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ENGINEERING PROJECT ANALYSIS AND INVESTMENT DECISION-MAKING UNDER UNCERTAINTY

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To Aurelio, Festina, Katia, and Laura,
who are my certain love

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Summary

The present thesis investigates innovative tools for the economic valuation of industrial, engineering, and financial investments and for the selection of technical alternatives, inspired by principles of economic rationality and value creation. Mingling concepts and techniques from engineering economics, mathematics, finance, and accounting, the current research introduces, among other results, i) the development of new theoretical and applicative evaluation models for investment projects under uncertainty, ii) the definition of new rational criteria and the implementation of new tools for value-creating investment decisions, iii) the application of the innovative logical framework introduced in Magni (2020) connecting operating estimates and financial decisions, iv) new applications of sensitivity analysis to investments for detecting the most influential economic and technical risk factors, and v) refined tools for the ex-post performance measurement considering interactions among value drivers. The thesis is structured as a collection of academic papers co-authored during the doctoral course.

Keywords. Engineering economics, investment decisions, value creation, financial efficiency, sensitivity analysis.

Title and summary in Italian

Titolo

Analisi di progetti ingegneristici e decisioni di investimento in condizioni di incertezza

Sommario

Questa tesi propone strumenti innovativi per la valutazione economica di investimenti industriali, ingegneristici e finanziari, ispirati ai principi di razionalità economica e creazione di valore. Fondendo nozioni e tecniche tradizionalmente studiate nelle discipline di ingegneria economica, matematica, finanza e contabilità, la presente ricerca introduce, tra gli altri risultati, i) lo sviluppo di nuovi modelli di valutazione sia teorici sia applicativi per progetti di investimento in condizioni di incertezza, ii) la definizione di nuovi criteri e l'implementazione di nuovi strumenti a supporto di decisioni economicamente razionali, iii) l'applicazione dell'innovativo sistema logico introdotto in Magni (2020) per interconnettere stime operative e decisioni finanziarie, iv) nuove applicazioni di analisi di sensibilità alla redditività degli investimenti per individuare i principali fattori di rischio tecnici ed economici, e v) il raffinamento della misurazione a posteriori dei risultati economici per considerare l'interazione tra le variabili del modello. La tesi si struttura come una raccolta di articoli scientifici co-autorati durante il dottorato.

Parole chiave. Ingegneria economica, decisioni di investimento, creazione di valore, efficienza finanziaria, analisi di sensibilità.

Introduction

The present thesis is the collection of academic papers co-authored by me during the doctoral course together with

- Carlo Alberto Magni, professor of Engineering Economics and Financial Management at the University of Modena and Reggio Emilia
- Moshe Ben-Horin, professor of Finance, president of Ono Academic College
- Yoram Kroll, professor of Finance at Ono Academic College and at Ruppin Academic Center
- Giovanni Mastroleo, assistant professor at the University of Salento
- Davide Baschieri, doctoral student at the University of Modena and Reggio Emilia and business analyst at GRAF Spa
- Stefano Malagoli, managing partner of Kaleidos Corporate Finance.

The papers develop and introduce innovative tools for supporting long-term investment decisions under uncertainty, inspired by principles of economic rationality and value creation. Mingling notions and methodologies from engineering economics, finance, accounting, and mathematics, three fundamental classes of tools have been conjunctionally applied in pursuing our research objective, with several degrees of integration in the various papers:

Valuation metrics. Many different metrics are used for evaluating investments and making decisions, including absolute metrics, relative metrics, and risk-adjusted performance ratios, conveying different information from one another: Absolute metrics, such as the net present value and net final value (also named value added), signal investors' wealth increase in monetary units (Brealey and Myers 2000, Ross, Westerfield and Jordan 2011, Hartman 2007); relative measures, such as rates of return and, among these, the internal rate of return and the average internal rate of return (AIRR, Magni 2010, 2013), quantify the economic profitability of a project per unit of invested capital; finally, the risk-adjusted performance ratios are typically defined as ratios of a portfolio's excess return to a quantitative risk measure, such as the Sharpe ratio (Sharpe 1964) which uses the standard deviation as the risk measure, and the downside-risk-adjusted performance ratios (see Nawrocki 1999 for a review).

Sensitivity analysis techniques. The evaluation of investment projects depends on large sets of parameters, so called value drivers, representing sources of variability affecting the investment results. Sensitivity analysis studies "how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input" (Saltelli et al. 2004), therefore, allowing to identify the most relevant value drivers for the investment success, with several applications to investment valuation, risk analysis, and performance measurement (among others, Borgonovo and Peccati 2006, Borgonovo, Gatti and Peccati 2010).

Fuzzy-logic expert systems. Expert systems are artificial intelligence techniques which automatically replicate the evaluation and decision processes performed by human experts, via a modular approach based on rules blocks and conditional implications. They are often employed along with the fuzzy logic, suited for providing analyses and for building models whose parameters are vague and difficult to express into precise real numbers, and for taking into account both quantitative and qualitative variables, with several applications to finance and management (among others, Magni et al. 2004, Magni et al. 2006, Malagoli et al. 2007).

I briefly describe the papers composing this thesis, classified in papers published or submitted to journals and papers presented at conferences.

Papers published or submitted to journals

- Marchioni, A., Magni, C.A. (2018). Investment decisions and sensitivity analysis: NPV-consistency of rates of return. *European Journal of Operational Research*, 268(1), 361-372.

  The paper introduces a new, stronger definition of NPV-consistency that takes into account the influence of value drivers on the metric output. A metric is strongly NPV-consistent if it signals value creation and the ranking of the value drivers in terms of impact is the same as that provided by the NPV. Finally, it shows that the average Return On Investment enjoys strong NPV-consistency under several methods of sensitivity analysis.

- Magni, C.A., Malagoli, S., Marchioni, A., Mastroleo, G. (2020). Rating firms and sensitivity analysis. *Journal of the Operational Research Society*, 71(12), 1940-1958.

  The paper introduces a model for rating a firm's default risk based on fuzzy logic and expert system and an associated model of sensitivity analysis for managerial purposes, allowing the decomposition of the historical variation of default risk, identifying the most relevant parameters for the risk variation, and suggesting actions to be undertaken for improving the firm's rating.

- Magni, C.A., Marchioni, A. (2020). Average rates of return, working capital, and NPV-consistency in project appraisal: A sensitivity analysis approach. *International Journal of Production Economics*, 229, 107769.

  The paper introduces the straight-line rate of return (SLRR), based on the notion of average rate of change, which overcomes all the problems encountered by average ROI and IRR taking into explicit account the role of working capital:

The SLRR always exists, is unique, strongly NPV-consistent for both accept-reject decisions and project ranking, and has an unambiguous financial nature.

- Kroll, Y., Marchioni, A., Ben-Horin, M. (2021). Coherent Portfolio Performance Ratios. *Quantitative Finance*. DOI: 10.1080/14697688.2020.1869293.

  The paper introduces an axiomatic foundation for coherent portfolio performance ratios, suggesting and analyzing four axioms: Monotonicty, size monotonicity, concavity, and portfolio riskless translation invariance (PRTI); then, it proves that performance ratios with fixed thresholds other than the risk-free rate do not satisfy PRTI; finally, it introduces a modified threshold eliminating the above shortcoming.

- Magni, C.A., Marchioni, A., Baschieri, D. (submitted). Value-based performance measurement with the Attribution Matrix and the Finite Change Sensitivity Index.

  The paper presents an innovative two-dimensional approach for performance measurement, based on a newly introduced Attribution Matrix, aimed at detecting the decision effects (measuring the impact of manager/investor choices on investment performance) and the period effects (measuring the impact of each period on investment performance).

- Magni, C.A., Baschieri, D., Marchioni, A. (submitted). Impact of financing and payout policy on the economic profitability of solar photovoltaic plants.

  The paper presents a comprehensive evaluation model for appraising an investment in a solar photovoltaic plant. It illustrates the intricate network of logical relations among technical (estimated) variables and financial (decision) variables and shows that establishing transparent links between the former and the latter enhances the accuracy and soundness of the model for correctly measuring shareholder value creation.

Papers presented at conferences

- Magni, C.A., Marchioni, A. (2018). Project appraisal and the Intrinsic Rate of Return. *4th International Conference on Production Economics and Project Evaluation*, ICOPEV, Guimaraes, Portugal, September 20-21.

  The paper proposes a new rate of return measuring a project's economic profitability, called the intrinsic rate of return, defined as the ratio of project return to project's intrinsic value.

- Magni, C.A., Marchioni, A. (2019). The accounting-and-finance of a solar photovoltaic plant: Economic efficiency of a replacement project. *4th International Conference on Energy and Environment*, ICEE, Guimaraes, Portugal, May 16-17.

The paper evaluates the economic profitability of a solar photovoltaic project whose cash-flow stream is nonnegative via the average ROI, which eliminates the shortcoming of the non-existent IRR.

- Magni, C.A., Marchioni, A. (2019). Performance measurement and decomposition of value added. *9th International Conference of the Financial Engineering and Banking Society*, Prague, Czech Republic, May 30-June 1.

  The paper decomposes the value added of an actively managed financial investment according to the influence of the investment choices (i.e., asset selection and allocation) made in the various periods.

- Kroll, Y., Marchioni, A., Ben-Horin, M. (2020). Sortino(γ): A Modified Sortino Ratio with Adjusted Threshold. *27th Annual Virtual Conference of the Multinational Finance Society*, June 28-29.

  The paper introduces a modified Sortino ratio, Sortino(γ), which is invariant with respect to the portfolio's risk-free vs. risky assets mix, eliminating a deficiency of the original Sortino ratio.

- Baschieri, D., Magni, C.A., Marchioni, A. (2020). Comprehensive financial modeling of solar PV systems. *37th European Photovoltaic Solar Energy Conference*, EU PVSEC, Lisbon, Portugal, September 7-11.

  The paper applies a sensitivity-analysis method, the Finite Change Sensitivity Index, to the economic evaluation of a real photovoltaic plant, identifying the contribution of any input factor to the value variation.

- Marchioni, A., Magni, C.A., Baschieri, D. (2020). Investment and financing perspectives for a solar photovoltaic project. *20th Management International Conference*, MIC, Ljubljana, Slovenia, November 12-15.

  The paper highlights the role of the distribution policy in the financial modeling of a solar photovoltaic plant, by underlining the strict logical connections between estimated data and decision variables.

Acknowledgements

This thesis concludes my doctoral course in the research fields of engineering economics and finance. My interest in these disciplines would have not risen without the enlightening lectures of Carlo Alberto Magni, who first taught me the principles and models of financial mathematics when I was an undergraduate student, some 10 years ago: I still remember, with admiration, the innovative outstanding contents of his lectures, including his freshly discovered definitions of average rates of return, offering to me as a young student a rare example of perfect conjunction between didactics and research. Then, few years later, Carlo Alberto, as supervisor of my bachelor's and master's theses, transmitted to me his strong passion for the academic investigation and his profound devotion to the logic of financial valuation, motivating and encouraging me to start this PhD course. Finally, having Carlo Alberto as doctoral supervisor has been my greatest fortune in the last three years: He has always donated to me a unique educational opportunity and a surprisingly broad intellectual incitement, with his important appreciation, benevolence, and affection. I am deeply grateful to him.

I am also very thankful to Moshe Ben-Horin and Yoram Kroll, models of academic excellence and rare altruism: They benevolently accepted me at their institution with welcoming hospitality, and taught me the mathematical foundation of portfolio performance valuation, with unique experience, vivid passion, and virtuous patience, generously helping me move the first steps in the capital market research field. Their help, encouragement, and friendship are very important to me. Furthermore, a sincere thank is due to Moshe Ben-Yorin and Lorenzo Peccati who carefully reviewed the present thesis, with expertise and generosity.

I thank Davide Baschieri for his natural gifts of analysis and synthesis of complex information which he openly shares with me, and Stefano Malagoli for his reassuring example of qualified professionalism bridging together financial practice and theory.

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The certain love of my family and the sure presence of my friends deserve a last, personal thank.

Papers published or submitted to journals

Investment decisions and sensitivity analysis: NPV-consistency of rates of return

Marchioni, A., Magni, C.A. (2018). Investment decisions and sensitivity analysis: NPV-consistency of rates of return. *European Journal of Operational Research*, 268(1), 361-372.

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Interfaces with Other Disciplines

Investment decisions and sensitivity analysis: NPV-consistency of rates of return

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ABSTRACT

Investment decisions may be evaluated via several different metrics/criteria, which are functions of a vector of *value drivers*. The economic significance and the reliability of a metric depend on its compatibility with the Net Present Value (NPV). Traditionally, a metric is said to be NPV-consistent if it is coherent with NPV in signaling value creation. This paper makes use of Sensitivity Analysis (SA) for measuring coherence between rates of return and NPV. In particular, it introduces a new, stronger definition of NPV-consistency that takes into account the influence of value drivers on the metric output. A metric is strongly NPV-consistent if it signals value creation *and* the ranking of the value drivers in terms of impact on the output is the same as that provided by the NPV. The degree of (in)coherence is calculated with Spearman (1904) correlation coefficient and Iman and Conover (1987) top-down coefficient. We focus on the class of AIRRs (Magni 2010, 2013) and show that the average Return On Investment (ROI) enjoys strong NPV-consistency under several (possibly all) methods of Sensitivity Analysis.

1. Introduction

In capital budgeting many different criteria are used for evaluating a project, measuring economic efficiency, and making decisions. Net Present Value (NPV) is considered the most theoretically reliable tool, since it correctly measures shareholder value creation (Brealey & Myers, 2000; Ross, Westerfield, & Jordan, 2011). However, in practice, many other metrics are used; in particular, relative measures of worth such as internal rate of return (IRR), profitability index (PI), modified internal rate of return (MIRR), Return On Investment (ROI), etc. Recently, a more general notion of rate of return, labeled AIRR (Average Internal Rate of Return) has been developed by Magni (2010, 2013), based on a capital-weighted mean of holding period rates. The AIRR approach consists in associating the capital amounts invested in each period with the corresponding period returns by means of a weighted arithmetic mean. Magni (2010, 2013) showed that any AIRR is NPV-consistent: decisions made by an investor who adopts NPV are the same as those made by an investor who adopts AIRR. Magni (2013) showed that many traditional metrics can be viewed as belonging to the class of AIRRs, including IRR, PI, MIRR. As a special case, this approach makes use of the Return On Investment (ROI) to get an *average ROI*, which is the ratio of the total project return to the total invested capital. Whatever the depreciation pattern, the average ROI exists and is unique, it has the unambiguous nature of investment rate, independent of the value drivers, and decomposes the economic value created into economic efficiency (the difference between average ROI and cost of capital) and investment scale (the sum of the committed amounts).

However, while traditional NPV-consistency is important, under uncertainty, an NPV or a rate of return are not the only factors that drive a decision. The investigation of the risk factors that mainly influence the value of the objective function is no less important.

Sensitivity analysis (SA) investigates the variation of an objective function under changes in the key inputs of a model, so aiming at identifying the most important risk factors affecting the output (and, therefore, the decision) and ranking them. There are many different SA techniques (see Pianosi et al., 2016 and Borgonovo & Plischke, 2016) and, given a technique, different objective functions may or may not lead to different results.

This paper positions itself in the interfaces of operational research (OR) and finance. The strict connections between oper-

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ations management and finance were recognized long since (e.g., Small, 1956, Weingartner, 1963, Adelson, 1965, Hespos & Strassman, 1965, Teichroew, Robichek, & Montalbano, 1965a, Teichroew, Robichek, & Montalbano, 1965b, Rivett, 1974, Ignizio, 1976) and scholarly contributions in the field have grown dramatically in the last decades (e.g., Rosenblatt and Sinuany-Stern, 1989, Grubbström & Ashcroft, 1991, Murthi, Choi, & Desai, 1997, Meier, Christofides, & Salkin, 2001, Gondzio & Kouwenberg, 2001, Baesens, Setiono, Mues, & Vanthienen, 2003, Steuer & Na, 2003, Xu & Birge, 2008, Koç et al., 2009, Fabozzi, Huang, & Zhou, 2010, Thomas, 2010, and Seifert, Seifert, & Protopappa-Siekec, 2013).

The relation between OR and finance is bidirectional. On one side, finance provides a rich toolkit of theories, criteria, and methodologies which enable operational managers to better understand the impact of their decisions so as to maximize the shareholders' wealth: "In order to make decisions managers need criteria of goodness, decision tools, and an understanding of the environment in which they operate ... The main elements of this are that the right criterion of goodness is the maximisation of shareholder wealth and that firms operate in something close to a perfect capital market." (Ashford, Berry, & Dyson, 1988). On the other side, operational research sets the aims and scope of financial modeling for managerial purposes: As opposed to finance theory which uses financial modeling for describing the behavior of the "average" investor and deriving the pricing process of financial assets, operational managers use financial modeling from the point of view of an individual decision maker with specific needs, constraints and preferences (Spronk & Hallerbach, 1997). Further, operations research itself provides techniques and tools that may be applied to several finance problems (Board, Sutcliffe, & Ziemba, 2003).

This paper is in line with the bidirectional relation between operations and finance. Specifically, it recognizes the fundamental roles of economic and financial measures of worth such as the NPV and the ROI for decision-making and, at the same time, applies an OR technique (SA) to such financial measures in order to investigate their compatibility. As such, it falls within that strand of the OR literature which makes use of various economic efficiency measures for managerial purposes, including the NPV (e.g., Yang, Talbot, & Patterson, 1993, Baroum & Patterson, 1996, Herroelen, Van Dommelen, & Demeulemeester, 1997, Cigola & Peccati, 2005, Borgonovo & Peccati, 2006a, Wiesemann, Kuhn, & Rustem, 2010, Leyman & Vanhoucke, 2017), the IRR (Nauss, 1988; Rapp, 1980, Hazen, 2003, Hazen, 2009, Hartman & Schafrick, 2004, Dhavale & Sarkis, 2018), the ROI (e.g., Danaher & Rust, 1996, Myung, Kim, & Tcha, 1997, Brimberg & ReVelle, 2000, Brimberg, Hansen, Laporte, Mladenovic, & Urosevic, 2008, Li, Min, Otake, & Van Voorhis, 2008, Menezes, Kim, & Huang, 2015, Magni, 2016) and the return to outlay (Kumbhakar, 2011). This work is strictly linked with some recent methodological papers within this field which evaluate rationality and robustness of various efficiency measures and/or their sensitivity to changes in the key parameters. Specifically, Magni (2015) showed that the average ROI (labeled average ROA) is reliable for measuring economic efficiency in industrial applications; Mørch, Fagerholta, Pantuso, and Rakkec (2017) used the average ROI as the objective function in a problem of renewal of shippings, and compared the results with those obtained from the traditional NPV maximization. Borgonovo and Peccati (2004, 2006b) studied the impact of the key drivers of an industrial project on NPV, IRR, and value at any time. Borgonovo, Gatti, and Peccati (2010) applied SA in a project financing transaction to assess the degree of coherence between NPV and debt service coverage ratio. Talavera, Nofuentes, and Aguilera (2010) applied SA to the IRR of photovoltaic grid-connected systems. Percoco and Borgonovo (2012) applied SA to IRR and NPV and studied the coherence between the two metrics in terms of importance of key drivers.

We investigate the coherence of average ROI and NPV and give a new, more stringent, definition of NPV-consistency (strong coherence), according to which a metric is strongly NPV-consistent under a given SA technique if it is NPV-consistent in the traditional sense and, in addition, the ranking of the project's value drivers (in terms of influence on the output) is the same. If a metric is not NPV-consistent, the degree of inconsistency may be measured by two alternative indices: Spearman (1904) coefficient or Iman and Conover (1987) top-down coefficient.

We find that the average ROI is strongly NPV-consistent under many techniques, even in a strict sense (the relevances of the parameters are the same). As a result, the average ROI is a reliable measure of worth which can coherently be associated with NPV in investment evaluation, assessment of economic efficiency, and decision-making.

The remaining part of the paper is structured as follows. Section 2 presents the average ROI and the notion of NPV-consistency. Section 3 briefly describes some known SA methods and Section 4 introduces the notion of pairwise coherence according to which any two functions are strongly coherent if the ranking of the model parameters coincides. This section shows that, under many SA techniques, a function f and an affine transformation of it share the same (ranking and) relevances of parameters, so they are strongly coherent in a strict sense. Section 5 shows that the average ROI is strongly NPV-consistent in a strict sense under many SA techniques. Some numerical examples are illustrated in Section 6. Some concluding remarks end the paper. (An Appendix is devoted to some other AIRRs, including non-strongly consistent ones such as IRR, MIRR and EAIRR.)

2. AIRR, average ROI, and NPV consistency

Let P be a project and let $\mathbf{F} = (F_0, F_1, \dots, F_p) \neq \mathbf{0}$ its estimated stream of free cash flows (FCFs), where $F_0 < 0$ is the investment cost and p is the lifetime of the project. Let τ be the tax rate, R_t be the revenues, O_t be the operating costs, and let Dep_t denote depreciation, $t = 1, 2, \dots, p$. Then,

$$\begin{aligned} F_t &= \overbrace{(R_t - O_t - \mathrm{Dep}_t)(1-\tau)}^{\text{operating profit}} + \mathrm{Dep}_t \\ &= (R_t - O_t)(1-\tau) + \tau \cdot \mathrm{Dep}_t. \end{aligned} \tag{1}$$

Revenues and costs are often estimated in terms of some key inputs such as prices, quantity produced and sold, unit costs, growth rates, etc. There may be several types of costs, such as energy, material, labor, selling, general, and administrative expenses, etc. For example,

$$F_t = \left(q \cdot p_0 (1+g_p)^t - \sum_{j=1}^{s} O_0^j (1+g_{Oj})^t \right)(1-\tau) + \tau \cdot \mathrm{Dep}_t \tag{2}$$

where p_0 denotes the initial price, q denotes the annual quantity sold, O_0^j denotes the initial amount of the j-th item of cost, g_p and g_{Oj} are the growth rates, and s is the number of cost items involved in the project under consideration. Let k be the (assumed constant) cost of capital (COC). We assume that the COC is exogenously fixed by the decision-maker/analyst. It is well-known that net present value (NPV) measures the economic value created: $\mathrm{NPV} = \sum_{t=0}^{p} F_t (1+k)^{-t}$. Therefore, the NPV decision criterion may be stated as follows:

Definition 1. (NPV criterion) *A project creates value (i.e., it is worth undertaking) if and only if the project NPV, computed at the discount rate k, is positive:* $\mathrm{NPV}(k) > 0$.

Let $\mathbf{C} = (C_0, C_1, \dots, C_n)$ be any vector representing some notion of capital, such that $C_0 = -F_0$ and $C_n = 0$ and let $I_t = F_t + C_t - C_{t-1}$ be the associated return. An AIRR, denoted as $\bar{\imath}$, is defined as the ratio of the overall return $I=\sum_{t=1}^{p} I_t(1+k)^{-(t-1)}$ earned by the investor to the overall capital committed $C=\sum_{t=1}^{p} C_{t-1}(1+k)^{-(t-1)}$:

$$\bar{\imath}=\frac{I}{C} \tag{3}$$

or, equivalently, as the weighted mean of period rates associated with the capital stream **C**:

$$\bar{\imath}=\frac{\sum_{t=1}^{p} i_t C_{t-1} d_{t-1}}{\sum_{t=1}^{p} C_{t-1} d_{t-1}}$$

where $d_t=(1+k)^{-t}$ is the discounting factor and $i_t=I_t/C_{t-1}$ is the period rate of return, $t=1,2,\dots,p$ (see Magni, 2010, 2013).

Magni (2010, 2013) defined a project a net investment if $C>0$ and a net financing if $C<0$. In such a way, the financial nature of any project (and its associated rate of return) can be identified as an investment project or a financing project (respectively, an investment rate or a financing rate).

Traditionally, it is widely accepted that a metric/criterion φ is said to be NPV-consistent if and only if a decision maker adopting φ makes the same decision suggested by the NPV criterion. We can formalize this standard notion as follows.

Definition 2. (NPV-consistency) *A metric/criterion φ is NPV-consistent if, given a cutoff rate k, the following statements are true:*

(i) *an investment project creates value if and only if $\varphi>k$*
(ii) *a financing project creates value if and only if $\varphi<k$.*

Magni (2010, 2013) showed that, given a cash-flow stream **F**, if $\varphi=\bar{\imath}$, then the metric is NPV-consistent, since, for any vector **C**, the following product structure holds:

$$\text{NPV}(1+k)=C(\bar{\imath}-k). \tag{4}$$

The above definition and Eq. (4) are particularly interesting because they show that the AIRR approach enables reframing the NPV in terms of product of a capital base C and an excess return $\bar{\imath}-k$. This means that the economic value created is determined by two factors: The project scale (C) and the project's economic efficiency, $\bar{\imath}-k$. The same NPV can be created either by investing a large capital amount at a small rate or investing a small capital at a high rate. Furthermore, the general definition stated above enables the analyst to understand whether value is created because capital is invested at a rate of return which is higher than the COC or because capital is borrowed at a financing rate which is smaller than the COC (see also Magni, 2015).

We now consider the special case of AIRR where $C_t=B_t$ is the capital which remains invested in the project at time t: $B_t=B_{t-1}-\text{Dep}_t$ and $B_0=-F_0$, so that I_t is the operating profit: $I_t=F_t+B_t-B_{t-1}=F_t-\text{Dep}_t=(R_t-O_t-\text{Dep}_t)(1-\tau)$. The associated period return rate is the Return on Investment (ROI):

$$\text{ROI}_t=\frac{\text{Operating profit}}{\text{Invested capital}}=\frac{(R_t-O_t-\text{Dep}_t)(1-\tau)}{B_{t-1}}. \tag{5}$$

Thus, the AIRR becomes

$$\bar{\imath}(B)=\frac{\text{Total Return}}{\text{Total Invested Capital}}=\frac{I}{B} \tag{6}$$

where $I=\sum_{t=1}^{p}((R_t-O_t-\text{Dep}_t)(1-\tau))\cdot d_{t-1}$ is the overall operating profit generated by the project and $B=\sum_{t=1}^{p} B_{t-1}d_{t-1}$ is the overall invested capital, expressing the size of the investment. As seen above, $\bar{\imath}(B)$ may be viewed as a weighted average of ROIs:

$$\bar{\imath}(B)=\alpha_1\text{ROI}_1+\alpha_2\text{ROI}_2+\dots+\alpha_p\text{ROI}_p \tag{7}$$

where $\alpha_t=B_{t-1}d_{t-1}/B$. We call $\bar{\imath}(B)$ average ROI.[1] As (4) holds for any **C** (and, therefore, for $\mathbf{B}=(B_0,B_1,\dots,B_n)$ as well), $\text{NPV}(1+k)=B\cdot(\bar{\imath}(B)-k)$ so the average ROI is NPV-consistent (see also Magni, 2015). It is also worth noting that the average ROI has the compelling property of existence and uniqueness for any project. Also, its financial nature does not depend on the value drivers nor the cost of capital: It is unambiguously determined as an investment rate, since $B_0=-F_0>0$ and $\text{Dep}_t>0$, which implies $B>0$. This makes it a good candidate as a reliable measure of worth.

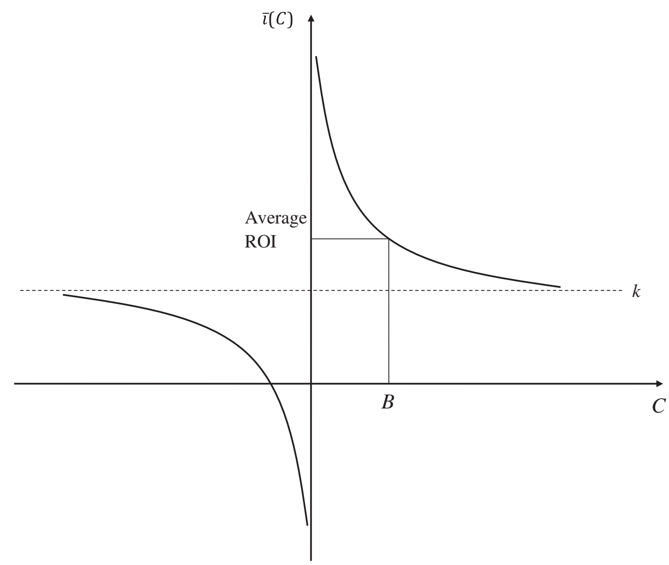

Fig. 1. Graph of the AIRR function for a positive-NPV project.

Owing to (1) and (2), the NPV is a function of several variables (the prospective revenues and costs). Practically, the analyst selects depreciation for every period, Dep_1, Dep_2, ..., Dep_p, then estimates the amount of sales, the initial price(s), the costs for labor, material, maintenance, energy, the growth rates, the tax rate, etc. These variables are risk factors, also known as *value drivers*, for they affect the FCFs. Hence, given the project COC, the project NPV is computed. For example, using (2),

$$\text{NPV}=F_0+\sum_{t=1}^{p}\frac{(q\cdot p_0(1+g_p)^t-\sum_{j=1}^{s}O_0^j(1+g_{O^j})^t)(1-\tau)+\tau\cdot\text{Dep}_t}{(1+k)^t}. \tag{8}$$

It is evident that the average ROI depends on these same value drivers, given that ROI_t depends on them. From (5),

$$\text{ROI}_t=\frac{\left(q\cdot p_0(1+g_p)^t-\sum_{j=1}^{s}O_0^j(1+g_{O^j})^t-\text{Dep}_t\right)(1-\tau)}{B_{t-1}}.$$

Exploiting (4), one can describe the AIRR as a function of the overall capital C:

$$\bar{\imath}=\bar{\imath}(C)=k+\frac{\text{NPV}}{C}(1+k). \tag{9}$$

Fig. 1 graphically describes the AIRR function $\bar{\imath}(C)$ for a value-creating project; each pair $(C,\bar{\imath}(C))$ represents an NPV-consistent rate of return; among the infinitely many AIRRs, we highlight the average ROI, which is the AIRR associated with the capital stream **B**.

