqslant \ldots \leqslant k_r$. Let $\mathbb{G}(k_i, n) \subset \mathbb{P}^{N_i}$, where $N_i = \binom{n+1}{k_i+1} - 1$, be the Grassmannian of k_i-dimensional linear subspaces of $\mathbb{P}(V)$ in its Plücker embedding. We have an embedding of the product of these Grassmannians

$$\mathbb{G}(k_1, n) \times \cdots \times \mathbb{G}(k_r, n) \subset \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r} \subset \mathbb{P}^N$$

where $N = \binom{n+1}{k_1+1} \cdots \binom{n+1}{k_r+1} - 1$.

The flag variety $\mathbb{F}(k_1, \ldots, k_r; n)$ is the set of flags, that is nested subspaces, $V_{k_1} \subset \cdots \subset V_{k_r} \subsetneq V$. This is a subvariety of the product of Grassmannians $\prod_{i=1}^{r} \mathbb{G}(k_i, n)$. Hence, via a product of Plücker embeddings followed by a Segre embedding we can embed $\mathbb{F}(k_1, \ldots, k_r; n)$

$$\mathbb{F}(k_1, \ldots, k_r; n) \hookrightarrow \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r} \hookrightarrow \mathbb{P}^N.$$

Consider natural numbers $a_1, \ldots, a_n$ such that $a_{k_1+1} = \cdots = a_{k_r+1} = 1$ and $a_i = 0$ for all $i \notin \{k_1 + 1, \ldots, k_r + 1\}$. Then, $\mathbb{F}(k_1, \ldots, k_r; n)$ generates the subspace

$$\mathbb{P}(\Gamma_{a_1,\ldots,a_n}) \subseteq \mathbb{P}\left(\bigwedge^{k_1+1} V \otimes \cdots \otimes \bigwedge^{k_r+1} V\right) \subseteq \mathbb{P}^N$$

where $\Gamma_{a_1,\ldots,a_n}$ is the irreducible representation of $\mathfrak{sl}_{n+1}\mathbb{C}$ with highest weight $(a_1 + \cdots + a_n)L_1 + \cdots + a_n L_n$, and $L_1 + \cdots + L_k$ is the highest weight of the irreducible representation $\bigwedge^k V$. We will denote $\Gamma_{a_1,\ldots,a_n}$ simply by Γ_a. By the Weyl character formula we have that

$$\dim \mathbb{P}(\Gamma_a) = \prod_{1 \leqslant i < j \leqslant n+1} \frac{(a_i + \cdots + a_{j-1}) + j - i}{j - i} - 1.$$

Furthermore

$$\dim \mathbb{F}(k_1, \dots, k_r; n) = (k_1 + 1)(n - k_1) + \sum_{j=2}^{r} (n - k_j)(k_j - k_{j-1})$$

and $\mathbb{F}(k_1, \dots, k_r; n) = \mathbb{P}(\Gamma_a) \cap \prod_{i=1}^{r} \mathbb{G}(k_i, n) \subset \mathbb{P}^N$, see [Ful97].

The geometry of these varieties has been investigated mostly from the point of view of Schubert calculus [Bri05] and dual defectiveness [Tev05]. Secant varieties of low dimensional flag varieties have been studied in [BD10] by taking advantage of the tropical approach to secant dimensions introduced by J. Draisma in [Dra08].

We investigate secant defectiveness of flag varieties following the machinery introduced in [MR17].

5.1 Osculating Spaces for products of Grassmannians

Consider the product $\mathbb{G}(k_1, n) \times \cdots \times \mathbb{G}(k_r, n) \subset \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_r} \subset \mathbb{P}^N$. Given a non-negative integer k define

$$\Lambda_k = \{I \subset \{0, \dots, n\} \mid |I| = k + 1\}.$$

For any $I = \{i_0, \dots, i_k\} \in \Lambda_k$ let $e_I \in \mathbb{G}(k, n)$ be the point corresponding to $e_{i_0} \wedge \cdots \wedge e_{i_k} \in \bigwedge^{k+1} \mathbb{C}^{n+1}$. We will denote by Z_I the Plücker coordinates on $\mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$.

From [MR17] we have a notion of distance in Λ_k given by

$$d(I, J) = |I| - |I \cap J| \tag{5.1.1}$$

for all $I, J \in \Lambda_k$. More generally, we define

$$\Lambda = \Lambda_{k_1} \times \cdots \times \Lambda_{k_r}.$$

Given $I = \{I^1, \dots, I^r\} \in \Lambda$ let $e_I \in \prod_{i=1}^{r} \mathbb{G}(k_i, n)$ be the point corresponding to $e_{I^1} \otimes \cdots \otimes e_{I^r} \in \mathbb{P}^N$, and by Z_I the corresponding homogeneous coordinates of $\mathbb{P}^N$. Furthermore, for all $I, J \in \Lambda$ with $I = \{I^1, \dots, I^r\}$ and $J = \{J^1, \dots, J^r\}$, we define their distance as

$$d(I, J) = \sum_{i=1}^{r} d(I^i, J^i)$$

where $d(I^i, J^i)$ is the distance defined in (5.1.1).

From now on we will assume that $n \geqslant 2k_r + 1$. Under this assumption Λ has diameter $r + \sum_{i=1}^{r} k_i$ with respect to this distance.

In the following, we give an explicit description of the osculating spaces of $\prod_{i=1}^{r} \mathbb{G}(k_i, n)$ at coordinate points.

Proposition 5.1.2. *For each $s \geqslant 0$*

$$\mathbb{T}^s_{e_I}\left(\prod_{i=1}^{r} \mathbb{G}(k_i, n)\right) = \langle e_J \,;\, d(I, J) \leqslant s\rangle = \{Z_J = 0 \,;\, d(I, J) > s\} \subset \mathbb{P}^N.$$

In particular, $\mathbb{T}^s_{e_I}\left(\prod_{i=1}^{r} \mathbb{G}(k_i, n)\right) = \mathbb{P}^N$ for $s \geqslant r + \sum_{i=1}^{r} k_i$.

Proof. Set $I = \{I^1, \ldots, I^r\} \in \Lambda$. We may assume that $I^i = \{0, \ldots, k_i\}$ for each $1 \leqslant i \leqslant r$ and consider the following parametrization of $\prod_{i=1}^r \mathbb{G}(k_i, n)$ in a neighborhood of e_I:

$$\begin{array}{cccc} \varphi & : & \prod_{i=1}^r \mathbb{C}^{(k_i+1)(n-k_i)} & \longrightarrow & \mathbb{P}^N \\ & & \left[I_{k_i+1} \quad (x^i_{l,m})\right]_{i=1,\ldots,r} & \longmapsto & \left(\prod_{i=1}^r \det(M_{J^i})\right)_{J=\{J^1,\ldots,J^r\}\in\Lambda} \end{array} \tag{5.1.3}$$

where M_{J^i} is the submatrix obtained from $\left[I_{k_i+1},\ (x^i_{l,m})_{\substack{0\leqslant l\leqslant k_i \\ k_i+1\leqslant m\leqslant n}}\right]$ by considering the columns indexed by J^i.

For each $J \in \Lambda$, we will denote $\prod_{i=1}^r \det(M_{J^l})$ simply by $\det(M_J)$. Note that each variable appears in degree at most one in the coordinates of φ. Therefore, deriving two times with respect to the same variable always gives zero. Furthermore, as $\det(M_J)$ has degree at most $r + \sum_{i=1}^r k_i$ all partial derivatives of order greater than or equal to $r + \sum_{i=1}^r k_i$ are zero. Thus, it is enough to prove the claim for $s \leqslant r + \sum_{i=1}^r k_i$.

Given $J = \{J^1, \ldots, J^r\} \in \Lambda$, let i, k, k' be integers such that $1 \leqslant i \leqslant r$, $k \in \{0 \ldots, k_i\}$ and $k' \in \{k_i + 1, \ldots, n\}$. Then

$$\frac{\partial \det(M_J)}{\partial x^i_{k,k'}} = \begin{cases} \pm \ldots \det(M_{J^{i-1}}) \det(M_{J^i,k,k'}) \det(M_{J^{i+1}}) \ldots & k' \in J^i \\ 0 & k' \notin J^i \end{cases}$$

where $M_{J^i,k,k'}$ is the submatrix obtained from M_{J^i} by deleting the column indexed by k' and the row indexed by k.

More generally, let $m_1, \ldots, m_r$ be non-negative integers such that their sum is bigger than one. For each $i = 1, \ldots, r$ consider

$$K_i = \{k^i_1, \ldots, k^i_{m_i}\} \subset \{0, \ldots, k_i\} \text{ and } K'_i = \{k'^i_1, \ldots, k'^i_{m_i}\} \subset \{k_i + 1, \ldots, n\}$$

with $|K_i| = |K'_i| = m_i$. Now, set $m = m_1 + \cdots + m_r$ and

$$K = \{K_1, \ldots, K_r\},\ K' = \{K'_1, \ldots, K'_r\}.$$

Therefore, denoting $\partial x^1_{k^1_1,k'^1_1} \cdots \partial x^r_{k^r_{m_r},k'^r_{m_r}}$ simply by $\partial^m K, K'$ we have

$$\frac{\partial^m \det(M_J)}{\partial^m K, K'} = \begin{cases} \pm \prod_{i=1}^r \det(M_{J^i,K_i,K'_i}) & \text{if } K' \subset J \text{ and } m \leqslant d(I,J) = \deg(\det(M_J)) \\ 0 & \text{otherwise} \end{cases}$$

for any $J \in \Lambda$, where $K' \subset J$ means that $\{k'^1_1, \ldots, k'^1_{m_1}\} \subset J^1, \ldots, \{k'^r_1, \ldots, k'^r_{m_r}\} \subset J^r$, and M_{J^i,K_i,K'_i} is the submatrix obtained from M_{J^i} deleting the columns indexed by K'_i and the rows indexed by K_i. Thus,

$$\frac{\partial^m \det(M_J)}{\partial^m K, K'}(0) = \begin{cases} \pm 1 & \text{if } J^i = K'_i \cup (\{I^i \backslash K_i\}) \text{ for each } i = 1, \ldots, r \\ 0 & \text{otherwise .} \end{cases}$$

Finally, let us denote by $J = K' \cup \{I \backslash K\}$ the element in Λ for which $J^i = K'_i \cup (\{I^i \backslash K_i\})$ for each $i = 1, \ldots, r$. Then, we have that

$$\frac{\partial^m \varphi}{\partial^m K, K'}(0) = \pm e_{K' \cup (\{I \backslash K\})}.$$

Note that $d(I, K' \cup \{I \backslash K\}) = m$, and any $J \in \Lambda$ with $d(I,J) = m$ may be written as $K' \cup \{I \backslash K\}$. Thus, we get that

$$\left\langle \frac{\partial^m \varphi}{\partial^m K, K'}(0) \mid m \leqslant s \right\rangle = \langle e_J \mid d(I,J) \leqslant s \rangle$$

which proves the claim. □

Now, it is immediate to compute the dimension of the osculating spaces of $\prod_{i=1}^r \mathbb{G}(k_i, n)$.

Corollary 5.1.4. *For any point* $p \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ *we have*

$$\dim \mathbb{T}_p^s(\textstyle\prod_{i=1}^r \mathbb{G}(k_i,n)) = \displaystyle\sum_{\substack{i=1,\dots,r\\ 0\leqslant s_i \leqslant k_i+1,\\ s_1+\dots+s_r\leqslant s}} \binom{n-k_1}{s_1}\binom{k_1+1}{s_1}\cdots\binom{n-k_r}{s_r}\binom{k_r+1}{s_r}$$

for any $0 \leqslant s < r + \sum_{i=1}^r k_i$ *while* $\mathbb{T}_p^s(\prod_{i=1}^r \mathbb{G}(k_i,n)) = \mathbb{P}^N$ *for any* $s \geqslant r + \sum_{i=1}^r k_i$.

Proof. Since the general linear group $GL(n+1)$ acts transitively on $\prod_{i=1}^r \mathbb{G}(k_i, n)$ the statement follows from Proposition 5.1.2. □

5.2 Osculating Projections

For a general point $p \in \prod_{i=1}^r \mathbb{G}(k_i,n)$, we will denote $\mathbb{T}_p^s(\prod_{i=1}^r \mathbb{G}(k_i,n))$ simply by $\mathbb{T}_p^s$. Now, take $0 \leqslant s \leqslant r + \sum_{i=1}^r k_i$ and $I \in \Lambda$. By Proposition 5.1.2 the linear projection of $\prod_{i=1}^r \mathbb{G}(k_i,n)$ from $\mathbb{T}_{e_I}^s$ is given by

$$\begin{array}{rccc} \Pi_{\mathbb{T}_{e_I}^s} : & \prod_{i=1}^r \mathbb{G}(k_i) & \dashrightarrow & \mathbb{P}^{N'_s} \\ & (Z_J)_{J\in\Lambda} & \longmapsto & (Z_J)_{J\in\Lambda \mid d(I,J)>s}. \end{array}$$

Moreover, given $I' \subset \{0,\dots,n\}$ with $|I'| = m$ we have the linear projection

$$\begin{array}{rccc} \pi_{I'} : & \mathbb{P}^n & \dashrightarrow & \mathbb{P}^{n-m} \\ & (x_i) & \longmapsto & (x_i)_{i\in\{0,\dots,n\}\backslash I'} \end{array}$$

which in turns induces the linear projection

$$\begin{array}{rccc} \Pi_{I'} : & \mathbb{G}(k,n) & \dashrightarrow & \mathbb{G}(k,n-m) \\ & V & \longmapsto & \langle \pi_{I'}(V) \rangle \\ & (Z_J)_{J\in\Lambda_k} & \longmapsto & (Z_J)_{J\in\Lambda_k \mid J\cap I'=\varnothing} \end{array}$$

whenever $k < n - m$.

Finally, let us fix $I = \{I^1,\dots,I^r\} \in \Lambda$ and take $m_1,\dots,m_r$ integers such that $m_i \leqslant k_i+1$ for each $i = 1,\dots,r$. Then, given $I'^1 \subset I^1,\dots,I'^r \subset I^r$, with $|I'^i| = m_i$, we have a projection

$$\begin{array}{rccc} \prod_{i=1}^r \Pi_{I'^i} : & \prod_{i=1}^r \mathbb{G}(k_i,n) & \dashrightarrow & \prod_{i=1}^r \mathbb{G}(k_i, n-m_i) \\ & V_1 \times \dots \times V_r & \longmapsto & \Pi_{I'^1}(V_1) \times \dots \times \Pi_{I'^r}(V_r). \end{array}$$

Note that a general fiber of $\prod_{i=1}^r \Pi_{I'^i}$ is isomorphic to $\prod_{i=1}^r \mathbb{G}(k_i, k_i+m_i)$. Indeed, let $x = \prod_{i=1}^r \Pi_{I'^i}((V_i)_{i=1}^r) \in \prod_{i=1}^r \mathbb{G}(k_i, n-m_l)$ be a general point. Then, we have

$$\overline{\left(\prod_{i=1}^r \Pi_{I'^i}\right)^{-1}(x)} = \left\{(W_i)_{i=1}^r \in \prod_{i=1}^r \mathbb{G}(k_i,n) \mid W_i \subset \langle V_i, e_{j_1^i}, \dots, e_{j_{m_i}^i}\rangle,\ i=1,\dots,r\right\}.$$

Lemma 5.2.1. *Let us fix $I = \{I^1,\dots,I^r\} \in \Lambda$. If $0 \leqslant s \leqslant r-2+\sum_{i=1}^r k_i$ and $I'^i \subset I^i$ with $|I'^i| = m_i$ for each $i=1,\dots,r$, then the rational map $\Pi_{\mathbb{T}^s_{e_I}}$ factors through $\prod_{i=1}^r \Pi_{I'^i}$ whenever $\sum_{i=1}^r m_i = s+1$.*

Proof. Since the diameter of Λ is $r+\sum k_i$ we have $\{J \in \Lambda \mid d(I,J) \leqslant s\} \subsetneq \Lambda$ and then $\Pi_{\mathbb{T}^s_{e_I}}$ is well-defined.

On the other hand, if $J = \{J^1,\dots,J^r\} \in \Lambda$ is such that $J^i \cap I'^i = \emptyset$ for all $i=1,\dots,r$, then $d(I,J) \geqslant \sum_{i=1}^r m_i > s$ which yields that the center of $\Pi_{\mathbb{T}^s_{e_I}}$ is contained in the center of $\prod_{i=1}^r \Pi_{I'^i}$. □

Proposition 5.2.2. *The rational map $\Pi_{\mathbb{T}^s_{e_I}}$ is birational for all $0 \leqslant s \leqslant r-2+\sum_{i=1}^r k_i$.*

Proof. Since $\mathbb{T}^s_{e_I}$ contains $\mathbb{T}^{s-1}_{e_I}$ it is enough to prove the statement for $s = r-2+\sum_{i=1}^r k_i$. Let us fix $m \in \{1,\dots,r\}$. By Lemma 5.2.1, for each subset $I'^m \subset I^m$ with $|I'^m| = k_m$ there is a rational map $\pi_{I'^m}$ that makes the following diagram commutative:

$$\begin{array}{ccc} \prod_{i=1}^r \mathbb{G}(k_i,n) & \overset{\Pi_{\mathbb{T}^s_{e_I}}}{\dashrightarrow} & \mathbb{P}^{N'_s} \\ & \searrow^{(\prod_{i\neq m}\Pi_{I^i})\times\Pi_{I'^m}} & \downarrow{\scriptstyle \pi_{I'^m}} \\ & & \left(\prod_{i\neq m}\mathbb{G}(k_i,n-k_i-1)\right)\times\mathbb{G}(k_m,n-k_m) \end{array}.$$

Let $x = \Pi_{\mathbb{T}^s_{e_I}}(\{V_i\}_{i=1}^r)$ be a general point and $X \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ be the fiber of $\Pi_{\mathbb{T}^s_{e_I}}$ over x. Set $x_{I'^m} = \pi_{I'^m}(x)$, and denote by $X_{I'^m} \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ the fiber of $\left(\prod_{i\neq m}\Pi_{I^i}\right)\times\Pi_{I'^m}$ over $x_{I'^m}$. Thus,

$$X \subset \bigcap_{I'^m} X_{I'^m}$$

where this intersection runs over all $I'^m \subset I^m$ with $|I'^m| = k_m$ and $m = 1,\dots,r$. Now, if $(W_i)_{i=1}^r$ is a general point in X then

$$W_m \subset \langle e_{j_1},\dots,e_{j_{k_m}}, V_m\rangle \text{ for any } I'^m = \{e_{j_1},\dots,e_{j_{k_m}}\} \subset I^m.$$

Therefore,

$$W_m \subset \bigcap_{I'^m}\langle e_{j_1},\dots,e_{j_{k_m}}, V_m\rangle = V_m.$$

This implies $W_m = V_m$ for every $m = 1,\dots,r$. Since we are working in characteristic zero, we conclude that $\Pi_{\mathbb{T}^s_{e_I}}$ is birational. □