The project's aim is to check whether the coherence of average ROI and NPV, which is guaranteed in a traditional sense, remains valid if changes in value drivers are considered. The analysis of

[1] Magni (2015) used the expression average ROA for this measure.

change in a model's inputs and the impact on the model output is the purpose of Sensitivity Analysis (SA).[2]

3. Sensitivity analysis

In the definition of Saltelli, Tarantola, Campolongo, and Ratto (2004, p. 45), sensitivity analysis (SA) is the "study of how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input."

Given a model and a set of inputs (parameters), the SA investigates the relevance of parameters in terms of variability of the model output. In the literature there exist many SA techniques (see Borgonovo & Plischke, 2016, and Pianosi et al., 2016, for review of SA methods). A model can be described as consisting of an objective function f defined on the *parameter space* A, which maps vector of inputs onto a model output y:

$$f : A \subset \mathbb{R}^n \to \mathbb{R}, \quad y = f(\alpha), \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n). \tag{10}$$

The vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in A \subset \mathbb{R}^n$ is the vector of inputs or parameters or key drivers and $y(\alpha)$ is the output of the model. Let $\alpha^0 = (\alpha_1^0, \alpha_2^0, \dots, \alpha_n^0) \in A$ be the *base-case*, a representative value (e.g., mean value, most probable value, etc.). The relevance of a parameter α_i (also known as importance measure) quantifies the impact of α_i on the output variation. Let $R^f = (R_1^f, R_2^f, \dots, R_n^f)$ be the vector of the relevances. The latter determines the ranking of the parameters in the following way. Input α_i is defined to be more relevant than α_j if and only if $|R_i^f| > |R_j^f|$. The parameters are equally relevant for f if $|R_i^f| = |R_j^f|$. The rank of α_i, denoted as r_i^f, depends on the importance measure: α_i has a higher rank (it has a greater impact on the output) than α_j if it has greater relevance. Let $r^f = (r_1^f, r_2^f, \dots, r_n^f)$ be the vector of ranks.

The average rank is $r_M^f = \frac{\sum_{i=1}^n i}{n} = \frac{\frac{n\cdot(n+1)}{2}}{n} = \frac{n+1}{2}$. The *high parameters* (or top parameters) are those whose rank is higher than the average rank r_M^f; the *low parameters* are those parameters whose rank is smaller than r_M^f.

Following we briefly describe some well-known (global and local) SA techniques.

(i) *Standardized regression coefficient (global SA)*

Let V denote variance and σ denote standard deviation. Consider the linear regression with dependent variable f and explanatory variables α_i, $\forall i = 1, \dots, n$, estimated with OLS method: $f = \beta_0^f + \sum_{i=1}^n \beta_i^f \cdot \alpha_i + u$. The standardized regression coefficient SRC_i^f measures the importance of α_i (Bring, 1994; Saltelli & Marivoet, 1990; Saltelli et al., 2008):

$$SRC_i^f = \frac{\beta_i^f \cdot \sigma(\alpha_i)}{\sigma(f)}. \tag{11}$$

(ii) *Sensitivity indices in variance-based decomposition methods (global SA)* In variance-based methods, the importance of a parameter is generally represented through the First Order Sensitivity Index (FOSI) and the Total Order Sensitivity Index (TOSI) (Saltelli et al., 2008). The FOSI, here denoted as $SI_i^{1,f}$, measures the individual effect of the parameter on the output variance:

$$SI_i^{1,f} = \frac{V(E(f|\alpha_i))}{V(f)}, \tag{12}$$

where $V(E(f|\alpha_i))$ is the variance of the expectation of f upon a fixed value of α_i.[3] The TOSI, here denoted as $SI_i^{T,f}$, measures the total contribution of α_i to the output variability, i.e., it is inclusive of the interaction effects with other parameters or groups of parameters. $SI_i^{T,f}$ can be calculated as (Saltelli et al., 2008)

$$SI_i^{T,f} = \frac{E(V(f|\alpha_{-i}))}{V(f)}, \tag{13}$$

where $f|\alpha_{-i} = f|\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n$ (see also Saltelli et al., 2008; Sobol', 1993; Sobol', 2001).

(iii) *Finite Change Sensitivity Indices (global SA)* The Finite Change Sensitivity Indices (FCSIs), introduced in Borgonovo (2010a, 2010b), focus on the output change due to a finite input change; there exist two versions of FCSIs: First Order FCSI and Total Order FCSI.

The First Order FCSI of a parameter measures the individual effect of the parameter's variation on f; the Total Order FCSI considers the total effect of a parameter's variation on f, including both the individual contribution and the interactions between a parameter and the other parameters.

Consider a change of parameters from α^0 to $\alpha^1 = (\alpha_1^1, \alpha_2^1, \dots, \alpha_n^1) \in A$. The output variation is $\Delta f = f(\alpha^1) - f(\alpha^0)$. Let $(\alpha_i^1, \alpha_{(-i)}^0) = (\alpha_1^0, \alpha_2^0, \dots, \alpha_{i-1}^0, \alpha_i^1, \alpha_{i+1}^0, \dots, \alpha_n^0)$ be obtained by varying the parameter α_i to the new value α_i^1, while the remaining $n-1$ parameters are fixed at α^0. The individual effect of α_i on Δf is $\Delta_i f = f(\alpha_i^1, \alpha_{(-i)}^0) - f(\alpha^0)$ and the First Order FCSI of α_i, denoted as $\Phi_i^{1,f}$, is (Borgonovo, 2010a):

$$\Phi_i^{1,f} = \frac{\Delta_i f}{\Delta f}. \tag{14}$$

Δf is equal to the sum of individual effects and interactions between parameters and groups of parameters. The total effect of the parameter α_i, denoted as $\Delta_i^T f$, is the sum of the individual effect of α_i and of the interactions that involve α_i. Borgonovo, (2010a, Proposition 1) showed that $\Delta_i^T f$ can be obtained as $\Delta_i^T f = f(\alpha^1) - f(\alpha_i^0, \alpha_{(-i)}^1)$ for all $i = 1, 2, \dots, n$, where $(\alpha_i^0, \alpha_{(-i)}^1)$ is the point with all the parameters equal to the new value α^1, except the parameter α_i, which is equal to α_i^0. The Total Order FCSI of the parameter α_i, denoted as $\Phi_i^{T,f}$, is (Borgonovo, 2010a):

$$\Phi_i^{T,f} = \frac{\Delta_i^T f}{\Delta f} = \frac{f(\alpha^1) - f(\alpha_i^0, \alpha_{(-i)}^1)}{\Delta f}. \tag{15}$$

(iv) *Helton's index (local SA)*

Helton (1993) proposed a variance decomposition of f based on Taylor approximation. He assumed parameters are not correlated, so the variance of f can be approximated by

$$\hat{V}(f) = \sum_{i=1}^n [f'_{\alpha_i}(\alpha^0)]^2 \cdot V(\alpha_i). \tag{16}$$

The impact of input α_i on $V(f)$ can be measured by

$$H_i^f(\alpha^0) = \frac{[f'_{\alpha_i}(\alpha^0)]^2 \cdot V(\alpha_i)}{\hat{V}(f)}. \tag{17}$$

(v) *Normalized Partial Derivatives* (*local SA*)

Helton (1993) also proposed the adoption of normalized partial derivatives as sensitivity measures. He defined two versions of normalized partial derivatives (NPDs):

$$NPD1_i^f(\alpha^0) = f'_{\alpha_i}(\alpha^0) \cdot \frac{\alpha_i^0}{f(\alpha^0)}, \tag{18}$$

$$NPD2_i^f(\alpha^0) = f'_{\alpha_i}(\alpha^0) \cdot \frac{\sigma(\alpha_i)}{\hat{\sigma}(f)}, \tag{19}$$

[2] Ekern (1981) and Foster and Mitra (2003) provide conditions under which a project's NPV is greater than a second project's NPV irrespective of the COC. Assuming that the second project is the null alternative, those conditions identify those projects which are robust under changes in the COC. Those conditions hold for any AIRR as well, given that any AIRR is NPV-consistent in the traditional sense (i.e., according to Definition 2). In this paper, we measure the robustness of the project with respect to the estimates of revenues and costs and focus on their impact on NPV and rate of return.

[3] It can be shown that $V(E(f|\alpha_i)) = V(f) - E[V(f|\alpha_i)]$ (see Saltelli et al., 2008).

where $\hat{\sigma}(f)$ is the square root of $\hat{V}(f)$ defined in (16). $NPD1_i^f(\alpha^0)$ measures the elasticity of f with respect to α_i in α^0 assuming that the relative change in α_i is fixed for $i = 1, 2, \ldots, n$ (Helton, 1993, p. 329). $|NPD2_i^f(\alpha^0)|$ is the square root of (17).

(vi) *Differential Importance Measure* (*local SA*)

The total variation $f(\alpha^0 + \mathrm{d}\alpha) - f(\alpha^0)$ of a differentiable function f due to a local change $\mathrm{d}\alpha$ can be approximated by the total differential $\mathrm{d}f = \sum_{i=1}^n f'_{\alpha_i}(\alpha^0) \cdot \mathrm{d}\alpha_i$. The Differential Importance Measure (DIM) of parameter α_i is the ratio of the partial differential of f with respect to α_i to the total differential of f (Borgonovo & Apostolakis, 2001; Borgonovo & Peccati, 2004):

$$DIM_i^f(\alpha^0, \mathrm{d}\alpha) = \frac{\mathrm{d}f_{a_i}}{\mathrm{d}f} = \frac{f'_{\alpha_i}(\alpha^0) \cdot \mathrm{d}\alpha_i}{\sum_{j=1}^n f'_{\alpha_j}(\alpha^0) \cdot \mathrm{d}\alpha_j}. \tag{20}$$

The DIM of a parameter represents the percentage of the function's variation due to the variation of that parameter (Borgonovo & Apostolakis, 2001; Borgonovo & Peccati, 2004).

4. Coherence between objective functions

Risk management problems are often characterized by the definition of more than one objective function (Borgonovo et al., 2010; Borgonovo & Peccati, 2006b). For a given technique, the analysis can be applied using different objective functions. A relevant aspect is the evaluation of the coherence (or compatibility) between the results of the sensitivity analysis for different functions.

We consider the objective functions $f, g : A \to \mathbb{R}$. The vector of importance measures are respectively $R^f = (R_1^f, R_2^f, \ldots, R_n^f)$ and $R^g = (R_1^g, R_2^g, \ldots, R_n^g)$; the ranking vectors are $r^f = (r_1^f, r_2^f, \ldots, r_n^f)$ and $r^g = (r_1^g, r_2^g, \ldots, r_n^g)$.

Definition 3. (Coherence) *Given a technique of SA and two objective functions f and g, they are coherent if the ranking vectors coincide: $r^f = r^g$. If, in addition, the vectors of the relevances coincide, $R^f = R^g$, they are strictly coherent.*

If two functions f and g are not coherent, the degree of incoherence can be alternatively measured through Spearman's rank correlation coefficient (Spearman, 1904) or top-down correlation coefficient (Iman & Conover, 1987).

Spearman's rank correlation coefficient (henceforth, Spearman's coefficient) between two stochastic variables is the correlation coefficient between the ranks of the stochastic variables (Spearman, 1904). In SA, Spearman's coefficient between two objective functions f and g, denoted as $\rho_{f,g}$, is the correlation coefficient of the ranking vectors r^f and r^g:

$$\rho_{f,g} = \frac{\mathrm{Cov}(r^f, r^g)}{\sigma(r^f) \cdot \sigma(r^g)} = \frac{\sum_{i=1}^n (r_i^f - r_M^f) \cdot (r_i^g - r_M^g)}{\sqrt{\sum_{i=1}^n (r_i^f - r_M^f)^2} \cdot \sqrt{\sum_{i=1}^n (r_i^g - r_M^g)^2}}, \tag{21}$$

where, as seen, $r_M^f = r_M^g = \frac{n+1}{2}$. The coefficient $\rho_{f,g}$ attributes the same weight to top and low parameters and lies in the interval $[-1, 1]$. The coefficient $\rho_{f,g}$ is equal to 1 if and only if f and g are coherent according to Definition 3. Therefore, a value of $\rho_{f,g}$ smaller than 1 signals incoherence between f and g: The smaller the value of $\rho_{f,g}$, the higher the degree of incoherence. The difference $1 - \rho_{f,g}$ can be taken as representative of the degree of incoherence.

Iman and Conover (1987) introduced the top-down correlation coefficient, a compatibility measure that attributes a higher weight to top parameters than to low parameters. This measure is based on *Savage Score* (Savage, 1956). The Savage score of parameter α_i is $S_i^f = \sum_{h=r_i^f}^n \frac{1}{h}$. The vector of Savage scores is $S^f = (S_1^f, S_2^f, \ldots, S_n^f)$.[4] The average Savage score is $S_M^f = \frac{\sum_{i=1}^n S_i^f}{n} = 1$.

The top-down correlation coefficient between the objective functions f and g, denoted as ρ_{S^f,S^g}, is the correlation coefficient between the Savage scores' vectors S^f and S^g (Iman & Conover, 1987):

$$\begin{aligned}\rho_{S^f,S^g} &= \frac{\mathrm{Cov}(S^f, S^g)}{\sigma(S^f) \cdot \sigma(S^g)} \\ &= \frac{\sum_{i=1}^n (S_i^f - S_M^f) \cdot (S_i^g - S_M^g)}{\sqrt{\sum_{i=1}^n (S_i^f - S_M^f)^2} \cdot \sqrt{\sum_{i=1}^n (S_i^g - S_M^g)^2}},\end{aligned} \tag{22}$$

where $S_M^f = S_M^g = 1$. The coefficient ρ_{S^f,S^g} measures the compatibility between the parameters' ranking of f and g: The accordance between top parameters has a remarkable influence on ρ_{S^f,S^g}, while the discordance between low parameters has a weak influence on ρ_{S^f,S^g} (Iman & Conover, 1987).

If the aim of the analysis is factor prioritization (i.e., identification of the most relevant parameters), the top-down coefficient should be preferred to Spearman's coefficient.

The maximum value of ρ_{S^f,S^g} is equal to 1. In case f and g have *no ties* (i.e., no relevance is equal), the minimum value is -1 for $n = 2$, it increases as n increases, and it tends to -0.645 as n tends to infinity (Iman & Conover, 1987).

ρ_{S^f,S^g} is equal to 1 if and only if f and g are coherent. Therefore, a value of ρ_{S^f,S^g} smaller than 1 signals incompatibility between f and g. The smaller the value of ρ_{S^f,S^g}, the higher the incoherence level. The degree of incoherence of f and g can then be measured by $1 - \rho_{S^f,S^g}$.

Borgonovo, Tarantola, Plischke, and Morris (2014) showed that an objective function f and a monotonic transformation g of it generate the same relevances of the parameters under suitable assumptions. This means that they are strictly coherent according to Definition 3.

We now show that, if g is an *affine* transformation of f, that is, $g(\alpha) = l \cdot f(\alpha) + q$ for all $\alpha \in A$, then f and g are strictly coherent under several techniques.

Proposition 1. *A function and an affine transformation of it are strictly coherent under the following techniques:*

(i) Standardized regression coefficient
(ii) Sensitivity Indices in variance-based decomposition methods
(iii) Finite Change Sensitivity Indices
(iv) Helton's index
(v) Normalized Partial Derivative ($NPD2_i^f$)
(vi) Differential Importance Measure.

Proof. By hypothesis, $g(\alpha) = l \cdot f(\alpha) + q$. Therefore,

(i) $g = l \cdot (\beta_0^f + \sum_{i=1}^n \beta_i^f \cdot \alpha_i + u) + q = (l \cdot \beta_0^f + q) + \sum_{i=1}^n (l \cdot \beta_i^f) \cdot \alpha_i + l \cdot u$, whence

$$\beta_0^g = l \cdot \beta_0^f + q,$$
$$\beta_i^g = l \cdot \beta_i^f$$

so that

$$SRC_i^g = \frac{\beta_i^g \cdot \sigma(\alpha_i)}{\sigma(g)} = \frac{l \cdot \beta_i^f \cdot \sigma(\alpha_i)}{l \cdot \sigma(f)} = SRC_i^f.$$

[4] For example, if $n = 5$ and $r^f = (1, 2, 3, 4, 5)$, then $S^f = (2.28\bar{3}, 1.28\bar{3}, 0.78\bar{3}, 0.45, 0.2)$.

(ii) Denoting as $f|\alpha_i$ (and $g|\alpha_i$) the function f (and g) conditional to a specific value of α_i, $g|\alpha_i = (l\cdot f + q)|\alpha_i = (l\cdot f)|\alpha_i + q = l\cdot f|\alpha_i + q$. Therefore,

$$SI_i^{1,g} = \frac{V(E(g|\alpha_i))}{V(g)} = \frac{V(E(l\cdot f|\alpha_i + q))}{V(l\cdot f + q)} = \frac{l^2\cdot V(E(f|\alpha_i))}{l^2\cdot V(f)} = SI_i^{1,f}.$$

Analogously, $g|\alpha_{-i} = (l\cdot f + q)|\alpha_{-i} = l\cdot f|\alpha_{-i} + q$. Hence,

$$SI_i^{T,g} = \frac{E(V(g|\alpha_{-i}))}{V(g)} = \frac{E(V(l\cdot f|\alpha_{-i} + q))}{V(l\cdot f + q)} = \frac{l^2\cdot E(V(f|\alpha_{-i}))}{l^2\cdot V(f)} = SI_i^{T,f}.$$

(iii) Since $\Delta g = g(\alpha^1) - g(\alpha^0) = l\cdot f(\alpha^1) + q - l\cdot f(\alpha^0) - q = l\cdot(f(\alpha^1) - f(\alpha^0)) = l\cdot\Delta f$, and

$$\begin{aligned}\Delta_i g &= g(\alpha_i^1, \alpha_{(-i)}^0) - g(\alpha^0) = l\cdot f(\alpha_i^1, \alpha_{(-i)}^0) \\ &\quad + q - l\cdot f(\alpha^0) - q \\ &= l\cdot(f(\alpha_i^1, \alpha_{(-i)}^0) - f(\alpha^0)) = l\cdot\Delta_i f,\end{aligned}$$

then

$$\Phi_i^{1,g} = \frac{\Delta_i g}{\Delta g} = \frac{l\cdot\Delta_i f}{l\cdot\Delta f} = \frac{\Delta_i f}{\Delta f} = \Phi_i^{1,f}.$$

As for the Total Indices,

$$\begin{aligned}\Delta_i^T g &= g(\alpha^1) - g(\alpha_i^0, \alpha_{(-i)}^1) = l\cdot f(\alpha^1) \\ &\quad + q - l\cdot f(\alpha_i^0, \alpha_{(-i)}^1) - q \\ &= l\cdot(f(\alpha^1) - f(\alpha_i^0, \alpha_{(-i)}^1)) = l\cdot\Delta_i^T f\end{aligned}$$

so that

$$\Phi_i^{T,g} = \frac{\Delta_i^T g}{\Delta g} = \frac{l\cdot\Delta_i^T f}{l\cdot\Delta f} = \frac{\Delta_i^T f}{\Delta f} = \Phi_i^{T,f}.$$

(iv) From (16),

$$\begin{aligned}\hat{V}(g) &= \sum_{i=1}^n [g'_{\alpha_i}(\alpha^0)]^2\cdot V(\alpha_i) \\ &= \sum_{i=1}^n [l\cdot f'_{\alpha_i}(\alpha^0)]^2\cdot V(\alpha_i) \\ &= l^2\cdot\sum_{i=1}^n [f'_{\alpha_i}(\alpha^0)]^2\cdot V(\alpha_i) \\ &= l^2\cdot\hat{V}(f).\end{aligned}$$

Hence,

$$H_i^g(\alpha^0) = \frac{[g'_{\alpha_i}(\alpha^0)]^2\cdot V(\alpha_i)}{\hat{V}(g)} = \frac{l^2\cdot[f'_{\alpha_i}(\alpha^0)]^2\cdot V(\alpha_i)}{l^2\cdot\hat{V}(f)} = H_i^f(\alpha^0).$$

(v) Straightforward, since $|NPD2_i^f|$ is the square root of $H_i^f(\alpha^0)$.[5]

(vi) From (20),

$$\begin{aligned}DIM_i^g(\alpha^0, \mathrm{d}\alpha) &= \frac{g'_{\alpha_i}(\alpha^0)\cdot\mathrm{d}\alpha_i}{\sum_{j=1}^n g'_{\alpha_j}(\alpha^0)\cdot\mathrm{d}\alpha_j} \\ &= \frac{l\cdot f'_{\alpha_i}(\alpha^0)\cdot\mathrm{d}\alpha_i}{\sum_{j=1}^n l\cdot f'_{\alpha_j}(\alpha^0)\cdot\mathrm{d}\alpha_j} \\ &= \frac{f'_{\alpha_i}(\alpha^0)\cdot\mathrm{d}\alpha_i}{\sum_{j=1}^n f'_{\alpha_j}(\alpha^0)\cdot\mathrm{d}\alpha_j} = DIM_i^f(\alpha^0, \mathrm{d}\alpha).\end{aligned}$$

(This result is independent of the structure of $\mathrm{d}\alpha$.) □

Remark 1. While we have proved that, for several SA techniques, a function and its affine transformation are coherent (even in a strict sense), it is intuitive to inductively believe that a function and its affine transformation share an *absolute* coherence, in that they are coherent for every existing SA technique. We leave the proof of this more general statement for future research.

5. Coherence between return rates and NPV

The investment risk can be defined as “the potential variability of financial outcomes” (White, Sondhi, & Fried, 1997). The future outcomes of an investment are stochastic and the investor has limited information. Referring to NPV and IRR, Joy and Bradley (1973, p. 1255) wrote: “It has often been suggested that capital budgeting theory has over-emphasized the development of such techniques with little regard for the typically poor data used in project evaluation and the effect that errors in capital budgeting inputs have on project profitability.” The practice of valuation criteria should be corroborated by a careful investment risk analysis.

Given an investment model based on a set of value drivers, SA allows the evaluator to identify the most relevant parameters in terms of variation of the value. The most relevant parameters are the risk factors that mainly influence the investment. After SA has been performed, the investment risk can be reduced through information insights on the main risk factors identified by the analysis; the collection of extra information on these parameters allows more precise estimates and a remarkable uncertainty reduction (Borgonovo & Peccati, 2006b). Furthermore, the potential investor is able to appreciate the convenience of possible hedging strategies.

As the NPV is the main decision criterion in capital budgeting theory, the analysis of the parameters’ relevance on NPV variability is fundamental. Any relative measure of worth should be consistent with NPV not only in terms of classical consistency but also in terms of output variability with respect to changes in the inputs.

Definition 4. (Strong NPV-consistency) *Given an analysis technique T, a metric φ (and its associated decision criterion) is strongly NPV-consistent (or coherent with NPV) under T if it fulfills Definition 2 and* NPV *and φ are coherent functions according to Definition 3. The metric φ is strictly NPV-consistent if the coherence is strict.*

If a metric/criterion possesses strong NPV-consistency, the investor can equivalently adopt NPV or such a criterion for measuring value creation under uncertainty. In case a metric is not strongly NPV-consistent, the degree of incompatibility can be measured through Spearman’s coefficient or through top-down coefficient, as seen in Section 4.

We now show that the average ROI possesses strong NPV-consistency. To this end, we maintain the symbol $\alpha = (\alpha_1, \ldots, \alpha_n)$ as the vector of the project’s value drivers and α^0 is the base value. We assume that the initial invested capital is exogenously given, as well as the COC (and p). The economic profitability of P depends on the realization of the value drivers, which affect the FCFs, as seen in Section 2: $F_t = F_t(\alpha)$,

[5] It is worth noting that f and g are coherent but not strictly coherent under $NPD1_i^f$ technique:

$$\begin{aligned}NPD1_i^g(\alpha^0) &= g'_{\alpha_i}(\alpha^0)\cdot\frac{\alpha_i^0}{g(\alpha^0)} = l\cdot f'_{\alpha_i}(\alpha^0)\cdot\frac{\alpha_i^0}{g(\alpha^0)}\cdot\frac{f(\alpha^0)}{f(\alpha^0)} \\ &= l\cdot\frac{f(\alpha^0)}{g(\alpha^0)}\cdot NPD1_i^f(\alpha^0)\end{aligned}$$

so that $|NPD1_i^f| > |NPD1_j^f|$ implies $|NPD1_i^g| > |NPD1_j^g|$. Therefore, the parameters’ ranking in f and g is equal: $r^f = r^g$.

$t = 1, 2, \dots p$. We now let $f(\alpha) = \mathrm{NPV}(\alpha) = -B_0 + \sum_{t=1}^{p} F_t(\alpha)(1+k)^{-t}$ and $g(\alpha)$ denotes the average ROI.

Proposition 2. *For any fixed k, p and* $\mathbf{Dep} = (\mathrm{Dep}_1, \dots, \mathrm{Dep}_p)$,[6] *average ROI and NPV are strongly consistent in a strict sense under the following techniques:*

(i) Standardized regression coefficient
(ii) Sensitivity Indices in variance-based decomposition methods
(iii) Finite Change Sensitivity Indices
(iv) Helton's index
(v) Normalized Partial Derivative ($NPD2_i^f$)
(vi) Differential Importance Measure.

Proof. The depreciation charge, Dep_t, does not depend on the value drivers; therefore, B does not depend on α and, using (9), one can write $g(\alpha) = q + l \cdot \mathrm{NPV}(\alpha)$ where $q = k$ and $l = (1+k)/B$. The thesis follows from Proposition 1.[7] □

The above proposition guarantees that the value drivers' effect on the variability of $\bar{\imath}(B)$ and NPV is the same, not only in terms of ranks ($r^{\mathrm{npv}} = r^{\bar{\imath}(B)}$) but also in terms of relevances ($R^{\mathrm{npv}} = R^{\bar{\imath}(B)}$). Therefore, $\rho_{\bar{\imath}(B),\mathrm{npv}} = \rho_{S^{\bar{\imath}(B)},S^{\mathrm{npv}}} = 1$. This means that an investor can equivalently employ average ROI or NPV to analyze an investment under uncertainty.

6. Worked examples

In the previous sections we have shown that, given a depreciation plan, the average ROI is strongly consistent with NPV. The aim of this section is to discuss two models. The first analyzes an example with straight-line depreciation. The second one is a real-life application, illustrated in Hartman (2007, p. 344) and is based on declining balance depreciation. We will accomplish a SA by focusing on two techniques: FCSI and DIM.

6.1. Straight-line depreciation

We discuss a simple model, consisting of a firm facing the opportunity of investing in a 4-period project whose estimated revenues and operating costs are R_t and O_t. We assume that the tax rate is zero, $\tau = 0$ (it is not a risk factor). This implies, from (1), $F_t = R_t - O_t$. The project's value drivers are then $\alpha_i = R_i$ for $i = 1, 2, 3, 4$ and $\alpha_i = O_{i-4}$ for $i = 5, 6, 7, 8$. Hence, the value driver's vector is $\alpha = \{R_1, R_2, R_3, R_4, O_1, O_2, O_3, O_4\}$. NPV is computed as:

$$\mathrm{NPV}(\alpha) = -B_0 + \frac{R_1 - O_1}{1+k} + \frac{R_2 - O_2}{(1+k)^2} + \frac{R_3 - O_3}{(1+k)^3} + \frac{R_4 - O_4}{(1+k)^4}.$$

We assume that straight-line (SL) depreciation is employed, which implies that the invested capital depreciates linearly with time: $\mathrm{Dep}_t = \gamma B_0$ where $\gamma = 1/p$. This means $B_t = B_0(1 - \gamma t)$ and, in turn, $B = B_0 \cdot \sum_{t=1}^{p}(1 - \frac{t-1}{p})(1+k)^{-(t-1)}$. This implies $\mathrm{ROI}_t = (R_t - O_t - \gamma B_0)/(B_0(1 - \gamma(t-1)))$. The average ROI can be computed as a weighted average of the ROIs or as the ratio of overall profit to overall capital, B. Equivalently, using NPV, one can compute it as the value obtained by the AIRR function at $C = B$. Specifically, $\bar{\imath}(B) = k + \mathrm{NPV}(1+k)/B$.

Example 1. Assume $B_0 = 750$ and $k = 10\%$. Table 1 describes the base value

$$\alpha^0 = (R_1^0, R_2^0, R_3^0, R_4^0, O_1^0, O_2^0, O_3^0, O_4^0)$$

and reports the corresponding Free Cash Flows and valuation metrics. The NPV is $157.37 = -750 + 380/1.1 + 270/(1.1)^2 + 360/(1.1)^3 + 100/(1.1)^4$. Considering that depreciation charge is $750/4 = 187.5$, the vector of capitals associated with the average ROI is $\mathbf{B} = (750, 562.5, 375, 187.5, 0)$ and $B = 1712.15 = 750 + 562.5/1.1 + 375/(1.1)^2 + 187.5/(1.1)^3$. Therefore, the average ROI is equal to $\bar{\imath}(B) = 10\% + 157.37/1712.15 \cdot 1.1 = 20.11\%$.

Table 1
Investment evaluated in α^0.

	0	1	2	3	4
R_t^0		580	570	560	400
O_t^0		200	300	200	300
F_t	−750	380	270	360	100
Valuation					
NPV	157.37				
$\bar{\imath}(B)$	20.11%				

Table 2
Investment evaluated in α^1.