The next step is to study linear projections from the span of several osculating spaces. In particular, we want to understand when such a projection is birational. First of all, note that the order of osculating spaces cannot exceed $r-2+\sum_{i=1}^r k_i$. Furthermore, in order to carry out the computations, we need to consider just the coordinates points of $\prod_{i=1}^r \mathbb{G}(k_i,n)$

such that the corresponding linear subspaces are linearly independent in $\mathbb{C}^{n+1}$, then we can use at most

$$\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$$

of them. Now, let us consider the points $e_{I_1}, \ldots, e_{I_\alpha} \in \prod_{i=1}^r \mathbb{G}(k_i, n)$, where

$$\begin{array}{l} I_1 = \{I_1^1 = \{0, \ldots, k_1\}, \ldots, I_1^r = \{0, \ldots, k_r\}\} \\ I_2 = \{I_2^1 = \{k_r+1, \ldots, k_r+k_1+1\}, \ldots, I_2^r = \{k_r+1, \ldots, k_r+k_r+1\}\} \\ \vdots \\ I_\alpha = \{\ldots, I_\alpha^i = \{(k_r+1)(\alpha-1), \ldots, (k_r+1)(\alpha-1)+k_i\}, \ldots\}. \end{array} \tag{5.2.3}$$

Let $s_1, \ldots, s_\alpha$ be integers such that $0 \leqslant s_m \leqslant r-2+\sum_{i=1}^r k_i$. Denote the linear subspace $\langle \mathbb{T}^{s_1}_{e_{I_1}}, \ldots, \mathbb{T}^{s_m}_{e_{I_m}} \rangle$ simply by $\mathbb{T}^{s_1,\ldots,s_m}_{e_{I_1},\ldots,e_{I_m}}$. Then, for $m \leqslant \alpha$ we have the linear projection

$$\begin{array}{rccc} \Pi_{\mathbb{T}^{s_1,\ldots,s_m}_{e_{I_1},\ldots,e_{I_m}}} : & \prod_{i=1}^r \mathbb{G}(k_i, n) & \dashrightarrow & \mathbb{P}^{N'_{s_1,\ldots,s_m}} \\ & (Z_J)_{J\in\Lambda} & \longmapsto & (Z_J)_{J\in\Lambda \mid d(J,I_1)>s_1,\ldots,d(J,I_m)>s_m}. \end{array}$$

Now, consider $I_1, \ldots, I_\alpha$ as in (5.2.3), and $I'^i_m \subset I^i_m$ with $|I'^i_m| = s^i_m$ for each $1 \leqslant m \leqslant \alpha$ and $i = 1, \ldots, r$, where s^i_m are non-negative integers. If I'^i denotes the union $\bigcup_{m=1}^\alpha I'^i_m$, then for each $i = 1, \ldots, r$ we have a linear projection of $\mathbb{P}^n$

$$\begin{array}{rccc} \pi_{I'^i} : & \mathbb{P}^n & \dashrightarrow & \mathbb{P}^{n-\sum_{m=1}^\alpha s^i_m} \\ & (x_i)_{0\leqslant i\leqslant n} & \longmapsto & (x_i)_{0\leqslant i\leqslant n \text{ and } i\notin I'^i} \end{array}$$

which in turns induces the following projection

$$\begin{array}{rccc} \Pi_{I'^i} : & \mathbb{G}(k_i, n) & \dashrightarrow & \mathbb{G}(k_i, n-\sum_{m=1}^\alpha s^i_m) \\ & V & \longmapsto & \langle \pi_{I'^i}(V) \rangle \\ & (Z_J)_{J\in\Lambda_{k_i}} & \longmapsto & (Z_J)_{J\in\Lambda_{k_i} \mid J\cap I'^i=\varnothing} \end{array}$$

whenever $n - \sum_{m=1}^\alpha s^i_m \geqslant k_i$. Finally, if $n - \sum_{m=1}^\alpha s^i_m \geqslant k_i$ for each $i = 1, \ldots, r$, then the projections above induce a projection

$$\begin{array}{rccc} \prod_{i=1}^r \Pi_{I'^i} : & \prod_{i=1}^r \mathbb{G}(k_i, n) & \dashrightarrow & \prod_{i=1}^r \mathbb{G}(k_i, n-\sum_{m=1}^\alpha s^i_m) \\ & (V_1, \ldots, V_r) & \longmapsto & (\Pi_{I'^1}(V_1), \ldots, \Pi_{I'^r}(V_r)). \end{array}$$

Lemma 5.2.4. *Let $I_1, \ldots, I_\alpha$ be as in (5.2.3), $m, s_1, \ldots, s_m$ integers such that $1 < m \leqslant \alpha$ and $0 \leqslant s_i \leqslant r-2+\sum_{i=1}^r k_i$. Now, consider $I'^i_1 \subset I^i_1, \ldots, I'^i_m \subset I^i_m$ with $|I'^i_j| = s^i_j$, where s^i_j is a non-negative integer for each $i = 1, \ldots, r$ and $1 \leqslant j \leqslant m$. For $j > m$ and $i = 1, \ldots, r$ set $I'^i_j = \varnothing \subset I^i_j$. Denote by I'^i the union $\bigcup_{j=1}^\alpha I'^i_j$ for each $i = 1, \ldots, r$ and assume that*

(i) $n - \sum_{j=1}^m s^i_j \geqslant k_i$ for each $i = 1, \ldots, r$;

(ii) $\sum_{i=1}^r s^i_j \geqslant s_j + 1$ for each $j = 1, \ldots, m$.

Then, the rational maps $\prod_{i=1}^{r} \Pi_{I'^i}$ *and* $\Pi_{\mathbb{T}^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ *are well-defined and the former factors through the latter.*

Proof. Note that $\Pi_{\mathbb{T}^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ is well-defined if and only if $\{J \in \Lambda | d(J, I_1) > s_1, \dots, d(J, I_m) > s_m\} \neq \varnothing$. From (i) we have that for each $1 \leqslant i \leqslant r$ the set $\{0, \dots, n\} \backslash I'^i$ has at least $k_i + 1$ elements. Therefore, we have a set $J^i \subset \{0, \dots, n\} \backslash I'^i$ of cardinality $k_i + 1$ and taking $J = \{J^1, \dots, J^r\} \in \Lambda$ we have

$$d(I_j, J) = \sum_{i=1}^{r} d(I_j^i, J^i) \geqslant \sum_{i=1}^{r} s_j^i = s_j + 1 > s_j$$

for each $1 \leqslant j \leqslant m$. Hence, $\Pi_{\mathbb{T}^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ is well-defined. Now, note that (i) yields that $\prod_{i=1}^{r} \Pi_{I'^i}$ is well-defined. Furthermore, if $J \in \Lambda$ and $J^i \cap I'^i = \varnothing$ for all $i = 1, \dots, r$, then $d(J, I_1) > s_1, \dots, d(J, I_m) > s_m$. Thus, the center of $\Pi_{\mathbb{T}^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ is contained in the center of $\prod_{i=1}^{r} \Pi_{I'^i}$. □

Proposition 5.2.5. *Let* $I_1, \dots, I_{\alpha-1}$ *be as in (5.2.3) and* $s_1, \dots, s_{\alpha-1}$ *be integers such that* $0 \leqslant s_j \leqslant s = r - 2 + \sum_{i=1}^{r} k_i$. *Then, the projection* $\Pi_{\mathbb{T}^{s_1,\dots,s_{\alpha-1}}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$ *is birational.*

Proof. Fix $m \in \{1, \dots, r\}$. For any $j = 1, \dots, \alpha - 1$ consider $I'^m_j \subset I^m_j$ with $|I'^m_j| = k_m$ and $I'^i_j = I^i_j$ for $i \neq m$. Set $I'^i = \bigcup_{j=1}^{\alpha-1} I'^i_j$, then

$$n - (\alpha - 1)(k_i + 1) \geqslant n - (\alpha - 1)(k_r + 1) \geqslant n - \frac{(n - k_r)}{k_r + 1}(k_r + 1) \geqslant k_r \geqslant k_i$$

and

$$n - (\alpha - 1)k_m \geqslant n - \frac{(n - k_r)}{k_r + 1} k_r = \frac{nk_r + n - nk_r + k_r^2}{k_r + 1} \geqslant \frac{2k_r + 1 + k_r^2}{k_r + 1} \geqslant k_m + 1.$$

Thus, our set of subsets I'^i_j satisfies (i) in Lemma 5.2.4. Furthermore, for each $j = 1, \dots, \alpha - 1$

$$\sum_{i=1}^{r} |I'^l_j| = k_m + \sum_{i \neq m} (k_i + 1) = r - 1 + \sum_{i=1}^{r} k_i = s + 1.$$

Therefore, by Lemma 5.2.4 there exists a rational map $\pi_{I'^m}$ that makes the following diagram commutative

$$\begin{array}{ccc} \prod_{i=1}^{r} \mathbb{G}(k_i, n) & \xrightarrow{\Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}} & \mathbb{P}^{N'_{s,\dots,s}} \\ & \searrow^{\prod_{i=1}^{r} \Pi_{I'^i}} & \downarrow \pi_{I'^m} \\ & & \prod_{i=1}^{r} \mathbb{G}(k_i, n - \sum_{j=1}^{\alpha-1} |I'^i_j|) \end{array}.$$

Now, let $x = \Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}(\{V_i\}_{i=1}^r)$ be a general point in the image of $\Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$, and $X \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ be the fiber of $\Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$ over x. Set $x_{I'^m} = \pi_{I'^m}(x)$ and denote by $X_{I'^m} \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ the fiber of $\prod_{i=1}^r \Pi_{I'^i}$ over $x_{I'^m}$. Therefore, $X \subset \bigcap_{I'^m} X_{I'^m}$ where this intersection runs over all subsets $I'^m = \bigcup_{j=1}^{\alpha-1} I'^m_j$ with $I'^m_j \subset I^m_j$ and $|I'^m_j| = k_m$. In particular, if $\{W_i\}_{i=1}^r \in X$ is a general point, then we must have $W_m \subset \langle e_i \mid i \in I'^m \,;\, V_m\rangle$ and hence $W_m \subset \bigcap_{I'^m}\langle e_i \mid i \in I'^m; V_m\rangle$. Now, since $|I'^m_j| = k_m$ we have $\bigcap_{I'^m}\langle e_i \mid i \in I'^m\rangle = \varnothing$ and then $V_m = \bigcap_{I'^m}\langle e_i \,|\, i \in I'^m\,;\, V_m\rangle$ which in turn yields $W_m = V_m$, for all $m = 1,\dots,r$. □

Now, we want to understand what is the largest integer s' for which $\Pi_{\mathbb{T}^{s,\dots,s,s'}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$ is birational.

Proposition 5.2.6. *Let $I_1,\dots,I_\alpha$ be as in (5.2.3) and $s = r - 2 + \sum_{i=1}^r k_i$. Consider $s'_i = \min\{k_i+1, n-\alpha(k_i+1)\}$ for $i \neq r$, $s'_r = \min\{k_r, n-\alpha k_r-1\}$, and set $s' = \sum_{i=1}^r s'_i - 1 \leqslant s$. Then,*

- $\Pi_{\mathbb{T}^{s,\dots,s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$ *is birational whenever $\alpha(k_r+1) - 1 < n < k_r^2 + 3k_r + 1$;*
- $\Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_\alpha}}}$ *is birational whenever $n \geqslant k_r^2 + 3k_r + 1$.*

Proof. First, let us assume that $s'_r < k_r$, that is $n - \alpha k_r - 1 < k_r$, or equivalently

$$n - \alpha k_r < k_r + 1 \Leftrightarrow n - \frac{(n+1)}{k_r+1}k_r < k_r + 1 \Leftrightarrow n < k_r^2 + 3k_r + 1.$$

Now, fix a pair of indexes $(l,m) \in \{1,\dots,\alpha-1\} \times \{1,\dots,r\}$ and consider subsets $I'^i_j \subseteq I^i_j$ with $|I'^i_j| = a_{j,i}$ for each $i \in \{1,\dots,r\}$ and $j \in \{1,\dots\alpha\}$ such that

$$a_{j,i} = \begin{cases} k_i & \text{if } \ i = m,\ j = l \ \text{ or } \ i = r,\ j \neq l,\alpha; \\ k_i + 1 & \text{if } \ i = r \neq m,\ j = l \ \text{ or } \ i \neq m,r \ \text{ or } \ l \neq m,\alpha; \\ s'_i & \text{if } \ j = \alpha,\ i \neq m; \\ s'_m - 1 & \text{if } \ j = \alpha,\ i = m. \end{cases}$$

Note that, since $\alpha(k_r+1) - 1 < n$ we have $a_{j,i} \geqslant 0$ for all $j \in \{1,\dots,\alpha\}$ and $i \in \{1,\dots,r\}$. Moreover, if $m \neq r$ then

$$n - \sum_{j=1}^{\alpha} |I'^m_j| = n-(\alpha-2)(k_m+1)-k_m-|I'^m_\alpha| \geqslant n-(\alpha-1)(k_m+1)-(n-\alpha(k_m+1)-1) = k_m+2$$

and $n - \sum_{j=1}^{\alpha} |I'^r_j| = n - (\alpha-2)k_r - (k_r+1) - |I'^r_\alpha| \geqslant n - (\alpha-1)k_r - 1 - (n - \alpha k_r - 1) = k_r$.
If $r = m$ we have

$$n - \sum_{j=1}^{\alpha} |I'^r_j| = n - (\alpha-2)k_r - (k_r+1) - |I'^r_\alpha| \geqslant n - (\alpha-1)k_r - 1 - (n - \alpha k_r - 2) = k_r + 1.$$

Finally, for $i \neq m, r$ we have

$$n - \sum_{j=1}^{\alpha} |I'^{i}_{j}| = n - (\alpha-1)(k_i+1) - |I'^{i}_{\alpha}| \geqslant n - (\alpha-1)(k_i+1) - (n - \alpha(k_i+1)) = k_i + 1.$$

This yields that (i) in Lemma 5.2.4 is satisfied by the sets I'^{i}_{j}. Moreover, (ii) is satisfied as well. Then, by Lemma 5.2.4 there exists a rational map $\pi_{I'^{m}_{l}, I'^{m}_{\alpha}}$ making the following diagram commutative

$$\begin{array}{ccc} \prod_{i=1}^{r} \mathbb{G}(k_i, n) & \xrightarrow{\Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}} & \mathbb{P}^{N'_{s,\dots,s'}} \\ & \searrow^{\prod_{i=1}^{r} \Pi_{I'^{i}}} & \downarrow \pi_{I'^{m}_{l}, I'^{m}_{\alpha}} \\ & & \prod_{i=1}^{r} \mathbb{G}(k_i, n - \sum_{j=1}^{\alpha} |I'^{i}_{j}|) \end{array}.$$

where $I'^{i} = \bigcup_{j=1}^{\alpha} I'^{i}_{j}$. Now, let $x = \Pi_{\mathbb{T}^{s,\dots,s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}(\{V_i\}_{i=1}^{r})$ be a general point in the image of $\Pi_{\mathbb{T}^{s,\dots,s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$, and $X \subset \prod_{i=1}^{r} \mathbb{G}(k_i, n)$ be the fiber of $\Pi_{\mathbb{T}^{s,\dots,s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$ over x. Set $x_{I'^{m}_{l}, I'^{m}_{\alpha}} = \pi_{I'^{m}_{l}, I'^{m}_{\alpha}}(x)$ and denote by $X_{I'^{m}_{l}, I'^{m}_{\alpha}} \subset \prod_{i=1}^{r} \mathbb{G}(k_i, n)$ the fiber of $\prod_{i=1}^{r} \Pi_{I'^{i}}$ over $x_{I'^{m}_{l}, I'^{m}_{\alpha}}$. Therefore, $X \subset \bigcap_{I'^{m}_{l}, I'^{m}_{\alpha}} X_{I'^{m}_{l}, I'^{m}_{\alpha}}$, where the intersection runs over all pairs of sets I'^{m}_{l} and I'^{m}_{α} with $|I'^{m}_{l}| = k_m$ and $|I'^{m}_{\alpha}| = s'_m - 1$, and for all pairs of indexes $(l, m) \in \{1, \dots, r\} \times \{1, \dots, \alpha - 1\}$. In particular, if $\{W_i\}_{i=1}^{r} \in X$ is a general point then for every $m \in \{1, \dots, r\}$ we have $W_m \subset \bigcap_{I'^{m}_{l}, I'^{m}_{\alpha}} \langle e_i \,|\, i \in I'^{m}; V_m \rangle$, where the intersection runs over all pair of sets I'^{m}_{l} and I'^{m}_{α} with $|I'^{m}_{l}| = k_m$ and $|I'^{m}_{\alpha}| = s'_m - 1$, and $l \in \{1, \dots, \alpha - 1\}$. Now, since $|I'^{m}_{l}| = k_m$, $s'_m - 1 \leqslant k_m$ and $l \in \{1, \dots, \alpha - 1\}$, we must have $\bigcap_{I'^{m}_{l}, I'^{m}_{\alpha}} \langle e_i \,|\, i \in I'^{m} \rangle = \varnothing$ and then $V_m = \bigcap_{I'^{m}} \langle e_i \,|\, i \in I'^{m} \,;\, V_m \rangle$ which yields $W_m = V_m$, for all $m = 1, \dots, r$.

Now, assume that $n \geqslant k_r^2 + k_r + 1$. In this case we have that

$$n - \alpha k_r - 1 \geqslant n - \frac{(n+1)}{k_r+1} k_r - 1 = \frac{n(k_r+1) - (n+1)k_r - (k_r+1)}{k_r+1} \geqslant k_r$$

and for $i < r$ we have

$$n - \alpha(k_i+1) \geqslant n - \alpha(k_r) \geqslant n - \frac{(n+1)}{k_r+1} k_r = \frac{n - k_r}{k_r+1} \geqslant \frac{k_r^2 + 3k_r + 1 - k_r}{k_r+1} = k_r + 1 > k_i + 1.$$

Now, for each pair of indexes $(l, m) \in \{1, \dots, \alpha\} \times \{1, \dots, r\}$ we can consider subsets $I'^{i}_{j} \subseteq I^{i}_{j}$ with $|I'^{i}_{j}| = a_{j,i}$ for each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots \alpha\}$ such that

$$a_{j,i} = \begin{cases} k_i & \text{if} \quad i = m,\ j = l \ \text{ or } \ i = r,\ j \neq l; \\ k_i + 1 & \text{if} \quad i = r \neq m,\ j = l \ \text{ or } \ i \neq m, r \ \text{ or } \ l \neq m. \end{cases}$$

Therefore, arguing as in the proof of the first claim we conclude that $\Pi_{\mathbb{T}^{s,\dots,s}_{e_{I_1},\dots,e_{I_\alpha}}}$ is birational.

$\square$

5.3 Degenerating tangential projections to osculating projections

Let us briefly recall, for the reader convenience, the notion of osculating regularity and strong osculating regularity.

Definition 5.3.1. Let $X \subset \mathbb{P}^N$ be a projective variety. We say that X has *m-osculating regularity* if the following property holds: given general points $p_1, \dots, p_m \in X$ and an integer $s \geqslant 0$, there exists a smooth curve C and morphisms $\gamma_j : C \to X$, $j = 2, \dots, m$, such that $\gamma_j(t_0) = p_1$, $\gamma_j(t_\infty) = p_j$, and the flat limit $\mathbb{T}_0$ in the Grassmannian of the family of linear spaces

$$\mathbb{T}_t = \left\langle \mathbb{T}^s_{p_1}, \mathbb{T}^s_{\gamma_2(t)}, \dots, \mathbb{T}^s_{\gamma_m(t)} \right\rangle,\ t \in C \backslash \{t_0\}$$

is contained in $\mathbb{T}^{2s+1}_{p_1}$. We say that $\gamma_2, \dots, \gamma_m$ realize the m-osculating regularity of X for $p_1, \dots, p_m$.

We say that X has *strong 2-osculating regularity* if the following property holds: given general points $p, q \in X$ and integers $s_1, s_2 \geqslant 0$, there exists a smooth curve $\gamma : C \to X$ such that $\gamma(t_0) = p$, $\gamma(t_\infty) = q$ and the flat limit $\mathbb{T}_0$ in the Grassmannian of the family of linear spaces

$$\mathbb{T}_t = \left\langle \mathbb{T}^{s_1}_p, \mathbb{T}^{s_2}_{\gamma(t)} \right\rangle,\ t \in C \backslash \{t_0\}$$

is contained in $\mathbb{T}^{s_1+s_2+1}_p$.

For a discussion on the notions of m-osculating regularity and strong 2-osculating regularity we refer to [MR17, Section 5] and [AMR19, Section 4].

Proposition 5.3.2. *The variety $\prod_{i=1}^r \mathbb{G}(k_i, n)$ has strong 2-osculating regularity.*

Proof. Let $p, q \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ be general points. We may assume that $p = e_{I_1}$ and $q = e_{I_2}$ with I_1, I_2 as in (5.2.3) and consider the degree $r + \sum_{i=1}^r k_i$ rational normal curve given by

$$\gamma(s:t) = \prod_{i=1}^r (se_0 + te_{k_r+1}) \wedge \dots \wedge (se_{k_i} + te_{k_r+k_i+1}).$$

We work on the affine chart $s = 1$ and set $t = (1:t) \in \mathbb{P}^1$. Now, consider the points

$$e_0, \dots, e_n, e^t_0 = e_0 + te_{k_r+1}, \dots, e^t_{k_r} = e_{k_r} + te_{2k_r+1}, e^t_{k_r+1} = e_{k_r+1}, \dots, e^t_n = e_n$$

and, for each $I = \{I^1, \dots, I^r\} \in \Lambda$, the corresponding points in $e^t_I = e^t_{I^1} \otimes e^t_{I^2} \otimes \dots \otimes e^t_{I^r} \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ where, setting $I^j = \{i^j_1, \dots, i^j_{k_j}\}$, $e^t_{I^j} = e^t_{i^j_1} \wedge \dots \wedge e^t_{i^j_{k_j}}$.

Given integers $s_1, s_2 \geqslant 0$, let us consider the family of linear spaces

$$\mathbb{T}_t = \langle \mathbb{T}^{s_1}_{e_{I_1}}, \mathbb{T}^{s_2}_{e_{\gamma(t)}} \rangle,\ \ t \in \mathbb{P}^1 \backslash \{0\}.$$

By Proposition 5.1.2 we have

$$\mathbb{T}_t = \langle e_J \mid d(I_1, J) \leqslant s_1 \,;\, e^t_J \mid d(I_2, J) \leqslant s_2 \rangle,\ t \neq 0$$

and

$$\mathbb{T}^{s_1+s_2+1}_{e_{I_1}} = \langle e_J \mid d(I_1, J) \leqslant s_1 + s_2 + 1\rangle = \{Z_J = 0 \mid d(I_1, J) > s_1 + s_2 + 1\}.$$

Now, let $\mathbb{T}_0$ be the flat limit of $\{\mathbb{T}_t\}_{t\in\mathbb{P}^1\backslash\{0\}}$, we want to show that $\mathbb{T}_0 \subset \mathbb{T}^{s_1+s_2+1}_p$. In order to do this it is enough to exhibit, for each index $I \in \Lambda$ with $d(I_1, I) > s_1+s_2+1$, a hyperplane H_I of type $Z_I + t\left(\sum_{J\neq I} f_J(t)Z_J\right) = 0$ where $f_J(t) \in \mathbb{C}[t]$ for every J. We define, for each $l \geqslant 0$ and $I = \{I^1, \dots, I^r\} \in \Lambda$,

$$\Delta(I, l) = \left\{\{(I^j\backslash J^j) \cup (J^j + k_r + 1)\}_{1\leqslant j\leqslant r} \mid J^j \subset I^j \cap I^j_1 \text{ and } \sum |J^j| = l\right\} \subset \Lambda.$$

Furthermore, for each $l > 0$ we define

$$\Delta(I, -l) = \{J \mid I \in \Delta(J, l)\};$$

$$s^+_I = \max_{l\geqslant 0}\{\Delta(I, l) \neq \varnothing\} \in \{0, \dots, \sum k_j + r\};$$

$$s^-_I = \max_{l>0}\{\Delta(I, -l) \neq \varnothing\} \in \{0, \dots, \sum k_j + r\};$$

$$\Delta(I)^+ = \bigcup_{0\leqslant l} \Delta(I, l) = \bigcup_{0\leqslant l\leqslant s^+_I} \Delta(I, l);$$

$$\Delta(I)^- = \bigcup_{0\leqslant l} \Delta(I, -l) = \bigcup_{0\leqslant l\leqslant s^-_I} \Delta(I, -l).$$

Now, let us write e^t_I with $d(I_1, I) \leqslant s_2$, in the basis e_J with $J \in \Lambda$. For any $I \in \Lambda$ we have

$$\begin{aligned} e^t_I &= e_I + t \sum_{J\in\Delta(I,1)} \text{sign}(J)e_J + \cdots + t^{s^+_I} \sum_{J\in\Delta(I,s^+_I)} \text{sign}(J)e_J \\ &= \sum_{l=0}^{s^+_I}\left(t^l \sum_{J\in\Delta(I,l)} \text{sign}(J)e_J\right) = \sum_{J\in\Delta(I)^+} t^{d(J,I)}\text{sign}(J)e_J \end{aligned}$$

where $\text{sign}(J) = \pm 1$. Note that $\text{sign}(J)$ depends on J but not on I, then we can write $e^t_I = \sum_{J\in\Delta(I)^+} t^{d(J,I)}e_J$. Therefore, we have

$$\mathbb{T}_t = \left\langle e_I \mid d(I_1, I) \leqslant s_1\,;\ \sum_{J\in\Delta(I)^+} t^{d(J,I)}e_J \mid d(I_1, I) \leqslant s_2\right\rangle.$$

Finally, we define

$$\Delta := \{I\ :\ d(I_1, I) \leqslant s_1\}\bigcup\left(\bigcup_{d(I_1,I)\leqslant s_2} \Delta(I)^+\right) \subset \Lambda.$$

Let $I \in \Lambda$ be an index such that $d(I_1, I) > s_1 + s_2 + 1$. If $I \notin \Delta$ then $\mathbb{T}_t \subset \{Z_I = 0\}$ for any $t \neq 0$ and we are done.

Assume that $I \in \Delta$. For any e_K^t with non-zero coordinate Z_I we have $I \in \Delta(K)^+$, that is $K \in \Delta(I)^-$. Now, it is enough to find a hyperplane H_I of type

$$F_I = \sum_{J \in \Delta(I)^-} t^{d(J,I)} c_J Z_J = 0$$

with $c_J \in \mathbb{C}$ and $c_I \neq 0$, and such that $\mathbb{T}_t \subset H_I$ for each $t \neq 0$. In the following, let us write $s_{i,I}^- = s$. Now, let us check what conditions we get by requiring $\mathbb{T}_t \subset \{F_I = 0\}$ for $t \neq 0$. Given $K \in \Delta(I)^-$ we have that $d(I, K) \leqslant s_K^+$ and

$$\begin{aligned} F_I(e_K^t) &= F_I\left(\sum_{J \in \Delta(K)^+} t^{d(J,K)} e_J\right) = \sum_{J \in \Delta(I)^-} t^{d(J,I)} c_J \left(\sum_{J \in \Delta(K)^+} t^{d(J,K)} e_J\right) \\ &= \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} t^{d(J,I)+d(J,K)} c_J = t^{d(I,K)} \left[\sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_J\right]. \end{aligned}$$

Therefore,

$$F_I(e_K^t) = 0 \ \forall\, t \neq 0 \Leftrightarrow \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_J = 0.$$

Note that this is a linear condition on the coefficients c_J, with $J \in \Delta(I)^-$. Hence

$$\begin{aligned} \mathbb{T}_t \subset \{F_I = 0\} \text{ for } t \neq 0 &\Leftrightarrow \begin{cases} F_I(e_K) = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_1] \\ F_I(e_K^t) = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_2] \end{cases} \\ &\Leftrightarrow \begin{cases} c_K = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_1] \\ \displaystyle\sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_J = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_2] \end{cases} \end{aligned} \tag{5.3.3}$$

where $B[J, l] := \{K \in \Lambda \mid d(J, K) \leqslant l\}$. The number of conditions on the c_J is then $c := |\Delta(I)^- \cap B[I_1, s_1]| + |\Delta(I)^- \cap B[I_1, s_2]|$.

The problem is now reduced to finding a solution of the linear system given by the c equations (5.3.3) in the $|\Delta(I)^-|$ variables c_J, $J \in \Delta(I)^-$ such that $c_I \neq 0$. Therefore, it is enough to find $s + 1$ complex numbers $c_I = c_0 \neq 0, c_1, \ldots, c_s$ satisfying the following conditions

$$\begin{cases} c_j = 0 & \forall\ j = s, \ldots, d - s_1 \\ \displaystyle\sum_{m=0}^{d(I,K)} |\Delta(I)^- \cap \Delta(K, l)| c_{d(I,K)-m} = 0 & \forall\ K \in \Delta(I)^- \cap B[I_1, s_2] \end{cases} \tag{5.3.4}$$

where $d = d(I_1, I) > s_1 + s_2 + 1$. Note that (5.3.4) can be written as

$$\begin{cases} c_j = 0 & \forall\ j = s, \ldots, d - s_1 \\ \displaystyle\sum_{m=0}^{j} \binom{j}{j-m} c_m = 0 & \forall\ j = s, \ldots, d - s_2 \end{cases}$$

that is

$$\begin{cases} c_s = 0 \\ \vdots \\ c_{d-s_1} = 0 \end{cases} \qquad \begin{cases} \binom{s}{0}c_s + \binom{s}{1}c_{s-1} + \cdots + \binom{s}{s-1}c_1 + \binom{s}{s}c_0 = 0 \\ \vdots \\ \binom{d-s_2}{0}c_{d-s_2} + \binom{d-s_2}{1}c_{d-s_2-1} + \cdots + \binom{d-s_2}{d-s_2-1}c_1 + \binom{d-s_2}{d-s_2}c_0 = 0. \end{cases} \tag{5.3.5}$$

Now, it is enough to show that the linear system (5.3.5) admits a solution with $c_0 \neq 0$. If, $s < d - s_2$ then the system (5.3.5) reduces to $c_s = \cdots = c_{d-s_1} = 0$ and then we can take $c_0 = 1$ and $c_1 = \cdots = c_s = 0$, since $d - s_1 > s_2 + 1 > 1$.

So, let us assume that $s \geqslant d - s_2$. Since $c_s = \cdots = c_{d-s_1} = 0$ our problem is translated into checking that the system (5.3.5) admits a solution involving the variables $c_{d-s_1-1}, \ldots, c_0$ with $c_0 \neq 0$. First of all, note that the system (5.3.5) can be rewritten as follows

$$\begin{cases} \binom{s}{s-(d-s_1-1)}c_{d-s_1-1} + \binom{s}{s-(d-s_2-2)}c_{d-s_1-2} + \cdots + \binom{s}{s-1}c_1 + \binom{s}{s}c_0 = 0 \\ \vdots \\ \binom{d-s_2}{d-s_2-(d-s_1-1)}c_{d-s_1-1} + \binom{d-s_2}{d-s_2-(d-s_1-2)}c_{d-s_1-2} + \cdots + \binom{d-s_2}{d-s_2-1}c_1 + \binom{d-s_2}{d-s_2}c_0 = 0. \end{cases}$$

Thus, it is enough to check that the $(s - d + s_2 + 1) \times (d - s_1 - 1)$ matrix

$$M = \begin{pmatrix} \binom{s}{s-(d-s_1-1)} & \binom{s}{s-(d-s_1-2)} & \cdots & \binom{s}{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d-s_2}{d-s_2-(d-s_1-1)} & \binom{d-s_2}{d-s_2-(d-s_1-2)} & \cdots & \binom{d-s_2}{d-s_2-1} \end{pmatrix}$$

has maximal rank. Now, note that $s \leqslant d$ and $d > s_1 + s_2 + 1$ yield $s - d + s_2 + 1 < s - s_1 \leqslant d - s_1$ and then $s - d + s_2 + 1 \leqslant d - s_1 - 1$. Therefore, we have to show that the $(s - d + s_2 + 1) \times (s - d + s_2 + 1)$ submatrix

$$M' = \begin{pmatrix} \binom{s}{s-d+s_2+1} & \binom{s}{s-d+s_2} & \cdots & \binom{s}{1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d-s_2}{s-d+s_2+1} & \binom{d-s_2}{s-d+s_2} & \cdots & \binom{d-s_2}{1} \end{pmatrix}$$

has non-zero determinant. Finally, since $d - s_2 > s_1 + 1 \geqslant 1$ [GV85, Corollary 2] yields that $\det(M') \neq 0$. □

Proposition 5.3.6. *Set* $\alpha = \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$. *Then, the variety* $\prod_{i=1}^r \mathbb{G}(k_i, n)$ *has* α*-osculating regularity.*

Proof. First of all, note that if $\alpha = 2$ then the statement follows from Proposition 5.3.2. Then we may assume $\alpha \geqslant 3$.

Let $p_1, \ldots, p_\alpha \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ be general points. We may assume that $p_j = e_{I_j}$ for $j = 1, \ldots, \alpha$. Each e_{I_j}, $j \geqslant 2$, is connected to e_{I_1} by the degree $r + \sum_{i=1}^r k_i$ rational normal curve defined by

$$\gamma_j(s:t) = \prod_{i=1}^r (se_0 + te_{(k_r+1)(j-1)}) \wedge \cdots \wedge (se_{k_i} + te_{(k_r+1)(j-1)+k_i}).$$

We work on the affine chart $s = 1$ and set $t = (1 : t)$. Now, given $s \geqslant 0$ we consider the family of linear subspaces

$$\mathbb{T}_t = \langle \mathbb{T}^s_{e_{I_1}}, \mathbb{T}^s_{\gamma_2(t)}, \dots, \mathbb{T}^s_{\gamma_\alpha(t)} \rangle, \ \ t \in \mathbb{P}^1 \backslash \{0\}.$$

Our goal is to show that the flat limit $\mathbb{T}_0$ of $\{\mathbb{T}_t\}_{t \in \mathbb{P}^1 \backslash \{0\}}$ in $\mathbb{G}(\dim(\mathbb{T}_t), N)$ is contained in $\mathbb{T}^{2s+1}_{e_{I_1}}$. In order to do this, let us consider the points

$$e_0, \dots, e_n, e_0^{j,t} = e_0 + te_{(k_r+1)(j-1)}, \dots, e_{k_r}^{j,t} = e_{k_r} + te_{(k_r+1)j-1}, e_{k_r+1}^{j,t} = e_{k_r+1}, \dots, e_n^{j,t} = e_n$$

and, for each $I = \{I^1, \dots, I^r\} \in \Lambda$ and $j = 2, \dots, \alpha$, the corresponding points in $e_I^{j,t} = e_{I^1}^{j,t} \otimes e_{I^2}^{j,t} \otimes \dots \otimes e_{I^r}^{j,t} \in \mathbb{P}^N$. By Proposition 5.1.2 we have

$$\mathbb{T}_t = \langle e_I \mid d(I_1, I) \leqslant s; e_I^{j,t} \mid d(I_j, I) \leqslant s, \ j = 2, \dots, \alpha \rangle, \ t \neq 0$$

and

$$\mathbb{T}^{2s+1}_{e_{I_1}} = \langle e_J \mid d(I_1, J) \leqslant 2s + 1 \rangle = \{Z_J = 0 \mid d(I_1, J) > 2s + 1\}.$$

In order to show that $\mathbb{T}_0 \subset \mathbb{T}^{2s+1}_p$, it is enough to exhibit, for each index $I \in \Lambda$ with $d(I_1, I) > 2s+1$, an hyperplane H_I of type $Z_I + t\left(\sum_{J \neq I} f_J(t) Z_J\right) = 0$ such that $\mathbb{T}_t \subset \{H_i = 0\}$ for $t \neq 0$.

For each $l \geqslant 0$, $j = 2, \dots, \alpha$ and $I = \{I^1, \dots, I^r\} \in \Lambda$ we define

$$\Delta(I, l)_j = \left\{ \{(I^k \backslash J^k) \cup (J^k + (j-1)(k_r + 1)\}_{1 \leqslant k \leqslant r} \mid J^k \subset I^k \cap I^k_1 \text{ and } \sum |J^k| = l \right\} \subset \Lambda$$

where $L + \lambda = \{i + \lambda \mid i \in L\}$ is the translation of the set L by the integer λ. For any $l > 0$ we define

$$\Delta(I, -l)_j = \{J \mid I \in \Delta(J, l)_j\};$$

$$s^+_{I,j} = \max_{l \geqslant 0} \{\Delta(I, l)_j \neq \varnothing\} \in \{0, \dots, \sum k_j + r\};$$

$$s^-_{I,j} = \max_{l > 0} \{\Delta(I, -l)_j \neq \varnothing\} \in \{0, \dots, \sum k_j + r\};$$

$$\Delta(I)^+_j = \bigcup_{0 \leqslant l} \Delta(I, l)_j = \bigcup_{0 \leqslant l \leqslant s^+_{I,j}} \Delta(I, l)_j;$$

$$\Delta(I)^-_j = \bigcup_{0 \leqslant l} \Delta(I, -l)_j = \bigcup_{0 \leqslant l \leqslant s^-_{I,j}} \Delta(I, -l)_j.$$

Note that for any l we have

$$J \in \Delta(I, l)_j \ \Rightarrow \ d(J, I) = |l| \ \text{ and } \ d(J, I_1) = d(I, I_1) + l. \tag{5.3.7}$$

We will write e_I^t with $d(I_1, I) \leqslant s$, in the basis e_J with $J \in \Lambda$. For any $I \in \Lambda$ we have

$$\begin{aligned} e_I^{j,t} &= e_I + t \sum_{J \in \Delta(I,1)_j} \operatorname{sign}(J) e_J + \cdots + t^{s^+_{I,j}} \sum_{J \in \Delta(I, s^+_{I,j})} \operatorname{sign}(J) e_J \\ &= \sum_{l=0}^{s^+_{I,j}} \left(t^l \sum_{J \in \Delta(I,l)_j} \operatorname{sign}(J) e_J \right) = \sum_{J \in \Delta(I)^+_j} t^{d(J,I)} \operatorname{sign}(J) e_J \end{aligned}$$

where $\operatorname{sign}(J) = \pm 1$. Note that $\operatorname{sign}(J)$ depends on J but not on I, then we can write $e_I^{j,t} = \sum_{J \in \Delta(I)^+_j} t^{d(J,I)} e_J$. Therefore, we have

$$\mathbb{T}_t = \left\langle e_I \mid d(I_1, I) \leqslant s \,;\, \sum_{J \in \Delta(I)^+_j} t^{d(J,I)} e_J \mid d(I_1, I) \leqslant s,\ 2 \leqslant j \leqslant \alpha \right\rangle .$$

Finally we define

$$\Delta := \{ I \,:\, d(I_1, I) \leqslant s \} \bigcup \left(\bigcup_{\substack{d(I_1,I) \leqslant s \\ 2 \leqslant j \leqslant \alpha}} \Delta(I)^+_j \right) \subset \Lambda .$$

Let $I \in \Lambda$ be an index such that $d(I_1, I) > 2s + 1$. If $I \notin \Delta$, then $\mathbb{T}_t \subset \{ Z_I = 0 \}$ for any $t \neq 0$ and we are done.