	0	1	2	3	4
R_t^1		800	810	780	630
O_t^1		350	250	380	600
F_t	−750	450	560	400	30
Valuation					
NPV	442.92				
$\bar{\imath}(B)$	38.46%				

Let α^1 be the vector of new values of revenues and costs (see Table 2), with the corresponding new values of F_t, NPV, and $\bar{\imath}(B)$. In α^1, NPV is 442.92, $\bar{\imath}(B)$ is 38.46%. The observed variations are: $\Delta\mathrm{NPV} = 285.55 = 442.92 - 157.37$; $\Delta\bar{\imath}(B) = 18.35\% = 38.46\% - 20.11\%$. Table 3 shows the First Order FCSIs ($\Phi_i^{1,f}$), the ranks (r_i^f), and the Savage scores of parameters (S_i^f) for NPV and $\bar{\imath}(B)$. The First Order FCSIs are equal: $\Phi_i^{1,\mathrm{npv}} = \Phi_i^{1,\bar{\imath}(B)}$. Hence, $\bar{\imath}(B)$ and NPV are strongly coherent in a strict sense and the degree of coherence is maximum: $\rho_{\bar{\imath}(B),\mathrm{npv}} = \rho_{S^{\bar{\imath}(B)},S^{\mathrm{npv}}} = 1$. (Note that, in this case, Total Order FCSIs and First Order FCSIs coincide, because the value drivers do not interact one another.)

We now illustrate one numerical example where the DIM technique is used. It is a local SA technique, so it measures the value drivers' impact on the objective function in a neighborhood of α^0. We assume that changes in the inputs are proportional to the base value ($d\alpha_i = \xi \cdot \alpha_i^0$ for some $\xi \neq 0$) so the resulting DIM is

$$DIM_i^f(\alpha^0) = \frac{f'_{\alpha_i}(\alpha^0) \cdot \xi \cdot \alpha_i^0}{\sum_{j=1}^{n} f'_{\alpha_j}(\alpha^0) \cdot \xi \cdot \alpha_j^0} = \frac{f'_{\alpha_i}(\alpha^0) \cdot \alpha_i^0}{\sum_{j=1}^{n} f'_{\alpha_j}(\alpha^0) \cdot \alpha_j^0} \tag{23}$$

(Borgonovo & Apostolakis, 2001, and Borgonovo & Peccati, 2004). In particular, the first partial derivatives of $\mathrm{NPV}(\alpha)$, evaluated in α^0, are

$$\mathrm{NPV}'_{\alpha_i}(\alpha^0) = \begin{cases} (1+k)^{-i}, & i = 1,2,3,4; \\ -(1+k)^{-(i-4)}, & i = 5,6,7,8. \end{cases} \tag{24}$$

The first partial derivatives of $\bar{\imath}(B)$, evaluated in α^0, are

$$\bar{\imath}(B)'_{\alpha_i}(\alpha^0) = \mathrm{NPV}'_{\alpha_i}(\alpha^0) \cdot \frac{1+k}{B}. \tag{25}$$

Example 2. Consider a four-period investment P, with $B_0 = 900$ and $k = 8\%$. Hence, Dep_t=225 which implies $B = 2089.41$. The base value is

$$\alpha^0 = (900,\ 1000,\ 1100,\ 1200,\ 600,\ 700,\ 800,\ 900).$$

The corresponding cash-flow vector is $\mathbf{F} = (-900, 300, 300, 300, 300)$ and $\mathrm{NPV} = 93.64$, $\bar{\imath}(B) = 12.84\%$. Table 4 shows the DIMs, the ranks, and the Savage scores. As expected, the two metrics share the same rank and even the same DIMs. Therefore, they are strictly coherent.

[6] Obviously, to fix **Dep** is equivalent to fixing **B**.
[7] Evidently, $\bar{\imath}(B)$ is strongly NPV-consistent under $NPD1_i^f$ as well but not in a strict sense.

Table 3
Finite change sensitivity indices.

Parameter	NPV			Average ROI		
	$\Phi_i^{T,\text{npv}} = \Phi_i^{1,\text{npv}}$	r_i^{npv}	S_i^{npv}	$\Phi_i^{T,\bar{\imath}(B)} = \Phi_i^{1,\bar{\imath}(B)}$	$r_i^{\bar{\imath}(B)}$	$S_i^{\bar{\imath}(B)}$
R_1	70.04%	2	1.718	70.04%	2	1.718
R_2	69.46%	3	1.218	69.46%	3	1.218
R_3	57.89%	4	0.885	57.89%	4	0.885
R_4	55.01%	5	0.635	55.01%	5	0.635
O_1	−47.76%	6	0.435	−47.76%	6	0.435
O_2	14.47%	8	0.125	14.47%	8	0.125
O_3	−47.36%	7	0.268	−47.36%	7	0.268
O_4	−71.76%	1	2.718	−71.76%	1	2.718
Correlations						
$\rho_{\bar{\imath}(B),\text{npv}}$	1					
$\rho_{S^{\bar{\imath}(B)},S^{\text{npv}}}$	1					

Table 4
Coherence under DIM technique.

Parameter	α^0	NPV			Average ROI		
		$DIM_i^{\text{npv}}(\alpha^0)$	r_i^{npv}	S_i^{npv}	$DIM_i^{\bar{\imath}(B)}(\alpha^0)$	$r_i^{\bar{\imath}(B)}$	$S_i^{\bar{\imath}(B)}$
R_1	900	83.87%	4	0.885	83.87%	4	0.885
R_2	1000	86.28%	3	1.218	86.28%	3	1.218
R_3	1100	87.88%	2	1.718	87.88%	2	1.718
R_4	1200	88.77%	1	2.718	88.77%	1	2.718
O_1	600	−55.91%	8	0.125	−55.91%	8	0.125
O_2	700	−60.40%	7	0.268	−60.40%	7	0.268
O_3	800	−63.91%	6	0.435	−63.91%	6	0.435
O_4	900	−66.58%	5	0.635	−66.58%	5	0.635
Correlations							
$\rho_{\bar{\imath}(B),\text{npv}}$	1						
$\rho_{S^{\bar{\imath}(B)},S^{\text{npv}}}$	1						

Table 5
Sunoco project: input data.

Stochastic (value drivers)			
Annual production	q	0.55	Million tons
Price	p_0	\$350	Per ton
Price growth rate	g_p	2%	
Materials	M	\$27.5	Million
Materials growth rate	g_m	2%	
Labor	L	\$75	Million
Labor growth rate	g_l	5%	
Energy	E	\$20	Million
Energy growth rate	g_e	3%	
Overhead	Ov	\$7	Million
Tax rate	τ	35%	
Non-stochastic			
Investment		\$140	Million
Salvage Value		\$0	Million
COC		12%	
Periods		15	Years
Dep Method		DDB-SL	

Table 6
Sunoco project evaluated in α^0 and α^1.

Parameter	α^0	α^1
q	0.55	0.57
p_0	\$350	\$340
g_p	2.0%	2.5%
M	\$27.5	\$35.0
g_m	2.0%	3.6%
L	\$75	\$68
g_l	5.0%	4.0%
E	\$20	\$25
g_e	3.0%	2.0%
O	\$7	\$10
τ	35.0%	38.0%
Valuation	α^0	α^1
NPV	\$120.61	\$128.53
$\bar{\imath}(B)$	34.60%	36.08%

6.2. Declining-balance depreciation

We discuss a model based on (2). In particular, we borrow from Hartman (2007, p. 344), a real-life application. In 2003, Sunoco Inc. agreed to build a coke-making plant with an annual capacity of 550,000 tons per year in order to supply plants of International Steel Group (ISG) Inc. The cost of the plant was \$140 million and ISG agreed to purchase the coke (needed for producing steel) for the next 15 years.

Table 5 collects the (stochastic and non-stochastic) relevant data affecting the project's economic profitability. The 11 stochastic parameters are evaluated in the base case α^0. We assume that the facility is depreciated in 15 years with a double-declining balance switching to SL depreciation (DDB-SL), that is, $\text{Dep}_t = \max(2/p \cdot C_{t-1}; C_{t-1}/(p-t+1))$. This implies that the depreciation schedule is

$$\mathbf{Dep} = (18.67, 16.18, 14.02, 12.15, 10.53, 9.13, 7.91, 6.86, 6.37, 6.37, 6.37, 6.37, 6.37, 6.37, 6.37).$$

From (2) the after-tax operating profit is obtained as

$$I_t = (q \cdot p_0(1+g_p)^t - M(1+g_m)^t - L(1+g_l)^t - E(1+g_e)^t - Ov - \text{Dep}_t)(1-\tau).$$

Table 6 describes the value drivers at α^0 and α^1 and the resulting value of NPV and average ROI. The individual and total contribution of the value drivers, as well as the ranking, are measured via the First Order FCSI and the Total Order FCSI respectively (Tables 7 and 8). Unlike the previous example, the two FCSIs are not equal, owing to nonzero interactions among the value

Table 7
Sunoco project: First Order FCSI.

Parameter	NPV			Average ROI		
	$\Phi_i^{1,\text{npv}}$	r_i^{npv}	S_i^{npv}	$\Phi_i^{1,\bar{\imath}(B)}$	$r_i^{\bar{\imath}(B)}$	$S_i^{\bar{\imath}(B)}$
q	442.23%	3	1.520	442.23%	3	1.520
p_0	−347.47%	6	0.737	−347.47%	6	0.737
g_p	383.60%	4	1.187	383.60%	4	1.187
M	−473.82%	2	2.020	−473.82%	2	2.020
g_m	−183.24%	8	0.427	−183.24%	8	0.427
L	534.84%	1	3.020	534.84%	1	3.020
g_l	357.37%	5	0.937	357.37%	5	0.937
E	−336.21%	7	0.570	−336.21%	7	0.570
g_e	81.31%	11	0.091	81.31%	11	0.091
O	−167.82%	9	0.302	−167.82%	9	0.302
τ	−107.70%	10	0.191	−107.70%	10	0.191
Correlations						
$\rho_{\bar{\imath}(B),\text{npv}}$	1					
$\rho_{S^{\bar{\imath}(B)},S^{\text{npv}}}$	1					

Table 8
Sunoco project: Total Order FCSI.

Parameter	NPV			Average ROI		
	$\Phi_i^{T,\text{npv}}$	r_i^{npv}	S_i^{npv}	$\Phi_i^{T,\bar{\imath}(B)}$	$r_i^{\bar{\imath}(B)}$	$S_i^{\bar{\imath}(B)}$
q	422.70%	3	1.520	422.70%	3	1.520
p_0	−354.32%	5	0.937	−354.32%	5	0.937
g_p	368.36%	4	1.187	368.36%	4	1.187
M	−499.62%	1	3.020	−499.62%	1	3.020
g_m	−222.45%	8	0.427	−222.45%	8	0.427
L	478.34%	2	2.020	478.34%	2	2.020
g_l	309.06%	6	0.737	309.06%	6	0.737
E	−301.30%	7	0.570	−301.30%	7	0.570
g_e	96.95%	11	0.091	96.95%	11	0.091
O	−160.07%	9	0.302	−160.07%	9	0.302
τ	−117.75%	10	0.191	−117.75%	10	0.191
Correlations						
$\rho_{\bar{\imath}(B),\text{npv}}$	1					
$\rho_{S^{\bar{\imath}(B)},S^{\text{npv}}}$	1					

Table 9
Sunoco project: DIM technique.

Parameter	α^0	NPV			Average ROI		
		$DIM_i^{\text{npv}}(\alpha^0)$	r_i^{npv}	S_i^{npv}	$DIM_i^{\bar{\imath}(B)}(\alpha^0)$	$r_i^{\bar{\imath}(B)}$	$S_i^{\bar{\imath}(B)}$
q	0.55	93.27%	1.5	2.520	93.27%	1.5	2.520
p_0	350	93.27%	1.5	2.520	93.27%	1.5	2.520
g_p	2%	11.54%	6	0.737	11.54%	6	0.737
M	27.5	−13.32%	5	0.937	−13.32%	5	0.937
g_m	2%	−1.65%	11	0.091	−1.65%	11	0.091
L	75	−43.95%	3	1.520	−43.95%	3	1.520
g_l	5%	−14.26%	4	1.187	−14.26%	4	1.187
E	20	−10.31%	7	0.570	−10.31%	7	0.570
g_e	3%	−1.95%	10	0.191	−1.95%	10	0.191
O	7	−3.00%	9	0.302	−3.00%	9	0.302
τ	35%	−9.64%	8	0.427	−9.64%	8	0.427
Correlations							
$\rho_{\bar{\imath}(B),\text{npv}}$	1						
$\rho_{S^{\bar{\imath}(B)},S^{\text{npv}}}$	1						

drivers. As expected, the effect of each parameter on average ROI is the same as its effect on NPV, in terms of both magnitude and direction, which means that the average ROI and the NPV are strictly coherent.[8]

We now use Sunoco's example to show the behavior of the two metrics with the DIM technique. The computation of DIMs is easy, given that the calculation of the partial derivatives of NPV with respect to each parameter is straightforward (see Appendix B for the list of derivatives) and the derivatives of the average ROI is obtained from (25).

The strict coherence obviously holds. It is interesting to note that, in this case, there are ties: The first rank is shared by two key drivers, the current price, p_0, and the quantity sold, q. An equal relative change of either parameter affects the average ROI (and the NPV) in the same way. The operating costs related to labor are top drivers (labor cost has rank 3 and its growth rate has rank 4). Much less impact have the growth rates in energy and materials (rank 10 and 11, respectively) (Table 9).

7. Concluding remarks

Many different investment criteria are available to managers, professionals and practitioners. NPV is considered a theoretically reliable measure of economic profitability. Industrial and financial investments are often evaluated through relative measures of worth as well. Recently, it has been introduced a new class of return rates named AIRR (Magni, 2010; 2013). This class includes the average ROI, which plays an important role in the appraisal of industrial investments (Magni, 2015; Mørch et al., 2017). The average ROI exists and is unique, and is coherent with NPV in the sense that it correctly signals value creation or value destruction, just like the NPV (and, therefore, the decision made using either metric is the same).

This work provides a new definition of NPV-consistency making use of sensitivity analysis (SA). Given an SA technique, a metric is strongly consistent or coherent with NPV if it fulfills the classical definition of NPV-consistency and generates the same ranking of the value drivers as that generated by the NPV. If, in addition, the parameters' relevances are equal to the ones associated with NPV, then the metric and NPV are strongly consistent in a strict form.

We assume that the COC is exogenously fixed by the decision maker, as well as the initial investment and the lifetime of the project. After proving that an affine transformation of a function preserves the ranking, we show that the average ROI, being an affine transformation of NPV, is strongly NPV-consistent under several (possibly, all) different techniques of SA.

We have illustrated some simple numerical examples using FCSI (Borgonovo, 2010a) and DIM (Borgonovo & Apostolakis, 2001; Borgonovo & Peccati, 2004), based on different depreciation plans (straight-line depreciation and accelerated depreciation). We have measured the degree of NPV-consistency via Spearman's (1904) coefficient and Iman and Conover's (1987) top-down coefficient. We have found that average ROI and NPV show perfect correlation and even strict consistency. However, we stress that not all AIRRs enjoy strong NPV-consistency, including the economic AIRR and the IRR, both showing degrees of incoherence that may be nonnegligible (see Appendix A).

The findings allow us to claim that the average ROI can be reliably associated with NPV, providing consistent pieces of information. Also, the average ROI is a good candidate for absolute NPV-consistency, to be intended as a strong coherence under any possible technique of SA (this should hold, given the affine relation between the average ROI and NPV). Future researches may be devoted to finding other relative measures of worth that enjoy strong NPV-consistency.

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[8] It is interesting to note that, while the change in both NPV and average ROI is not so large, the effect of each parameter on the two metrics is extremely high. In this model, the NPV and the average ROI are highly sensitive to the contributions of each driver but, overall, the parameters' effects reciprocally compensate, in such a way that the resulting change is "smoothed".

Appendix A. (Non)strong consistency of other AIRRs

In principle, the class of AIRRs consists of infinitely many rates of return (albeit most of them non-economically significant), so it is no wonder that many of them are not strongly-consistent. In this appendix we briefly focus on four special cases of AIRR, three of which are not strongly consistent with NPV.

Internal rate of return (IRR). Magni (2010, 2013) shows that the internal rate of return (IRR) is a special case of AIRR. Specifically, the IRR is a weighted mean of generally time-varying period rates, generated by any vector **C** fulfilling the following condition: $C=\sum_{t=1}^{p}\sum_{k=t}^{p}F_k(1+x)^{-(p-t+1)}\cdot(1+k)^{-(t-1)}$.[9] While the IRR is traditionally NPV-consistent (Hazen, 2003), it suffers from some difficulties that have been extensively investigated in the literature. A part of it has been concerned with the necessary and sufficient condition for existence and uniqueness (e.g., Bernhard, 1980; De Faro, 1978; Soper, 1959. See also Magni, 2010, and references therein) or with project ranking (see Ekern, 1981 and Foster & Mitra, 2003 for ranking of risk-free projects. See Ben-Horin and Kroll, 2017, for ranking of nonequivalent-risk projects). In particular, Ekern (1981) and Foster and Mitra (2003) can be interpreted as supplying conditions of (non)existence of IRR in the interval $(0,+\infty)$, assuming that a project is ranked against the null alternative. Therefore, they provide a tool to measure the robustness of a value-creating project under changes in the COC and, at the same time, the conditions where IRR does not exist and cannot then be employed for ranking value drivers.[10] Percoco and Borgonovo (2012) show that, if the IRR exists and is unique, the ranking of value drivers provided by IRR is not equal to the ranking provided by the NPV, which means that the IRR is not strongly NPV-consistent. It is easy to see that its degree of NPV-inconsistency, as measured by Spearman's correlation coefficient or Iman and Conover's top-down coefficient, may be not negligible.[11] Also, the financial nature of the IRR is not unambiguously determined: An investment project may well turn to a financing project if value drivers change, which makes SA meaningless.

Economic AIRR (EAIRR). Another relevant AIRR is the economic AIRR, based on market values (Barry & Robison, 2014; Bosch-Badia, Montllor-Serrats, & Tarrazon-Rodon, 2014; Magni, 2013, 2014, 2016). It is generated by picking $C_t=\sum_{k=t+1}^{p}F_k(1+k)^{t-k}$ for all $t=1,\dots,p-1$ (while $C_0=-F_0$), which represents the economic value of the project at time t. The EAIRR is NPV-consistent in a traditional sense and, unlike the IRR, this AIRR always exists and is unique. However, just like the IRR, its financial nature may change under changes in the value drivers.

Strong consistency with NPV is not guaranteed because C_t (and, therefore, C) depends on **F** which, in turn, depends on the value drivers. Hence, it is not an affine transformation of NPV. The degree of inconsistency may be rather high.[12]

Modified internal rate of return (MIRR). The MIRR approach, also known as the *external-rate-of-return* approach, consists of modifying project P by discounting and/or compounding some or all of its cash flows at an external rate so as to generate a modified project P' (with a modified cash-flow stream $\mathbf{F}'$) bearing the same NPV as P but such that $\mathbf{F}'$ has only one change in sign for the cash-flow stream. This guarantees that the IRR of P' (i.e., the MIRR of P) exists and is unique. The MIRR suffers from some ambiguities of definition: (i) it is not clear what the external rate should be, (ii) there are many ways to modify the project (resulting in different MIRRs), none of which seems to deserve a privileged status, and (iii) it does not actually measure P's rate of return (see Brealey & Myers, 2000; Magni, 2015; Ross et al., 2011).

Being an IRR of P', the MIRR is an AIRR of P' and is not strongly consistent. Further, the external rate from which it depends adds a source of uncertainty in the valuation process (it may be equal or different from the COC). This implies that the MIRR may not be NPV-consistent, not even in the traditional sense (see Magni, 2015, Appendix).

Profitability Index (PI). PI is defined as $PI=\text{NPV}/B_0$. It is a well-known and widespread metric that measures the NPV per unit of initial investment. It is strongly consistent with NPV in a strict sense, as it is an affine transformation of NPV. It is also easily seen that the PI is strictly linked with AIRR; namely, if cash-flow accounting is used, that is, assets are expensed immediately (whence $B_t=0$ for $t>0$), then $B=B_0$ and, from (9), $\bar{\imath}(B_0)=k+\text{NPV}(1+k)/B_0$ whence $PI=(\bar{\imath}(B_0)-k)/(1+k)$. Therefore, PI is an affine transformation of the average ROI that is associated with a cash-flow-accounting depreciation system.

Appendix B. Partial derivatives

$$\text{NPV}'_q(\alpha^0)=p_0\cdot(1-\tau)\cdot\sum_{t=1}^{15}\left(\frac{1+g_p}{1+k}\right)^t,$$

$$\text{NPV}'_{p_0}(\alpha^0)=q\cdot(1-\tau)\cdot\sum_{t=1}^{15}\left(\frac{1+g_p}{1+k}\right)^t,$$

$$\text{NPV}'_{g_p}(\alpha^0)=p_0\cdot q\cdot(1-\tau)\cdot\sum_{t=1}^{15}\frac{t\cdot(1+g_p)^{t-1}}{(1+k)^t},$$

$$\text{NPV}'_M(\alpha^0)=-(1-\tau)\cdot\sum_{t=1}^{15}\left(\frac{1+g_m}{1+k}\right)^t,$$

$$\text{NPV}'_{g_m}(\alpha^0)=-M\cdot(1-\tau)\cdot\sum_{t=1}^{15}\frac{t\cdot(1+g_m)^{t-1}}{(1+k)^t},$$

$$\text{NPV}'_L(\alpha^0)=-(1-\tau)\cdot\sum_{t=1}^{15}\left(\frac{1+g_l}{1+k}\right)^t,$$

$$\text{NPV}'_{g_l}(\alpha^0)=-L\cdot(1-\tau)\cdot\sum_{t=1}^{15}\frac{t\cdot(1+g_l)^{t-1}}{(1+k)^t},$$

$$\text{NPV}'_E(\alpha^0)=-(1-\tau)\cdot\sum_{t=1}^{15}\left(\frac{1+g_e}{1+k}\right)^t,$$

$$\text{NPV}'_{g_e}(\alpha^0)=-E\cdot(1-\tau)\cdot\sum_{t=1}^{15}\frac{t\cdot(1+g_e)^{t-1}}{(1+k)^t},$$

$$\text{NPV}'_O(\alpha^0)=-(1-\tau)\cdot\sum_{t=1}^{15}\frac{1}{(1+k)^t},$$

$$\text{NPV}'_\tau(\alpha^0)=-\sum_{t=1}^{15}\frac{I_t}{(1-\tau)\cdot(1+k)^t}.$$

[9] This implies that the assumption $i_t=x$ for all t is sufficient but not necessary to generate a rate of return equal to IRR.

[10] It is usually believed that the case of no-IRR is very rare. However, in some engineering projects it is not infrequent that disposal and remedial costs occur at the terminal date, which is a necessary condition for inexistence of IRR. Most recently, Lima, Silva, Sobreiro, and Kimura (2017) focus on a very common transaction where the case of (multiple IRRs and) no IRR is the rule rather than the exception.

[11] For instance, in Example 1, the ranking generated by IRR with Total Order FCSIs is (1, 2, 4, 6, 3, 8, 7, 5) and the top-down coefficient is $\rho_{S_{npv},S_{irr}}=0.409$. In Example 2, where the DIM technique is used, the parameters ranking supplied by IRR is (1, 2, 3, 4, 8, 7, 5, 6) and the top-down coefficient is $\rho_{S_{npv},S_{irr}}=0.309$.

[12] In Example 1, the Total Order FCSIs for EAIRR generate the parameters' ranking (1, 2, 4, 6, 3, 8, 5, 7) and $\rho_{S_{npv},S_{eairr}}=0.239$. In Example 2 the ranking is (1, 2, 3, 4, 8, 7, 5, 6) (equal to the ranking of IRR) and, therefore, the top-down coefficient is equal as well: $\rho_{S_{npv},S_{eairr}}=0.309$.

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Rating firms and sensitivity analysis

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Rating firms and sensitivity analysis

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Abstract

This paper introduces a model for rating a firm's default risk based on fuzzy logic and expert system and an associated model of sensitivity analysis (SA) for managerial purposes.

The rating model automatically replicates the evaluation process of default risk performed by human experts. It makes use of a modular approach based on rules blocks and conditional implications. The SA model investigates the change in the firm's default risk under changes in the model inputs and employs recent results in the engineering literature of Sensitivity Analysis. In particular, it (i) allows the decomposition of the historical variation of default risk, (ii) identifies the most relevant parameters for the risk variation, and (iii) suggests managerial actions to be undertaken for improving the firm's rating.

Keywords. Credit rating, default risk, fuzzy logic, fuzzy expert system, sensitivity analysis.

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1 Introduction

This paper presents a model for rating firms combined with a model for accomplishing sensitivity analysis (SA). The rating model is based on a fuzzy expert system, the SA model is based on recent results aiming at quantifying the impact of the model's input factors on the model's output change.

Several recent works analyze credit rating under managerial and financial perspectives (Bonsall IV et al. 2017, Griffin, Hong and Ryou 2018, Kouvelis and Zhao 2017, Kisgen 2006, Karampatsas et al. 2014, An and Chan 2008, Lim et al. 2017).

The evaluation of credit rating may be performed via several different quantitative methods (Hwang 2013a, 2013b, Pfeuffer et al. 2019, Doumpos et al. 2015, Doumpos and Zopounidis 2011, Angilella and Mazzù 2019). However, financial (quantitative) data are often insufficient or even unreliable for measuring the credit rating of an enterprise where judgmental, qualitative information is to be considered (Angilella and Mazzù 2015). Fuzzy logic is suited for providing financial analyses and for building rating models whose functioning is influenced by human judgment and whose parameters are vague and difficult to express into precise real numbers (Chen and Chiou 1999, Syau et al. 2001, Jiao et al. 2007. See also Levy et al. 1991, Peña et al. 2018, Bai et al. 2019).

Fuzzy logic is often employed along with techniques of artificial intelligence. Typically, expert systems, artificial neural networks, machine learning, and hybrid intelligence systems are applied to almost every area of management (see Ignizio 1990 for an overview of expert systems). Several studies show that artificial intelligence achieves high performance in predicting credit rating, in terms of explanatory power and stability (e.g., Lee 2007, Kim and Ahn 2012, Huang et al. 2004). As for finance, applications of artificial intelligence are numerous (Brown et al. 1990, Matsatsinis et al. 1997, Bahrammirzaee 2010, Dirks et al. 1995, Ferreira et al. 2019, Dawood 1996, Volberda and Rutges 1999, Lincy and John 2016, Chen and Li 2014).

Fuzzy expert systems have been advanced as well in several areas of finance and management (Magni et al. 2004, Marzouk and Aboushady 2018, Magni et al. 2006, Malagoli et al. 2007, Cheng et al. 2013, Doumpos and Figueira 2018, Vassiliou 2013, Agliardi and Agliardi 2009).

We present a rating model which is an input-output model formally represented by a fuzzy expert system: It automatically provides a firm's default risk (model output) and its associated credit rating on the basis of 18 selected key drivers (model inputs). The latter are aggregated in a modular approach via "if-then" implications applied to fuzzy numbers. As such, it is capable of taking into account both quantitative and qualitative financial and managerial variables. The proposed rating model is a judgmental expert-based system for credit risk assessment, differing from widely adopted statistical and machine learning approaches. Statistical models are based on mathematical descriptions aiming at representing the patterns in the economic data via selecting an optimal method a priori; machine learning techniques are computational-based, data-driven algorithms, less relying on assumptions about data (Galindo and Tamayo 2000). In contrast, judgmental expert-based systems reproduce the evaluation and decision processes performed by human experts, through logical inference, knowledge base, and heuristics. More specifically, in comparing machine learning with expert systems, both belong to the artificial intelligence techniques class, but machine learning is an adaptive information processing system using learning and generalization capabilities whereas an expert system is a computer system containing a well-structured, static body of knowledge imitating expert skills, capable to solve difficult problems requiring significant human expertise (Bahrammirzaee 2010).

In addition, we associate the fuzzy expert system with a sensitivity analysis (SA) model which enables performing a detailed financial and managerial analysis, proposing a combination method which has been analogously applied to other research areas of management and policy making, such as the assessment of ecological and human sustainability of countries (Grigoroudis et al. 2014, Andriantiatsaholiniaina et al. 2004). In particular, given a change in the output of a model and given two associated sequences of input parameters, a SA technique enables measuring the impact of each input parameter on the output change. Also, it enables ranking the parameters according to their importance. In such a way, it is possible to understand the reasons why the output change has occurred and the appropriate actions that may lead the decision maker toward an improvement in the output change by a proper management of the key drivers.

SA techniques are widely employed in various areas of finance and management (Huefner 1972, Luo et al. 2015, Donders et al. 2018, Madu 1988, Borgonovo and Peccati 2004, 2006, Borgonovo et al. 2010, Délèze and Korkeamäki 2018, Talavera et al. 2010, Percoco and Borgonovo 2012, Marchioni and Magni 2018, Chapman et al. 1984, Vázquez-Abad and LeQuoc 2001, Parnes 2010).

Among the various SA techniques, a recent approach is based on the notion of Finite Change Sensitivity Index (FCSI) (Borgonovo 2010a, 2010b), which we employ in our model. The FCSI represents a powerful analytical tool, which is used for studying a finite change in the model output. We aim at applying this SA technique to the rating model in order to identify the causes of variation in the default risk and then analyze the effects of different financial and managerial actions on the prospective rating.