Now, assume that $I \in \Delta$. We will show that $\Delta(K_1)^+_{j_1} \cap \Delta(K_2)^+_{j_2} = \varnothing$ whenever $K_1, K_2 \in \Lambda$ with $d(K_1, I_1), d(K_2, I_2) \leqslant s$ and $2 \leqslant j_1, j_2 \leqslant \alpha$ with $j_1 \neq j_2$.

In fact, suppose that $\Delta(K_1)^+_{j_1} \cap \Delta(K_2)^+_{j_2} \neq \varnothing$, that is there exists $I \in \Lambda$ such that

$$I \in \Delta(K_1, l_1)_{j_1} \cap \Delta(K_2, l_2)_{j_2} \text{ for some } l_1 \text{ and } l_2 .$$

Now, consider the following sets

$$\begin{aligned} I^0 &:= I \cap I_1 ; \\ I^1 &:= I \cap \{ K_1 + (j_1 - 1)(k_r + 1) \} ; \\ I^2 &:= I \cap \{ K_2 + (j_2 - 1)(k_2 + 1) \} ; \\ I^3 &:= I \backslash (I^0 \cup I^1 \cup I^2) . \end{aligned}$$

Since $I \in \Delta(K_1, l_1)_{j_1} \cap \Delta(K_2, l_2)_{j_2}$ we have $|I^1| = l_1$ and $|I^2| = l_2$. Set $|I^3| = u$, then

$$d(I, I_1) = l_1 + l_2 + u \leqslant l_1 + l_2 + 2u \overset{(5.3.7)}{=} d(K_1, I_1) + d(K_1, I_1) \leqslant 2s$$

contradicting $d(I_1, I) > 2s + 1$. Therefore we conclude that there is a unique j_I for which

$$I \in \bigcup_{d(I_1, I) \leqslant s} \Delta(I)^+_{j_I} .$$

Now, let $J \in \Lambda$ such that $d(J, I_1) \leqslant s$ and $I \in \Delta(J)^+_{j_I}$. Note that

$$d(I, I_1) - s(I)^-_{j_I} \leqslant d(I, I_1) - d(I, J) = d(J, I_1) \leqslant s \Rightarrow s + 1 - D + s(I)^-_{j_I} > 0$$

where $D = d(I, I_1) > 2s + 1$. We define

$$\Gamma(I) = \sum_{0 \leqslant l \leqslant s+1-D+s(I)^-_{j_I}} \Delta(I, -l)_{j_I} \subset \Gamma.$$

Our aim now is to find a hyperplane of the form

$$H_I = \left\{ \sum_{J \in \Gamma(I)} t^{d(J,I)} c_J Z_J = 0 \right\} \tag{5.3.8}$$

such that $\mathbb{T}_t \subset H_I$ and $c_I \neq 0$. First, note that

$$J \in \Gamma(I) \Rightarrow J \notin \bigcup_{\substack{d(I_1,K) \leqslant s \\ 2 \leqslant j \leqslant \alpha\,;\, j \neq j_I}} \Delta(K)^+_j. \tag{5.3.9}$$

In fact, suppose that $J \in \Delta(I, -l)_{j_I} \cap \Delta(K, m)_j$, for some $K \in \Lambda$ with $d(K, I_1) \leqslant s$, and $0 \leqslant j \leqslant s + 1 - D + s(I)^-_{j_I}$ with $j \neq j_I$. Then, since $J \in \Delta(I, -l)_{j_I}$ we have

$$|J \cap I_{j_I}| = |I \cap I_{j_I}| - l \geqslant s(I)^-_{j_I} - l \geqslant D - k - 1 > s.$$

On the other hand, since $J \in \Delta(K, m)_j$ with $j \neq j_I$ we have

$$|J \cap I_{j_I}| = |K \cap I_{j_I}| \leqslant d(K, I_1) \leqslant s$$

which is a contradiction. Now, note that if $K \in \Lambda$ is such that $d(K, I_1) \leqslant s$ and $K \in \Gamma(I)$, then

$$d(K, I_1) = d(I, I_1) - d(I, K) > 2s + 1 - (s + 1 - D + s(I)^-_{j_I}) > s + D - s(I)^-_{j_I} > s.$$

Thus (5.3.9) yields that the hyperplane H_I given by (5.3.8) is such that

$$\left\langle e_K \mid d(I_1, K) \leqslant s\,;\ \sum_{J \in \Delta(K)^+_j} t^{d(J,K)} e_J \mid d(I_1, K) \leqslant s,\ 2 \leqslant j \leqslant \alpha\,;\ j \neq j_I \right\rangle \subset H_I,\ t \neq 0.$$

Therefore

$$\mathbb{T}_t \subset H_I,\ t \neq 0 \Leftrightarrow \left\langle \sum_{J \in \Delta(K)^+_{j_I}} t^{d(J,K)} e_J \mid d(I_1, K) \leqslant s \right\rangle \subset H_I,\ t \neq 0.$$

Now, arguing as in the proof of Proposition 5.3.2 we get

$$\mathbb{T}_t \subset H_I,\ t \neq 0 \Leftrightarrow \sum_{J \in \Delta(K)^+_{j_I} \cap \Gamma(I)} c_J = 0, \quad \forall\, K \in \Delta(I)^-_{j_I} \cap B[I_1, s]. \tag{5.3.10}$$

So, the problem is reduced to finding a solution $(c_J)_{J\in\Gamma(I)}$ for the linear system (5.3.10) such that $c_I \neq 0$. We set $c_J = c_{d(I,J)}$ and reduce, as in the proof of Proposition 5.3.2, to the linear system

$$\sum_{l=0}^{s+1+D-s(I)^-_{j_I}} \binom{D-i}{D-i-l} c_l, \;\; D - S(I)^-_{j_I} \leqslant i \leqslant k. \tag{5.3.11}$$

We have $s+2+D-s(I)^-_{j_I}$ variables $c_0,\dots,c_{s+1+D-s(I)^-_{j_I}}$ and $s+1+D-s(I)^-_{j_I}$ equations. Finally, the argument used in the last part of the proof of Proposition 5.3.2 shows that the linear system (5.3.11) admits a solution with $c_0 \neq 0$. □

5.4 On secant defectiveness of products of Grassmannians

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension n and let $\mathbb{S}ec_h(X) \subset \mathbb{P}^N$ be the $h-$secant variety of X. Recall that if the tangential projection

$$\tau_{X,h} : X \dashrightarrow \mathbb{P}^{N_h}$$

is generically finite then X is not $(h+1)-$defective.

Remark 5.4.1. Note that if $\delta_h(X) = 0$ and $\mathbb{S}ec_h(X) = \mathbb{P}^N$ then the tangential projection $\tau_{X,h}$ is not generically finite even if X is not $(h+1)$-defective. On the other hand, the techniques we use allow us to study h-tangential projections only for values of h such that the expected dimension of $\mathbb{S}ec_h(X)$ is strictly smaller than N.

Let us briefly recall the function $h_m : \mathbb{N}_{\geqslant 0} \longrightarrow \mathbb{N}_{\geqslant 0}$ counting how many tangent spaces can be degenerated into a higher order osculating space (see Definition 1.5.4).

Given an integer $m \geqslant 0$ we define a function

$$h_m : \mathbb{N}_{\geqslant 0} \longrightarrow \mathbb{N}_{\geqslant 0}$$

as follows: for $h_m(0) = 0$ and for any $k > 0$ write

$$k+1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geqslant 1$ and $\varepsilon \in \{0,1\}$, then

$$h_m(k) = m^{\lambda_1 - 1} + m^{\lambda_2 - 1} + \cdots + m^{\lambda_l - 1}.$$

Recall finally the theorem that enable us to relate the $h-$defectiveness with the osculating projection $\Pi^{s_1,\dots,s_l}_{p_1,\dots,p_l}$:

Theorem 5.4.2. *[MR17, Theorem 5.3] Let $X \subset \mathbb{P}^N$ be a projective variety having m-osculating regularity and strong 2-osculating regularity. Let $s_1,\dots,s_l \geqslant 1$ be integers such that the general osculating projection $\Pi^{s_1,\dots,s_l}_{p_1,\dots,p_l}$ is generically finite. If*

$$h \leqslant \sum_{j=1}^{l} h_m(s_j)$$

then X is not $(h+1)$-defective.

Now, we are ready to prove our main result on non-defectiveness of a product of Grassmannians. We follow the notation introduced in the previous sections.

Theorem 5.4.3. *Assume that* $n \geqslant 2k_r + 1$. *Set*

$$\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$$

and let h_α *be as in Definition 1.5.4. Assume that*

- *either* $n \geqslant k_r^2 + 3k_r + 1$ *and* $h \leqslant \alpha h_\alpha(\sum_{i=1}^r k_i + r - 2)$ *or*
- $\alpha(k_r+1) - 1 < n < k_r^2 + 3k_r + 1$ *and* $h \leqslant (\alpha-1)h_\alpha(\sum_{i=1}^r k_i + r - 2) + h_\alpha(s')$

where $s' = \sum_{i=1}^r s_i - 2$ *with* $s'_i = \min\{k_i+1, n-\alpha(k_i+1)\}$ *for* $i \neq r$ *and* $s'_r = \min\{k_r, n - \alpha k_r - 1\}$. *Then* $\prod_{i=1}^r \mathbb{G}(k_i, n)$ *is not* $(h+1)$*-defective.*

Proof. We have shown in Propositions 5.3.6, 5.3.2 that $\prod_{i=1}^r \mathbb{G}(k_i, n)$ has respectively α-osculating regularity for $\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$, and strong 2-osculating regularity. The statement then follows immediately from Proposition 5.2.6 and Theorem 5.4.2. □

For the reader convenience let us show an explicit computation:

Example 5.4.4. Consider $X = \mathbb{G}(1,23) \times \mathbb{G}(2,23) \times \mathbb{G}(3,23)$. Since $23 > 7$ we are in the hypothesis of Theorem 5.4.3. Here $k_r = k_3 = 3$ with $\alpha = 6$. Now $23 > (3)^2 + 3 \cdot 3 + 1$ and so X is not $(h+1)-$defective for $h \leqslant 6h_6(7)$. Now, $7 + 1 = 2^3$ and $h_6(7) = 6^2 = 36$. Therefore, Theorem 5.4.3 yields that X is not $217-$defective.

Corollary 5.4.5. *The variety* $\prod_{i=1}^r \mathbb{G}(k_i, n)$ *is not* $(h+1)$*-defective for*

$$h \leqslant \left(\frac{n+1}{k_r+1} \right)^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}.$$

Proof. We may write

$$\sum_{i=1}^r k_i + r - 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon \tag{5.4.6}$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geqslant 1$ and $\varepsilon \in \{0,1\}$. Then $h_\alpha(\sum_{i=1}^r k_i + r - 2) = \alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1}$.

The first bound in Theorem 5.4.3 gives $h \leqslant \alpha^{\lambda_1} + \cdots + \alpha^{\lambda_l}$. Furthermore, considering just the first summand in the second bound in Theorem 5.4.3 we get that $\prod_{i=1}^r \mathbb{G}(k_i, n)$ is not $(h+1)$-defective for $h \leqslant (\alpha - 1)(\alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1})$.

Finally, from (5.4.6) we get that $\lambda_1 = \lfloor \log_2(r - 1 + \sum k_i) \rfloor$. Hence, asymptotically we have $h_\alpha(\sum k_j + r - 2) \sim \alpha^{\lfloor \log_2(r-1+\sum k_i) \rfloor - 1}$, and by Theorem 5.4.3 if $h \leqslant \left(\frac{n+1}{k_r+1} \right)^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}$ then the variety $\prod_{i=1}^r \mathbb{G}(k_i, n)$ is not $(h+1)$-defective. □

5.5 On secant defectiveness of Flag varieties

Our goal is to compute the higher osculating spaces of $\mathbb{F}(k_1,\dots,k_r;n)$. In order to do this, we will use the notion of osculating well-behaved variety, see Definition 1.5.8.

Let us denote by M_i the following $(k_i+1)\times(n+1)$ matrix

$$M_i = \begin{bmatrix} I_{k_1+1} & \cdots & \cdots & (x^1_{l,m})_{\substack{0\leqslant l\leqslant k_1\\ k_1+1\leqslant m\leqslant n}} \\ 0 & I_{k_2-k_1} & \cdots & (x^2_{l,m})_{\substack{k_1+1\leqslant l\leqslant k_2\\ k_2+1\leqslant m\leqslant n}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_{k_i-k_{i-1}} & (x^i_{l,m})_{\substack{k_{i-1}+1\leqslant l\leqslant k_i\\ k_i+1\leqslant m\leqslant n}} \end{bmatrix}$$

and consider the map

$$\begin{array}{rccl} \varphi': & \prod_{i=1}^r \mathbb{C}^{(k_1+1)(n-k_1)+\sum_{j=2}^i (n-k_j)(k_j-k_{j-1})} & \longrightarrow & \mathbb{P}^N \\ & (M_1,\dots,M_r) & \longmapsto & \left(\prod_{i=1}^r \det(M_{J^i})\right)_{J=\{J^1,\dots,J^r\}\in\Lambda} \end{array}$$

where M_{J^i} is the submatrix obtained from M_i by considering only the columns indexed by J^i.

For each $2\leqslant i\leqslant r$ and $m\leqslant k_l$, let us take $x^i_{l,m}=0$ in M_i. Then φ' becomes the parametrization φ of $\prod_{i=1}^r \mathbb{G}(k_i,n)$ in (5.1.3).

Now, set $x^i_{l,m}=x^r_{l,m}$ in M_i for each $i=1,\dots,r-1$ and $1\leqslant l<m\leqslant n$. Hence φ becomes the parametrization of $\mathbb{F}(k_1,\dots,k_r;n)$ given by

$$\begin{array}{rccl} \overline{\varphi}: & \mathbb{C}^{(k_1+1)(n-k_1)+\sum_{j=2}^r (n-k_j)(k_j-k_{j-1})} & \longrightarrow & \mathbb{P}(\Gamma_a)\subset\mathbb{P}^N \\ & M_r & \longmapsto & \varphi\left(\overline{M}_1,\dots,\overline{M}_r\right) \end{array}$$

where $\overline{M}_i$ is the submatrix obtained from M_r by considering only the first k_i+1 rows.

Lemma 5.5.1. *Let* $\mathbb{T}^s_{\varphi'}\left(\prod_{i=1}^r \mathbb{G}(k_i,n)\right) := \left\langle \dfrac{\partial^{|I|}\varphi'}{\partial x_{|I|}}(0) \mid |I|\leqslant s \right\rangle$ *be the* s*-osculating space of* $\prod_{i=1}^r \mathbb{G}(k_i,n)$ *with respect to* φ'*. Then* $\mathbb{T}^s_{\varphi'}\left(\prod_{i=1}^r \mathbb{G}(k_i,n)\right) = \mathbb{T}^s\left(\prod_{i=1}^r \mathbb{G}(k_i,n)\right)$ *for every* $s\leqslant r+\sum_{i=1}^r k_i$*. In particular,*

$$\frac{\partial^s\varphi'}{\partial x_{|I|}}(0) = \frac{\partial^{|J|}\varphi}{\partial x_{|J|}}(0)$$

for some J *with* $|J|\leqslant|I|$*.*

Proof. First, note that if for any $x^i_{l,m}\in x_{|I|}$ we have $m>k_i$, then $\frac{\partial^s\varphi'}{\partial x_{|I|}}(0)=\frac{\partial^{|I|}\varphi}{\partial x_{|I|}}(0)$ and we are done.

Now, let $2\leqslant i\leqslant r$ and consider a derivative $\frac{\partial^{|I|}\varphi'}{\partial x_{|I|}}(0)$ such that $x^i_{l,m}\in x_{|I|}$ with $m\leqslant k_i$. Therefore, to prove the statement it is enough to show that this partial derivative can be written in terms of another partial derivative $\dfrac{\partial^{|J|}\varphi'}{\partial x_{|J|}}(0)$ with $x^i_{l,m}\notin x_{|J|}$, $m\leqslant k_i$ and $|J|<|I|$.

Fix $2 \leqslant i \leqslant r$ and let $x^i_{l_1,m_1},...,x^i_{l_h,m_h},x^i_{l_{h+1},m_{h+1}},...,x^i_{l_b,m_b} \in x_{|I|}$ such that $m_a \leqslant k_i$ for every $a = 1,...,h$ and $b \leqslant k_i + 1$.

If $\frac{\partial^b \varphi'}{\partial x^i_{l_1,m_1} \cdots \partial x^i_{l_h,m_h} \partial x^i_{l_{h+1},m_{h+1}} \cdots \partial x^i_{l_b,m_b}}(0) \neq 0$ consider the minor M_{J^i} of M_i such that the monomial

$$x^i_{l_1,m_1} \dots x^i_{l_h,m_h} x^i_{l_{h+1},m_{h+1}} \dots x^i_{l_b,m_b}$$

appears in the expression of $\det(M_{J^i})$. Then, there exist variables

$$x^i_{\sigma_{J^i}(l_{h+1}),\sigma_{J^i}(m_{h+1})},...,x^i_{\sigma_{J^i}(l_b),\sigma_{J^i}(m_b)}$$

such that $x^i_{\sigma_{J^i}(l_{h+1}),\sigma_{J^i}(m_{h+1})} \cdots x^i_{\sigma_{J^i}(l_b),\sigma_{J^i}(m_b)}$ is also a monomial in $\det(M_J)$, where σ_{J^i} is a permutation on the indexes such that $\sigma_{J^i}(m_a) > k_i$ for all $h+1 \leqslant a \leqslant b$.