However, while the FCSIs provide the correct ranking of the input factors in terms of their impact on the output change, they are not aimed at providing an exact decomposition of the output change, in the sense that the sum of the contributions of the input factors to the output change is not equal to the output change, owing to some double-counting of interactions among variables. In other words, given a change in the default risk and given a set of n economic parameters that affect the model output, the FCSI provides the parameter's contribution to an output change which includes individual contribution and joint interactions with the other model inputs. However, the sum of all the FCFIs does not equate the output change.[1] For this reason, we fine-tune the FCSI notion via a duplication-free procedure and supply a "clean FCSI". We apply it to the rating model for managerial and financial analysis for exactly decomposing the contributions

[1] For example, suppose the selected inputs are $n = 3$. It might turn out that 45% of the output change has been generated by the change of parameter 1, 35% has been generated by the change of parameter 2, and 30% has been generated by the change of parameter 3. The sum of the contributions is $0.45 + 0.35 + 0.30 = 1.1 \neq 1$.

of the input factors to the output change. We call the combined model (fuzzy expert system + SA model) the "Default Risk & Sensitivity Model" (DRSM): It rates the firm and, at the same time, ranks the parameters affecting the risk change in terms of their importance. We show how the DRSM may be applied for (i) rating a firm automatically, based on a given set of input parameters, (ii) identifying the causes of the change of the default risk in two different years, (iii) decomposing the change in the default risk and ranking the key drivers in terms of impact on such a change. For illustrative purposes, we also apply the DRSM to an Italian-controlled industrial company. We provide its rating in various years and analyze the change of the default risk and the change in rating in different years. Furthermore, while DRSM merges a fuzzy expert system with a model of sensitivity analysis, we stress that the proposed SA application for credit rating can be usefully combined to any approach for rating firms such as statistical and machine learning techniques adopting analogous fuzzy-logic models; in particular, SA may be applied as a tool enhancing the interpretability and comprehensibility of fuzzy models, whose comprehension is often hard because of the adoption of complex rule bases. Furthermore, SA is helpful for testing and validating the representativeness of the underlying credit scoring model: Additional simulation runs which measure the sensitivity of the output under changes in the various inputs may corroborate the model or reveal the need for revising some of the choices made in the model setup (Pianosi et al. 2016).

The remainder of the paper is structured as follows. Section 2 presents the fuzzy expert system. Section 3 illustrates the basic notions of sensitivity analysis and defines the FCSI and its use. Section 4 fine-tunes the FCSI via a duplication-free procedure and provides an exact decomposition of the output change of a model. Section 5 applies the DRSM (rating model + clean FCSI) to an Italian-controlled industrial company and shows some possible uses of it. Some remarks conclude the paper.

2 Fuzzy-logic expert system for credit rating

The current work introduces a credit rating model based on fuzzy logic and expert system, which derives the default risk and the rating class of a corporation according to rules blocks based on conditional implications. Our fuzzy-logic rating model considers a set of 18 economic and financial variables (the model inputs), both quantitative (such as Leverage, OCF-to-Debt, EBITDA on Sales) and qualitative (as Product Positioning and Industry Prospects), which are grouped under a managerial and financial perspective in first-level intermediate variables which are in turn gathered to form second-level intermediate variables which are in turn grouped to form a third level of intermediate variables. Finally, the latter determine the firm's default risk (model output). Figure 1 represents the conceptual map of variables aggregation from the model inputs to the Default Risk through the various intermediate steps (see also descriptions of input and intermediate variables in the Appendix).[2]

[2]The inputs may themselves be considered 0-level intermediate variables, determined by lower-level basic parameters. For example, the Return On Investment (ROA) is a function of three parameters: NOPAT, R&D and invested capital (see Appendix).

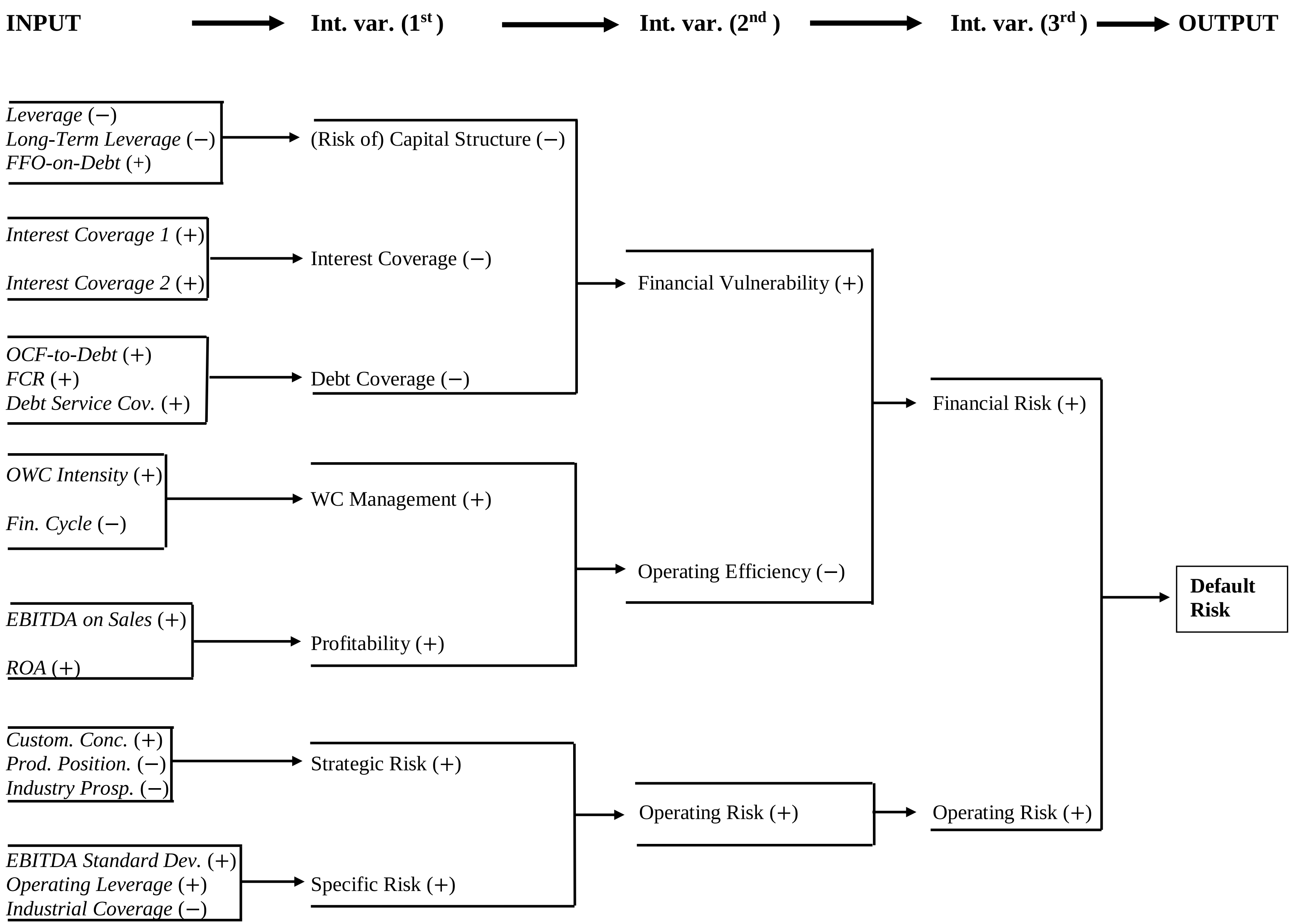

Figure 1: Conceptual map of variables aggregation

0-level		1st-level		2nd-level		3rd-level		4th-level
$\vec{x}$	$\longrightarrow$	$f_1(\vec{x})$	$\longrightarrow$	$f_2(f_1(\vec{x}))$	$\longrightarrow$	$f_3\left(f_2(f_1(\vec{x}))\right)$	$\longrightarrow$	$f_4\left(f_3\left(f_2(f_1(\vec{x}))\right)\right)$

$$
\begin{bmatrix}
x_1 = Leverage \\
x_2 = Long - Term\ Leverage \\
x_3 = FFO - on - Debt \\
x_4 = Interest\ Coverage\ 1 \\
x_5 = Interest\ Coverage\ 2 \\
x_6 = OCF - to - Debt \\
x_7 = FCR \\
x_8 = Debt\ Service\ Coverage \\
x_9 = OWC\ Intensity \\
x_{10} = Financial\ Cycle \\
x_{11} = EBITDA\ on\ Sales \\
x_{12} = ROA \\
x_{13} = Customer\ Concentration \\
x_{14} = Product\ Positioning \\
x_{15} = Industry\ Prospects \\
x_{16} = EBITDA\ Standard\ Deviation \\
x_{17} = Operating\ Leverage \\
x_{18} = Industrial\ Coverage
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
(Risk\ of)\ Capital\ Structure \\
Interest\ Coverage \\
Debt\ Coverage \\
WC\ Management \\
Profitability \\
Strategic\ Risk \\
Specific\ Risk
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
Financial\ Vulnerability \\
Operating\ Efficiency \\
Operating\ Risk
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
Financial\ Risk \\
Operating\ Risk
\end{bmatrix}
\longrightarrow
y = Default\ Risk
$$

Figure 2: The expert system as a composed function

The approach is then modular and gives rise to an evaluation tree that is run from branches to trunk. The link between the set of the input parameters and the output may be represented as a function of the 18 variables, $x_i, i = 1, 2, \ldots, 18$ affecting the dependent variable, y (Default risk), so that $y = f(\vec{x})$, where $\vec{x} = (x_1, x_2, \ldots, x_{18})$. For any given value of $\vec{x}$, the model automatically provides the default risk. Mathematically, the model is a composed function. There are 4 composing functions, whose values represent the four steps through which the inputs are processed and the output is fleshed out:

$$\vec{x} \to f_1(\vec{x}) \to f_2(f_1(\vec{x})) \to f_3(f_2(f_1(\vec{x}))) \to f_4(f_3(f_2(f_1(\vec{x})))) = y.$$

As shown in Figure 2, starting from 18 parameters, one gets a vector of 7 components (via f_1), then a vector of 3 components (via f_2), then a vector of 2 components (via f_3) and, finally one single component, the model's output (via f_4). Figure 1 is the representation of the fuzzy expert system as a conceptual map, Figure 2 is the same expert system described as a composed function.

Each composing function is either monotonically increasing or monotonically decreasing with respect to prior-level intermediate variables. Figure 1 indicates monotonicity via plus (+) or minus (−) sign. Specifically, a given variable z may affect the next-level variable q positively (+) or negatively (−). Variable z affects variable q positively if q increases (decreases) whenever z increases (decreases); it affects q negatively if q decreases (increases) whenever z increases (decreases).

Each variable of the model (inputs, intermediate variables, model output) can be associated with several attributes, which are represented graphically by fuzzy numbers and a membership function. For instance, the input factor Fixed Charge Ratio (FCR, defined in the Appendix) is characterized by the membership function reported in Figure 3. The horizontal axis collects the numerical values of FCR, while the vertical axis reports the membership degrees (or activation levels) of each linguistic attribute. For each value of FCR, all the attributes are activated at a certain degree, ranging from 0 to 1. For example, a FCR equal to 1.05 is at the same time:

- low at degree 0;
- medium low at degree 0.22;
- medium high at degree 0.78;
- high at degree 0.

The intermediate variables at any level and the model output Default Risk are evaluated using rules blocks built upon conditional ("if-then") implications which map the variables attributes at the previous level onto the attributes of the next level through a modular approach. For instance, in Table 1 we report the rules block for determining (the risk of) the Capital Structure (first-level intermediate variable), depending on the input variables Leverage, Long-Term Leverage, and FFO-on-Debt. For example, the first four-rule block informs about the degree of risk of the capital structure under changes in the FFO-on-Debt while Leverage and Long-Term Leverage are kept at low levels. Note that, for increasing value of FFO-on-Debt, the risk level of the Capital Structure increases (i.e.,

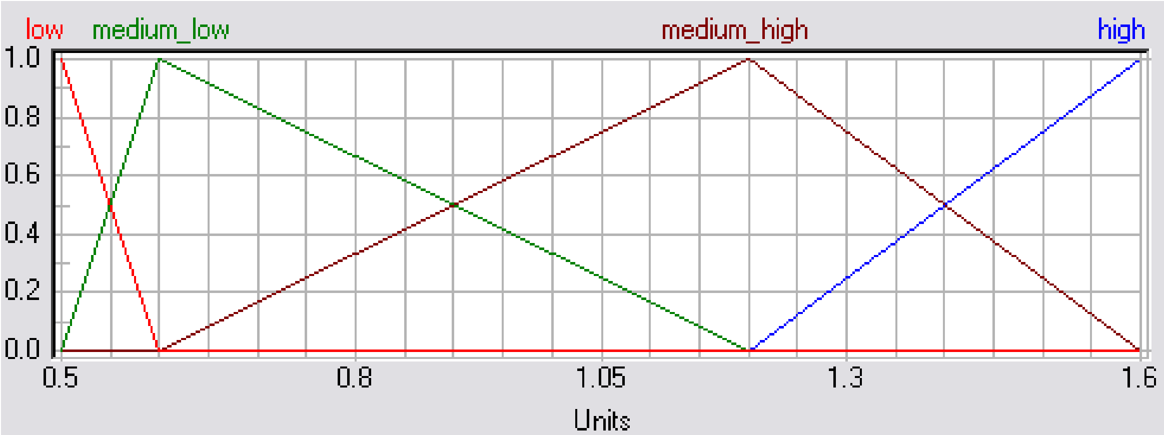

Figure 3: Membership function of FCR

the capital structure becomes riskier), meaning that the weight of debt becomes higher. For example, the fourth rule may be read as follows:

IF	
Leverage	is Low
Long-Term Leverage	is Low
FFO-on-Debt	is High
THEN	
(Risk of) Capital Structure	is Very High

Each rule of any block is activated simultaneously, at a certain degree, precisely because each variable has a certain membership degree for each attribute. All variables (including the output, Default Risk) are fuzzy numbers which are associated with membership degrees.

Reading Figure 1 backward from output to inputs, one can see that the Default Risk depends on two variables, Financial Risk and Operating Risk. Financial Risk depends in turn by Financial Vulnerability and Operating Efficiency, each of which in turn depends on other variables; specifically, Financial Vulnerability depends on (Risk of) Capital Structure, Interest Coverage, Debt Coverage while Operating Efficiency depends on WC Managament and Profitability. In turn, each of the latter depends on some group of inputs. Likewise, Operating Risk depends on Strategic Risk and Specific Risk,[3] which in turn depend on different groups of inputs.

Whenever the input vector is selected, the output (Default Risk) is automatically provided. Table 2 reports the rules block for Default Risk, conditionally to Financial Risk and Operating Risk (e.g., focusing on the fourth rule, if Financial Risk is AAA and Operating Risk is BBB, then the Default Risk is evaluated at AA). Note that the Financial Risk, the Operating Risk, and the Default Risk are described in terms of eight rating classes, from the safest one, AAA, to the riskiest one, CC. The output provided by the rule block, the Default Risk, is a fuzzy number. Through a defuzzification procedure, the default risk is automatically converted into a crisp (real) number in the normalized

[3]Operating Risk is associated to Financial Risk to determine the Default Risk, so it is repeated as a 2$^{\text{nd}}$-level and 3$^{\text{rd}}$-level intermediate variable. In terms of composing function, one may interpret it as an identity function.

Table 1: Rules block for (Risk of) Capital Structure

IF			THEN
Leverage	**Long-Term Leverage**	**FFO-on-Debt**	**(Risk of) Capital Structure**
low	low	low	high
low	low	medium-low	high
low	low	medium-high	very-high
low	low	high	very-high
medium-low	low	low	medium-high
medium-low	low	medium-low	high
medium-low	low	medium-high	high
medium-low	low	high	very-high
medium-high	low	low	medium-low
medium-high	low	medium-low	medium-high
medium-high	low	medium-high	high
medium-high	low	high	high
high	low	low	medium-low
high	low	medium-low	medium-low
high	low	medium-high	medium-high
high	low	high	high
low	medium-low	low	medium-high
low	medium-low	medium-low	high
low	medium-low	medium-high	high
low	medium-low	high	very-high
medium-low	medium-low	low	medium-low
medium-low	medium-low	medium-low	medium-high
medium-low	medium-low	medium-high	high
medium-low	medium-low	high	high
medium-high	medium-low	low	medium-low
medium-high	medium-low	medium-low	medium-low
medium-high	medium-low	medium-high	medium-high
medium-high	medium-low	high	high
high	medium-low	low	low
high	medium-low	medium-low	medium-low
high	medium-low	medium-high	medium-low
high	medium-low	high	medium-high
low	medium-high	low	medium-low
low	medium-high	medium-low	medium-high
low	medium-high	medium-high	high
low	medium-high	high	high
medium-low	medium-high	low	medium-low
medium-low	medium-high	medium-low	medium-low
medium-low	medium-high	medium-high	medium-high
medium-low	medium-high	high	high
medium-high	medium-high	low	low
medium-high	medium-high	medium-low	medium-low
medium-high	medium-high	medium-high	medium-low
medium-high	medium-high	high	medium-high
high	medium-high	low	very-low
high	medium-high	medium-low	low
high	medium-high	medium-high	medium-low
high	medium-high	high	medium-low
low	high	low	medium-low
low	high	medium-low	medium-low
low	high	medium-high	medium-high
low	high	high	high
medium-low	high	low	low
medium-low	high	medium-low	medium-low
medium-low	high	medium-high	medium-low
medium-low	high	high	medium-high
medium-high	high	low	very-low
medium-high	high	medium-low	low
medium-high	high	medium-high	medium-low
medium-high	high	high	medium-low
high	high	low	very-low
high	high	medium-low	very-low
high	high	medium-high	low
high	high	high	medium-low

Table 2: Rules block for Default Risk

IF		THEN
Financial Risk	Operating Risk	Default Risk
AAA	AAA	AAA
AAA	AA	AAA
AAA	A	AAA
AAA	BBB	AA
AAA	BB	AA
AAA	B	AA
AAA	CCC	AA
AAA	CC	AA
AA	AAA	AA
AA	AA	AA
AA	A	AA
AA	BBB	A
AA	BB	A
AA	B	A
AA	CCC	A
AA	CC	A
A	AAA	A
A	AA	A
A	A	A
A	BBB	A
A	BB	BBB
A	B	BBB
A	CCC	BBB
A	CC	BBB
BBB	AAA	A
BBB	AA	BBB
BBB	A	BBB
BBB	BBB	BBB
BBB	BB	BBB
BBB	B	BB
BBB	CCC	BB
BBB	CC	BB
BB	AAA	BBB
BB	AA	BB
BB	A	BB
BB	BBB	BB
BB	BB	BB
BB	B	BB
BB	CCC	B
BB	CC	B
B	AAA	BB
B	AA	BB
B	A	BB
B	BBB	BB
B	BB	BB
B	B	BB
B	CCC	BB
B	CC	B
CCC	AAA	B
CCC	AA	B
CCC	A	B
CCC	BBB	CCC
CCC	BB	CCC
CCC	B	CCC
CCC	CCC	CCC
CCC	CC	CCC
CC	AAA	CCC
CC	AA	CCC
CC	A	CCC
CC	BBB	CC
CC	BB	CC
CC	B	CC
CC	CCC	CC
CC	CC	CC

interval $[0, 1]$.[4]

Finally, a conversion table (Table 3) converts the (crisp) default risk into a rating class. Given a sequence of inputs, there automatically corresponds a firm's default risk and, hence, a class of rating. The logical chain is then as follows:

Inputs (fuzzy numbers)	$\vec{x}$
$\Longrightarrow$ first-level intermediate variables (fuzzy numbers)	f_1
$\Longrightarrow$ second-level intermediate variables (fuzzy numbers)	f_2
$\Longrightarrow$ third-level intermediate variables (fuzzy numbers)	f_3
$\Longrightarrow$ Default Risk (fuzzy number)	$f_4 = y$
$\Longrightarrow$ Default Risk (crisp number)	defuzzification
$\Longrightarrow$ Rating class (letter)	conversion

Table 3: Conversion table from default risk to rating class

Default risk	**Rating class**
[0, 0.125)	AAA
[0.125, 0.25)	AA
[0.25, 0.375)	A
[0.375, 0.5)	BBB
[0.5, 0.625)	BB
[0.625, 0.75)	B
[0.75, 0.875)	CCC
[0.875, 1]	CC

The 18 attributes selected represent a minimum set of meaningful risk components and profiles. The choice depends on our operational experience in corporate finance practice (debt restructuring in particular) and on the fact that they are commonly used by rating agencies' models. Therefore, the choice of this minimum set reflects the knowledge base of the experts. However, the fuzzy expert system is flexible enough for customization: It may be augmented with other appropriate input factors, which may be aggregated via if-then rules in a modular approach, as previously seen.

In general, statistical data might be collected and processed to determine and tune memberships degrees and decisions rules. Industry prospects, for example, might be based on data available from accredited sources; product positioning might be based on data from interviews to a statistically significant sample of customers; accounting data such as ROA might be compared with a sample of comparable firms of the same sector and membership degrees might be evaluated on the basis of the sample mean. Even in our model, study sectors and comparisons with industry means as well as our expertise have been relevant for determining the membership degrees. Decision rules in our model are based on our expertise as advisors and academics, but automatic extensions may be conceived in several ways, with the purpose of automatically infer the fuzzy

[4] The defuzzification procedure applied to the Default Risk uses the Center of Maximum method (CoM) (von Altrock 1995).

rules based on large samples of historic data. Large amounts of historical data make it possible to use different types of approaches, based on the knowledge or technology or types of analysis software available; for example, neuro-fuzzy models, used to model the membership functions as well as to create the blocks of rules, or genetic algorithms or the widely employed fuzzy-clustering methods. This is particularly important if the model is enriched with a high number of inputs, which would make the work of the experts extremely burdensome and characterized by a significant degree of inaccuracy. In this respect, there may be a trade-off between interpretability and automatic learning methods and several authors have dealt with the problem of rule generation (see Guillaume 2001, Gómez-Skarmeta et al. 1999, Zhang et al. 2009, Xiao and Liu 2005). In Guillaume and Charnomordic (2011) a free software is proposed, available on the web, which allows the interpretation of systems built automatically from the data, in all phases of design.

3 Sensitivity Model and FCSI

In this section, we associate the fuzzy expert system described above with a model of sensitivity analysis (SA). The expert system and the SA model form what we call the Default Risk & Sensitivity Model (DRSM).

SA is the "study of how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input" (Saltelli et al. 2004, p. 45). Given a model and a set of inputs (parameters), SA measures the parameters' influence in terms of variability of the model output. Specifically, SA models aim to investigate the variation of the objective function (in our case, the firm's default risk) under changes in the model inputs, also aiming at identifying the most influential risk factors affecting the model output.

Many SA techniques are defined in the literature (see Borgonovo and Plischke 2016, Pianosi et al. 2016, for review of SA methods) and the choice of technique depends on several factors, among which the purpose of the analysis and the size of the variation of the parameters.

In our case, the default risk variation caused by changes in key drivers or groups of key drivers, is analyzed in both chronological and managerial perspectives:

- DRSM decomposes the historical realized variation of the enterprise default risk into the effects of key parameters and identifies the main reasons of rating variation across time
- DRSM suggests managerial actions which should be undertaken for improving the rating, especially for increasing the success of complex financial operations such as bond issues, mergers and acquisitions, and debt restructuring.

The scope of DRSM is multiple and concerns several dimensions of analysis:

- it supports the evaluator in identifying the effects of each parameter on the rating variation
- it enables accomplishing a selective analysis in terms of groups of parameters. For example, it measures the impact of the following main groups on the default risk

profile: (i) Financial Vulnerability, (ii) Operating Efficiency, (iii) Operating Risk

- it enables ranking any group of variables according to their relevance on the default risk variation
- it enables identifying the maximum effect of a variable or group of variables on the default risk
- it supports the financial manager in her/his activities of financial planning and optimization, and in functions of programming, control and capital structuring
- it offers managerial actions for improving and controlling the credit risk profile of the enterprise.

It is worth noting that

- the DRSM can be performed even starting from primitive, 0-level economic and financial variables as they result from the operations (such as revenues, COGS, long-term debt), not just from worked drivers such as indices and ratios (e.g. Leverage, Long-Term Leverage, FFO-on-Debt. See also footnote 2)
- the application of SA is independent of the adopted rating model: While we present it in conjunction with the fuzzy expert system illustrated in the previous sections, the SA model is readily available for any algorithm and any set of parameters defining any possible rating model (i.e., the SA model does not depend on the credit rating model).

Finite Change Sensitivity Indices (FCSIs; Borgonovo 2010a, 2010b) represent a Sensitivity Analysis technique focusing on the output change due to a finite variation of the inputs. The FCSI technique is applicable for whatever parameters variation; it does not require any peculiar variation scheme or sufficiently small parameters changes.[5]

Let f be the objective function, defined on the *parameter space* X, which maps the vector of inputs (or parameters or key drivers) $x = (x_1, x_2, \ldots, x_n) \in X$ onto the model output $y(x)$:

$$f : X \subset \mathbb{R}^n \to \mathbb{R}, \quad y = f(x), \quad x = (x_1, x_2, \ldots, x_n)\,. \tag{1}$$

Let $x^0 = (x_1^0, \ldots, x_n^0)$ be the base (or initial) value of the parameters and $f(x^0)$ be the corresponding model output. The parameters vary from x^0 to $x^1 = \left(x_1^1, x_2^1, \ldots, x_n^1\right) \in X$, the so-called realized value, and the related output is $f(x^1)$. The output variation is $\Delta f = f(x^1) - f(x^0)$.

Let $(x_i^1, x_{(-i)}^0) = (x_1^0, x_2^0, \ldots, x_{i-1}^0, x_i^1, x_{i+1}^0, \ldots, x_n^0)$ be obtained by varying the parameter x_i to the new value x_i^1, while the remaining $n-1$ parameters are fixed at x^0. Similarly, $(x_i^1, x_j^1, x_{(-i,j)}^0) = (x_1^0, x_2^0, \ldots, x_{i-1}^0, x_i^1, x_{i+1}^0, \ldots, x_{j-1}^0, x_j^1, x_{j+1}^0, \ldots, x_n^0)$ is the vector of inputs assuming that x_i and x_j are set to the new values, while the remaining $n-2$ are unvaried, and so forth for all j-tuples of inputs, $j = 1, 2, \ldots, n$.

Two viable definitions of Finite Change Sensitivity Indices are First Order FCSI and Total Order FCSI. The First Order FCSI of parameter x_i considers the individual effect

[5] FCSIs are based on the properties of functional ANOVA decomposition for finite changes (Rabitz and Alis 1999, Borgonovo 2010b).

of x_i on the variation of f (Borgonovo 2010b):

$$\Delta_i^1 f = f(x_i^1, x_{(-i)}^0) - f(x^0) \tag{2}$$

and, in normalized version, $\Phi_i^{1,f} = \frac{\Delta_i f}{\Delta f}$.

The Total Order FCSI of a parameter, instead, measures the total effect of the input on f, including both the individual contribution and the interactions between the parameter and the other parameters. The interaction between x_i and x_j, denoted as $\Delta_{i,j} f$, is the portion of $f(x_i^1, x_j^1, x_{(-i,j)}^0) - f(x^0)$ that is not explained by the individual effects $\Delta_i^1 f$ and $\Delta_j^1 f$: $\Delta_{i,j} f = f(x_i^1, x_j^1, x_{(-i,j)}^0) - f(x^0) - \Delta_i^1 f - \Delta_j^1 f$. Likewise, the interaction between the triplet of inputs x_i, x_j and x_h, identified as $\Delta_{i,j,h} f$, is the portion of $f(x_i^1, x_j^1, x_h^1, x_{(-i,j,h)}^0) - f(x^0)$ that is not explained by the individual effects and by the interactions between any pair of inputs x_i, x_j and x_h:

$$\Delta_{i,j,h} f = f(x_i^1, x_j^1, x_h^1, x_{(-i,j,h)}^0) - f(x^0) - \Delta_i^1 f - \Delta_j^1 f - \Delta_h^1 f - \Delta_{i,j} f - \Delta_{i,h} f - \Delta_{j,h} f$$

(analogously for a group of $s > 3$ parameters). The variation of f between the base and the realized case, Δf, can be written as the sum of individual effects and interactions between parameters and groups of parameters (Borgonovo 2010b):[6]

$$\Delta f = \overbrace{\sum_{i=1}^{n} \Delta_i^1 f}^{\text{individual effects}} + \underbrace{\overbrace{\sum_{i_1<i_2} \Delta_{i_1,i_2} f}^{\text{pairs}} + \overbrace{\sum_{i_1<i_2<i_3} \Delta_{i_1,i_2,i_3} f}^{\text{triplets}} + \cdots + \overbrace{\sum_{i_1<i_2\cdots<i_s} \Delta_{i_1,i_2,\ldots,i_s} f}^{s\text{-tuples}} + \cdots + \overbrace{\Delta_{i_1,i_2,\ldots,i_n} f}^{n\text{-tuple}}}_{\text{interactions}}, \tag{3}$$

where the general term $\sum_{i_1<i_2\cdots<i_s} \Delta_{i_1,i_2,\ldots,i_s} f$ is the sum of the interactions between groups of s parameters.

The Total Order FCSI of x_i, denoted as $\Delta_i^T f$, is defined as the sum of the individual effect of x_i and the interaction effect of x_i, which is the sum of any interaction involving x_i, identified as $\Delta_i^I f$:

$$\Delta_i^T f = \Delta_i^1 f + \Delta_i^I f = \Delta_i^1 f + \sum_{\substack{i_1<i_2 \\ i\in\{i_1,i_2\}}} \Delta_{i_1,i_2} f + \cdots + \sum_{\substack{i_1<i_2\cdots<i_s \\ i\in\{i_1,i_2,\ldots,i_s\}}} \Delta_{i_1,i_2,\ldots,i_s} f + \cdots + \Delta_{i_1,i_2,\ldots,i_n} f \tag{4}$$

and the normalized Total Order FCSI is $\Phi_i^T = \frac{\Delta_i^T f}{\Delta f}$.

Borgonovo (2010b, Proposition 1) showed that $\Delta_i^T f$ is also obtained as

$$\Delta_i^T f = f(x^1) - f(x_i^0, x_{(-i)}^1), \ \forall i = 1, 2, \ldots, n, \tag{5}$$

where $(x_i^0, x_{(-i)}^1)$ is the point with each parameter equal to the realized value x^1, except for the parameter x_i which is equal to x_i^0.

Considering a subset of parameters $S_k = \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$, the relevance of the subset is defined from the notion of importance measures of a single parameter in (2) and (5). The First Order FCSI of S_k is $\Delta_{S_k}^1 f = f(x_{(i_1,i_2,\ldots,i_s)}^1, x_{(-(i_1,i_2,\ldots,i_s))}^0) - f(x^0)$, that

[6] Each interaction between parameters and group of parameters is counted only once in this formula.

we denote as $f(x^1_{S_k}, x^0_{(-S_k)}) - f(x^0)$, and the Total Order FCSI is $\Delta^T_{S_k} f = f(x^1) - f(x^0_{(i_1,i_2,\dots,i_s)}, x^1_{(-(i_1,i_2,\dots,i_s))})$, which can be denoted as $f(x^1) - f(x^0_{S_k}, x^1_{(-S_k)})$.
Furthermore, given a pair of disjoint subsets of parameters (i.e. whose intersection is the empty set), here denoted as S_k and S_l, the interaction between S_k and S_l is $\Delta_{S_k,S_l} f = f(x^1_{S_k}, x^1_{S_l}, x^0_{(-S_k,S_l)}) - f(x^0) - \Delta^1_{S_k} f - \Delta^1_{S_l} f$; the interaction between an increasing group of disjoint subsets (e.g., a triplet of subsets) can be calculated similarly to an increasing group of parameters.

Finally, consider a group of d disjoint subsets whose union is the whole set of parameters; Δf can be decomposed in the sum of individual effects of any subset and the interactions between any group of subsets, similarly to (3). The Total Order FCSI of the subset S_k, $\Delta^T_{S_k} f$, can be calculated as the sum of its individual effect $\Delta^1_{S_k} f$ and its interaction effect $\Delta^I_{S_k} f$, defined as the sum of any interaction involving S_k, consistently with equation (4):

$$\begin{aligned}\Delta^T_{S_k} f = \Delta^1_{S_k} f + \Delta^I_{S_k} f = \Delta^1_{S_k} f + &\sum_{\substack{k_1<k_2\\ k\in\{k_1,k_2\}}} \Delta_{S_{k_1},S_{k_2}} f + \dots \\ + &\sum_{\substack{k_1<k_2\dots<k_s\\ k\in\{k_1,k_2,\dots,k_s\}}} \Delta_{S_{k_1},S_{k_2},\dots,S_{k_s}} f + \dots + \Delta_{S_{k_1},S_{k_2},\dots,S_{k_d}} f.\end{aligned}$$

Despite its usefulness, the definition of Total Order FCSI does not provide a clean decomposition of the output change in terms of Total FCSIs. In other words, the sum of the parameters' effects is not equal to the function variation.

The reason is that (4) includes duplications of the interactions between pairs, triplets, s-tuples. More precisely, the summand $\sum_{i_1<i_2} \Delta_{i_1,i_2} f$ includes twice the interaction between any pair of parameters, the summand $\sum_{i_1<i_2<i_3} \Delta_{i_1,i_2,i_3} f$ contains three times the interaction between any triplet of parameters, and, in general, $\sum_{i_1<i_2\dots<i_s} \Delta_{i_1,i_2,\dots,i_s} f$ contains s times the interactions between any s-tuple of parameters. Conversely, in (3), the interaction terms only appears once. As a result:

$$\Delta^T_1 f + \Delta^T_2 f + \dots + \Delta^T_n f \neq \Delta f$$

or, dividing by Δf,

$$\Phi^T_1 + \Phi^T_2 + \dots + \Phi^T_n \neq 1.$$

This means that the Total FCSIs do not sum up to 100% of the output change: It either explains less or more than 100%.

Example 1. Let f be the market value of the equity of a firm, depending on the share price p and the number of shares q. The vector of inputs is $x = (p, q)$ and the equity market value is $f(p, q) = p \cdot q$. We assume that the initial state is $x^0 = (p^0, q^0) = (10,\ 200)$, which implies that the equity value is $f(p^0, q^0) = p^0 \cdot q^0 = 10 \cdot 200 = 2,000$; we also assume that, after one year, price and number of share have changed to $x^1 = (p^1, q^1) = (13,\ 300)$, so that the market value of equity is $f(p^1, q^1) = 13 \cdot 300 = 3,900$. The change in the equity value is then $\Delta f = f(x^1) - f(x^0) = 3,900 - 2,000 = 1,900$. We aim at identifying the relevance of the share price and the number of share in terms of the variation of the market value of equity. From eq. (2), the First Order FCSI of share price is $\Delta^1_p f = f(p^1, q^0) - f(p^0, q^0) = 13 \cdot 200 - 10 \cdot 200 = 600$ and the First Order FCSI of q is $\Delta^1_q f = f(p^0, q^1) - f(p^0, q^0) = 10 \cdot 300 - 10 \cdot 200 = 1,000$. The interaction between p and q, $\Delta_{p,q} f$ is equal to the interaction effect of both the parameters:

$$\begin{aligned}\Delta_{p,q} f &= f(p^1, q^1) - f(p^0, q^0) - \Delta^1_p f - \Delta^1_q f \\ &= 13 \cdot 300 - 10 \cdot 200 - 600 - 1,000 \\ &= 300 \\ &= \Delta^I_p f = \Delta^I_q f.\end{aligned}$$

However, from (4), the Total Order FCSI of the share price is $\Delta^T_p f = \Delta^1_p f + \Delta^I_p f = 600+300 = 900$ and the Total Order FCSI of the number of shares is $\Delta^T_q f = \Delta^1_q f + \Delta^I_q f = 1,000 + 300 = 1,300$.[7] Therefore, the sum of the Total Order FCSIs is different from Δf:

$$\Delta^T_p f + \Delta^T_q f = 900 + 1,300 = 2,200 \neq 1,900 = \Delta f.$$

The reason is that the interaction term between price and number of shares is included in both $\Delta^T_p f$ and $\Delta^T_q f$, so there is double-counting that prevents the correct decomposition of the output change. Equivalently, one may write

$$\Phi^T_p + \Phi^T_q = (900/1,900) + (1,300/1,900) = 0.4737 + 0.6842 = 1.1579 \neq 1.$$

In this case, the Total FCSI explains too much. ◊

We now solve the problem by introducing a duplication-cleaning procedure which eliminates the redundant, multiple interactions and allows a complete and exact decomposition of the output change through the Clean Total Order FCSIs.

4 Cleaning the Total Order FCSI

We fine-tune the FCSI by defining the *clean* interaction effect of parameter x_i, as the interaction effect $\Delta^I_i f$ multiplied for a special corrective factor α. Denoting as $\Delta^{I*}_i f$ the clean interaction effect:

$$\Delta^{I*}_i f = \Delta^I_i f \cdot \alpha, \tag{6}$$

where we define α as

$$\alpha = \frac{\sum_{j_1<j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_n} f}{\sum_{j=1}^{n} \Delta^I_j f}. \tag{7}$$

Since α is the ratio of the sum of the true interaction effects over the total imputed interaction effect, it measures the degree of redundancy (if it is smaller than 1) or deficiency

[7] The Total Order FCSIs can also be determined from (5): $\Delta^T_p f = f(p^1, q^1) - f(p^0, q^1) = 13 \cdot 300 - 10 \cdot 300 = 900$ and $\Delta^T_q f = f(p^1, q^1) - f(p^1, q^0) = 13 \cdot 300 - 13 \cdot 200 = 1,300$.

(if it is greater than 1) of the Total Order FCSI. From (3), α can be rewritten as

$$\alpha = \frac{\Delta f - \sum_{j=1}^{n} \Delta_j^1 f}{\sum_{j=1}^{n} \Delta_j^I f} \tag{8}$$

whence

$$\Delta_i^{I*} f = \Delta_i^I f \cdot \frac{\Delta f - \sum_{j=1}^{n} \Delta_j^1 f}{\sum_{j=1}^{n} \Delta_j^I f} = \overbrace{\frac{\Delta_i^I f}{\sum_{j=1}^{n} \Delta_j^I f}}^{\text{interaction imputed to parameter } i} \cdot \overbrace{\Big(\Delta f - \sum_{j=1}^{n} \Delta_j^1 f\Big)}^{\text{overall interaction}} \tag{9}$$

The clean interaction effect $\Delta_i^{I*} f$ can then be interpreted as the component of $\Delta f - \sum_{j=1}^{n} \Delta_j^1 f$ according to the proportion of $\Delta_i^I f$ over the sum of $\Delta_j^I f$ for any parameter.

We can now define the Clean Total Order FCSI of parameter x_i, $\Delta_i^{T*} f$, as the sum of individual contribution and clean interaction effect of x_i:

$$\Delta_i^{T*} f = \Delta_i^1 f + \Delta_i^{I*} f \tag{10}$$

and, in normalized version, $\Phi_i^{T*} = \frac{\Delta_i^{T*} f}{\Delta f}$. We now show that the clean indeces perfectly decompose the output change, explaining the 100% of the variation.

Proposition 1. *The sum of Clean Total Order FCSIs is equal to the variation of the model output f: $\sum_{i=1}^{n} \Delta_i^{T*} f = \Delta f$. In normalized version, $\sum_{i=1}^{n} \Phi_i^{T*} = 1$.*

Proof. From (9),

$$\sum_{i=1}^{n} \Delta_i^{I*} f = \sum_{i=1}^{n} \frac{\Delta_i^I f}{\sum_{j=1}^{n} \Delta_j^I f} \cdot \Big(\Delta f - \sum_{j=1}^{n} \Delta_j^1 f\Big) = \Delta f - \sum_{j=1}^{n} \Delta_j^1 f. \tag{11}$$

From (10) and (11),

$$\sum_{i=1}^{n} \Delta_i^{T*} f = \sum_{i=1}^{n} \Delta_i^1 f + \sum_{i=1}^{n} \Delta_i^{I*} f = \sum_{i=1}^{n} \Delta_i^1 f + \Delta f - \sum_{i=1}^{n} \Delta_i^1 f = \Delta f. \tag{12}$$

Diving both terms of the equality by Δf, one gets $\sum_{i=1}^{n} \Phi_i^{T*} = 1$. □

The duplication-cleaning procedure is applicable not also for measuring the relevance of single drivers but also for determining the importance of disjoint subsets of parameters. The clean interaction effect of a subset S_k, denoted as $\Delta_{S_k}^{I*} f$, can be obtained from (6) and (9) just considering interactions between subsets, interaction effect and individual effect of the subset, instead of the effects of single parameters:

$$\begin{aligned} \Delta_{S_k}^{I*} f &= \Delta_{S_k}^{I} f \cdot \frac{\sum_{l_1<l_2} \Delta_{S_{l1},S_{l2}} f + \cdots + \sum_{l_1<l_2\cdots<l_s} \Delta_{S_{l1},S_{l2},\ldots,S_{ls}} f + \cdots + \Delta_{S_{l1},S_{l2},\ldots,S_{ld}} f}{\sum_{l=1}^{d} \Delta_{S_l}^I f} \\ &= \frac{\Delta_{S_k}^I f}{\sum_{l=1}^{d} \Delta_{S_l}^I f} \cdot \Big(\Delta f - \sum_{l=1}^{d} \Delta_{S_l}^1 f\Big). \end{aligned} \tag{13}$$

Similarly, the Clean Total Order FCSI of S_k, represented as $\Delta_{S_k}^{T*} f$, can be determined from (10) by summing up the individual effect and the clean interaction effect of the subset:

$$\Delta_{S_k}^{T*} f = \Delta_{S_k}^{1} f + \Delta_{S_k}^{I*} f. \tag{14}$$

Example 2. Consider Example 1. From (9), the clean interaction effect attributable to the price, p, is

$$\begin{aligned}
\Delta_p^{I*} f &= \frac{\Delta_p^I f}{\Delta_p^I f + \Delta_q^I f} \cdot \left(\Delta f - \Delta_p^1 f - \Delta_q^1 f\right) \\
&= \frac{300}{300 + 300} \cdot \Big((13 \cdot 300 - 10 \cdot 200) - 600 - 1,000\Big) \\
&= 150
\end{aligned}$$

and is equal to the clean interaction effect of q: $\Delta_q^{I*} f = \Delta_p^{I*} f = 150$. From (10), the clean Total Order FCSI of p is $\Delta_p^{T*} f = \Delta_p^1 f + \Delta_p^{I*} f = 600 + 150 = 750$ and the clean Total Order FCSI of q is $\Delta_q^{T*} f = \Delta_q^1 f + \Delta_q^{I*} f = 1,000 + 150 = 1,150$.[8] The sum of the clean Total Order FCSIs is equal to the variation of f: $\Delta_p^{T*} f + \Delta_q^{T*} f = 750 + 1,150 = 1,900 = \Delta f$.

5 A case study

We apply DRSM to an Italian-controlled industrial company, mainly operating in the automotive business. We have used real, publicly available, consolidated financial statements of the company in recent years. We denote as 0 the base year, and rating has been determined for four years: 0, 3, 5, and 6. The vector of inputs $x = (x_1, x_2, \ldots, x_n) \in X$ consists of the 18 economic and financial variables (rating model inputs) which we have described in Section 2. The model output $y(x)$ is the default risk. We calculate the default risk and the credit rating of the company in the four periods via the application of the fuzzy-logic expert rating model introduced in this work and determine the changes in default risk from period to period. The evolution of the default risk, rating and risk variation across time is summarized in the following table:

Year	Default Risk	Rating	Risk variation
0	0.8185	CCC	–
3	0.7143	B	−0.1042
5	0.5079	BB	−0.2064
6	0.5714	BB	+0.0635

In the first two intervals (0, 3) and (3, 5) the company has reduced its default risk and improved the credit rating from class CCC (in 0) to B (in 3) and from class B (in 3) to BB (in 5); in the last interval (5, 6) the default risk has increased, but the rating class has not varied.

[8] In this trivial case, interaction effect is split up in half, but this is not so in general.

Decomposition of the change in default risk and ranking of parameters. We focus on the decrease in default risk from year 0 to year 3. Specifically, the change in default risk in this time interval has been $\Delta f = -0.1042$ (see Table 4). The (clean) importance measures of the 18 key parameters are reported in Table 5. Note that

- many variables have no individual effect whatsoever nor interaction effects (e.g. FFO-on-Debt and Interest Coverage 2): Their influence on the change in default risk is zero
- for all inputs (except Interest Coverage 1), First Order FCSI and interaction effect have opposite sign, which means that they tend to offset each other
- one input (Interest Coverage 1) has no First Order effect but (slightly) affects the change in default risk via the interaction effect.

As now evident, the sum of Clean Total Order FCSIs is equal to the variation of the default risk ($\Delta f = -0.1042$). Table 5 ranks the input variables according to their relevance on the change in default risk. It turns out that

- the decrease in default risk (and the related rating improvement) is mainly determined by the increase of OFC-to-Debt (rank 2), which improves the financial vulnerability profile, by the increase of OWC Intensity (rank 3), which determines an efficiency enhancement, and by reduction of Leverage and Long-Term Leverage (ranks 4 and 5), which contribute to decrease the financial vulnerability of the firm
- the improvement in rating is smoothed by the increase of EBITDA Standard Deviation, which increases the Operating Risk via the Specific Risk. The standard deviation of EBITDA is the most relevant variable of the set of parameters (rank 1). The improvement in rating is also negatively affected by the decrease of Operating Leverage and Interest Coverage 1 (however, their effect on the output change is very mild)
- all the remaining variables have no influence on the default risk variation.

Figure 4 is the graphical representation of Table 5. The parameters are reported on the horizontal axis, sorted by decreasing influence on rating variation (hence, rank of parameters decreases from left to right); as for the vertical dimension, the Clean Total Order FCSIs ($\Delta_i^{T*} f$) are reported: A bar above the axis informs that the parameter has increased the default risk, while a bar below the axis informs that the parameter has decreased the default risk.

Impact on output change of one key driver as opposed to the residual drivers. As the most determinant parameter for risk reduction in $(0, 3)$ is OCF-to-Debt, a viable application of DRSM is to investigate the role of OCF-to-Debt as compared with the residual input factors. To this end, we divide the set of parameters into OCF-to-Debt, on one side, and the subset of the residual 17 drivers, on the other side. We determine (i) the individual contribution of OCF-to-Debt, (ii) the individual effect of the above mentioned subset, and (iii) the interaction between OCF-to-Debt and the subset. It is worth noting that the individual effect of the subset consisting of the residual drivers

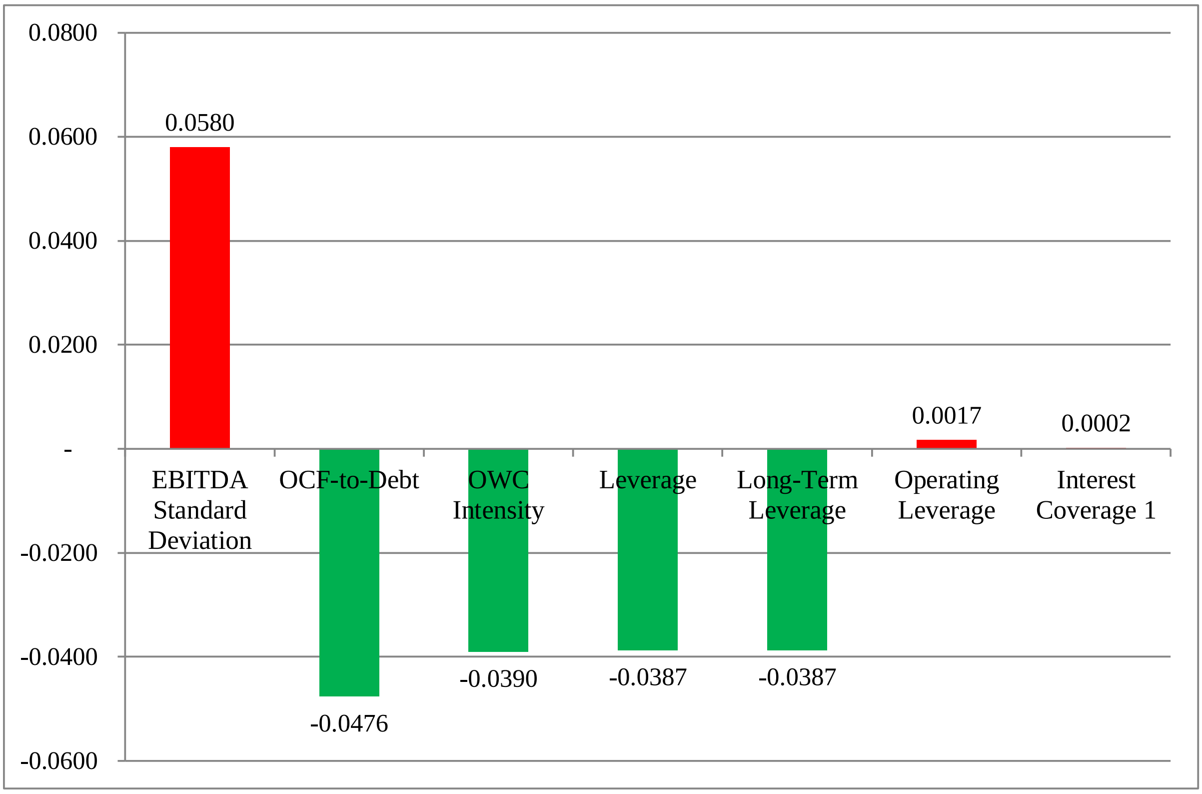

Figure 4: (Clean) Total Order FCSIs of the parameters ($\Delta_i^{T*} f$)

Table 4: Values of the parameters in 0 and 3

	Variable	0	3	Variation
1	Leverage	0.8589	0.7249	−0.1340
2	Long-Term Leverage	0.8126	0.5295	−0.2831
3	FFO-on-Debt	0.0959	0.1143	0.0184
4	Interest Coverage 1	0.7292	2.0779	1.3487
5	Interest Coverage 2	−0.2502	0.5485	0.7987
6	OCF-to-Debt	−0.0518	0.0919	0.1437
7	FCR	0.2790	0.3118	0.0328
8	Debt Service Coverage	−0.2915	0.2869	0.5784
9	OWC Intensity	0.0404	0.3073	0.2669
10	Financial Cycle	0.6250	0.6438	0.0188
11	EBITDA on Sales	0.0399	0.0182	−0.0217
12	ROA	−0.0122	0.0256	0.0378
13	Customer Concentration	0.6063	0.6250	0.0187
14	Product Positioning	0.6438	0.6250	−0.0188
15	Industry Prospects	0.5188	0.6063	0.0875
16	EDITDA Standard Deviation	0.3550	0.7313	0.3763
17	Operating Leverage	0.3550	0.3750	0.0200
18	Industrial Coverage	2.0836	2.2674	0.1838
	Output			
	Default Risk	0.8185	0.7143	−0.1042

quantifies the change in the default risk in case all variables except OCF-to-Debt vary from the initial value at time 0 to the realized value at time 3 (with OCF-to-Debt kept constant at its initial value at 0). Table 6 shows that the individual variation of OCF-to-Debt explains the 62.76% of the change in default risk in the interval $(0, 3)$, while the individual effect of the other 17 variables, taken together, determines the 89.44% of the default risk variation. Therefore, the OCF-to-Debt has a relative impact equal to 70.17% = 62.76%/89.44% of the impact of the other 17 parameters considered together, thereby confirming the crucial influence of OCF-to-Debt on risk variation.

Analysis of groups of variables. A further useful application of the DRSM consists of analyzing the role of selective groups of variables bearing special importance, aiming at identifying the influence of different areas of financial management on default risk variation. This analysis aims at pointing out the most effective managerial actions and policies for the evolution of the enterprise credit risk across time. For example, referring to Figure 1, consider the following areas pinpointed by the second-level intermediate variables, namely, Financial Vulnerability, Operating Efficiency, and Operating Risk:

- Financial Vulnerability represents the degree at which the firm is exposed to risk owing to an excessive debt. It is a second-level intermediate variable and is affected by 8 input factors. We denote it as V;
- Operating Efficiency represents the degree at which the firm is able to manage the operations in an efficient way. It is a second-level intermediate variable which has

Table 5: (Clean) importance measures and ranks of the parameters

	Variable	**First Order FCSI**	**Interaction**	**Total Order FCSI**	**Normalized Total Order FCSI**	**Rank**
16	EBITDA Standard Deviation	0.0782	−0.0202	0.0580	−55.62%	1
6	OCF-to-Debt	−0.0654	0.0177	−0.0476	45.71%	2
9	OWC Intensity	−0.0535	0.0145	−0.0390	37.44%	3
1	Leverage	−0.0531	0.0144	−0.0387	37.13%	4
2	Long-Term Leverage	−0.0531	0.0144	−0.0387	37.13%	5
17	Operating Leverage	0.0023	−0.0006	0.0017	−1.61%	6
4	Interest Coverage 1	0	0.0002	0.0002	−0.19%	7
3	FFO-on-Debt	0	0	0	0%	8
5	Interest Coverage 2	0	0	0	0%	8
7	FCR	0	0	0	0%	8
8	Debt Service Coverage	0	0	0	0%	8
10	Financial Cycle	0	0	0	0%	8
11	EBITDA on Sales	0	0	0	0%	8
12	ROA	0	0	0	0%	8
8	Customer Concentration	0	0	0	0%	8
14	Product Positioning	0	0	0	0%	8
15	Industry Prospects	0	0	0	0%	8
18	Industrial Coverage	0	0	0	0%	8
	Sum	−0.1446	0.0404	−0.1042	100.00%	

Table 6: The role of OCF-to-Debt

Effect	Description	Change in risk	%
Individual effect of OCF-to-Debt	OCF-to-Debt varies, residual drivers are constant	-0.0654	$+62.76\%$
Individual effect of residual drivers	OCF-to-Debt is constant, residual drivers vary	-0.0932	$+89.44\%$
Interaction effect	Interaction between OCF-to-Debt and residual drivers	$+0.0544$	-52.20%
	Sum	-0.1042	100.00%

to do with the economic profitability (EBITDA, ROA) and the ability of collecting cash from customers early and delaying payments to suppliers (operating cycle, cash cycle). It is affected by 4 input factors. We denote this group as E;[9]

- Operating Risk joins two kinds of risk: The strategic risk, related to such drivers as the customer concentration, the product positioning, the industry prospects, and the specific risk, referred to specific features of the firm under analysis (standard deviation of EBITDA, operating leverage, industrial coverage). It is a second-level intermediate variable which is affected by 6 key drivers. We denote it as R.

For each subset S_k we determine the First Order FCSI ($\Delta^1_{S_k} f$) and any interaction involving S_k. For instance, the individual effect of the Financial Vulnerability on the risk change from 0 to 3 is $\Delta^1_V f = f(x^1_V, x^0_{(-V)}) - f(x^0) = -0.1078$, meaning that it has played a positive role. As for the pairwise interaction, the interaction of this group with the Operating Efficiency is $\Delta_{V,E} f = f(x^1_V, x^1_E, x^0_R) - f(x^0) - \Delta^1_V f - \Delta^1_E f = 0.0535$, meaning that it has negatively (albeit very slightly) affected the rating; the interaction with the Operating Risk has acted positively, since $\Delta_{V,R} f = f(x^1_V, x^1_R, x^0_E) - f(x^0) - \Delta^1_V f - \Delta^1_R f = -0.0798$. The interaction between the three groups is $\Delta_{V,E,R} f = f(x^1) - f(x^0) - \Delta^1_V f - \Delta^1_E f - \Delta^1_R f - \Delta_{V,E} f - \Delta_{V,R} f - \Delta_{E,R} f = 0.0388$. Individual effects and interactions are collected in Table 7. Using the duplication-cleaning procedure, we perfectly decompose the change in default risk. Indeed, the sum of any contribution (individual effect and interaction), counted just once, is equal to the change in risk, $\Delta f = -0.1042$. The ranking is shown in Table 8 and in Figure 5. As can be gleaned from inspection of table and figure,

- the better rating at time 3 is primarily driven by the reduction in the Financial Vulnerability (V), which is the most influential area of financial management in the analysis, and by the decrease of the Operating Risk (R), which represents the second most relevant subset of parameters
- the better rating is curbed by the worsening of the Operating Efficiency (E), which is, however, the least influential management area.

Maximum effect of a variable. Another possible use of DRSM is the study of the maximum effect of a variable or a subset of variables on the default risk. For example,

[9]Note that Financial Vulnerability and Operating Efficiency are the antecedents of the (third-level intermediate variable) Financial Risk.

Table 7: First Order FCSIs and interactions of the subsets of parameters

First Order FCSIs	
$\Delta^1_V f$	-0.1078
$\Delta^1_E f$	-0.0535
$\Delta^1_R f$	0.0834
Interactions	
$\Delta_{V,E} f$	0.0535
$\Delta_{V,R} f$	-0.0798
$\Delta_{E,R} f$	-0.0388
$\Delta_{V,E,R} f$	0.0388
Sum $= \Delta f$	-0.1042

Table 8: Ranking of the subsets of parameters

	Subset	First Order FCSI	Interaction	Total Order FCSI	Normalized Total Order FCSI	Rank
V	Financial Vulnerability	-0.1078	0.0242	-0.0836	80.25%	1
R	Operating Risk	0.0834	-0.1533	-0.0700	67.13%	2
E	Operating Efficiency	-0.0535	0.1029	0.0494	-47.38%	3
	Sum of contributions	-0.0780	-0.0262	-0.1042	100.00%	

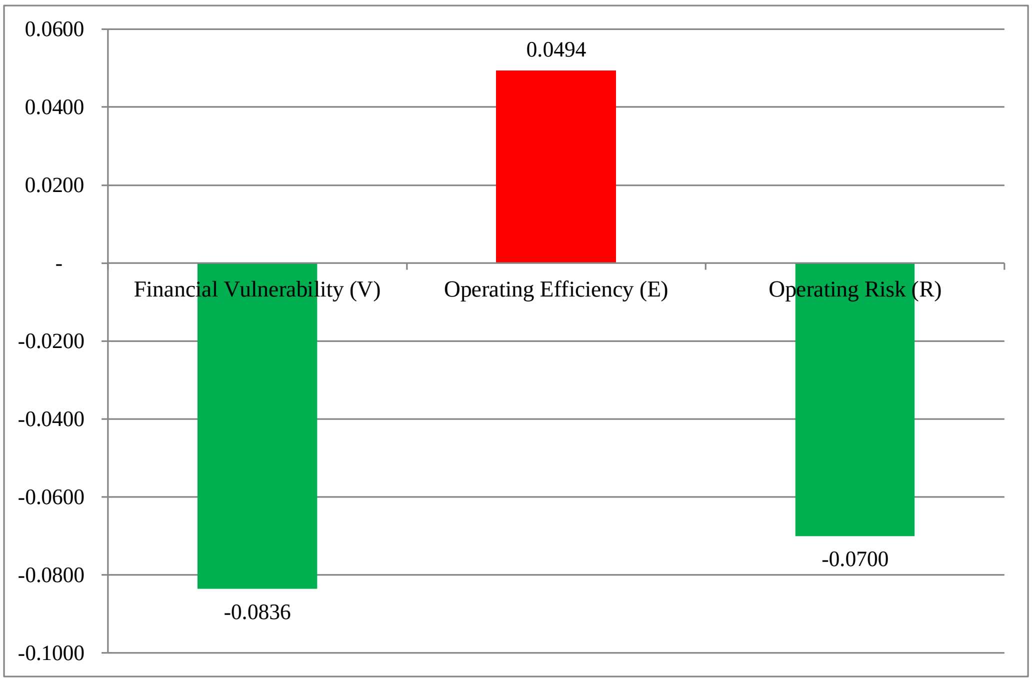

Figure 5: Group Analysis: (Clean) Total Order FCSIs ($\Delta^{T*}_{S_k} f$)

consider the parameter Fixed Charge Ratio (FCR), which represents a significant synthetic ratio of the firm's capacity to service the debt and, probably, the most informative index of financial stability. We analyze the effects of the improvement in FCR, compared to the base-case year 0, while all the residual parameters are fixed at their initial value in 0. Table 9 collects the levels of default risk and credit rating corresponding to increasing values of FCR. The first line of the table describes the base-case and reports the values of FCR, default risk and credit rating in year 0; in the second line FCR is evaluated in 3, while all the residual parameters are equal to their initial value; the following lines are obtained by increasing FCR by 0.06 starting from the base case, with all the other variables evaluated in 0. From inspection of the table,

- other things equal, the improvement in FCR is able to decrease the default risk to a minimum of 0.57142 and increase the rating to a maximum of BB, corresponding to a 30.19% risk reduction and a two-classes rating improvement
- values of FCR greater than 1.659 are uninfluential, so that a further improvement in rating must be accomplished by improving some other input factors.