This shows that

$$\frac{\partial^m \varphi'}{\partial x^i_{l_1,m_1} \cdots \partial x^i_{l_h,m_h} \partial x^i_{l_{h+1},m_{h+1}} \cdots \partial x^i_{l_b,m_b}}(0) = \frac{\partial^m \varphi'}{\partial x^i_{\sigma_{J^i}(l_{h+1}),\sigma_{J^i}(m_{h+1})},...,\partial x^i_{\sigma_{J^i}(l_b),\sigma_{J^i}(m_b)}}(0).$$

We have decreased the number of variables involved and thus lowered the order of the derivatives. Finally, since $\frac{\partial \varphi}{\partial x^i_{l,m}}(0) = \frac{\partial \varphi'}{\partial x^i_{l,m}}(0)$ for $m > k_i$ we are done. □

Lemma 5.5.2. *Since $\overline{\varphi}$ is a sub-parametrization of φ' by the chain rule we have*

$$\frac{\partial^s \overline{\varphi}}{\partial x_{|I|}}(0) = \sum_{|K|} \frac{\partial^s \varphi'}{\partial x_{|K|}}(0) = \sum_{|J|} \frac{\partial^s \varphi}{\partial x_{|J|}}(0)$$

where $|K| = |I| = s$ and $|J| \leqslant |I|$. Let $\dfrac{\partial^s \overline{\varphi}}{\partial x_{|I|}}(0) \neq 0$ with $|I| = s$ such that for each $x^i_{l,m} \in x_{|I|}$ we have that $m > k_i$. Then, in the above decomposition there is at least a vector $\dfrac{\partial^s \varphi}{\partial x_{|J|}}(0)$ with $|J| = s$.

Proof. For any $x^i_{l,m} \in x_{|I|}$ let $h(m)$ be the maximum index in $\{1,...,r\}$ such that $m > k_{h(m)}$. Since for each $x^i_{l,m} \in x_{|I|}$ we have that $m > k_i$ and $\frac{\partial^s \overline{\varphi}}{\partial x_{|I|}}(0) \neq 0$, we get that any $x^i_{l,m} \in x_{|I|}$ appears at most $h(m)$ times in $x_{|I|}$.

Now, for any $s \leqslant h(m)$, the chain rule expression of $\frac{\partial^s \overline{\varphi}}{(\partial x^i_{l,m})^s}(0)$ contains the factor

$$\frac{\partial^s \varphi'}{\partial x^1_{l,m} \partial x^2_{l,m} \cdots \partial x^{h(m)}_{l,m}}(0) = \frac{\partial^s \varphi}{\partial x^1_{l,m} \partial x^2_{l,m} \cdots \partial x^{h(m)}_{l,m}}(0).$$

Repeating this argument for all indexes $x^i_{l,m} \in x_{|I|}$ we conclude. □

Proposition 5.5.3. *The flag variety is osculating well-behaved, that is*

$$\mathbb{T}^s_p \mathbb{F}(k_1,\dots,k_r;n) = \mathbb{T}^s_p \prod_{i=1}^r \mathbb{G}(k_i,n) \cap \mathbb{P}(\Gamma_a)$$

for any $p \in \mathbb{F}(k_1,\dots,k_r;n)$ and non-negative integer s.

Proof. We may assume that $p = e_I$ where $I = \{I^1, \dots, I^r\}$ and $I^l = \{0, \dots, k_l\}$ for each $1 \leqslant l \leqslant r$. Let us first assume that $s = r + \sum_{i=1}^r k_i$. Note that s is the smallest integer for which $\mathbb{T}_p^s\mathbb{F}(k_1, \dots, k_r; n) = \mathbb{P}(\Gamma_a)$ and $\mathbb{T}_p^s \prod_{i=1}^s \mathbb{G}(k_i, n) = \mathbb{P}^N$, in this case $\mathbb{T}_p^s\mathbb{F}(k_1, \dots, k_r; n) = \mathbb{P}(\Gamma_a) = \mathbb{P}(\Gamma_a) \cap \mathbb{P}^N = \mathbb{P}(\Gamma_a) \cap \mathbb{T}_p^s \prod_{i=1}^s \mathbb{G}(k_i, n)$ and we are done. Now, assume $s < r + \sum_{i=1}^r k_i$. Let

$$v = \sum_{|I| \leqslant s-1} \alpha_{|I|} \frac{\partial^{|I|}\varphi}{\partial x_{|I|}}(0) \tag{5.5.4}$$

be a general vector in $\mathbb{T}_p^{s-1} \prod_{i=1}^r \mathbb{G}(k_i, n)$, and assume that

$$v \in \mathbb{T}_p^{s-1} \prod_{i=1}^r \mathbb{G}(k_i, n) \cap \mathbb{P}(\Gamma_a) \subset \mathbb{T}_p^s \prod_{i=1}^r \mathbb{G}(k_i, n) \cap \mathbb{P}(\Gamma_a) = \mathbb{T}_p^s\mathbb{F}(k_1, \dots, k_r; n)$$

this yields that v can be written as

$$v = \sum_{|I| \leqslant s-1} \alpha_{|I|} \frac{\partial^{|I|}\overline{\varphi}}{\partial x_{|I|}}(0) + \sum_{|I|=s} \beta_{|I|} \frac{\partial^{s}\overline{\varphi}}{\partial x_{|I|}}(0). \tag{5.5.5}$$

Now, recall that for any I such that there are variables $x_{l,m}^i \in x_{|I|}$ with $m \leqslant k_i$ we can find another set J for which $|J| < |I|$ and

$$\frac{\partial^{s}\varphi'}{\partial x_{|I|}}(0) = \frac{\partial^{|J|}\varphi}{\partial x_{|J|}}(0).$$

Therefore, we can assume that any set I in the second summand of (5.5.5) is such that $m > k_i$ for any $x_{l,m}^i \in x_{|I|}$. Thus, by Lemma 6.2.6, we will have an equality in (5.5.4) and (5.5.5) if and only if $\beta_{|I|} = 0$ for any set I such that $m > k_i$ for all $x_{l,m}^i \in x_{|I|}$. Hence $v \in \mathbb{T}_p^{s-1}\mathbb{F}(k_1, \dots, k_r; n)$. □

5.6 Osculating Projections

Let $s_1, \dots, s_\alpha$ be integers such that $0 \leqslant s_m \leqslant r - 2 + \sum_{i=1}^r k_i$. Denote $\mathbb{T}_p^s\mathbb{F}(k_1, \dots, k_r; n)$ simply by $\mathbb{T}_p^s\mathbb{F}$ and the linear subspace $\langle \mathbb{T}_{e_{I_1}}^{s_1}\mathbb{F}, \dots, \mathbb{T}_{e_{I_m}}^{s_m}\mathbb{F}\rangle$ by $\mathbb{T}_{e_{I_1}, \dots, e_{I_m}}^{s_1, \dots, s_m}\mathbb{F}$. Then, for $m \leqslant \alpha$ we have the linear projection

$$\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_m}}^{s_1, \dots, s_m}\mathbb{F}} : \mathbb{F}(k_1, \dots, k_r; n) \dashrightarrow \mathbb{P}^{N_{s_1, \dots, s_m}}.$$

Proposition 5.6.1. *Let $I_1, \dots, I_\alpha$ be as in (5.2.3) and $s = r - 2 + \sum_{i=1}^r k_i$. Then,*

- $\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_{\alpha-1}}}^{s, \dots, s}\mathbb{F}}$ *is birational;*

- $\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_\alpha}}^{s, \dots, s}\mathbb{F}}$ *is birational whenever $n \geqslant k_r^2 + 3k_r + 1$.*

Proof. Since $\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_{\alpha-1}}}^{s, \dots, s}}$ factors trough $\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_{\alpha-1}}}^{s, \dots, s}\mathbb{F}}$, it is enough to show that the restriction of $\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_{\alpha-1}}}^{s, \dots, s}}$ to $\mathbb{F}(k_1, \dots, k_r)$ is birational.

For any $i \neq r$ and $1 \leqslant j \leqslant \alpha - 1$ consider $I'^i_j = I^i_j$ and $I'^r_j \subset I^r_j$ of cardinality k_r. Since $n \geqslant 2k_r + 1$ and $k_r \geqslant k_i$ we must have

$$n - \sum_{j=1}^{\alpha-1} |I'^i_j| \quad = \quad n - (\alpha - 1)(k_i + 1) \geqslant n - (\alpha - 1)k_r \geqslant k_r + 1 \leqslant k_i + 1 \ .$$

Now, let us denote by I'^i the union $\bigcup_{j=1}^{\alpha-1} I'^i_j$. Then, by Lemma 5.2.4 there exists a rational map $\pi_{I'^r}$ making the following diagram commutative

$$\begin{array}{ccc} \mathbb{F}(k_1, \ldots, k_r; n) & \xrightarrow{\Pi_{\mathbb{T}^{s,\ldots,s,s'}_{e_{I_1},\ldots,e_{I_{\alpha-1}},e_{I_\alpha}}}} & \mathbb{P}^{N'_{s,\ldots,s}} \\ & \searrow^{\prod_{i=1}^r \Pi_{I'^i}} & \downarrow \pi_{I'^r} \\ & & \prod_{i=1}^r \mathbb{G}(k_i, n - \sum_{j=1}^{\alpha} |I'^i_j|) \end{array}$$

Now, let $x = (\{V_i\}_{i=1}^r)$ be a general point in the image of $\Pi_{\mathbb{T}^{s,\ldots,s}_{e_{I_1},\ldots,e_{I_{\alpha-1}}}}$ and $X \subset \mathbb{F}(k_1, \ldots, k_r; n)$ be the fiber of $\Pi_{\mathbb{T}^{s,\ldots,s}_{e_{I_1},\ldots,e_{I_{\alpha-1}}}}$ over x. Set $x_{I'^r} = \pi_{I'^r}(x)$ and denote by $X_{I'^r} \subset \mathbb{F}(k_1, \ldots, k_r; n)$ the fiber of $\prod_{i=1}^r \Pi_{I'^i}$ over $x_{I'^r}$.

Therefore, $X \subset \bigcap_{I'^r} X_{I'^r}$, where the intersection runs over all sets $I'^r = \bigcup_{j=1}^{\alpha-1} I'^r_j$ with $I'^r_j \subset I^r_j$ and $|I'^r_j| = k_r$ for $1 \leqslant j \leqslant \alpha - 1$.

Now, note that if $\{W_i\}_{i=1}^r \in X$ is a general point, then we must have $W_i \subset \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^i_j \ ; \ V_i \rangle$ for any choice of $\bigcup_{j=1}^{\alpha} I'^i_j$. Hence,

$$W_i \subset \bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^i_j \ ; \ V_i \rangle. \tag{5.6.2}$$

In particular, $W_r \subset \bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^r_j ; V_r \rangle$. Now, since $|I'^r_j| \leqslant k_r$ we must have $\bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^r_j \rangle = \varnothing$ and then $V_r = \bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^r_j \ ; \ V_r \rangle$ which yields $W_r = V_r$.

Now, set $i \leqslant r - 1$. Since $\{V_i\}_{i \in K}$ is general in $\mathbb{F}(k_1, \ldots, k_r; n)$ and $n - \sum_{j=1}^{\alpha-1} |I'^i_j| \geqslant k_r + 1$ we have $V_r \cap \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I^i_j \rangle = \varnothing$. On the other hand $W_i \subset W_r = V_r$ for all $i \leqslant r - 1$, then $W_i \cap \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I^i_j \rangle = \varnothing$. Hence, by (5.6.2) we must have $W_i = V_i$ for any $i \leqslant r - 1$.

Now, assume that $n \geqslant k_r^2 + 3k_r + 1$ then

$$\begin{aligned} n - \alpha(k_i + 1) \geqslant n - \alpha k_r \quad & \geqslant \quad n - \frac{(n+1)}{k_r + 1} k_r = \frac{n(k_r + 1) - (n+1)k_r}{k_r + 1} \\ & = \quad \frac{n - k_r}{k_r + 1} \geqslant \frac{k_r^2 + 3k_r + 1 - k_r}{k_r + 1} = k_r + 1. \end{aligned}$$

Then, arguing as in the proof of the first case, for any choice of subsets $I'^i_j \subset I^i_j$, $I'^i_j = I^i_j$ with $i \neq r$ and $1 \leqslant j \leqslant \alpha - 1$, $I'^r_j \subsetneq I^r_j$ of cardinality k_r we get, by Lemma 5.2.4, a rational map $\pi_{I'^r}$ making the following diagram commutative

$$\begin{array}{ccc} \mathbb{F}(k_1, \dots, k_r; n) & \overset{\Pi_{\mathbb{T}_{e_{I_1}, \dots, e_{I_\alpha}}}}{\dashrightarrow} & \mathbb{P}^{N'_{s,\dots,s}} \\ & \searrow^{\prod_{i=1}^r \Pi_{I'^i}} & \downarrow^{\pi_{I'^r}} \\ & & \prod_{i=1}^r \mathbb{G}(k_i, n - \sum_{j=1}^\alpha |I'^i_j|) \end{array}$$

where $I'^i = \bigcup_{j=1}^\alpha I'^i_j$, $i = 1, \dots, r$. Now, to conclude, it is enough to follow the same argument used in the end of the proof of the first claim. □

5.7 Non-Secant defectiveness of Flag varieties

We recall [FMR20, Proposition 4.4] which describes how the notion of osculating regularity behaves under linear sections (see Chapter 1).

Proposition 5.7.1. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and $Y = \mathbb{P}^k \cap X$ a linear section of X that is osculating well-behaved. Assume that given general points $p_1, \dots, p_m \in Y$ one can find smooth curves $\gamma_j : C \to X, j = 2, \dots, m$, realizing the m-osculating regularity of X for $p_1, \dots, p_m$ such that $\gamma_j(C) \subset Y$. Then Y has m-osculating regularity as well. Furthermore, the analogous statement for strong 2-osculating regularity holds as well.*

Proposition 5.7.2. *The flag variety $\mathbb{F}(k_1, \dots, k_r; n)$ has strong 2-osculating regularity and α-osculating regularity, where $\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$.*

Proof. The statement follows immediately from Propositions 5.3.2, 5.3.6, 5.7.1. □

Now, we are ready to prove our main result on non-defectiveness of flag varieties.

Theorem 5.7.3. *Assume that $n \geqslant 2k_r + 1$. Set*

$$\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$$

and let h_α be as in Definition 1.5.4. If either

- $n \geqslant k_r^2 + 3k_r + 1$ *and* $h \leqslant \alpha h_\alpha(\sum k_j + r - 2)$ *or*
- $n < k_r^2 + 3k_r + 1$ *and* $h \leqslant (\alpha - 1)h_\alpha(\sum k_j + r - 2)$.

Then, $\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective. In particular, if

$$h \leqslant \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}$$

then $\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective.

Proof. The first part is an immediately consequence of Propositions 5.7.1, 5.6.1 and Theorem 5.4.2. For the last claim note that if we write

$$\sum k_j + r - 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon \tag{5.7.4}$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geqslant 1$ and $\varepsilon \in \{0,1\}$. Then

$$h_\alpha(\sum k_j + r - 2) = \alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1}.$$

Therefore, the first bound in Theorem 5.7.3 yields

$$h \leqslant \alpha^{\lambda_1} + \alpha^{\lambda_2} + \cdots + \alpha^{\lambda_l}.$$

Furthermore, by the second bound in Theorem 5.7.3 we get that $\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective for

$$h \leqslant (\alpha - 1)(\alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1}).$$

Finally, by (5.7.4) we get that $\lambda_1 = \lfloor \log_2(\sum k_j + r - 1) \rfloor$. Hence, asymptotically we have $h_\alpha(\sum k_j + r - 2) \sim \alpha^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}$, and by Theorem 5.7.3 for $h \leqslant \alpha^{\lfloor \log_2(\sum k_j + r - 1) \rfloor} \leqslant \left(\dfrac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}$ the flag variety $\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective. □

Remark 5.7.5. Now, given a flag $\mathbb{F}(k_1, \dots, k_r; n)$ with $n < 2k_r + 1$. Assume that $n \geqslant 2k_j + 1$ for some index j and let l be the maximum among these j's. Then we have a natural projection

$$\begin{array}{rccc} \pi & : & \mathbb{F}(k_1, \dots, k_r; n) & \longrightarrow & \mathbb{F}(k_1, \dots, k_l; n) \\ & & \{V_i\}_{i=1,\dots,r} & \longmapsto & \{V_i\}_{i=1,\dots,l}. \end{array}$$

The fiber of π over a general point in $\mathbb{F}(k_1, \dots, k_l; n)$ is isomorphic to $\mathbb{F}(k_{l+1} - k_l - 1, \dots, k_r - k_l - 1; n - k_l - 1)$. Now let $p_1, \dots, p_h \in \mathbb{F}(k_1, \dots, k_l; n)$ be general points, and $\mathbb{T}_{p_i}\mathbb{F}(k_1, \dots, k_l; n)$ be the tangent space at p_i. Then, we have

$$\mathbb{T}_{\pi^{-1}(p_i)}\mathbb{F}(k_1, \dots, k_r; n) = \left\langle \mathbb{T}_{p_i}\mathbb{F}(k_1, \dots, k_l; n), \mathbb{T}_{\pi^{-1}(p_i)}\mathbb{F}(k_{l+1} - k_l, \dots, k_r - k_l; n - k_l) \right\rangle$$

and $\mathbb{T}_{p_i}\mathbb{F}(k_1, \dots, k_l; n) \cap \mathbb{T}_{\pi^{-1}(p_i)}\mathbb{F}(k_{l+1} - k_l, \dots, k_r - k_l; n - k_l) = \varnothing$.

Now, observe that if $\mathbb{T}_{\pi^{-1}(p_i)}\mathbb{F}(k_1, \dots, k_r; n) \cap \mathbb{T}_{\pi^{-1}(p_j)}\mathbb{F}(k_1, \dots, k_r; n) \neq \varnothing$ then

$$\dim \left\langle \mathbb{T}_{\pi^{-1}(p_j)}\mathbb{F}(k_1, \dots, k_r; n) \; ; \; j = 1, \dots, h \right\rangle \leqslant h \dim \mathbb{F}(k_1, \dots, k_l; n) + h - 2.$$

Since $\mathbb{T}_{\pi^{-1}(p_i)}\mathbb{F}(k_{l+1} - k_l - 1, \dots, k_r - k_l - 1; n - k_l - 1)$ is contracted by π for any $j = 1, \dots, h$ we have that

$$\begin{array}{rcl} \dim \pi(T) & \leqslant & h \dim \mathbb{F}(k_1, \dots, k_r; n) + h - 2 - h \dim \mathbb{F}(k_{l+1} - k_l, \dots, k_r - k_l; n - k_l) \\ & = & h \dim \mathbb{F}(k_1, \dots, k_l; n) + h - 2 \end{array}$$

where $T = \left\langle \mathbb{T}_{\pi^{-1}(p_i)}\mathbb{F}(k_1, \dots, k_r; n) \; ; \; i = 1, \dots, h \right\rangle$.

In particular, by Terracini's lemma [Ter11] we have that if $\mathbb{F}(k_1, \dots, k_l; n)$ is not h-defective, then $\mathbb{F}(k_1, \dots, k_r; n)$ is not h-defective.