The relationship between the increase of FCR and the decrease in default risk is represented in Figure 6. The relatively high impact of FCR on credit risk resulted from this analysis (assuming other things equal) is not surprising if one considers that FCR is an important measure of financial stability. While we have shown the impact of a single key driver, the analysis may be extended to considering the maximum effect of a subset of the parameters. This is accomplished by improving each variable belonging to the subset while all the parameters outside the relevant subset are kept fixed at their initial value in 0. The analysis becomes less trivial (because interaction effects among the group's variables occur) but the DRSM easily manages any such case and the change in risk may be exactly decomposed.

6 Concluding remarks

This paper introduces a credit rating model based on fuzzy logic and expert system, able to replicate and attribute logical consistency to the evaluation process of default risk and credit rating which is usually performed by human experts on the basis of available data. The expert system uses available data (knowledge base) and an inferential engine to produce the output. We consider a set of 18 economic and financial variables, both quantitative and qualitative. The system determines the default risk and the rating class in various years through a modular approach which aggregates the variables under a managerial and financial perspective.

We associate the rating system with the Finite Change Sensitivity Indices (Borgonovo 2010a, 2010b), a recent addition to the techniques of Sensitivity Analysis (SA) which aims at measuring the impact of the model inputs on the output change occurred passing from a base value (e.g., the output value at a given date) to a realized value (the output value at a subsequent date). We fine-tune FCSIs by eliminating some duplication effects and provide a clean, exact decomposition of the output change.

Table 9: Maximum effect of FCR on default risk and rating

	FCR	Default risk	Rating
(year 0)	0.279	0.8185	CCC
(year 3)	0.312	0.8185	CCC
	0.339	0.8185	CCC
	0.399	0.8185	CCC
	0.459	0.8185	CCC
	0.519	0.7784	CCC
	0.579	0.75574	CCC
	0.639	0.73256	B
	0.699	0.73256	B
	0.759	0.73256	B
	0.819	0.73256	B
	0.879	0.73256	B
	0.939	0.73256	B
	0.999	0.73256	B
	1.059	0.73256	B
	1.119	0.73256	B
	1.179	0.72640	B
	1.239	0.71428	B
	1.299	0.71428	B
	1.359	0.71428	B
	1.419	0.71428	B
	1.479	0.70386	B
	1.539	0.67882	B
	1.599	0.57834	BB
	1.659	0.57142	BB
	> 1.659	0.57142	BB

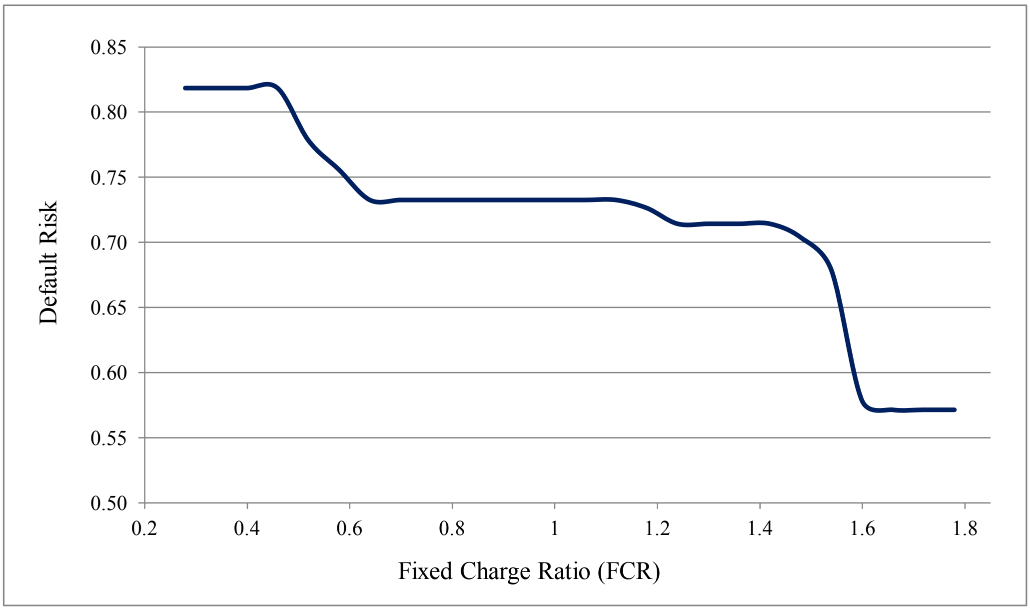

Figure 6: Increase of FCR and decrease in default risk

We use the results for giving rise to the Default Risk Sensitivity Model (DRSM) which investigates the variation of the enterprise default risk under changes in the model inputs for ex post analysis and for managerial decision-making. As for the former perspective, the DRSM allows the decomposition of the historic change of default risk and identifies the most relevant parameters which generated the change; as for the latter perspective, it suggests suitable managerial actions to be undertaken for improving the prospective rating and/or increasing the success of complex financial operations that are to be taken. Overall, a sensitivity analysis module, such as the one presented in this work, is a valuable tool to enhance the understanding of a complex fuzzy-logic model by providing insights into how the inputs affect the outputs of such models, thus strengthening the confidence of credit analysts in using such method in practice. From this point of view, sensitivity analysis is also crucial in model testing/validation: Additional simulation runs may be used for corroborating and, when necessary, calibrating the model.

Several categories of companies may benefit from the application of DRSM: Firms aiming at controlling and/or reducing their credit risk profile, enterprises needing a dynamic and mindful debt management, and public companies which are willing to inform the financial markets about the firm's present economic results and future prospects.

We have applied the DRSM to an Italian-controlled industrial company. We have identified the effects of parameters on the default risk and the rating change through time, then have determined the aggregate effects of groups of variables (specifically, Financial Vulnerability, Operating Efficiency, Operating Risk), have analyzed the impact of the ratio of the operating cash flow to the debt amount as opposed to the impact of the other variables taken together, and have calculated the maximum effect of a variable (FCR) on default risk.

Finally, it is worth noting that the sensitivity model is detached from the expert system: They are reciprocally autonomous in that each of them may be used independently. In particular, the sensitivity model presented does not depend on the fuzzy expert system: It is suitable for applications with any possible rating model and, therefore, any set of parameters (symmetrically, the rating model may also be adopted in association with other SA techniques). A potential scenario of future development is the combination of sensitivity analysis with automatic machine-learning algorithms for rating firms, aiming at melting the high learning and generalization capabilities of adaptive, computational-based, data-driven system with the promising feature of increasing the comprehensibility of complex models via the application of sensitivity analysis. Future researches may also be conducted for formal testing/validation of machine-learning approaches using data-driven schemes.

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Appendix

This Appendix reports a legend of accounting and financial terminology, a list of primary relations involving the main dimensions of the analysis, the description of the 18 model inputs, and the structure of the intermediate variables of the rating system.

Legend

COGS	= Cost of Goods Sold
D&A	= Depreciation and Amortization Expenditures
EBIT	= Earnings Before Interest and Taxes
EBITDA	= Earnings Before Interest, Taxes, Depreciation and Amortization
FCR	= Fixed Charge Ratio
FE	= Financial Expenses
FFO	= Funds from Operations
NI	= Net Income
NOPAT	= Net Operating Profit After Taxes
OCF	= Operating Cash Flow
OWC	= Operating Working Capital
PBT	= Profit Before Taxes
R&D	= Research and Development expenses
ROA	= Return on Assets
SG&A	= Selling, General and Administrative Expenses
T	= Taxes

Primary relations

Gross Profit	= Revenues − COGS
EBITDA	= Gross Profit − SG&A
EBIT	= EBITDA − D&A
NOPAT	= EBIT·(1−tax rate)
PBT	= EBIT − FE + Interest Income ± Extraordinary Items
NI	= PBT − T
OCF	= EBIT + D&A − investments + disposals − ΔOWC

Model inputs

The definition of the model inputs is reported in the following table:

	INPUT	DESCRIPTION
1	Leverage	Total Debt/(Total Debt + Equity Value)
2	Long-Term Leverage	Long-Term Debt/(Long-Term Debt + Equity Value)
3	FFO-on-Debt	(NOPAT + D&A + other noncash items)/Total Debt
4	Interest Coverage 1	(EBITDA − R&D)/ Financial Expenses
5	Interest Coverage 2	(EBIT − R&D)/ Financial Expenses
6	OCF-to-Debt	OCF/Total Debt
7	Fixed Charge Ratio (FCR)	(EBITDA − R&D)/(Debt Service + Taxes + Capital Expenditures)
8	Debt Service Coverage	OCF/Debt Service
9	OWC Intensity	*qualitative*
10	Financial Cycle	*qualitative*
11	EBITDA on Sales (Adjusted)	(EBITDA − R&D)/Revenues
12	ROA (Adjusted)	(EBIT − R&D)/Total Assets
13	Customer Concentration	*qualitative*
14	Product Positioninig	*qualitative*
15	Industry Prospects	*qualitative*
16	EBITDA Standard Deviation	Standard deviation of EBITDA (last five years)
17	Operating Leverage	Fixed Costs/Total Costs
18	Industrial Coverage	Gross Profit/Capital Expenditure

Notes

1. R&D are deducted from EBIT and EBITDA only if they are capitalized.
2. Total assets in ROA is net of minority participations.
3. Industry prospects is related to the market risk.
4. OWC intensity is the average amount of accounts receivable as opposed to the accounts payable.
5. Product positioning is an indicator of the product quality as perceived by the customers.
6. Customer concentration refers to the percentage of revenues of the 4 most important buyers (the higher the concentration, the higher the risk). Depending on the sector and on the case at hand, an index of geographic concentration may also be used in place or along with this metric.
7. Financial cycle is the lapse of time between when cash is paid to suppliers and cash is received from customers.

Intermediate variables

Figure 7 represents the sequence of the intermediate variables of the rating system until reaching the model output Default Risk (see also Figure 1).

Intermediate variables (1-st level)	(Risk of) Capital Structure
	Interest Coverage
	Debt Coverage
	WC Management
	Profitability
	Strategic Risk
	Specific Risk
Intermediate variables (2-nd level)	Financial Vulnerability
	Operating Efficiency
	Operating Risk
Intermediate variables (3-rd level)	Financial Risk
	Operating Risk
Model output (4-th level)	Default Risk

Figure 7: Intermediate variables

Average rates of return, working capital, and NPV-consistency in project appraisal: A sensitivity analysis approach

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Average rates of return, working capital, and NPV-consistency in project appraisal: A sensitivity analysis approach[☆]

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ABSTRACT

In project appraisal under uncertainty, the economic reliability of a measure of financial efficiency such as a rate of return depends on its strong NPV-consistency, meaning that the performance metric (i) supplies the same recommendation in accept–reject decisions as the NPV, (ii) ranks competing projects in the same way as the NPV, (iii) has the same sensitivity to perturbations in the input data as the NPV. In real-life projects, financial efficiency is greatly affected by the management of the working capital. Using a sensitivity analysis approach and taking into explicit account the role of working capital, we show that the *average return on investment* (ROI) is not strongly NPV-consistent in accept–reject decisions if the working capital is uncertain and changes under changes in revenues and costs. Also, it is not strongly NPV-consistent in project ranking. We also show that the *internal rate of return* (IRR) is not strongly NPV-consistent and economic analysis may even turn out to be impossible, owing to possible nonexistence and multiplicity caused by perturbations in the input data, as well as to possible shifts in the financial meaning of IRR under changes in the project's value drivers. We introduce the *straight-line rate of return* (SLRR), based on the notion of average rate of change, which overcomes all the problems encountered by average ROI and IRR: It always exists, is unique, strongly NPV-consistent for both accept–reject decisions and project ranking, and has an unambiguous financial nature.

1. Introduction

In capital asset projects, economic profitability may be measured with absolute metrics, such as the net present value (NPV), expressing value increase in monetary units, or relative metrics, expressing rates of return or profitability indices which aim at identifying a project's financial efficiency.

The preference for absolute metrics or relative metrics in practice may depend on several factors. Capital rationing is one such factor. It may occur in several different forms; for example, the firm may face an upper limit to borrow from banks; headquarters may impose budget limits on expenditures of a division; the firm may have more positive NPVs that it can finance; the firm's owners may exclude issuance of new shares to avoid loss of the firm's control; a given amount of monetary resources may be freed out of current operations and be available for new investments. Other kinds of constraints (limits in management time, skilled labor, equipment, know-how, etc.) and agency conflicts are also frequent in capital investment decisions. These (soft or hard) constraints often induce managers to focus on relative metrics measuring the marginal efficiency of capital (see Pike and Ooi, 1988; Berkovitch and Israel, 2004; Ross et al., 2011; Brealey et al., 2011).

Functional areas and educational background of decision makers play also a role. For instance, practitioners seem to be at ease with the intuitive appeal of a rate of return (Evans and Forbes, 1993; Graham and Harvey, 2001; Sandahl and Sjögren, 2003; Lindblom and Sjögren, 2009). Managers with a strong financial background generally do not encounter difficulties in using absolute metrics, whereas managers with a traditional accounting or engineering imprinting may be more confident in using rates of return instead of monetary values.

Therefore, the coherence or incoherence between absolute and relative metrics is, comprehensibly, an important theoretical and applicative issue. Net-present-value (NPV) consistency of a performance metric means that the decisions recommended by the metric are the same as the ones recommended by the NPV criterion. The literature on NPV-consistent (or NPV-compatible) measures is enormous and spans

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over several decades (e.g., see Hajdasinski, 1995, 1997; Hartman, 2000; Hartman and Schafrick, 2004; Pfeiffer, 2004; Gow and Reichelstein, 2007; Lindblom and Sjögren, 2009; Chiang et al., 2010; Pasqual et al., 2013).

Recent studies take a different view on NPV-consistency. Percoco and Borgonovo (2012) and Borgonovo and Peccati (2004, 2006) analyze the influence on the NPV and the internal rate of return (IRR) of the value drivers (also called key parameters or input data, which are the sources of investment risk) via the application of Sensitivity Analysis (SA). They show that the parameters whose uncertainty is most influential on NPV are not the same as the IRR's. More recently, using the average-internal-rate-of-return (AIRR) approach Marchioni and Magni (2018) (henceforth, MM, 2018) proposed a relative metric, the average Return On Investment (ROI), which enjoys strong NPV-consistency, in the sense that changes in the key parameters have the same effects on NPV and on average ROI, overcoming the deficiency of IRR described in Percoco and Borgonovo (2012) and Borgonovo and Peccati (2004, 2006). However, all these authors implicitly assumed a working capital equal to zero throughout the project's life. Also, they did not cope with project ranking.

The influence of working capital (WC) management on financial performance is suggested by several recent works. Among others, Caballero et al. (2014) find a significant link between working capital management and corporate performance. Chauhan (2019) highlights the long-term role of working capital management, as opposed to the traditional short-term view of working capital. Bian et al. (2018) study the effect of working capital requirements on the company's financial situation via a discounted cash flow model over the planning horizon, and Luciano and Peccati (1999) present the application of adjusted present value techniques to an inventory management problem. Huang et al. (2020) analyze the role of the supply chain finance to alleviate financing problems of small and medium enterprises and the beneficial effect of efficient working capital management on the selection of reasonable financing modes. Song et al. (2020) analyze the role of supply chain finance in reducing information asymmetry and increasing the possibility to raise WC. Pirttilä et al. (2020) underline the importance of the supply chain finance on the competitive advantage in the Russian automotive industry. Furthermore, Peng and Zhou (2019) propose three different models describing different level of cooperations into the supply chain and suggest to manage WC according to a supply chain-oriented solution. Moreover, Protopappa-Sieke and Seifert (2010) investigate the advantages of interrelating operational and financial aspects in decision-making about supply chain and working capital. In addition, Wetzel and Hofmann (2019) realize an exploratory network analysis about supply chain finance, financial constraints and corporate performance. Wu et al. (2019) consider the role of the payment term and of the payment approach on the financial performance of the supplier and retailer through cash flow optimization.

We build upon the SA literature as a tool for managing risk and we specifically focus on the recent subset of papers which study the reciprocal consistency of different performance metrics. At the same time, we take into explicit account the role of working capital management in selecting an economically significant and reliable measure of efficiency for making financial analyses and capital investment decisions. In particular, we

- show that the average ROI is not strongly NPV-consistent in presence of WC
- introduce a new performance metric, the Straight-Line Rate of Return (SLRR), which allows for nonzero (uncertain) WC while retaining strong NPV-consistency
- extend the notion of strong NPV-consistency to project ranking, showing that the SLRR's ranking is strongly NPV-consistent, if the initial outlays are equal
- measure the degree of inconsistency of the average ROI and the IRR and show that the SLRR outperforms these indices
- introduce some previously unknown pitfalls of the IRR.

Specifically, we show that, if one relaxes the assumption of zero WC, the average ROI is strongly NPV-consistent in accept–reject only if the WC is exogenous, that is, it does not change under changes in the value drivers. However, this case is not frequent, given the strong link which usually occurs between accounts receivable and revenues, between accounts payable and operating costs, and between inventory and production and sales. Also, the average ROI is not strongly NPV-consistent in project ranking. Moreover, albeit a rare case, the average ROI might not exist.

We use the notion of Chisini mean (Chisini, 1929) to find possible substitutes for the average ROI: The internal rate of return and the straight-line rate of return. We prove, via several counterexamples, that the IRR is not strongly NPV-consistent (see also Battaglio et al., 1996; Borgonovo and Peccati, 2006; Percoco and Borgonovo, 2012 on divergence between IRR and NPV) with non-negligible degrees of inconsistency, as measured via Spearman's (1904) correlation coefficient and Iman and Conover's (1987) top-down coefficient. We discover new, previously unknown deficiencies of IRR in project appraisal under uncertainty: Even in those cases where it exists and is unique, a simple perturbation of the key parameters may cause the IRR to disappear or generate multiple IRRs, with the unpleasant implication of making it impossible to assess the impact of a change in value drivers on the IRR; furthermore, the IRR may change its financial nature (investment rate versus financing rate) under changes in the key parameters, which makes IRR unhelpful.

In contrast, we find that the SLRR is strongly NPV-consistent, even in a strict sense (the relevances of the value drivers are the same as the NPV's) in accept–reject decisions and, if the competing projects share the same initial investment, in project raking. Also, it always exists, is unique, and has an unambiguous meaning.

The remaining part of the paper is structured as follows. Section 2 recalls the definition of strong NPV-consistency proposed in MM (2018) for accept–reject decisions, based on sensitivity analysis, and shows that the strong NPV-consistency of the average ROI rests on the assumption of zero WC or, alternatively, the assumption that WC is exogenously determined (i.e., it does not depend on revenues and costs); without either assumption, strong NPV-consistency of average ROI is not guaranteed. Section 3 uses the notion of Chisini mean to find alternative candidates enjoying strong NPV-consistency. Chisini's invariance requirement supplies the internal rate of return and the straight-line rate of return. The SLRR is shown to exist, be unique, and be strongly NPV-consistent in a strict form for accept–reject decisions, whereas the IRR is not. In Sections 4 and 5 we introduce new types of difficulties suffered by IRR under uncertainty. Section 6 proves, via counterexamples, that, in general, the average ROI is not strongly NPV-consistent under uncertain WC, and it further measures its level of inconsistency. Section 7 extends the notion of strong NPV-consistency to project ranking and shows that, unlike average ROI and IRR, the SLRR fulfills it if the projects' initial investment is the same. Some concluding remarks end the paper and summarize the difference among the three performance metrics.

2. Accept–reject decisions and NPV-consistency of average ROI

2.1. Economic setting of investment decisions

Consider a capital asset project, P, and let $F = (F_0, F_1, \ldots, F_p)$, $F_0 \neq 0$, be its estimated stream of free cash flows (FCFs), where p is the number of periods in which the firm operates the project. A positive cash flow means that the capital providers (i.e., shareholders and debtholders) receive money from the firm (i.e., money *flows out* of the firm), a negative cash flow means that the capital providers contribute money to the firm (i.e., money *flows in* the firm). The project's net present value (NPV) is the algebraic sum of the discounted cash flows, and represents the *economic value created*: NPV $= \sum_{t=0}^{p} F_t(1+k)^{-t}$. The discount rate k is the so-called *cost of capital* (COC) (or minimum attractive rate of return).[1]

Definition 1 (*NPV Citerion for Accept/reject Decisions*)**.** A project creates value (i.e., it is worth undertaking) if and only if NPV > 0.

Following we define the classical notion of NPV-consistency for a rate of return. It provides a notion of weak NPV-consistency based on the decision recommended by a given metric.

Definition 2 (*Weak NPV-consistency for Accept/reject Decisions*)**.** A rate of return φ is weakly NPV-consistent if and only if a decision maker adopting φ makes the same decision suggested by the NPV criterion. In formal terms, φ is NPV-consistent if, given a cutoff rate k, the following statements are true:

– an investment project creates value if and only if $\varphi > k$
– a financing project creates value if and only if $\varphi < k$.

In real-life applications, to evaluate a project and make a decision on project acceptability, the analyst draws, for each period, the project's pro forma financial statements (balance sheets and income statements) where prospective incomes and book values are determined. More precisely, the analyst estimates, for every $t = 0, 1, \dots, p$, the incomes, I_t, and the book values, b_t, which represents the amount of invested capital at the beginning of period $[t, t+1]$. The initial book value coincides with the initial investment (i.e., $b_0 = -F_0$) and the terminal book value (after liquidation) is equal to zero (i.e., $b_p = 0$). After estimating incomes and book values, the analyst derives the cash flows, often called *free cash flows* (FCF), by subtracting the changes in book value from the incomes:

$$F_t = I_t - \Delta b_t, \tag{1}$$

where $\Delta b_t = b_t - b_{t-1}$. The pro forma financial statements along with Eq. (1) represent a standard tool in finance and in industry and are the basis for the financial modeling of capital asset projects.[2] Hence, the NPV may be framed in terms of incomes and changes in book value: $\text{NPV} = -b_0 + \sum_{t=1}^{p} (I_t - \Delta b_t)/(1+k)^t$.

Magni (2010) proved that, for *any* stream $\boldsymbol{C} = (C_0, C_1, C_2, \dots, C_{p-1})$ of capital amounts such that $C_0 = -F_0$ and any stream $\boldsymbol{J} = (0, J_1, J_2, \dots, J_p)$ of profits such that

$$F_t = J_t - \Delta C_t, \tag{2}$$

the following equality holds:

$$\text{NPV}(1+k) = C(\bar{\imath} - k) \tag{3}$$

where

$$\bar{\imath} = \frac{J}{C} \tag{4}$$

is an Average Internal Rate of Return (AIRR) and $C = \sum_{t=1}^{p} C_{t-1}(1+k)^{-(t-1)}$ and $J = \sum_{t=1}^{p} J_t(1+k)^{-(t-1)}$ (see also Magni, 2013).

If $C > 0$ the project is defined a net investment, whereas if $C < 0$ the project is defined a net financing (Magni, 2010, 2013). Therefore, the financial nature of any project (and its associated average ROI) can be identified as an investment project or a financing project (respectively, an investment rate or a financing rate).

Eq. (1) is a special case of (2). MM (2018) precisely used eq. (4) picking up the book value capitals invested in the project (i.e., $C_t = b_t$) and the vector of pro forma accounting incomes (i.e., $J_t = I_t$).

With this choice, (4) becomes the so-called *average Return On Investment* (ROI), here denoted as $\bar{\imath}(b)$:

$$\bar{\imath}(b) = \frac{I}{b} = \frac{\text{Total profit}}{\text{Total invested capital}} \tag{5}$$

where $I = \sum_{t=1}^{p} I_t(1+k)^{-(t-1)}$ represents the overall profit which the project is expected to generate and $b = \sum_{t=1}^{p} b_{t-1}(1+k)^{-(t-1)}$ represents the total invested capital (pro forma book values).

It is important to stress that, in an industrial project, the invested capital, quantified by b_t, may consist of net fixed assets or working capital (or both):

– net fixed assets (NFA) are depreciable assets (property, plant and equipment)
– working capital (WC) is made up of inventories and accounts receivables, net of accounts payable.

Therefore, $b_t = \text{NFA}_t + \text{WC}_t$.

Let R_t and OpC_t be the revenues and operating costs, respectively (excluding depreciation and taxes); let $\text{Dep}_t = -\Delta\text{NFA}_t$ be the depreciation charge for the fixed assets with $\Delta\text{NFA}_t = \text{NFA}_t - \text{NFA}_{t-1}$, and let τ be the company tax rate.[3] Therefore, the project's (operating) income, I_t, is equal to

$$I_t = (\text{Rev}_t - \text{OpC}_t - \text{Dep}_t)(1-\tau).$$

This income is often called in finance *net operating profit after taxes* (NOPAT). Using (1), the FCF is

$$F_t = \overbrace{(\text{Rev}_t - \text{OpC}_t - \text{Dep}_t)(1-\tau)}^{I_t} - \overbrace{(\Delta\text{NFA}_t + \Delta\text{WC}_t)}^{\Delta b_t} \tag{6}$$

where $\Delta\text{WC}_t = \text{WC}_t - \text{WC}_{t-1}$, $\Delta\text{WC}_0 = \text{WC}_0$. According to eq. (6), the NPV depends on several key parameters, including the working capital (via ΔWC_t). However, in their formulation of the book value capital, MM (2018) implicitly assumed that the working capital is zero, implying that $b_t = \text{NFA}_t$ and

$$\begin{aligned} F_t &= (\text{Rev}_t - \text{OpC}_t - \text{Dep}_t)(1-\tau) - \overbrace{\Delta\text{NFA}_t}^{\Delta b_t} \\ &= (\text{Rev}_t - \text{OpC}_t - \text{Dep}_t)(1-\tau) + \text{Dep}_t \end{aligned} \tag{7}$$

which is eq. (6) with $\Delta\text{WC}_t = 0$ (see MM, 2018, eq. (1)).[4]

[1] The COC can be determined in various way, using some asset pricing models, which may be integrated by (or even replaced by) subjectively determined thresholds (see Magni, 2010, 2020). In finance, the recommended COC is the *weighted average cost of capital* (WACC). Its significance, estimation and relation with the cost of equity and the cost of debt have been extensively investigated in the literature (see, for example, Arditti and Levy, 1977; Miles and Ezzell, 1980; Cigola and Peccati, 2005, Block, 2011, Massari et al., 2008; Dempsey, 2013. See also Magni, 2020 and references therein). Consistently with MM (2018), we assume k is exogenously given and time-invariant (a usual assumption in finance).

[2] "The first thing we need when we begin evaluating a proposed investment is a set of pro forma, or projected, financial statements. Given these, we can develop the projected cash flows from the project. Once we have the cash flows, we can estimate the value of the project" (Ross et al., 2011, p. 271); "free cash flow is the total amount of cash available for distribution to the creditors who have loaned money to finance the project and to the owners who have invested in the equity of the project. In practice this cash flow information is compiled from pro forma financial statements" (Titman et al., 2011, p. 383). Eq. (1) is also known as *clean surplus relation* (Anon, 1996).

[3] The rate τ is the company's *marginal* tax rate, which is applied to the incremental gross operating profit generated by the project. If it is positive, it means that the project-with-the-firm will pay additional taxes as opposed to the firm-without-the-project; if it is negative, it means that, the firm-with-the-project will pay less taxes than the firm-without-the-project.

[4] As opposed to the zero-WC case, and assuming other things unvaried, nonzero WC affects cash flows (and, therefore, NPV) in the following way. If, in a given period $[t-1, t]$, WC increases (i.e., $\Delta\text{WC}_t > 0$), the FCF is smaller than in the zero-WC case. In contrast, if WC decreases (i.e., $\Delta\text{WC}_t < 0$), the FCF is greater than in the zero-WC case. Overall, the role of working capital on NPV depends on the timeline of signs and magnitudes of changes, $(\Delta\text{WC}_0, \Delta\text{WC}_1, \dots, \Delta\text{WC}_p)$ with $\Delta\text{WC}_0 = \text{WC}_0$.

2.2. Strong NPV-consistency of rates of return

MM (2018) introduced a stronger definition of NPV-consistency presented by taking into account the sources of investment risk. Their definition is based upon the project's value drivers and sensitivity analysis (SA). Specifically, let f be a valuation metric defined on the *parameter space* A, which maps the vector of inputs (or parameters or value drivers) $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in A \subset \mathbb{R}^n$ onto the model output $y(\alpha)$:

$$f : A \subset \mathbb{R}^n \to \mathbb{R}, \quad y = f(\alpha), \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n). \tag{8}$$

The vector of value drivers, α, collects the key assumptions on sales revenues and costs, including labor costs, energy costs, materials, selling, general, and administrative expenses, etc. Let $\alpha^0 = (\alpha_1^0, \alpha_2^0, \dots, \alpha_n^0) \in A$ be the *base-case* value, a representative value for the parameters. The relevance of a parameter α_i, also known as *importance measure*, quantifies the effect on y of a change in α_i. Let R_i^f be the relevance of parameter α_i and let $R^f = \left(R_1^f, R_2^f, \dots, R_n^f\right)$ be the vector of the relevances: If $|R_i^f| > |R_j^f|$, then parameter α_i has a rank higher than α_j. We denote as r_i^f the rank of parameter α_i and denote as $r^f = \left(r_1^f, r_2^f, \dots, r_n^f\right)$ the rank vector.

Example 1. Consider the NPV of a project and let φ be a different valuation metric. Assume the vector of relevances are

$$R^{\text{npv}} = (0.1, -0.3, 0.2, 0.05, 0.35)$$

for NPV and

$$R^{\varphi} = (0.07, 0.35, 0.15, 0.03, 0.40)$$

for φ. Since the rank is determined by the absolute value of the importance measure, NPV and φ determine the same ranking: $r^{\text{npv}} = r^{\varphi} = (4, 2, 3, 5, 1)$, which means that parameter 5 has the highest rank, followed by parameter 2, then parameter 3, parameter 1, and, finally, parameter 4, which has the smallest impact. ⋄

MM (2018) supplied the following definition of strong NPV-consistency.

Definition 3 (*Strong NPV-consistency for Accept–reject Decisions*)**.** Given an SA technique, a metric φ (and its associated decision criterion) is strongly NPV-consistent if

– φ is weakly NPV-consistent (Definition 2)
– the rank vector of φ is equal to the rank vector of NPV: $r^{\text{npv}} = r^{\varphi}$.

If φ is strongly NPV-consistent and, in addition, the vectors of the relevances coincide, $R^{\text{npv}} = R^{\varphi}$, then φ is strictly NPV-consistent.

In Example 1, φ is strongly NPV-consistent, since $r^{\text{npv}} = r^{\varphi}$. However, it is not strictly NPV-consistent, for the relevances are different. For instance, focusing on parameter 1, the relevance is $R_1^{\text{npv}} = 0.1$ for NPV and $R_1^{\varphi} = 0.07$ for φ.

There are many ways of defining a vector of relevances, each one associated with a specific SA technique (see Borgonovo and Plischke, 2016; Pianosi et al., 2016, for review of SA methods). MM (2018) coped with several different techniques. The authors showed that, if φ is an affine transformation of NPV, that is, $\varphi(\alpha) = m \cdot \text{NPV}(\alpha) + q$ for all $\alpha \in A$ with $m, q \in \mathbb{R}$, then φ is strictly NPV-consistent under the following techniques: (i) Standardized regression coefficient (ii) Sensitivity Indices in variance-based decomposition methods (iii) Finite Change Sensitivity Indices (iv) Helton's index (v) Normalized Partial Derivative (NP2) (vi) Differential Importance Measure.

Finally, the authors showed that the average ROI, $\bar{\imath}(b)$, is an affine transformation of NPV. Precisely, they showed that

$$\bar{\imath}(b) = k + \frac{\text{NPV}(\alpha)(1+k)}{b} \tag{9}$$

where NPV(α) highlights the dependence of NPV on α, the vector of value drivers. Therefore, they concluded that the average ROI is strictly NPV-consistent.

However, note that the typical stream of value drivers α in a capital asset project may be partitioned into three groups:

– sales revenues (prices, quantity, growth rates)
– cost of goods sold (labor costs, material, energy, overhead, etc.)
– selling, general and administrative costs.

All these items affect cash flows. In many cases, working capital is present, either because inventory is needed (e.g., manufacturing firms) and/or because purchases of material is made on credit (so that accounts payable are nonzero) and/or because sales are made on credit (so that accounts receivable are nonzero). If WC is present, it may or may not be affected by the above mentioned value drivers. Overall, there are three possibilities:

1. WC is zero for all t
2. WC is nonzero for some t and is unaffected by revenues and costs (i.e., it is, so to say, *exogenous*)
3. WC is nonzero for some t and is affected by revenues and/or costs (i.e., it is, so to say, *endogenous*).

As mentioned above, MM (2018) assumed zero WC (case 1), which implies that $b = \sum_{t=1}^{p} \text{NFA}_t (1+k)^{-(t-1)}$ does not depend on α. Case 2 might occur, for example, when WC is estimated to be a given percentage of NFA. Or, alternatively, when WC is managed so as to remain constant until the liquidation date (e.g., Hartman, 2007). In the latter case, $\Delta\text{WC}_t = 0$ for all t (except $t = 0$ and $t = p$). Case 3 may occur, for example, whenever inventory and accounts payable are estimated to be a percentage of operating costs, while accounts receivable are a percentage of the sales revenues (e.g., see Titman and Martin, 2011). In this case, FCF is obtained as

$$F_t = (\text{Rev}_t - \text{OpC}_t - \text{Dep}_t)(1-\tau) - \overbrace{(\Delta\text{NFA}_t + \Delta\text{WC}_t(\alpha))}^{\Delta b_t(\alpha)}. \tag{10}$$

Note that in this case the book value depends on α: $b_t = b_t(\alpha) = \text{NFA}_t + \text{WC}_t(\alpha)$. This means that the average ROI,

$$\bar{\imath}(b) = \bar{\imath}\big(b(\alpha)\big) = k + \frac{\text{NPV}(\alpha)(1+k)}{b(\alpha)} \tag{11}$$

ceases to be an affine transformation of NPV, since $(1+k)/b(\alpha)$ is not constant under changes in α. Therefore, strong NPV-consistency of average ROI is not guaranteed. Also, note that, regardless of dependence on α, the overall book value may be equal to zero. In this case, the average ROI does not exist.

Contrary to MM (2018), we allow for the more general case of nonzero working capital ($\text{WC}_t \neq 0$) and, in the next section, we investigate a performance metric which is strongly NPV-consistent.

3. Searching for strongly NPV-consistent measures: IRR and SLRR

The strong NPV-consistency of a rate of return, φ, introduced in MM (2018), enables the analyst to enrich the economic analysis or even replace NPV with a measure which precisely quantifies the economic efficiency of the project, something which the NPV is not capable to convey.[5] Therefore, the use of rates of return and, in general, relative measures, is especially suitable for project valuation and selection under budget constraints, where capital amounts are managed as scarce resources (see also the Introduction). However, contrary to MM (2018), we now allow for nonzero WC and, in particular, for the case where WC is endogenous, meaning that it depends on revenues and costs, which is a most usual case in industrial applications.

[5] NPV is affected by the project scale and correctly provides the shareholders wealth increase, but it does not tell how efficiently money is managed. For this, one needs a rate of return.

Since the average ROI does not guarantee strong NPV-consistency in the presence of uncertain WC, in this work we search for alternative valuation metrics. To this end, we consider the possibility of using the average rate of change of the book value to build an economically significant capital base and a related rate of return which may be strongly NPV-consistent, as opposed to the average ROI, whenever WC is nonzero and is not exogenously determined. In this respect, we stress that the rate of change in pro forma book values is time-varying.

To this end, we make use of Chisini's (1929) invariance requirement: Given a function $g(y_1, y_2, \dots, y_p)$ of p data, one replaces the p data with a unique value $\bar{y}$ such that the value of the function remains unvaried: $g(y_1, y_2, \dots, y_p) = g(\bar{y}, \bar{y}, \dots, \bar{y})$. The number $\bar{y}$ is called the *Chisini mean* of $y_1, y_2, \dots, y_p$.[6]

We consider the rate of change of the book value between $t-1$ and t. Now, the initial invested capital is $C_0 = b_0$ and there are (at least) two ways to formalize the rate of change of the invested capital, in geometric or linear shape. In the former case, the rate of change, denoted as x_t, is such that $E_t = C_{t-1}(1+x_t)$, where $E_t = C_t + \text{FCF}_t$ is the end-of-period capital value; in the latter case, the rate of change, denoted as λ_t, is such that $C_t = C_{t-1} - \lambda_t C_0 = C_{t-1} - \lambda_t b_0$. These two mutually exclusive framings imply, respectively,

1. $C_p = -\sum_{t=0}^{p} F_t(1+x_{t+1}) \cdot (1+x_{t+2}) \cdot \ldots \cdot (1+x_p)$
2. $C_p = b_0(1 - \lambda_1 - \lambda_2 - \dots - \lambda_p)$.

Applying Chisini invariance requirement upon both, one gets the equations

$$\sum_{t=0}^{p} F_t(1+x_{t+1}) \cdot (1+x_{t+2}) \cdot \ldots \cdot (1+x_p) = \sum_{t=0}^{p} F_t(1+x)^{p-t}$$

$$b_0(1 - \lambda_1 - \lambda_2 - \dots - \lambda_p) = b_0 \underbrace{(1 - \lambda - \lambda - \dots - \lambda)}_{1-p\lambda}.$$

The first equation is not solvable analytically. However, recalling that $C_p = 0$, it may be rewritten as

$$\sum_{t=0}^{p} F_t(1+x)^{-t} = 0. \tag{12}$$

The solution of this equation, x, is the well-known internal rate of return (IRR). As a result, the first candidate for replacing the average ROI is the IRR. We denote the associated overall average capital as $C^x = \sum_{t=1}^{p} \sum_{j=t}^{p} F_j(1+x)^{t-1-j} \cdot (1+k)^{-(t-1)}$.

As for the second equation, it has a (unique) solution, λ, such that

$$\lambda = \frac{\sum_{t=1}^{p} \lambda_t}{p} = \frac{1}{p}.$$

This means that the average capital, denoted as C_t^{sl}, is $C_t^{sl} = C_{t-1}^{sl} - b_0/p = b_0(1 - t/p)$. Hence, the overall average capital is $C^{sl} = \sum_{t=1}^{p} b_0\left(1 - \frac{t-1}{p}\right)(1+k)^{-(t-1)}$. Picking $C_t = C_t^{sl}$ in (2), and denoting as I_t^{sl} the corresponding "average" profit J_t,[7] one gets

$$I_t^{sl} = C_t^{sl} + F_t - C_{t-1}^{sl} = F_t - \lambda b_0 = F_t - \frac{b_0}{p}.$$

Following Eq. (4), one divides the overall profit I^{sl} by the total average capital C^{sl}. The result is the second candidate for substituting the average ROI:

$$\bar{\imath}(C^{sl}) = \frac{I^{sl}}{C^{sl}} = \frac{\sum_{t=1}^{p}(F_t - \frac{b_0}{p}) \cdot (1+k)^{-(t-1)}}{\sum_{t=1}^{p} b_0 \cdot \left(1 - \frac{t-1}{p}\right)(1+k)^{-(t-1)}}. \tag{13}$$

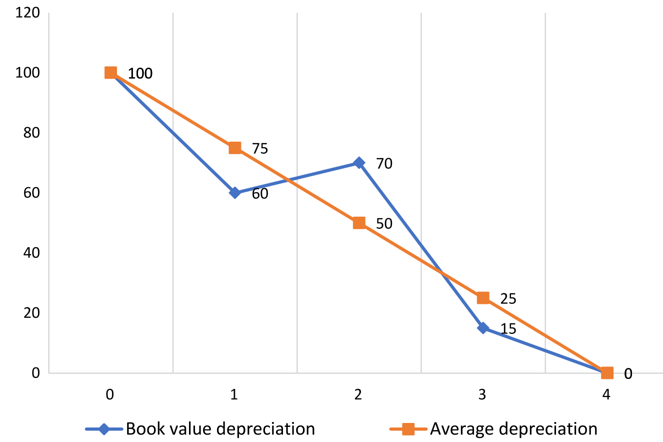

Fig. 1. Average depreciation.

We call $\bar{\imath}(C^{sl})$ the *average, straight-line rate of return* (SLRR). For simplicity, we henceforth denote it with the symbol $\bar{\imath}^{sl}$.

Example 2. A 4-period investment project has book value capitals represented by the vector $\mathbf{b} = (100, 60, 70, 15, 0)$. Therefore, in linear shape the period depreciation rates are $\lambda_1 = 40\%$, $\lambda_2 = -10\%$, $\lambda_3 = 55\%$, $\lambda_4 = 15\%$. The invested capital at time 0 is $b_0 = -F_0 = 100$ and the average rate of change is the Chisini mean of period depreciation rates: $\lambda = 25\% = (40\% - 10\% + 55\% + 15\%)/4 = 1/4$; the average capital is then $\boldsymbol{C}^{sl} = (100, 75, 50, 25, 0)$. Fig. 1 represents the dynamics of the book value and the average capital. ⋄

As (3) holds for any $\boldsymbol{C}$ and associated $\boldsymbol{J}$, both IRR and SLRR are weakly NPV-consistent (see Hazen, 2003; Magni, 2010).

This means that both are good candidates as substitutes for the average ROI whenever WC depends on the value drivers.

We now need analyze whether they are strongly NPV-consistent or not and, if not, we aim at measuring their degree of inconsistency, which is a signal of their reliability.

However, we anticipate that, regardless of strong NPV-consistency, IRR is known to be subject to some difficulties. Among others, owing to the way it is derived, it may not exist or multiple IRRs may arise: For instance, engineering projects with considerable length and numerous changes in sign of cash flows, possibly due to disposal and remediation costs, may have no IRR or multiple IRRs (Magni, 2013; Hartman, 2007). More simply, any project which does not require investment in equity (i.e., outflows are financed with either debt or liquid assets or both) has no IRR for shareholders.[8]

Also, the financial nature of the IRR depends upon the COC, k, as C^x is not necessarily invariant under changes in k (see Magni, 2013 for a compendium).

Contrary to IRR and average ROI, the SLRR has the nice property of existence. It always exists, because $b_0 = -F_0 \neq 0$.[9] Also, contrary to IRR, it is unique, since it is derived from a linear equation. Furthermore,

[6] For example, in financial mathematics the compounding factor for a three-period investment is $g(y_1, y_2, y_3) = (1+y_1)(1+y_2)(1+y_3)$, where y_i is the capital growth rate in period i. The Chisini mean of y_1, y_2, y_3 with respect to g is that unique value $\bar{y}$, named average growth rate, such that $(1+y_1)(1+y_2)(1+y_3) = (1+\bar{y})^3$ that is, $\bar{y} = \sqrt[3]{(1+y_1)(1+y_2)(1+y_3)} - 1$.

[7] More precisely, this is the profit which is associated with the average capital.

[8] For example, suppose a firm purchases a piece of equipment for an amount of \$10 in order to increase production and sales. Suppose it is financed by withdrawing cash from the firm's bank account (or by selling some marketable securities). Incremental cash flows are expected to be equal to \$3, \$6, \$12 at times 1, 2, and 3, respectively. Suppose the firm's liquid assets are currently invested at 1%. Therefore, there is no incremental outflow for the firm's shareholders (\$10 − \$10 = 0) and the prospective incremental inflows for shareholders will be \$3, \$6, and \$1.7 (= $12 - 10(1.01)^3$). The resulting cash-flow stream is $(0, 3, 6, 1.7)$, which possesses no real-valued IRR.

[9] Even if $F_0 = 0$, one may redefine b_0 as the first nonzero book value and neglect the previous zero cash flows. For example, if $\boldsymbol{F} = (0, 0, 0, -200, 100, 140)$, one may reframe the cash-flow stream as $\boldsymbol{F} = (-200, 100, 140)$ and set $b_0 = 200$.

its financial nature is not affected by the revenues and costs, being unambiguously determined by the sign of b_0, which coincides with the sign of C^{sl} for any given k: $C^{sl} > 0$ if and only if $b_0 > 0$.

Example 3. Consider a project P such that $F = (-10, 23, -17, 24, -22)$ and a COC equal to $k = 32\%$. The NPV is $0.86 = -10 + 23 \cdot 1.32^{-1} - 17 \cdot 1.32^{-2} + 24 \cdot 1.32^{-3} - 22 \cdot 1.32^{-4}$; therefore the project is worth undertaking. Two IRRs exist: $x_1 = 11.2\%$ and $x_2 = 67\%$. The former is associated with the stream $\boldsymbol{C}^{x_1} = (10, -6.3, 6.5, -13.2, 0)$, the latter is associated with the stream $\boldsymbol{C}^{x_2} = (10, -11.9, 3.8, -19.8, 0)$. The overall capital associated with x_1 is $C^{x_1} = 2.4 > 0$, the overall capital associated with x_2 is $C^{x_2} = -4.1 < 0$. Therefore, IRR does not unambiguously determine the financial nature of the project: According to the first IRR, the project is an investment, according to the second IRR the project is a financing. The first IRR is a rate of return, the second IRR is a rate of cost. Conversely, the SLRR exists and is unique in any case, and unambiguously identifies the project as an investment, since the associated capital stream is $\boldsymbol{C}^{sl} = (10, 7.5, 5, 2.5, 0)$ so that the total average capital is $C^{sl} = 14.9 > 0$. The SLRR is then $\bar{\imath}^{sl} = 0.32 + 0.86(1 + 0.32)/14.9 = 37.8\%$. ⋄

We now show that SLRR is strongly NPV-consistent, in a strict sense.

Proposition 1. *For any fixed k, C_0, and p, SLRR is strictly NPV-consistent for accept–reject decisions.*

Proof. Recalling that (3) holds irrespective of the capital stream $\boldsymbol{C}$ and picking $C = C^{sl}$, one gets $\text{NPV}(\alpha)(1 + k) = C^{sl}(\bar{\imath}^{sl} - k)$ where $C^{sl} = \sum_{t=1}^{p}\big(b_0(1 - (t-1)/p)(1+k)^{-(t-1)}\big)$ does not depend on α. This implies

$$\bar{\imath}^{sl} = k + \frac{\text{NPV}(\alpha)(1+k)}{C^{sl}}. \tag{14}$$

This means $\varphi = q + m \cdot \text{NPV}(\alpha)$ where $\varphi = \bar{\imath}^{sl}$, $q = k$ and $m = (1+k)/C^{sl}$. Therefore, the SLRR is an affine transformation of NPV. The thesis follows from MM (2018, Proposition 1). □

The proposition above shows that SLRR and NPV are identically influenced by the variation of the project's value drivers, not only in terms of ranks ($r^{\text{npv}} = r^{\text{slrr}}$) but also in terms of relevances ($R^{\text{npv}} = R^{\text{slrr}}$). This ensures the equivalence of NPV and SLRR criteria for investment decisions even when working capital is nonzero and is estimated on the basis of revenues and costs.

As for IRR, note that it is an implicit function of the value drivers, since it depends on revenues and costs, both directly (via Rev_t and OpC_t) irrespective of whether WC is zero or not and irrespective of how it is estimated:

$$\sum_{t=0}^{p}\big((\text{Rev}_t - \text{OpC}_t - \text{Dep}_t)(1-\tau) - (\text{NFA}_t - \text{NFA}_{t-1}) - (\text{WC}_t - \text{WC}_{t-1})\big)(1+x)^{-t} = 0.$$

Therefore, in general, it is not possible to determine an analytical relationship between NPV and IRR (see also Borgonovo and Peccati, 2006, 2004; Percoco and Borgonovo, 2012). Indeed, let $\alpha^* \in A$ be a given value of parameters and x^* be the associated IRR, such that $\text{NPV}(\alpha^*, x^*) = 0$.[10] If there exists a neighborhood of α^* where function $\text{NPV}(\alpha, k)$ is a continuously differentiable function and $\frac{\partial \text{NPV}}{\partial k}(\alpha^*, x^*) \neq 0$, then there exists a neighborhood $V(\alpha^*) \subset A$ and a neighborhood $W(x^*) \subset \mathbb{R}$ such that $x(\alpha): V \to W$ is the implicitly-defined function from the equation $\text{NPV}(\alpha, k) = 0$ and

$$x(\alpha^*) = x^*,$$

$$\text{NPV}(\alpha, x(\alpha)) = 0,\ \forall \alpha \in V,$$

$$\frac{\partial x}{\partial \alpha_i}(\alpha) = -\frac{\frac{\partial \text{NPV}}{\partial \alpha_i}(\alpha, x(\alpha))}{\frac{\partial \text{NPV}}{\partial k}(\alpha, x(\alpha))},\ \forall \alpha \in V.$$

In particular,

$$\frac{\partial x}{\partial \alpha_i}(\alpha^*) = -\frac{\frac{\partial \text{NPV}}{\partial \alpha_i}(\alpha^*, x^*)}{\frac{\partial \text{NPV}}{\partial k}(\alpha^*, x^*)}. \tag{15}$$

Therefore, IRR is not an affine transformation of NPV. In the next section, we demonstrate, via some counterexamples, that IRR may not be used for accomplishing *ex ante* risk analysis or *ex post* performance measurement for several different reasons:

- it is not strongly NPV-consistent
- it may not exist in some scenario
- multiple IRRs may arise
- the financial nature of IRR may change under changes in the value drivers.

In contrast, the SLRR always exists, is unique, possesses an unambiguous financial nature, and enjoys strong NPV-consistency.

For reasons of space, we limit the analysis to two SA techniques: The Finite Change Sensitivity Index (FCSI) (Borgonovo, 2010a) and Differential Importance Measure (DIM) (Borgonovo and Apostolakis, 2001; Borgonovo and Peccati, 2004). The FCSI index is particularly useful when two different scenarios for the value drivers are compared, namely, α^0 (base value or base case) and α^1 (perturbed value). It may be used for *ex ante* analysis, when the analyst aims to compare a base case and a possible different scenario or, more compellingly, for *ex post* auditing, when the analyst wants to investigate the source of variation of the actual performance (α^1) with respect to the expected one (α^0). The DIM is useful when not-so-large deviations around the base value are assumed; therefore, it is most useful in ex ante decision-making to measure the major sources of risk in terms of key parameters.

Furthermore, we need avail ourselves of a measure for quantifying the degree of NPV-inconsistency of average ROI or IRR: The higher the degree of inconsistency, the smaller the reliability of average ROI or IRR. We comply with MM's (2018) choice of the Spearman's rank correlation coefficient (Spearman, 1904) and top-down correlation coefficient (Iman and Conover, 1987). Spearman's coefficient is the correlation coefficient of the rank vectors r^{npv} and r^{φ}: $\rho_{\text{npv},\varphi} = \frac{\text{Cov}(r^{\text{npv}}, r^{\varphi})}{\sigma(r^{\text{npv}}) \cdot \sigma(r^{\varphi})}$. The top-down correlation coefficient, introduced by Iman and Conover (1987), attributes a higher weight to top parameters than to low parameters, based on *Savage Score* (Savage, 1956). The Savage score of parameter α_i is $S_i^{\text{npv}} = \sum_{h=r_i^{\text{npv}}}^{n} \frac{1}{h}$. For example, considering a vector of $n = 8$ value drivers, such that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ and assuming α_2 has rank $r_2^{\text{npv}} = 3$, then its Savage score will be

$$S_2^{\text{npv}} = \sum_{h=3}^{8} \frac{1}{h} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = 1.218.$$

In general, the Savage scores' vector of f is $S^f = \left(S_1^f, S_2^f, \ldots, S_n^f\right)$. The top-down correlation coefficient between NPV and φ is the correlation coefficient between the Savage scores' vectors S^{npv} and S^{φ} (Iman and Conover, 1987): $\rho_{S^{\text{npv}}, S^{\varphi}} = \frac{\text{Cov}(S^{\text{npv}}, S^{\varphi})}{\sigma(S^{\text{npv}}) \cdot \sigma(S^{\varphi})}$.

The coefficients $\rho_{\text{npv},\varphi}$ and $\rho_{S^{\text{npv}}, S^{\varphi}}$ are equal to 1 if and only if φ is strongly NPV-consistent. The smaller the value of $\rho_{\text{npv},\varphi}$ and $\rho_{S^{\text{npv}}, S^{\varphi}}$, the higher the degree of NPV-inconsistency. The differences $1 - \rho_{\text{npv},\varphi}$ and $1 - \rho_{S^{\text{npv}}, S^{\varphi}}$ can be taken as representative of the degree of inconsistency.

4. Comparison of SLRR and IRR using FCSI

In this section, as well as in Section 5, we assume that working capital is equal to zero (e.g., customers pay in cash, suppliers are paid in cash, and no inventory exists) (in Section 6 we will remove

[10] Let α be a generic value belonging to a neighborhood of α^*. $\text{NPV}(\alpha, k)$ is the NPV calculated with discount rate k.

Table 1
Investment evaluated in α^0.

	0	1	2	3	4
Rev_t^0		580	570	560	400
OpC_t^0		200	300	200	300
F_t	−750	380	270	360	100
Valuation					
NPV	157.37				
$\bar{\imath}^{sl}$	20.11%				
x	20.86%				

Table 2
Investment evaluated in α^1.

	0	1	2	3	4
Rev_t^1		800	810	780	630
OpC_t^1		350	250	380	600
F_t	−750	450	560	400	30
Valuation					
NPV	442.92				
$\bar{\imath}^{sl}$	38.46%				
x	41.12%				

this assumption). We also assume $\tau = 0$. Therefore, $\mathrm{FCF}_t = \mathrm{Rev}_t - \mathrm{OpC}_t$, $\forall t > 0$. We focus on the FCSI technique (see Eqs. (17)–(18) in the Appendix of this paper) and illustrate four numerical applications, aimed at presenting the problems of the IRR:

1. in the first application, IRR exists and is unique but is not strongly NPV-consistent[11]
2. in the second application, despite IRR exists and is unique in the base case α^0, it does not exist in α^1 (or vice versa), making it impossible to perform the SA
3. in the third application, multiple IRRs arise for $\alpha = \alpha^1$
4. in the fourth application, IRR changes its financial nature from investment rate (in α^0) to financing rate (in α^1).

No such problems will arise with (average ROI and) SLRR, which is strictly NPV-consistent.[12]

We will consider the simple model described in MM (2018), consisting of a firm facing the opportunity of investing in a 4-period project whose estimated revenues and costs are denoted as Rev_t and OpC_t. As anticipated, the FCF is $\mathrm{FCF}_t = \mathrm{Rev}_t - \mathrm{OpC}_t$. The project's value drivers are then $\alpha_i = \mathrm{Rev}_i$ for $i = 1, 2, 3, 4$ and $\alpha_i = \mathrm{OpC}_{i-4}$ for $i = 5, 6, 7, 8$. Hence, the value drivers' vector for the base case is

$$\alpha^0 = (\mathrm{Rev}_1^0, \mathrm{Rev}_2^0, \mathrm{Rev}_3^0, \mathrm{Rev}_4^0, \mathrm{OpC}_1^0, \mathrm{OpC}_2^0, \mathrm{OpC}_3^0, \mathrm{OpC}_4^0)$$

while the value drivers' vector for the alternative (perturbed) case is

$$\alpha^1 = (\mathrm{Rev}_1^1, \mathrm{Rev}_2^1, \mathrm{Rev}_3^1, \mathrm{Rev}_4^1, \mathrm{OpC}_1^1, \mathrm{OpC}_2^1, \mathrm{OpC}_3^1, \mathrm{OpC}_4^1).$$

NPV is computed as:

$$\mathrm{NPV}(\alpha) = -C_0 + \frac{\mathrm{Rev}_1 - \mathrm{OpC}_1}{1+k} + \frac{\mathrm{Rev}_2 - \mathrm{OpC}_2}{(1+k)^2} + \frac{\mathrm{Rev}_3 - \mathrm{OpC}_3}{(1+k)^3} + \frac{\mathrm{Rev}_4 - \mathrm{OpC}_4}{(1+k)^4}.$$

Example 4 (*NPV Inconsistency*)**.** Assume $C_0 = 750$ and $k = 10\%$. Table 1 describes the base value α^0 and reports the corresponding FCFs and valuation metrics. The NPV is $157.37 = -750 + 380/1.1 + 270/(1.1)^2 + 360/(1.1)^3 + 100/(1.1)^4$, the vector of average capitals is $\boldsymbol{C}^{sl} = (750, 562.5, 375, 187.5, 0)$ and the overall average capital is $C^{sl} = 1{,}712.15 = 750 + 562.5/1.1 + 375/(1.1)^2 + 187.5/(1.1)^3$. Therefore, SLRR is equal to $\bar{\imath}^{sl} = 10\% + 157.37/1{,}712.15 \cdot 1.1 = 20.11\%$. The IRR exists and is unique, $x = 20.86\%$.

Table 2 reports the alternative scenario α^1 and the corresponding new values of F_t, NPV, SLRR, and IRR. In α^1, NPV is 442.92, SLRR is 38.46%, IRR is 41.12% (it exists and is unique). The observed variations are: $\Delta\mathrm{NPV} = 285.55 = 442.92 - 157.37$; $\Delta\bar{\imath}^{sl} = 18.35\% = 38.46\% - 20.11\%$; $\Delta x = 20.25\% = 41.12\% - 20.86\%$.

Table 3 shows the First Order FCSIs ($\Phi_i^{1,f}$), the ranks (r_i^f), and the Savage Scores (S_i^f) for NPV, SLRR and IRR. The (ranks and) importance measures of NPV and SLRR are equal, $\Phi_i^{1,\mathrm{npv}} = \Phi_i^{1,\mathrm{slrr}}$, meaning that SLRR is strictly NPV-consistent. The relevances of NPV and IRR are different, $\Phi_i^{1,\mathrm{npv}} \neq \Phi_i^{1,\mathrm{irr}}$, as well the ranks, $r^{\mathrm{npv}} \neq r^{\mathrm{irr}}$, implying that the IRR is not strongly NPV-consistent according to Definition 3. The degree of NPV-inconsistency, measured via (one minus) Spearman's coefficient or top-down coefficient, is $1 - \rho_{\mathrm{irr,npv}} = 1 - 0.857 = 0.143$ and $1 - \rho_{S^{\mathrm{irr}},S^{\mathrm{npv}}} = 1 - 0.77 = 0.23$.