Theorem 5.7.6. *Consider a flag variety $\mathbb{F}(k_1, \ldots, k_r; n)$ with $n < 2k_r + 1$. Assume that $n \geqslant 2k_j + 1$ for some index j and let l be the maximum among these j's. Then, for*

$$h \leqslant \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1) \rfloor}$$

$\mathbb{F}(k_1, \ldots, k_r; n)$ is not $(h+1)$-defective.

Proof. It is an immediate consequence of Theorem 5.7.3 and Remark 5.7.5. □

5.8 On identifiability of products of Grassmannians and Flag varieties

Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerated variety. A point $p \in \mathbb{P}^N$ is said to be h-identifiable, with respect to X, if it lies on a unique $(h-1)$-plane h-secant to X. Furthermore, X is said to be h-identifiable if a general point of $\mathbb{S}ec_h(X)$ is h-identifiable.

Now, we combine our bounds on non-secant defectiveness of products of Grassmannians and flag varieties and [CM19, Theorem 3] to get the following.

Corollary 5.8.1. *Consider the product of Grassmannians $\prod_{i=1}^{r} \mathbb{G}(k_i, n)$. Assume that*

$$2\prod_{i=1}^{r}(k_i+1)(n-k_i) - 1 \leqslant \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_i + r - 1) \rfloor}$$

Then, $\prod_{i=1}^{r} \mathbb{G}(k_i, n)$ is h-identifiable for $h \leqslant \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_i + r - 1) \rfloor}$.

Furthermore, let us suppose that $n \geqslant 2k_j + 1$ for some index j and consider l the maximum among these j's. Assume that

$$2((k_1+1)(n-k_1) + \sum_{j=2}^{i}(n-k_j)(k_j-k_{j-1})) - 1 \leqslant \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1) \rfloor}$$

Then $\mathbb{F}(k_1, \ldots, k_r; n)$ is h-identifiable for $h \leqslant \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1) \rfloor}$.

Proof. It is enough to apply Corollary 5.4.5, Theorem 5.7.6 and [CM19, Theorem 3]. □

5.9 On the chordal variety of $\mathbb{F}(0, k; n)$

In this section we consider the particular case of flag varieties parametrizing chains of type $p \in H^k \subset \mathbb{P}^n$.

Proposition 5.9.1. *Let us consider the flag variety $\mathbb{F}(0, k; n) \subset \mathbb{P}(\Gamma) \subset \mathbb{P}^N$, where $0 < k < n$. Then, $\mathbb{S}ec_2\mathbb{F}(0, k; n)$ has always the expected dimension except when $k = n-1$, in this case $\mathbb{F}(0, n-1; n)$ is 2-defective with 2-defect $\delta_2(\mathbb{F}(0, n-1; n)) = 1$.*

Proof. Let $p, q \in \mathbb{F}(0,k;n)$ be two general points, without loss of generality we can assume that $p = e_{0,\{0,\dots,k\}} = e_{0,I_0}$ and $q = e_{n,\{n-k,\dots,n\}} = e_{n,I_1}$.

Now, Proposition 5.5.3 yields that

$$\mathbb{T}_{e_{0,I_0}}\mathbb{F}(0,k;n) \quad = \quad \langle e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1\rangle \cap \mathbb{P}(\Gamma)$$

and

$$\mathbb{T}_{e_{n,I_1}}\mathbb{F}(0,k;n) \quad = \quad \langle e_{i,I} \mid d((i,I),(n,I_n)) \leqslant 1\rangle \cap \mathbb{P}(\Gamma).$$

Note that $d((i,I),(0,I_0)) = 1$ if and only if either $i \neq 0$ and $I = I_0$ or $i = 0$ and $|I \cap I_0| = k$. Similarly, $d((i,I),(n,I_1)) = 1$ if and only if either $i \neq n$ and $I = I_1$ or $i = n$ and $|I \cap I_1| = k$. Therefore, since $n \neq 0$ and $I_1 \neq I_0$ we have that $e_{i,I} \in \{e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1\} \cap \{e_{i,I} \mid d((i,I),(n,I_1)) \leqslant 1\}$ if and only if either $I = I_0$ and $i = n$ or $I = I_1$ and $i = 0$.

Now, assume that $I = I_0$ and $i = n$, this is $e_{i,I} \in \mathbb{T}_{e_{0,I_0}}\mathbb{F}(0,k;n) \cap \mathbb{T}_{e_{n,I_1}}\mathbb{F}(0,k;n)$, in particular we have $|I \cap I_1| = |I_0 \cap I_1| = k$ and hence $\{1,\dots,k\} \subset I_1$ once $0 \notin I_1$. So we must have $k = n-1$. Similarly, if $I = I_1$ and $i = 0$ we conclude that $k = n-1$.

Therefore, if $k < n-1$, we get

$$\{e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1\} \cap \{e_{i,I} \mid d((i,I),(n,I_1)) \leqslant 1\} = \varnothing$$

and hence

$$\{e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1\} \cap \{e_{i,I} \mid d((i,I),(n,I_1)) \leqslant 1\} \cap \mathbb{P}(\Gamma) = \varnothing$$

which implies that

$$\dim \left\langle \mathbb{T}_{e_{0,I_0}}\mathbb{F}(0,k;n), \mathbb{T}_{e_{n,I_1}}\mathbb{F}(0,k;n) \right\rangle = 2\dim \mathbb{F}(0,k;n) + 1.$$

So, Terracini's lemma [Ter11] yields that $\mathbb{S}ec_2\mathbb{F}(0,k;n)$ has the expected dimension whenever $k < n-1$.

Now, assume that $k = n-1$. In this case we have

$$\{e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1\} \cap \{e_{i,I} \mid d((i,I),(n,I_1)) \leqslant 1\} = \{e_{0,\{1,\dots,n\}}, e_{n,\{0,\dots,n-1\}}\}.$$

Furthermore, $\mathbb{F}(0,n-1;n)$ is the hypersurface cut out in $\mathbb{P}^n \times \mathbb{P}^{n*}$ by

$$\sum_{i=0}^{n} (-1)^i Z_{i,I_n\setminus\{i\}} = 0$$

where $I_n = \{0,\dots,n\}$.

Therefore, we get that $\mathbb{T}_{e_{0,I_0}}\mathbb{F}(0,n-1;n) = \langle e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1\rangle \cap \mathbb{P}(\Gamma)$ is given by

$$\left\langle e_{0,\{1,\dots,n\}} + (-1)^{n+1} e_{n,\{0,\dots,n-1\}}\ ;\ e_{i,I} \mid d((i,I),(0,I_0)) \leqslant 1 \text{ and } i,I \neq \left\{ \begin{array}{l} 0,\{1,\dots,n\} \\ n,\{1,\dots,n-1\} \end{array} \right. \right\rangle$$

and $\mathbb{T}_{e_{n,I_1}}\mathbb{F}(0,n-1;n) = \langle e_{i,I} \mid d((i,I),(n,I_1)) \leqslant 1\rangle \cap \mathbb{P}(\Gamma)$ is given by

$$\left\langle e_{0,\{1,\dots,n\}} + (-1)^{n+1} e_{n,\{0,\dots,n-1\}}\ ;\ e_{i,I} \mid d((i,I),(n,I_1)) \leqslant 1 \text{ and } i,I \neq \left\{ \begin{array}{l} 0,\{1,\dots,n\} \\ n,\{1,\dots,n-1\} \end{array} \right. \right\rangle.$$

Therefore,

$$\dim\left\langle \mathbb{T}_{e_{0,I_0}}\mathbb{F}(0, n-1; n), \mathbb{T}_{e_{n,I_1}}\mathbb{F}(0, n-1; n)\right\rangle = 2\dim\mathbb{F}(0, n-1; n) < \operatorname{expdim}\mathbb{S}ec_2\mathbb{F}(0, n-1; n).$$

Finally, since $\operatorname{expdim}\mathbb{S}ec_2\mathbb{F}(0, k; n) = 2\dim\mathbb{F}(0, n-1; n) + 1$ we have that $\mathbb{F}(0, n-1, n)$ is 2-defective with 2-defect $\delta_2(\mathbb{F}(0, n-1; n)) = 1$. $\square$

Chapter 6

On (h,s)-tangential weak defectiveness

Identifiability has been related to the concept of weak defectiveness in [Mel06], and more recently to the notion of tangential weak defectiveness in [CO12].

We introduce the concept of (h,s)*-tangential weakly defectiveness*, where h,s are positive integers. A variety $X \subset \mathbb{P}^N$ is (h,s)-tangentially weakly defective if a general linear subspace of dimension s, which is tangent to X at h general points $x_1,\dots,x_h \in X$, is tangent to X along a positive dimensional subvariety of X containing at least one of the x_i. In particular, when $s = \dim\langle \mathbb{T}_{x_1}X,\dots,\mathbb{T}_{x_h}X\rangle$ we recover the notion of *h-tangential weak defectiveness* while for $s = N-1$ we get the notion of *h-weak defectiveness*.

6.1 Osculating regularity and $(h,s)-$tangential weak defectiveness

We start introducing a notion that measures how much a h-weakly defective variety is far from being h-tangentially weakly defective.

Definition 6.1.1. Let $x_1,\dots,x_h \in X$ be general points and $\Pi \subset \mathbb{P}^N$ a linear subspace of dimension s containing $\langle \mathbb{T}_{x_1}X,\dots,\mathbb{T}_{x_h}X\rangle$. The (h,s)-tangential contact locus $\Gamma_{x_1,\dots,x_h,\Pi}$ of X with respect to $x_1,\dots,x_h,\Pi$ is the closure in X of the union of all the irreducible components which contain at least one of the x_i of the locus of points of X where Π is tangent to X. Let $\gamma_{x_1,\dots,x_h,\Pi}$ be the largest dimension of the components of $\Gamma_{x_1,\dots,x_h,\Pi}$. If $\gamma_{x_1,\dots,x_h,\Pi} > 0$ for Π general, we say that X is (h,s)-tangentially weakly defective.

In particular, when $s = \dim\langle \mathbb{T}_{x_1}X,\dots,\mathbb{T}_{x_h}X\rangle$ from Definition 6.1.1 we recover the notion of h-tangential weak defectiveness while for $s = N-1$ we get the notion of h-weak defectiveness.

We begin by proving a simple result on the behavior of contact loci under flat degenerations.

Lemma 6.1.2. *Let $X \subset \mathbb{P}^N$ be a projective variety, $\Delta \subset \mathbb{C}$ a complex disk around the origin and $\{\Pi_t\}_{t\in\Delta}$ a family of linear subspaces of $\mathbb{P}^N$. Then*

$$\dim(\mathrm{Sing}(\Pi_0 \cap X)) \geqslant \dim(\mathrm{Sing}(\Pi_t \cap X))$$

for $t \in \Delta$.

Furthermore, let $\{\Gamma_t\}_{t\in\Delta}$ be a family of linear subspaces $\Gamma_t \subset \mathbb{P}^N$, $\Lambda \subset \mathbb{P}^N$ a linear subspace containing Γ_0, and Π a linear subspace containing Λ. Then

$$\dim(\mathrm{Sing}(\widetilde{\Pi}_t \cap X)) \leqslant \dim(\mathrm{Sing}(\Pi \cap X))$$

where $\widetilde{\Pi}_t$ is a general linear subspace of dimension $\dim(\Pi)$ containing Γ_t.

Proof. For the first claim it is enough to consider the variety

$$Y = \{(x,t) \mid x \in \mathrm{Sing}(X \cap \Pi_t)\} \subset X \times \Delta$$

with projection $\pi_2 : Y \to \Delta$ and to conclude by semi-continuity.

For the second part note that since $\Gamma_0 \subseteq \Lambda$ we have that $\Gamma_0 \subseteq \Pi$. Let $\Gamma' \subset \Pi$ be a subspace such that $\Pi = \langle \Gamma_0, \Gamma' \rangle$, $\Gamma' \cap \Gamma_0 = \varnothing$, and set $\Pi_t = \langle \Gamma_t, \Gamma' \rangle$. Then $\{\Pi_t\}_{t\in\Delta}$ is a family of linear subspace such that $\Gamma_t \subset \Pi_t$ for all $t \in \Delta$. By the first part of the proof we have $\dim(\mathrm{Sing}(\Pi \cap X)) \geqslant \dim(\mathrm{Sing}(\Pi_t \cap X))$ for all $t \in \Delta$. Now, consider the Grassmannian $\mathbb{G}(\dim(\Pi)-\dim(\Gamma_t)-1, N-\dim(\Gamma_t)-1)$ parametrizing $\dim(\Pi)$-dimensional linear subspaces of $\mathbb{P}^N$ containing Γ_t, and the variety

$$Z = \{(x, \widetilde{\Pi}_t) \mid x \in \mathrm{Sing}(\widetilde{\Pi}_t \cap X)\} \subseteq X \times \mathbb{G}(\dim(\Pi) - \dim(\Gamma_t) - 1, N - \dim(\Gamma_t) - 1)$$

with projection $\pi_2 : Z \to \mathbb{G}(\dim(\Pi)-\dim(\Gamma_t)-1, N-\dim(\Gamma_t)-1)$. Again by semi-continuity we have

$$\dim(\mathrm{Sing}(\widetilde{\Pi}_t \cap X)) \leqslant \dim(\mathrm{Sing}(\Pi_t \cap X))$$

for $\widetilde{\Pi}_t \in \mathbb{G}(\dim(\Pi) - \dim(\Gamma_t) - 1, N - \dim(\Gamma_t) - 1)$ general, and hence $\dim(\mathrm{Sing}(\Pi \cap X)) \geqslant \dim(\mathrm{Sing}(\Pi_t \cap X)) \geqslant \dim(\mathrm{Sing}(\widetilde{\Pi}_t \cap X))$. □

We are ready to prove the main result of this section relating osculating regularity to tangential weak defectiveness.

Theorem 6.1.3. *Let $X \subset \mathbb{P}^N$ be a projective variety having m-osculating regularity and strong 2-osculating regularity. Assume that there exist integers $l, k_1, \dots, k_l \geqslant 1$, general points $p_1, \dots, p_l \in X$ and a linear subspace of dimension s containing $\langle \mathbb{T}^{k_1}_{p_1}, \dots, \mathbb{T}^{k_l}_{p_l} \rangle$ that is not tangent to X along a positive dimensional subvariety. Set*

$$h := \sum_{j=1}^{l} h_m(k_j)$$

Then X is not (h,s)-tangentially weakly defective.

Proof. Let us consider the linear span

$$T=\left\langle \mathbb{T}^1_{p_1^1},\dots,\mathbb{T}^1_{p_1^{h_m(k_1)}},\dots,\mathbb{T}^1_{p_l^1},\dots,\mathbb{T}^1_{p_l^{h_m(k_l)}}\right\rangle$$

and $p_1^1=p_1,\dots,p_l^1=p_l$. For seek of notational simplicity along the proof we will assume $l=1$. For the general case it is enough to apply the same argument l times.

Let us begin with the case $k_1+1=2^\lambda$. Then $h_m(k_1)=m^{\lambda-1}$. Since X has m-osculating regularity we can degenerate T, in a family parametrized by a smooth curve, to a linear space U_1 contained in

$$V_1=\left\langle \mathbb{T}^3_{p_1^1},\mathbb{T}^3_{p_1^{m+1}},\dots,\mathbb{T}^3_{p_1^{m^{\lambda-1}-m+1}}\right\rangle$$

Again, since X has m-osculating regularity we may specialize, in a family parametrized by a smooth curve, the linear space V_1 to a linear space U_2 contained in

$$V_2=\left\langle \mathbb{T}^7_{p_1^1},\mathbb{T}^7_{p_1^{m^2+1}},\dots,\mathbb{T}^7_{p_1^{m^{\lambda-1}-m^2+1}}\right\rangle$$

Proceeding recursively in this way in last step we get a linear space $U_{\lambda-1}$ which is contained in

$$V_{\lambda-1}=\mathbb{T}^{2^\lambda-1}_{p_1^1}$$

Now, more generally, let us assume that

$$k_1+1=2^{\lambda_1}+\cdots+2^{\lambda_a}+\varepsilon$$

with $\varepsilon\in\{0,1\}$, and $\lambda_1>\lambda_2>\cdots>\lambda_a\geqslant 1$. Then

$$h_m(k_1)=m^{\lambda_1-1}+\cdots+m^{\lambda_a-1}$$

By applying a times the argument for $k_1+1=2^\lambda$ in the first part of the proof we may specialize T to a linear space U contained in

$$V=\left\langle \mathbb{T}^{2^{\lambda_1}-1}_{p_1^1},\mathbb{T}^{2^{\lambda_2}-1}_{p_1^{m^{\lambda_1-1}+1}},\dots,\mathbb{T}^{2^{\lambda_a}-1}_{p_1^{m^{\lambda_1-1}+\cdots+m^{\lambda_{a-1}-1}+1}}\right\rangle$$

Finally, using that X has strong 2-osculating regularity $a-1$ times we specialize V to a linear space $U^{'}$ contained in

$$V^{'}=\mathbb{T}^{2^{\lambda_1}+\cdots+2^{\lambda_a}-1}_{p_1^1}$$

Note that $\mathbb{T}^{2^{\lambda_1}+\cdots+2^{\lambda_a}-1}_{p_1^1}=\mathbb{T}^{k_1}_{p_1^1}$ if $\varepsilon=0$, and $\mathbb{T}^{2^{\lambda_1}+\cdots+2^{\lambda_a}-1}_{p_1^1}=\mathbb{T}^{k_1-1}_{p_1^1}\subset\mathbb{T}^{k_1}_{p_1^1}$ if $\varepsilon=1$. In any case, since by hypothesis there is an s-dimensional linear subspace containing $\langle\mathbb{T}^{k_1}_{p_1},\dots,\mathbb{T}^{k_l}_{p_l}\rangle$ that is not tangent to X along a positive dimensional subvariety we conclude by Lemma 6.1.2. □

6.2 On tangential weak defectiveness of Segre-Veronese varieties

Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers, with $n_1 \leqslant \cdots \leqslant n_r$ and $d = d_1 + \cdots + d_r \geqslant 3$. Let $SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$, where $N(\boldsymbol{n},\boldsymbol{d}) = \prod_{i=1}^{r} \binom{n_i+d_i}{d_i} - 1$, be the corresponding Segre-Veronese variety that is the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ embedded by the complete linear system $\left|\mathcal{O}_{\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}}(d_1, \ldots, d_r)\right|$. We recall the notion of distance for Segre-Veronese varieties given in [AMR19, Definition 2.4].

Definition 6.2.1. Let n and d be positive integers, and set

$$\Lambda_{n,d} = \{I = \{i_1, \ldots, i_d\}, 0 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant n\}$$

For $I, J \in \Lambda_{n,d}$, we define their distance $d(I,J)$ as the number of different coordinates. More precisely, write $I = \{i_1, \ldots, i_d\}$ and $J = \{j_1, \ldots, j_d\}$. There are $r \geqslant 0$ distinct indexes $\lambda_1, \ldots, \lambda_r \subset \{1, \ldots, d\}$ and distinct indexes $\tau_1, \ldots, \tau_r \subset \{1, \ldots, d\}$ such that $i_{\lambda_k} = j_{\tau_k}$ for every $1 \leqslant k \leqslant r$, and

$$\{i_\lambda \mid \lambda \neq \lambda_1, \ldots, \lambda_r\} \cap \{j_\tau \mid \tau \neq \tau_1, \ldots, \tau_r\} = \varnothing$$

Then $d(I,J) = d - r$. Now, set

$$\Lambda = \Lambda_{\boldsymbol{n},\boldsymbol{d}} = \Lambda_{n_1,d_1} \times \cdots \times \Lambda_{n_r,d_r}$$

For $I = (I^1, \ldots, I^r), J = (J^1, \ldots, J^r) \in \Lambda$, we define their distance as

$$d(I,J) = d(I^1, J^1) + \cdots + d(I^r, J^r)$$

Such a distance, called the Hamming distance, was defined in [CGG02, Section 2] for Segre varieties. We will denote the homogeneous coordinates and the corresponding coordinate points of $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ by X_J and e_J respectively, for $J \in \Lambda$.

Proposition 6.2.2. *Let $p_0, \ldots, p_{n_1} \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ be general points. If $d := \min\{d_1, \ldots, d_r\} \geqslant 2$ then a general hyperplane $H \subset \mathbb{P}^N$ containing $T = \langle \mathbb{T}_{p_0}^{d-1} SV_{\boldsymbol{d}}^{\boldsymbol{n}}, \ldots, \mathbb{T}_{p_{n_1}}^{d-1} SV_{\boldsymbol{d}}^{\boldsymbol{n}} \rangle$ is not tangent to $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ along a positive dimensional subvariety.*

Proof. Since $PGL(n_1+1) \times \cdots \times PGL(n_r+1)$ acts transitively on $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ we may assume that $p_i = e_{I_i}$, where $I_i = (\{i, \ldots, i\}, \ldots, \{i, \ldots, i\})$. By [AMR19, Proposition 2.5] $\mathbb{T}_{e_{I_i}}^{d-1} = \langle e_J \mid d(I_i, J) \leqslant d-1 \rangle$, and hence

$$\begin{aligned} \langle \mathbb{T}_{e_{I_0}}^{d-1}, \ldots, \mathbb{T}_{e_{I_{n_1}}}^{d-1} \rangle &= \langle e_J \mid d(I_i, J) \leqslant d-1 \text{ for some } i = 0, \ldots n_1 \rangle \\ &= \{X_J = 0 \mid d(I_i, J) > d-1 \text{ for all } i = 0, \ldots n_1\} \end{aligned}$$

Now, let $H \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ be a general hyperplane containing T. We have that H is given by an equation of type

$$\sum_{J \in \Lambda \mid d(I_i,J) > d-1, \forall i = 0, \ldots, n_1} \alpha_J X_J = 0, \ \ \alpha_J \in \mathbb{C} \tag{6.2.3}$$

Let us denote by $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-\dim(T)-1}$ the projective space whose homogeneous coordinates are the α_J with $J \in \Lambda$ and $d(I_i, J) > d-1$ for all $i = 0, \dots, n_1$. Now, for each fixed $i = 0, \dots, n_1$ we consider the following subset of Λ: for each $1 \leqslant l \leqslant r$ and $0 \leqslant j \leqslant n_l$ with $j \neq i$ let

$$J_{i,j,l} = (J_1, \dots, J_r) \in \Lambda \text{ where } J_l = \{j, \dots, j\} \text{ and } J_k = \{i, \dots, i\} \text{ for } k \neq l$$

and set $\Lambda_i = \{J_{i,j,l} \in \Lambda \mid \text{ for all } 1 \leqslant l \leqslant r \text{ and } 0 \leqslant j \leqslant n_l \text{ with } j \neq i\}$.

Observe that, since $d = \min\{d_i\}$ and $j \neq i$, each $J \in \Lambda_i$ satisfies $d(I_i, J) \geqslant d > d-1$ for all $i = 0, \dots, n_1$. Consider the projection

$$\begin{array}{cccc} \pi_i & : & \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-\dim(T)-1} & \dashrightarrow & \mathbb{P}^{\sum_{i\neq j} n_j} \\ & & (\alpha_J)_{J\in\Lambda \mid d(I_l,J)>d-1\, l=0,\dots,n_1} & \longmapsto & (\alpha_J)_{J\in\Lambda_i} \end{array}$$

the point $[1 : \dots : 1] \in \mathbb{P}^{\sum_{j\neq i} n_j}$ and let $H \in \pi_i^{-1}([1 : \dots : 1])$ be the hyperplane given by $\sum_{J\in\Lambda_i} X_J = 0$. The intersection $H \cap SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ corresponds to the hypersurface

$$\sum_{J\in\Lambda_i} X_{1,i}^{d_1} \cdots X_{l,j}^{d_l} \cdots X_{r,i}^{d_r} = 0 \tag{6.2.4}$$

where $X_{l,j}$ for $j = 0, \dots, n_l$ are the homogeneous coordinates on $\mathbb{P}^{n_l}$. Thus, in the affine chart $X_{1,i} = \dots = X_{r,i} = 1$ equation (6.2.4) becomes

$$\sum_{\substack{1\leqslant l\leqslant r \\ 0\leqslant j\leqslant n_l,\, j\neq i}} X_{l,j}^{d_l} = 0 \tag{6.2.5}$$

The singular locus of $H \cap SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ in the affine chart $X_{1,i} = \dots = X_{r,i} = 1$ is given by the following system of equations

$$\{d_l X_{l,j}^{d_l-1} = 0\}_{1\leqslant l\leqslant r,\, 0\leqslant j\leqslant n_l,\, j\neq i}$$

The only solution of this system is $X_{l,j} = 0$, and so the hypersurface (6.2.5) is singular only at $p_0 = (0, \dots, 0)$. Therefore, we conclude that the intersection of $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ with a general hyperplane H containing T is singular, in a neighborhood of p_0, only at p_0. Since this argument holds for each $i = 0, \dots, n_1$ using Lemma 6.1.2 we get the claim. □

Proposition 6.2.6. *Let $p_0, \dots, p_{n_1} \in SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ be general points and assume that $d = d_1 \leqslant d_i-2$ for each $i \neq 1$. Then a general hyperplane $H \subset \mathbb{P}^N$ containing $T = \langle \mathbb{T}^d_{p_0} SV^{\boldsymbol{n}}_{\boldsymbol{d}}, \dots, \mathbb{T}^d_{p_{n_1}} SV^{\boldsymbol{n}}_{\boldsymbol{d}}\rangle$ is not tangent to $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ along a positive dimensional subvariety.*

Proof. As in Proposition 6.2.2 we may assume that $p_i = e_{I_i}$, with $I_i = (\{i, \dots, i\}, \dots, \{i, \dots, i\})$. By [AMR19, Proposition 2.5] $\mathbb{T}^d_{e_{I_i}} = \langle e_J \mid d(I_i, J) \leqslant d\rangle$. Hence

$$\begin{array}{rcl} \langle \mathbb{T}^d_{e_{I_0}}, \dots, \mathbb{T}^d_{e_{I_{n_1}}} \rangle & = & \langle e_J \mid d(I_i, J) \leqslant d \text{ for some } i = 0, \dots n_1\rangle \\ & = & \{X_J = 0 \mid d(I_i, J) > d \text{ for all } i = 0, \dots n_1\} \end{array}$$

Now, let $H \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ be a general hyperplane containing T. We have that H is given by an equation of type

$$\sum_{J\in\Lambda \mid d(I_i,J)>d, \forall\, i=0,\dots,n_1} \alpha_J X_J = 0, \ \alpha_J \in \mathbb{C}$$

Let us denote by $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-\dim(T)-1}$ the projective space whose homogeneous coordinates are the α_J with $J \in \Lambda$ and $d(I_i,J) > d$ for all $i = 0,\dots,n_1$. Now, for each fixed $i = 0,\dots,n_1$ we consider the following subset of Λ: for each $2 \leqslant l \leqslant r$ and $0 \leqslant j \leqslant n_l$ with $j \neq i$ set

$$J_{i,j,l} = (J_1,\dots,J_r) \in \Lambda \ \text{ where } \ J_l = \{i,j,\dots,j\},\ J_k = \{i,\dots,i\} \ \text{ for } \ k \neq l$$

and $\Lambda_{i,1} = \{J_{i,j,l} \in \Lambda \mid \text{ for all } \ j,l \neq i\}$.

Moreover, we also consider another subset of Λ defined as follows: for each $0 \leqslant j \leqslant n_1$ with $j \neq i$ let

$$J_{i,j} = (J_1,\dots,J_r) \in \Lambda \text{ where } \ J_1 = \{j,\dots,j\},\ \ J_2 = \{j,i,\dots,i\},\ J_k = \{i,\dots,i\} \ \text{ for } \ k \neq 1,2$$

and $\Lambda_{i,2} = \{J_{i,j,l} \in \Lambda \mid \text{ for all } \ j,l \neq i\}$, $\Lambda_i = \Lambda_{i,1} \cup \Lambda_{i,2}$.

Observe that, since $d = d_1 < d_i - 2$ for $i \neq 1$ and $j \neq i$, each $J \in \Lambda_i$ satisfies $d(I_l,J) \geqslant d+1 > d$ for all $l = 0,\dots,n_1$. Therefore, we have a projection

$$\begin{array}{cccc} \pi_i & : & \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-\dim(T)-1} & \dashrightarrow & \mathbb{P}^{\sum_{i\neq j} n_j} \\ & & (\alpha_J)_{J\in\Lambda \,\mid\, d(I_l,J)>d\, l=0,\dots,n_1} & \longmapsto & (\alpha_J)_{J\in\Lambda_i} \end{array}$$

Now, consider the point $[1:\cdots:1] \in \mathbb{P}^{\sum_{j\neq i} n_j}$ and let $H \in \pi_i^{-1}([1:\cdots:1])$ be the hyperplane given by

$$\sum_{J\in\Lambda_i} X_J = 0$$

The intersection $H \cap SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ corresponds to the hypersurface

$$\sum_{J\in\Lambda_{i,1}} X_{1,i}^{d_1}\cdots X_{l,i}X_{l,j}^{d_l-1}\cdots X_{r,i}^{d_r} + \sum_{J\in\Lambda_{i,2}} X_{1,j}^{d_1}X_{2,j}X_{2,i}^{d_2-1}X_{3,i}^{d_3}\cdots X_{r,i}^{d_r} = 0 \tag{6.2.7}$$

where $X_{j,i}$, $i = 0,\dots,n_j$, are the homogeneous coordinates on $\mathbb{P}^{n_j}$. Thus, in the affine chart $X_{1,i} = \cdots = X_{r,i} = 1$ the equation (6.2.7) becomes

$$F = \sum_{\substack{2\leqslant l\leqslant r\\ 0\leqslant j\leqslant n_l,\, j\neq i}} X_{l,j}^{d_l-1} + \sum_{0\leqslant j\leqslant n_1,\, j\neq i} X_{1,j}^{d_1}X_{2,j} = 0 \tag{6.2.8}$$

The system of the partial derivatives of F is given by

$$\begin{cases} d_1X_{1,j}^{d_1-1}X_{2,j} = 0 \\ (d_2-1)X_{2,j}^{d_2-2} + X_{1,j}^{d_1} = 0 \\ (d_l-1)X_{l,j}^{d_l-2} = 0,\ l = 3,\dots,r \ \text{ and } \ j \neq i \end{cases}$$

This system has a solution only when all the coordinates $X_{l,j}$ vanish, and so the hypersurface $\{F = 0\}$ in (6.2.8) is singular only at $p_0 = (0,\dots,0)$. Therefore, we conclude that for a general hyperplane H containing T the hypersurface $H \cap SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ is singular, in a neighborhood of p_0, only at p_0. Since this argument holds for each $i = 0,\dots,n_1$ using Lemma 6.1.2 we get the statement. □

Theorem 6.2.9. *Set* $d := \min\{d_1, \dots, d_r\}$*. If*

- $h \leqslant (n_1+1)h_{n_1+1}(d-1)$ *or*
- $h \leqslant (n_1+1)h_{n_1+1}(d)$ *and* $d = d_1 \leqslant d_i - 2$ *for each* $2 \leqslant i \leqslant r$

then $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ *is not h-weakly defective.*

Proof. Since by [AMR19, Propositions 5.1, 5.10] the Segre-Veronese variety $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ has strong 2-osculating regularity and (n_1+1)-osculating regularity, the statement follows immediately from Propositions 6.2.2, 6.2.6 and Theorem 6.1.3. □

Remark 6.2.10. Write $d = 2^{\lambda_1} + 2^{\lambda_2} \dots + 2^{\lambda_s} + \epsilon$ with $\lambda_1 > \lambda_2 > \dots > \lambda_s \geqslant 1$ and $\epsilon \in \{0,1\}$, so that $\lambda_1 = \lfloor \log_2(d) \rfloor$. The first part of Theorem 6.2.9 says that $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ is not h-weakly defective for $h \leqslant (n_1+1)((n_1+1)^{\lambda_1-1} + (n_1+1)^{\lambda_2-1} + \dots + (n_1+1)^{\lambda_s-1})$.

Now, write $d+1 = 2^{\lambda_1} + 2^{\lambda_2} \dots + 2^{\lambda_s} + \epsilon$ with $\lambda_1 > \lambda_2 > \dots > \lambda_s \geqslant 1$ and $\epsilon \in \{0,1\}$, hence $\lambda_1 = \lfloor \log_2(d+1) \rfloor$. The second part of Theorem 6.2.9 yields that $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ is not h-weakly defective for $h \leqslant (n_1+1)((n_1+1)^{\lambda_1-1} + (n_1+1)^{\lambda_2-1} + \dots + (n_1+1)^{\lambda_s-1})$. Therefore, we have that asymptotically for

$$h \leqslant (n_1+1)^{\lfloor \log_2(d) \rfloor}$$

$SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ is not h-weakly defective.

6.3 On 1-weak defectiveness of Segre-Veronese varieties

In this section we give condition ensuring that Segre-Veronese varieties are not 1-weakly defective. Note that this yields that their dual varieties are hypersurfaces.

Proposition 6.3.1. *If* $n_r \leqslant \sum_{i=1}^{r-1} n_i$ *then* $SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ *is not* 1*-weakly defective.*

Proof. First of all, let us consider the Segre embedding of $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$, that is $\boldsymbol{d} = (1, \dots, 1)$. Let $p \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}$ be a general point, without loss of generality we may assume that $p = e_{0,\dots,0}$. Hence $\mathbb{T}_p(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r}) = \langle e_J \mid d(J, (\{0\}, \dots, \{0\})) \leqslant 1 \rangle$. Thus, a general hyperplane containing $\mathbb{T}_p(\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r})$ is given by an equation of type

$$\sum_{J \in \Lambda \mid d(J,(\{0\},\dots,\{0\})) \geqslant 2} \alpha_J X_J = 0$$

where Λ is the set of indexes of the standard Segre variety. On the affine chart $X_{1,0} = \dots = X_{r,0} = 1$, where $X_{i,0}, \dots, X_{i,n_i}$ are homogeneous coordinates of $\mathbb{P}^{n_i}$, we have that $H \cap (\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r})$ is the hypersurface in $\mathbb{C}^{\sum n_i}$ given by

$$\sum_{J=(\{j_1\},\dots,\{j_r\}) \in \Lambda \mid d(J,(\{0\},\dots,\{0\})) \geqslant 2} \alpha_J X_{1,j_1} \cdots X_{r,j_r} = 0 \qquad (6.3.2)$$

where in the above formula whenever some of the variables $X_{1,0}, \dots, X_{r,0}$ appear we set them equal to one. Note that for a general choice of the α_J the hypersurface defined by 6.3.2 has 0-dimensional singular locus, since by an easy computation the Segre variety $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ is not 1-weakly defective under the numerical assumption $n_r \leqslant \sum_{i=1}^{r-1} n_i$.

From now on Λ will be the set of indexes of a Segre-Veronese variety. Let $p \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$. As before without loss of generality we can assume that $p = e_{I_0}$. By [AMR19, Theorem 2.1] $\mathbb{T}_p SV_{\boldsymbol{d}}^{\boldsymbol{n}} = \langle e_J \mid d(I_0, J) \leqslant 1\rangle$. Observe that for each $J = (\{j_1\},\dots,\{j_r\})$ such that $d(J, (\{0\},\dots,\{0\})) \geqslant 2$ we can consider $J' = (J_1,\dots,J_r) \in \Lambda$ where $J_i = \{0,\dots,0,j_i\}$. Therefore, considering the hyperplane H given by

$$\sum_{J'} \alpha_J X_{J'} = 0$$

where we set $X_{1,0} = \cdots = X_{r,0} = 1$ whenever these variables appear in the expression above, we see that in the affine chart $X_{1,0} = \cdots = X_{r,0} = 1$ the hypersurface $H \cap SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ in $\mathbb{C}^{\sum n_i}$ is given by (6.3.2). Thus, the statement follows from the first part of the proof. □

Proposition 6.3.3. *Assume that* $n_r > \sum_{i=1}^{r-1} n_i$.

- *If* $d_r \geqslant 2$ *then* $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ *is not* (n_1+1)*-weakly defective.*
- *If* $d_r = 1$ *then* $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ *is* 1*-weakly defective.*

Proof. Let $p_0,\dots,p_{n_1} \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ be general points. Without loss of generality, we can suppose that $p_i = e_{I_i}$. By [AMR19, Proposition 2.5] $\mathbb{T}_{e_{I_i}} SV_{\boldsymbol{d}}^{\boldsymbol{n}} = \langle e_J \mid d(I_i, J) \leqslant 1\rangle$, and hence

$$\begin{aligned} T = \langle \mathbb{T}^1_{e_{I_0}},\dots,\mathbb{T}^1_{e_{I_{n_1}}}\rangle &= \langle e_J \mid d(I_i,J) \leqslant 1 \ \text{ for some } \ i = 0,\dots n_1\rangle \\ &= \{X_J = 0 \mid d(I_i,J) > 1 \ \text{ for all } \ i = 0,\dots n_1\} \end{aligned}$$

Now, let $H \subset \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ be a general hyperplane containing $\langle \mathbb{T}^1_{p_0},\dots,\mathbb{T}^1_{p_{n_1}}\rangle$. Then H is given by an equation of type

$$\sum_{J\in\Lambda \,\mid\, d(I_i,J)>1, \forall\, i=0,\dots,n_1} \alpha_J X_J = 0, \ \ \alpha_J \in \mathbb{C}$$

Let us denote by $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-\dim(T)-1}$ the projective space whose homogeneous coordinates are the α_J with $J \in \Lambda$ and $d(I_i, J) > d$ for all $i = 0,\dots,n_1$.

To prove the first claim let us fix $l \in \{0,\dots,n_1\}$. We will discuss in detail the case $l = 0$, the argument for the remaining values of l is analogous.

Let us consider the subset $\Lambda' \subset \Lambda$ given by the set of indexes $J' = (J_1,\dots,J_r)$ where for each pair i,j with $i \in \{1,\dots,r-1\}$ and $1 \leqslant j \leqslant n_i$ we set

$$J_i = \{0,\dots,0,j\},\ J_r = \left\{0,\dots,0,1+j+\sum_{l<i} n_l\right\} \ \text{ and } J_k = \{0,\dots,0\} \ \text{ for } \ k \neq i,r$$

Furthermore, consider the subset $\Lambda'' \subset \Lambda$ given by the set of indexes $J'' = J_j = (J_1,\dots,J_r)$ such that

$$J_r = \{j,\dots,j\}, \ \text{ and } J_k = \{0,\dots,0\} \ \text{ for } \ k \neq r$$

for each $2 + \sum_{l\leqslant r-1} n_l \leqslant j \leqslant n_r$ and $j = 1$.

Since $1 \leqslant j < 1 + j + \sum_{l<i} n_l$, each $J \in \Lambda_0 = \Lambda' \cup \Lambda''$ satisfies $d(I_i, J) > 1$ for all $i = 0, \dots, n_1$. Thus, we have a natural projection

$$\begin{array}{rccc} \pi_l & : & \mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})-\dim(T)-1} & \dashrightarrow & \mathbb{P}^{n_r} \\ & & (\alpha_J)_{J\in\Lambda \mid d(I_i,J)>1\, i=0,\dots,n_1} & \longmapsto & (\alpha_J)_{J\in\Lambda_0} \end{array}$$

Now, consider the point $[1 : \dots : 1] \in \mathbb{P}^{n_r}$ and let $H \in \pi_l^{-1}([1 : \dots : 1])$ be the hyperplane given by

$$\sum_{J\in\Lambda_0} X_J = 0$$

In the affine chart $X_{1,0} = \dots = X_{r,0} = 1$, where for each $i \in \{1, \dots, r\}$, $X_{i,0}, \dots, X_{i,n_i}$ are the homogeneous coordinates on $\mathbb{P}^{n_i}$, we have that $H \cap SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ is the hypersurface in $\mathbb{C}^{\sum n_i}$ given by

$$\sum_{\substack{1\leqslant i\leqslant r-1\\ 1\leqslant j\leqslant n_i}} X_{i,j}X_{r,j+1+\sum_{l<i} n_l} + \sum_{2+\sum_{l\leqslant r-1} n_l\leqslant j\leqslant n_r} X_{r,j}^{d_r} + X_{r,1}^{d_r} = 0$$

Looking at the system of the partial derivatives we see that this hypersurface is singular only at $(0, \dots, 0)$. Therefore, using Lemma 6.1.2 we prove the first claim. For the second part, let us consider a general hyperplane H that contains $\mathbb{T}_{e_{I_0}} SV^{\boldsymbol{n}}_{\boldsymbol{d}}$. Hence, H is the zero locus of a polynomial F of the form

$$F = \sum_{J\in\Lambda \mid d(J,I_0)\geqslant 2} \alpha_J X_J, \;\; \alpha_J \in \mathbb{C}$$

In the affine chart $X_{1,0} = \dots = X_{r,0} = 1$ the intersection $H \cap SV^{\boldsymbol{n}}_{\boldsymbol{d}}$ is the hypersurface in $\mathbb{C}^{\sum n_i}$ given by

$$\widetilde{F} = \sum_{J=(J_1,\dots,J_{r-1},\{j\})\in\Lambda \mid d(J,I_0)\geqslant 2} \alpha_J X_{1,J_1} \cdots X_{r,j} = 0$$

where with X_{1,J_k} we denote the product of powers of the homogeneous coordinates on $\mathbb{P}^{n_l}$ with exponents given by the J_k. Observe that for each $1 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant n_i$ we have

$$\frac{\partial \widetilde{F}}{\partial X_{i,j}} = (\sum_{k=1}^{n_r} \alpha_{i,j}^k X_{r,k} + G_k(X_{1,1}, \dots, X_{r-1,n_{r-1}})X_{r,k}) + G(X_{1,1}, \dots, X_{r-1,n_{r-1}})$$

and for each $1 \leqslant k \leqslant n_r$ we have

$$\frac{\partial \widetilde{F}}{\partial X_{r,k}} = G'(X_{1,1}, \dots, X_{r-1,n_{r-1}})$$

with $G_k(X_{1,1}, \dots, X_{r-1,n_{r-1}})$, $G(X_{1,1}, \dots, X_{r-1,n_{r-1}})$ and $G'(X_{1,1}, \dots, X_{r-1,n_{r-1}})$ polynomials with no constant terms since by assumption $d_r = 1$.

Now, note that the locus given by $X_{1,1} = X_{1,2} = \dots = X_{r-1,n_{r-1}-1} = X_{r-1,n_{r-1}} = 0$ and

$$\sum_{k=1}^{n_r} \alpha_{1,1}^k X_{r,k} = \sum_{k=1}^{n_r} \alpha_{1,2}^k X_{r,k} = \dots = \sum_{k=1}^{n_r} \alpha_{r-1,r-1}^k X_{r,k} = 0$$

is contained in the singular locus of $\{\widetilde{F}=0\}$. Therefore, we get a linear system in n_r variables and $\sum_{i=1}^{r-1} n_i$ equations. Since $n_r > \sum_{i=1}^{r-1} n_i$ we conclude that the singular locus of $H \cap SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ contains at least a linear space of dimension $n_r - \sum_{i=1}^{r-1} n_i > 0$ yielding that $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is 1-weakly defective. □

By Proposition 6.3.3 we have that $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ with $\boldsymbol{n}=(1,n)$ and $\boldsymbol{d}=(d,1)$ is 1-weakly defective. Now, we determine the smallest dimension of a linear subspace tangent to $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ along a positive dimensional subvariety.

Proposition 6.3.4. *Let $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ with $\boldsymbol{n}=(1,n)$ and $\boldsymbol{d}=(d,1)$. Then $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not $(1,s)$-tangentially weakly defective if and only if $s \leqslant d(n+1)$.*

Proof. Let $p \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ be a general point, without loss the generality we can suppose that $p = e_{\{0,\dots,0\},\{0\}}$. Then we have $\mathbb{T}_p SV_{\boldsymbol{d}}^{\boldsymbol{n}} = \langle e_J \mid d(J,(\{0,\dots,0\},\{0\})) \leqslant 1\rangle$.

Now, let $\Pi \subset \mathbb{P}^{dn+d+n}$ be a general linear subspace of dimension s such that $\mathbb{T}_p SV_{\boldsymbol{d}}^{\boldsymbol{n}} \subset \Pi$. Therefore, we may write $\Pi = \bigcap_{i=1,\dots,dn+d+n-s} H_i$, where the H_i are general hyperplanes tangent to $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ at p. We have that $\Pi \cap SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is given by

$$\begin{cases} F_1 = \sum_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leqslant n}} \alpha_{i,j}^1 X_0^{d-i}X_1^iY_j + \sum_{2\leqslant i\leqslant d}\alpha_{i,0}^1 X_0^{d-i}X_1^iY_0 = 0\\ \vdots \\ F_{dn+d+n-s} = \sum_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leqslant n}} \alpha_{i,j}^{dn+d+n-s} X_0^{d-i}X_1^iY_j + \sum_{2\leqslant i\leqslant d}\alpha_{i,0}^{dn+d+n-s} X_0^{d-i}X_1^iY_0 = 0\end{cases}$$

and working on the affine chart $X_0=Y_0=1$ we reduce to

$$\begin{cases} F_1 = \sum_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leqslant n}} \alpha_{i,j}^1 X_1^iY_j + \sum_{2\leqslant i\leqslant d}\alpha_{i,0}^1 X_1^i = 0\\ \vdots \\ F_{dn+d+n-s} = \sum_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leqslant n}} \alpha_{i,j}^{dn+d+n-s} X_1^iY_j + \sum_{2\leqslant i\leqslant d}\alpha_{i,0}^{dn+d+n-s} X_1^i = 0\end{cases} \tag{6.3.5}$$

Then, $\mathrm{Sing}(H_1 \cap \cdots \cap H_{dn+d+n-s} \cap SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ contains the variety cut out by the following equations

$$\begin{cases} \sum_{1\leqslant j\leqslant n} \alpha_{1,j}^1 Y_j = 0\\ \vdots \\ \sum_{1\leqslant j\leqslant n} \alpha_{1,j}^{dn+d+n-s} Y_j = 0\\ X_1 = 0\end{cases} \tag{6.3.6}$$

and, for a general choice of the $\alpha_{i,j}^k$ we have that this is a linear space in the hyperplane $X_1=0$ of dimension $s-d(n+1)$.

Now, consider a special linear space Π such that (6.3.5) takes the following form

$$\begin{cases} F_1 = \sum_{1\leqslant j\leqslant n} \alpha_{1,j}^1 X_1Y_j = 0\\ \vdots \\ F_{dn+d+n-s} = \sum_{1\leqslant j\leqslant n} \alpha_{1,j}^{dn+d+n-s} X_1Y_j = 0\end{cases}$$

Then $\{F_1 = \cdots = F_{dn+d+n-s} = 0\}$ splits as

$$\{X_1 = 0\} \cup \{ \sum_{1\leqslant j\leqslant n} \alpha_{1,j}^1 Y_j = \cdots = \sum_{1\leqslant j\leqslant n} \alpha_{1,j}^{dn+d+n-s} Y_j = 0\}$$

and its singular locus is exactly given by (6.3.6). Now, Lemma 6.1.2 yields that a general linear space of dimension s containing $\mathbb{T}_p SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ has contact locus of dimension at most $s - d(n+1)$. Hence, $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not $(1,s)$-tangentially weakly defective for $s \leqslant d(n+1)$. □

Following the line of proof of Proposition 6.3.4 we can prove the following result on $(1,s)$-tangential weak defectiveness.

Proposition 6.3.7. *Consider $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ with $\boldsymbol{n} = (n_1, \dots, n_r)$ and $\boldsymbol{d} = (d_1, \dots, d_{r-1}, 1)$, and assume that $n_r > \sum_{i=1}^{r-1} n_i$. If*

$$s \leqslant \prod_{i=2}^{r} \binom{n_i + d_i}{n_i} - n_r \sum_{i=1}^{r-1} n_i$$

then $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is not $(1,s)$-tangentially weakly defective.

Proof. Without loss of generality we can assume as usual that $p = e_{J_0} \in SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ where $J_0 = (\{0,\dots,0\},\dots,\{0,\dots,0\})$. A basis for the linear system of the hyperplanes containing $\mathbb{T}_p SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is given by

$$\{X_{1,J_1} \dots X_{r-1,J_{r-1}} X_{r,j} = 0\}_{J=\{J_1,\dots,J_{r-1},\{j\}\}\in\Lambda \,|\, d(J,I_0)\geqslant 2}$$

Now let us consider hyperplane sections of the form

$$F_{i,j,l} = X_{1,0}^{d_1} \dots X_{i,j} X_{i,0}^{d_i - 1} \dots X_{r,l} = 0$$

for $1 \leqslant i \leqslant r-1, 1 \leqslant j \leqslant n_i$ and $1 \leqslant l \leqslant n_r$.

In the affine chart $\mathbb{C}^{\sum_{i=1}^r n_i}$ defined by $X_{1,0} = \cdots = X_{r,0} = 1$ the partial derivatives of $F_{i,j,l}$ are given by

$$\frac{\partial (X_{1,0}^{d_1} \dots X_{i,j} X_{i,0}^{d_i-1} \dots X_{r,l})}{\partial X_{i,j}} = X_{r,l}, \quad \frac{\partial (X_{1,0}^{d_1} \dots X_{i,j} X_{i,0}^{d_i-1} \dots X_{r,l})}{\partial X_{r,l}} = X_{i,j}$$

Then the Jacobian matrix of the $F_{i,j,l}$ has rank zero if and only if all the coordinates $X_{i,j}$ with $1 \leqslant j \leqslant n_i$ vanish. In particular, the intersection of the special hyperplane sections

$$X_{1,0}^{d_1} \dots X_{i,j} X_{i,0}^{d_i-1} \dots X_{r,l} = 0$$

has a singularity spanning the whole of $\mathbb{C}^{\sum_{i=1}^r n_i}$ only at $(0,\dots,0)$. Now, to conclude it is enough to note that the number of these special hyperplane sections is $n_r \sum_{i=1}^{r-1} n_i$ and to apply Lemma 6.1.2. □

Finally, we have the following classification of 1-weakly defective Segre-Veronese varieties.

Theorem 6.3.8. *The Segre-Veronese $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is 1-weakly defective if and only if $d_r = 1$ and $n_r > \sum_{i=1}^{r-1} n_i$.*

Proof. It is an immediate consequence of Propositions 6.3.1, 6.3.3. □

6.4 On tangential weak defectiveness of products

In this section we study tangential weak defectiveness for varieties that can be written as a product of a smaller dimensional variety and the projective line.

Lemma 6.4.1. *Let $W \subseteq \mathbb{P}^m$ be a non-degenerated irreducible projective variety, and consider the Segre embedding of $X = W \times \mathbb{P}^r \subseteq \mathbb{P}^m \times \mathbb{P}^r \to \mathbb{P}^N$ with $N = rm + r + m$. Fix a point $p \in \mathbb{P}^r$ and a hyperplane $H \subset \mathbb{P}^r$ not passing through p. Let $Z = W \times \{p\}$, $Y = W \times H$, and denote by $H_Z = \langle Z \rangle$, $H_Y = \langle Y \rangle$ their linear spans. Then H_Z and H_Y are complementary subspaces of $\mathbb{P}^N$, and $X \cap H_Z = Z$, $X \cap H_Y = Y$.*

Proof. Since $W \subseteq \mathbb{P}^m$ is non-degenerated we have that $H_Z = \langle \mathbb{P}^m \times \{p\} \rangle$ and $H_Y = \langle \mathbb{P}^m \times H \rangle$. Consider homogeneous coordinates $[x_0 : \cdots : x_r]$ on $\mathbb{P}^r$ and $[y_0 : \cdots : y_m]$ on $\mathbb{P}^m$. Without loss of generality we may assume that $p = [1 : 0 : \cdots : 0]$ and $H = \{x_0 = 0\}$. Hence, $H_Z = \{z_{0,1} = \cdots = z_{m,r} = 0\}$ and $H_Y = \{z_{0,0} = \cdots = z_{m,0} = 0\}$, where $z_{i,j}$ is the homogeneous coordinate on $\mathbb{P}^N$ corresponding to $y_i x_j$. Hence H_Z and H_Y are complementary subspaces of $\mathbb{P}^N$.

Now, assume that there is a point $q \in X \cap H_Z$ with $q \notin Z$. Since $X = W \times \mathbb{P}^r$ the point q lies on a fiber $\mathbb{P}^r_w$ over a point $w \in W$. Such fiber intersects Z in a points $z \in Z$ with $z \neq q$ and hence $\mathbb{P}^r_w$ intersects H_Z in at least two distinct points. On the other hand, note that $H_Z = \langle \mathbb{P}^m \times \{p\} \rangle$ is the fiber $\mathbb{P}^m_p$ over p of the projection $\mathbb{P}^m \times \mathbb{P}^r \to \mathbb{P}^r$. A contradiction.

Similarly, assume that there is a point $q \in X \cap H_Y$ with $q \notin Y$. The point q lies on a fiber $\mathbb{P}^r_w$ over a point $w \in W$. Hence $\mathbb{P}^r_w$ intersects Y in a hyperplane H_w of $\mathbb{P}^r_w$ not containing q, and H_Y contains the fiber $\mathbb{P}^r_w = \langle q, H_w \rangle$. A contradiction. □

Proposition 6.4.2. *Let $W \subseteq \mathbb{P}^m$ be a non-degenerated irreducible projective variety, and consider the Segre embedding of $X = W \times \mathbb{P}^r \subseteq \mathbb{P}^m \times \mathbb{P}^r \to \mathbb{P}^N$ with $N = rm + r + m$.*

If $p, q \in X$ are two distinct points lying on the same fiber of $\pi : X \to W$ over a smooth point $w \in W$ then the span of the tangent spaces $\langle \mathbb{T}_p X, \mathbb{T}_q X \rangle$ is tangent to X along the line $\langle p, q \rangle$.

Proof. Let $w \in W$ be a smooth point. We can parametrize W in a neighborhood of W as

$$\begin{array}{cccc} \varphi : & \mathbb{C}^d & \longrightarrow & \mathbb{C}^m \\ & (x_1, \ldots, x_d) & \longmapsto & (\phi_1(x_1, \ldots, x_d), \ldots, \phi_m(x_1, \ldots, x_d)) \end{array}$$

where $d = \dim(W)$ and $\phi(0) = w$. Hence, a parametrization of X is given by

$$\begin{array}{cccc} \psi : & \mathbb{C}^d \times \mathbb{C}^r & \longrightarrow & \mathbb{C}^N \\ & ((x_1, \ldots, x_d), (1, y_1, \ldots, y_r)) & \longmapsto & (\phi_1, \ldots, \phi_m, \phi_1 y_1, \ldots, \phi_m y_r) \end{array}$$

Let us set $a_{i,j} = \frac{\partial \phi_i}{\partial x_j}(0)$ and $b_k = \phi_k(0)$. Without loss of generality we may assume that $p = \psi((0, \ldots, 0), (1, 0, \ldots, 0))$ and $p = \psi((0, \ldots, 0), (1, \ldots, 1))$ so that the line $\langle p, q \rangle$ is parametrized by $\gamma(t) = \psi((0, \ldots, 0), (1, t, \ldots, t))$. Now, the tangent space of X at $\gamma(t)$ is spanned by the rows of the following matrix

$$A(t) = \begin{pmatrix} a_{1,1}t & \dots & a_{1,1}t & a_{2,1}t & \dots & a_{2,1}t & \dots & \dots & a_{m,1}t & \dots & a_{m,1}t & a_{1,1} & \dots & a_{m,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1,d}t & \dots & a_{1,d}t & a_{2,d}t & \dots & a_{2,d}t & \dots & \dots & a_{m,d}t & \dots & a_{m,d}t & a_{1,d} & \dots & a_{m,d} \\ b_1 & \dots & 0 & b_2 & \dots & 0 & \dots & \dots & b_m & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & b_1 & 0 & \dots & b_2 & \dots & \dots & 0 & \dots & b_m & 0 & \dots & 0 \end{pmatrix}$$

and to conclude it is enough to observe that $A(t) = tA(1) - (t-1)A(0)$. □

The main goal now is to use our technique of the study of $(h, s)-$tangential weak defectiveness in order to improve the known bound on identifiability for varieties of the type $W \times \mathbb{P}^1$, where $W \subset \mathbb{P}^m$ is a smooth non-degenerated irreducible projective variety and

$$W \times \mathbb{P}^1 \subset \mathbb{P}^{2(m+1)-1}$$

is Segre embedded.

We are working on an improved version of a theorem inspired by [BCO14, Theorem 5.4], which makes a clever use of the osculating degeneration technique in order to ensure the $(h, s)-$tangential weak defectiveness of W and hence its $h-$ identifiability.

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