Table 4 shows Total Order FCSIs ($\Phi_i^{T,f}$), ranks (r_i^f), and Savage scores (S_i^f) for the three metrics. The (ranks and) Total Order FCSIs of NPV and SLRR are equal, $\Phi_i^{T,\mathrm{npv}} = \Phi_i^{T,\mathrm{slrr}}$, therefore SLRR is strictly NPV-consistent, whereas the ranks (and relevances) of NPV and IRR are different, implying that the IRR is not strongly NPV-consistent with degree of incoherence equal to $1 - \rho_{\mathrm{irr,npv}} = 1 - 0.667 = 0.333$ and $1 - \rho_{S^{\mathrm{irr}},S^{\mathrm{npv}}} = 1 - 0.409 = 0.591$. This is especially due to the ranking distortion of OpC_4, with rank 1 according to NPV and SLRR, and rank 5 in terms of IRR. ⋄

Example 5 (*Nonexistence of IRR in* α^1)**.** Consider a project P such that $C_0 = 750$ and $k = 10\%$. Hence $C^{sl} = 1{,}712.15$. The base value is described in the revenue–cost vector $\alpha^0 = (630, 740, 850, 600, 180, 390, 490, 550)$; the revenue–cost vector for the perturbed scenario is $\alpha^1 = (600, 700, 800, 500, 200, 400, 500, 850)$, a worse situation in terms of both revenues and costs. Table 5 reports cash flows, NPV, SLRR, and IRR. In α^0 IRR exists, is unique, and is equal to 28.52%. In α^1 IRR does not exist. This implies that the sensitivity analysis cannot be applied for IRR: Δx is not defined, hence the First Order and Total Order FCSIs of IRR do not exist.

SLRR does not suffer from this problem because it always exists and is unique. Table 6 shows the First Order and Total Order FCSIs of NPV and SLRR: As expected, SLRR is strictly NPV-consistent.

The opposite case may also occur, whereby the IRR does not exist in α^0 while it exists in α^1, resulting in the same kind of pitfall (e.g., just reverse the base-case value and the perturbed value of this example). ⋄

Example 6 (*Nonuniqueness of IRR*)**.** Consider a project P, with $C_0 = 800$ and $k = 15\%$. Therefore, $C^{sl} = 1{,}755.70$. The base value is described in the input vector $\alpha^0 = (2300, 1100, 1400, 2000, 1300, 1200, 1600, 1300)$; the input vector in the perturbed state is $\alpha^1 = (2960, 500, 400, 2300, 600, 1440, 2750, 550)$. Table 7 shows the cash flows and the valuation metrics in α^0 and α^1. In α^0, the IRR function supplies a unique value and is equal to 36.72%. For α^1, there exist three different IRRs: $x_1(\alpha^1) = 8.07\%$, $x_2(\alpha^1) = 25.0\%$, $x_3(\alpha^1) = 61.93\%$ so the sensitivity analysis is problematic: It is not clear which one IRR should be the relevant one, if any.

Table 8 shows the First Order and Total Order FCSIs of NPV and SLRR: As obvious, SLRR is strictly NPV-consistent. ⋄

Example 7 (*Financial Nature of IRR*)**.** Consider a project P such that $C_0 = 500$ and $k = 5\%$. Therefore $C^{sl} = 1{,}191.88$. The base case is described in the input vector $\alpha^0 = (800, 2{,}150, 950, 850, 1{,}500, 805, 915, 510)$. The perturbed vector is $\alpha^1 = (600, 2{,}000, 800, 800, 1{,}000, 305, 415, 2{,}010)$. The difference between α^0 and α^1 lies in lower revenues for α^1 and in intertemporal cost allocation: The total amount of costs is the

[11] Examples of this kind of shortcoming for IRR are also described in Borgonovo and Peccati (2004, 2006), Percoco and Borgonovo (2012), which show that parameter rankings for NPV and IRR are different.

[12] To compute SLRR, one may either use the definition in (13) or the shortcut in (14).

Table 3
First Order FCSI.

Parameter	NPV			SLRR			IRR		
	$\Phi_i^{1,\mathrm{npv}}$	r_i^{npv}	S_i^{npv}	$\Phi_i^{1,\mathrm{slrr}}$	r_i^{slrr}	S_i^{slrr}	$\Phi_i^{1,\mathrm{irr}}$	r_i^{irr}	S_i^{irr}
Rev_1	70.04%	2	1.718	70.04%	2	1.718	79.78%	1	2.718
Rev_2	69.46%	3	1.218	69.46%	3	1.218	64.05%	3	1.218
Rev_3	57.89%	4	0.885	57.89%	4	0.885	45.56%	5	0.635
Rev_4	55.01%	5	0.635	55.01%	5	0.635	37.68%	7	0.268
OpC_1	−47.76%	6	0.435	−47.76%	6	0.435	−46.93%	4	0.885
OpC_2	14.47%	8	0.125	14.47%	8	0.125	13.68%	8	0.125
OpC_3	−47.36%	7	0.268	−47.36%	7	0.268	−45.25%	6	0.435
OpC_4	−71.76%	1	2.718	−71.76%	1	2.718	−76.83%	2	1.718
Correlations									
$\rho_{\mathrm{slrr},\ \mathrm{npv}}$	1								
$\rho_{S^{\mathrm{slrr}},\ S^{\mathrm{npv}}}$	1								
$\rho_{\mathrm{irr},\ \mathrm{npv}}$	0.857								
$\rho_{S^{\mathrm{irr}},\ S^{\mathrm{npv}}}$	0.770								

Table 4
Total Order FCSI.

Parameter	NPV			SLRR			IRR		
	$\Phi_i^{T,\mathrm{npv}}$	r_i^{npv}	S_i^{npv}	$\Phi_i^{T,\mathrm{slrr}}$	r_i^{slrr}	S_i^{slrr}	$\Phi_i^{T,\mathrm{irr}}$	r_i^{irr}	S_i^{irr}
Rev_1	70.04%	2	1.718	70.04%	2	1.718	75.79%	1	2.718
Rev_2	69.46%	3	1.218	69.46%	3	1.218	65.33%	2	1.718
Rev_3	57.89%	4	0.885	57.89%	4	0.885	44.78%	4	0.885
Rev_4	55.01%	5	0.635	55.01%	5	0.635	34.09%	6	0.435
OpC_1	−47.76%	6	0.435	−47.76%	6	0.435	−57.78%	3	1.218
OpC_2	14.47%	8	0.125	14.47%	8	0.125	13.18%	8	0.125
OpC_3	−47.36%	7	0.268	−47.36%	7	0.268	−31.29%	7	0.268
OpC_4	−71.76%	1	2.718	−71.76%	1	2.718	−34.93%	5	0.635
Correlations									
$\rho_{\mathrm{slrr},\ \mathrm{npv}}$	1								
$\rho_{S^{\mathrm{slrr}},\ S^{\mathrm{npv}}}$	1								
$\rho_{\mathrm{irr},\ \mathrm{npv}}$	0.667								
$\rho_{S^{\mathrm{irr}},\ S^{\mathrm{npv}}}$	0.409								

Table 5
IRR not existing in α^1.

	α^0	α^1
Cash flows		
F_0	−750	−750
F_1	450	400
F_2	350	300
F_3	360	300
F_4	50	−350
Valuation		
NPV	252.97	−152.09
$\bar{\imath}^{sl}$	26.25%	0.23%
x	28.52%	–

Table 7
Multiple IRR in α^1.

	α^0	α^1
Cash flows		
F_0	−800	−800
F_1	1,000	2,360
F_2	−100	−940
F_3	−200	−2,350
F_4	700	1,750
Valuation		
NPV	262.67	−3.20
$\bar{\imath}^{sl}$	32.21%	14.79%
x	36.72%	8.07%; 25.0%; 61.93%

Table 6
IRR not existing in α^1: First Order and Total Order FCSIs.

Parameter	NPV		SLRR		IRR	
	$\Phi_i^{T,\mathrm{npv}} = \Phi_i^{1,\mathrm{npv}}$	r_i^{npv}	$\Phi_i^{T,\mathrm{slrr}} = \Phi_i^{1,\mathrm{slrr}}$	r_i^{slrr}	$\Phi_i^{T,\mathrm{irr}}$	r_i^{irr}
Rev_1	6.73%	5	6.73%	5	–	–
Rev_2	8.16%	4	8.16%	4	–	–
Rev_3	9.27%	3	9.27%	3	–	–
Rev_4	16.86%	2	16.86%	2	–	–
OpC_1	4.49%	6	4.49%	6	–	–
OpC_2	2.04%	7	2.04%	7	–	–
OpC_3	1.85%	8	1.85%	8	–	–
OpC_4	50.59%	1	50.59%	1	–	–

Table 8
Multiple IRR in α^1: First Order and Total Order FCSIs.

Parameter	NPV		SLRR		IRR	
	$\Phi_i^{T,\mathrm{npv}} = \Phi_i^{1,\mathrm{npv}}$	r_i^{npv}	$\Phi_i^{T,\mathrm{slrr}} = \Phi_i^{1,\mathrm{slrr}}$	r_i^{slrr}	$\Phi_i^{T,\mathrm{irr}}$	r_i^{irr}
Rev_1	−215.86%	4	−215.86%	4	–	–
Rev_2	170.64%	5	170.64%	5	–	–
Rev_3	247.31%	2	247.31%	2	–	–
Rev_4	−64.51%	8	−64.51%	8	–	–
OpC_1	−228.94%	3	−228.94%	3	–	–
OpC_2	68.26%	7	68.26%	7	–	–
OpC_3	284.40%	1	284.40%	1	–	–
OpC_4	−161.29%	6	−161.29%	6	–	–

same in the two cases, but in α^1 costs are highly concentrated in period 4 (one may assume remedial costs at the end of the project have been paid). Table 9 shows the project's cash flows and the corresponding NPV, SLRR, and IRR in α^0 and α^1. In the base case IRR exists, is unique, and is equal to 22.17% and the IRR-implied capital vector is $\boldsymbol{C}^{\boldsymbol{x}}$ = (500, 1,310.85, 256.45, 278.30, 0) whence $C^x(\alpha^0)$ = 2,221.44; therefore, IRR is an investment rate in α^0. In α^1, IRR exists, is unique, and is equal to 10%, associated with the vector $\boldsymbol{C}^{\boldsymbol{x}}$ = (500, 950, −650, −1,100, 0), implying $C^x(\alpha^1) = -135.03 < 0$ which means that the IRR is a financing rate in α^1. This proves that a change in the value drivers' vector may

Table 9
IRR changes its financial nature.

	α^0	α^1
Cash flows		
F_0	−500	−500
F_1	−700	−400
F_2	1,345	1,695
F_3	35	385
F_4	340	−1,210
Valuation		
NPV	363.24	−6.43
$\bar{\imath}^{sl}$	37.00%	4.43%
x	22.17%	10.00%
x	(Investment rate)	(Financing rate)

cause IRR to change financial nature (from investment rate to financing rate or vice versa). The decomposition of the output variation with FCSIs is economically dubious, as the model output does not merely change in quantitative terms, but it changes in meaning: No more a rate of return but a financing rate.

SLRR does not suffer from this problem, because its financial nature only depends on the sign of C_0. In this case, SLRR is an investment rate, regardless of changes in the value drivers.

It is worth noting that two or more of the above mentioned problems may occur simultaneously. For instance, IRR changes financial nature from α^0 to α^1 and, at the same time, the importance measure of one of the value drivers, namely the costs in period 4, suffers from a problem of nonexistence: $x(\alpha^1_8, \alpha^0_{(-8)})$ is not defined because the associated cash flows vector (−500, −700, 1,345, 35, −1,160) does not admit any real IRR > -1, therefore $\Phi^{1,\text{irr}}_8$ does not exist. Consequently, the parameters ranking for IRR is not possible and correlation coefficients are not computable (see Table 10). ⋄

5. Comparison of IRR and SLRR using DIMs

In this section, we analyze the behavior of IRR and SLRR under the DIM technique, which presupposes small perturbations in the input data and makes use of derivatives (see Eq. (21)). In this model, the first partial derivatives of NPV(α), evaluated in α^0, are

$$\frac{\partial \text{NPV}}{\partial \alpha_i}(\alpha^0) = \begin{cases} (1+k)^{-i}, & i = 1,2,3,4; \\ -(1+k)^{-(i-4)}, & i = 5,6,7,8 \end{cases} \tag{16}$$

(see also MM, 2018). Using (14), the first partial derivatives of SLRR, evaluated in α^0, are

$$\frac{\partial \bar{\imath}^{sl}}{\partial \alpha_i}(\alpha^0) = \text{NPV}'_{\alpha_i}(\alpha^0) \cdot \frac{(1+k)}{C^{sl}}.$$

This implies that SLRR and NPV share the same DIMs and, therefore, SLRR is strictly NPV-consistent, as already stated in Proposition 1.

The case with IRR is more problematic. From (15) and (16),

$$\frac{\partial x}{\partial \alpha_i}(\alpha^0) = \begin{cases} -(1+x^0)^{-i} \cdot (\text{NPV}'_k(\alpha^0, x^0))^{-1}, & i = 1,2,3,4; \\ (1+x^0)^{-(i-4)} \cdot (\text{NPV}'_k(\alpha^0, x^0))^{-1}, & i = 5,6,7,8 \end{cases}$$

where

$$\frac{\partial \text{NPV}}{\partial k}(\alpha^0, x^0) = -\frac{\text{Rev}^0_1 - \text{OpC}^0_1}{(1+x^0)^2} - 2 \cdot \frac{\text{Rev}^0_2 - \text{OpC}^0_2}{(1+x^0)^3} - 3 \cdot \frac{\text{Rev}^0_3 - \text{OpC}^0_3}{(1+x^0)^4} - 4 \cdot \frac{\text{Rev}^0_4 - \text{OpC}^0_4}{(1+x^0)^5}.$$

This suggests that IRR is not strongly NPV-consistent.

We now illustrate a numerical application of DIM technique which, being a counterexample, shows that the IRR is indeed NPV-inconsistent under DIM according to Definition 3.

Example 8. We consider an investment P, with $C_0 = 900$ and $k = 8\%$. Therefore $C^{sl} = 2{,}089.41$. The base value is $\alpha^0 = (900, 1{,}000, 1{,}100, 1\,1{,}200, 600, 700, 800, 900)$. The corresponding cash-flow vector is $F = (-900, 300, 300, 300, 300)$ and $\text{NPV}(\alpha^0) = 93.64$, $\bar{\imath}^{sl}(\alpha^0) = 12.84\%$, $x(\alpha^0) = 12.59\%$. Table 11 shows the DIMs, the ranks, and the Savage scores. The DIMs for NPV and IRR are different: $DIM^{\text{npv}}_i(\alpha^0) \neq DIM^{\text{irr}}_i(\alpha^0)$. Not even the ranking is equal, therefore IRR is NPV-inconsistent according to Definition 3 and, since $1 - \rho_{\text{irr,npv}} = 0.262$ and $1 - \rho_{\mathcal{S}^{\text{irr}},\mathcal{S}^{\text{npv}}} = 0.691$, the degree of NPV-inconsistency is remarkable when using top-down coefficient. ⋄

6. Non-strong NPV-consistency of average ROI

In the previous sections, we have shown, by means of counterexamples, that the IRR is not strongly NPV-consistent, even though the WC is not present. With this assumption, the average ROI is strictly NPV-consistent, as shown in MM (2018).

In this section, we deal with nonzero WC and assume it depends on value drivers. This implies that the average ROI is not an affine transformation of NPV. It is then natural to make the conjecture that the average ROI is not strongly NPV-consistent. To prove the conjecture, it suffices to provide one counterexample. For illustrative purposes, we will deal with the FCSI technique and will illustrate two simple applications, where we compare average ROI, IRR, and SLRR:

1. in the first application, working capital is exogenous. Average ROI and SL rate of return are both strictly NPV-consistent; IRR is not strongly NPV-consistent
2. in the second application, working capital is endogenous (it changes under change in α). Average ROI and IRR are not strongly consistent with NPV, whereas SLRR is strictly NPV-consistent.

(Importance measures, ranks, and correlation coefficients inherent to average ROI are denoted with the superscript “roi”.)

Example 9 (*Exogenous WC*). Consider a project P with initial investment in fixed assets equal to $\text{NFA}_0 = 500$. Depreciation is equal to $\text{Dep}_1 = 250$, $\text{Dep}_2 = 100$, $\text{Dep}_3 = 50$, and $\text{Dep}_4 = 100$ so that $\text{NFA}_1 = 250$, $\text{NFA}_2 = 150$, $\text{NFA}_3 = 100$. The working capital is assumed to be 50% of the net fixed assets in each period, $\text{WC}_t = 50\% \cdot \text{NFA}_t$. Therefore $\text{WC}_0 = 250$, $\text{WC}_1 = 125$, $\text{WC}_2 = 75$, $\text{WC}_3 = 50$. Hence, the vector of book value capitals is $\mathbf{b} = (750, 375, 225, 150, 0)$, while the vector of average capital is $\boldsymbol{C}^{sl} = (750, 562.5, 375, 187.5, 0)$. Assuming that cost of capital is $k = 6\%$, the overall book value capital is $b = 1{,}429.97$ and the overall SL capital is $\boldsymbol{C}^{sl} = 1{,}771.84$. Revenues and costs in the base case and in the perturbed case are $\alpha^0 = (420, 460, 480, 520, 300, 290, 280, 260)$ and $\alpha^1 = (450, 428, 512, 487, 329, 321, 249, 292)$, respectively. From the estimates of book value capitals and incomes, the cash flow streams in α^0 and α^1 are calculated via (6) and reported in Table 12. Average ROI, SL rate of return, and IRR are calculated from (5), (14), and (12) respectively. The book value of working capital (and, hence, the book value of invested capital) does not depend on revenues and costs, which implies, from (9), that the average ROI is an affine transformation of NPV and, therefore, from MM (2018, Proposition 1), is strictly NPV-consistent under FCSI and DIM. The same applies to SL rate of return, since C^{sl} does not depend on the value drivers. Results of the analysis via Total Order FCSI are shown in Table 13. Since average ROI and SLRR are strictly NPV-consistent, their correlation with NPV is equal to 1 (with Spearman's and top-down coefficients): $\rho_{\text{roi,npv}} = \rho_{\mathcal{S}^{\text{roi}},\mathcal{S}^{\text{npv}}} = \rho_{\text{slrr,npv}} = \rho_{\mathcal{S}^{\text{slrr}},\mathcal{S}^{\text{npv}}} = 1$. As expected, IRR is not strongly NPV-consistent, with $\rho_{\text{irr,npv}} = 0.857$ and $\rho_{\mathcal{S}^{\text{irr}},\mathcal{S}^{\text{npv}}} = 0.611$. ⋄

Example 10 (*Endogenous WC*). We consider an investment project P with initial investment in fixed assets equal to $\text{NFA}_0 = 500$. Revenues and costs in the base case and perturbed case are, respectively,

$\alpha^0 = (420, 460, 480, 520, 300, 290, 280, 260)$

Table 10
First Order FCSI: IRR changes its financial nature.

Parameter	NPV			SLRR			IRR		
	$\Phi_i^{1,\text{npv}}$	r_i^{npv}	S_i^{npv}	$\Phi_i^{1,\text{slrr}}$	r_i^{slrr}	S_i^{slrr}	$\Phi_i^{1,\text{irr}}$	r_i^{irr}	S_i^{irr}
Rev_1	51.53%	5	0.635	51.53%	5	0.635	79.64%	–	–
Rev_2	36.80%	6	0.435	36.80%	6	0.435	53.16%	–	–
Rev_3	35.05%	7	0.268	35.05%	7	0.268	45.59%	–	–
Rev_4	11.13%	8	0.125	11.13%	8	0.125	12.17%	–	–
OpC_1	−128.81%	2	1.718	−128.81%	2	1.718	−268.88%	–	–
OpC_2	−122.68%	3	1.218	−122.68%	3	1.218	−175.42%	–	–
OpC_3	−116.84%	4	0.885	−116.84%	4	0.885	−127.90%	–	–
OpC_4	333.82%	1	2.718	333.82%	1	2.718	–	–	–
Correlations									
$\rho_{\text{slrr, npv}}$	1								
$\rho_{S^{\text{slrr}},\ S^{\text{npv}}}$	1								
$\rho_{\text{irr, npv}}$	–								
$\rho_{S^{\text{irr}},\ S^{\text{npv}}}$	–								

Table 11
Coherence under DIM technique.

Parameter	α^0	NPV			SLRR			IRR		
		$DIM_i^{\text{npv}}(\alpha^0)$	r_i^{npv}	S_i^{npv}	$DIM_i^{\text{slrr}}(\alpha^0)$	r_i^{slrr}	S_i^{slrr}	$DIM_i^{\text{irr}}(\alpha^0)$	r_i^{irr}	S_i^{irr}
Rev_1	900	83.87%	4	0.885	83.87%	4	0.885	88.82%	1	2.718
Rev_2	1000	86.28%	3	1.218	86.28%	3	1.218	87.65%	2	1.718
Rev_3	1100	87.88%	2	1.718	87.88%	2	1.718	85.64%	3	1.218
Rev_4	1200	88.77%	1	2.718	88.77%	1	2.718	82.97%	4	0.885
OpC_1	600	−55.91%	8	0.125	−55.91%	8	0.125	−59.21%	8	0.125
OpC_2	700	−60.40%	7	0.268	−60.40%	7	0.268	−61.36%	7	0.268
OpC_3	800	−63.91%	6	0.435	−63.91%	6	0.435	−62.28%	5	0.635
OpC_4	900	−66.58%	5	0.635	−66.58%	5	0.635	−62.23%	6	0.435
Correlations										
$\rho_{\text{slrr, npv}}$	1									
$\rho_{S^{\text{slrr}},\ S^{\text{npv}}}$	1									
$\rho_{\text{irr, npv}}$	0.738									
$\rho_{S^{\text{irr}},\ S^{\text{npv}}}$	0.309									

Table 12
Exogenous WC: Average ROI, SLRR, and IRR.

	α^0	α^1
Cash flows		
F_0	−750	−750
F_1	245	246
F_2	220	157
F_3	225	288
F_4	310	245
Valuation		
NPV	111.39	57.68
$\bar{\imath}$ (b)	14.26%	10.28%
$\bar{\imath}^{sl}$	12.66%	9.45%
x	12.08%	9.22%

Table 13
Exogenous WC: Total Order FCSIs of average ROI, SLRR, and IRR.

Parameter	NPV		Average ROI		SLRR		IRR	
	$\Phi_i^{T,\text{npv}}$	r_i^{npv}	$\Phi_i^{T,\text{roi}}$	r_i^{roi}	$\Phi_i^{T,\text{slrr}}$	r_i^{slrr}	$\Phi_i^{T,\text{irr}}$	r_i^{irr}
Rev_1	−52.69%	2	−52.69%	2	−52.69%	2	−56.26%	1
Rev_2	53.02%	1	53.02%	1	53.02%	1	55.75%	3
Rev_3	−50.02%	5	−50.02%	5	−50.02%	5	−51.76%	5
Rev_4	48.66%	6	48.66%	6	48.66%	6	47.03%	7
OpC_1	50.93%	4	50.93%	4	50.93%	4	55.99%	2
OpC_2	51.36%	3	51.36%	3	51.36%	3	54.01%	4
OpC_3	−48.45%	7	−48.45%	7	−48.45%	7	−50.12%	6
OpC_4	47.19%	8	47.19%	8	47.19%	8	45.64%	8
Correlations								
$\rho_{\text{roi, npv}}$	1							
$\rho_{S^{\text{aroi}},\ S^{\text{npv}}}$	1							
$\rho_{\text{slrr, npv}}$	1							
$\rho_{S^{\text{slrr}},\ S^{\text{npv}}}$	1							
$\rho_{\text{irr, npv}}$	0.857							
$\rho_{S^{\text{irr}},\ S^{\text{npv}}}$	0.611							

and

$\alpha^1 = (450, 428, 513, 487, 329, 321, 249, 292)$.

The NFA is assumed to depreciate uniformly, that is, $\text{Dep}_t = 500/7 = 62.5$. The initial investment in working capital is $\text{WC}_0 = 250$. In the following periods, the working capital is equal to 20% of revenues: $\text{WC}_t = 20\% \cdot \text{Rev}_t$, with $0 < t < p$. With such an assumption, the working capital (and, hence the book value of assets) changes under changes in the value drivers: $b_t = b_t(\alpha)$. Cost of capital is assumed to be $k = 10\%$. Tables 14 and 15 report the book values, b_t (sum of fixed assets and working capital), the average capitals, C_t^{sl}, the FCFs, F_t, and the valuation metrics in the base case and perturbed case, respectively. The FCF streams in α^0 and α^1 are derived from the estimates of incomes and book value capitals. Results of the analysis via Total Order FCSI are collected in Table 16, which shows that average ROI and IRR are not strongly NPV-consistent. The degree of NPV-inconsistency of IRR is higher than the inconsistency of average ROI: $1 - \rho_{\text{irr,npv}} = 0.286$, $1 - \rho_{S^{\text{irr}},S^{\text{npv}}} = 0.646$, $1 - \rho_{\text{aroi,npv}} = 0.048$, and $1 - \rho_{S^{\text{aroi}},S^{\text{npv}}} = 0.201$. As expected, the SLRR is strictly NPV-consistent. ⋄

7. Strong NPV-consistency for project ranking

In this section we deal with the ranking of independent projects available to the firm. We first recall the NPV criterion.

Definition 4 (*NPV Criterion for Project Ranking*)**.** Consider a bundle of N projects which share the same risk. Project j is preferable to project

Table 14
Endogenous WC: Average ROI, SLRR, and IRR in α^0.

	0	1	2	3	4
Capital amounts					
b_t	750	334	242	146	0
NFA_t	500	250	150	50	0
WC_t	250	84	92	96	0
C^{sl}_t	750	562.5	375	187.5	0
Overall capital					
b	1,403.06				
C^{sl}	1,771.84				
Cash flows					
F_t	−750	286	162	196	356
Valuation					
NPV	110.54				
$\bar{\imath}$ (b)	14.35%				
$\bar{\imath}^{sl}$	12.61%				
x	12.02%				

Table 15
Endogenous WC: Average ROI, SLRR, and IRR in α^1.

	0	1	2	3	4
Capital amounts					
b_t	750	340	235.6	152.6	0
NFA_t	500	250	150	50	0
WC_t	250	90	85.6	102.6	0
C^{sl}_t	750	562.5	375	187.5	0
Overall capital					
b	1,408.56				
C^{sl}	1,771.84				
Cash flows					
F_t	−750	281	111.4	247	297.6
Valuation					
NPV	57.35				
$\bar{\imath}$ (b)	10.32%				
$\bar{\imath}^{sl}$	9.43%				
x	9.18%				

h if and only if the NPV of j is greater than the NPV of h: $\text{NPV}^j > \text{NPV}^h$, $j, h \in \{1, 2, \dots, N\}$.

The notion of weak NPV-consistency for project ranking may be stated as follows.

Definition 5 (*Weak NPV-consistency for Project Ranking*)**.** A rate of return φ is weakly NPV-consistent for project ranking if and only if the ranks of projects derived from φ is the same as the ranks of projects derived from NPV. Formally, φ is NPV-consistent for project ranking if the following statements are true:

– for every pair of investment projects j and h, $\text{NPV}^j > \text{NPV}^h$ if and only if $\varphi^j > \varphi^h$
– for every pair of financing projects j and h, $\text{NPV}^j > \text{NPV}^h$ if and only if $\varphi^j < \varphi^h$.

We now define strong NPV-consistency for project ranking and then show that, contrary to IRR and average ROI, the SLRR fulfills it under suitable assumptions.

Definition 6 (*Strong NPV-consistency for Project Ranking*)**.** Given an SA technique, a metric φ (and its associated decision criterion) is strongly NPV-consistent for project ranking if

– φ is weakly NPV-consistent for project ranking (Definition 5)
– the parameters' rank vector of φ is equal to the parameters' rank vector of NPV for every project: $r^{\text{npv}^j} = r^{\varphi^j}, j \in \{1, 2, \dots, N\}$.

Table 16
Endogenous WC: Total Order FCSIs (Average ROI, SLRR, IRR)

Parameter	NPV		Average ROI		SLRR		IRR	
	$\Phi_i^{T,\text{npv}}$	r_i^{npv}	$\Phi_i^{T,\text{roi}}$	r_i^{roi}	$\Phi_i^{T,\text{slrr}}$	r_i^{slrr}	$\Phi_i^{T,\text{irr}}$	r_i^{irr}
Rev_1	−52.61%	2	−51.96%	1	−52.61%	2	−55.56%	2
Rev_2	52.94%	1	51.87%	2	52.94%	1	54.84%	3
Rev_3	−51.50%	4	−50.87%	5	−51.50%	4	−52.73%	5
Rev_4	49.14%	6	48.75%	6	49.14%	6	47.21%	7
OpC_1	51.44%	5	51.02%	4	51.44%	5	56.14%	1
OpC_2	51.87%	3	51.45%	3	51.87%	3	54.17%	4
OpC_3	−48.94%	7	−48.54%	7	−48.94%	7	−50.22%	6
OpC_4	47.66%	8	47.27%	8	47.66%	8	45.81%	8
Correlations								
$\rho_{\text{roi, npv}}$	0.952							
$\rho_{S^{\text{roi}}, S^{\text{npv}}}$	0.799							
$\rho_{\text{slrr, npv}}$	1							
$\rho_{S^{\text{slrr}}, S^{\text{npv}}}$	1							
$\rho_{\text{irr, npv}}$	0.714							
$\rho_{S^{\text{irr}}, S^{\text{npv}}}$	0.354							

If φ is strongly NPV-consistent for project ranking and, in addition, the vectors of the relevances coincide, $R^{\text{npv}^j} = R^{\varphi^j}, j \in \{1, 2, \dots, N\}$, then φ is strictly NPV-consistent for project ranking.

It is worth noting that, if the metric φ is not weakly NPV-consistent, the degree of NPV-(in)consistency is irrelevant. That is, even if the degree of NPV-consistency is 1, the fact that the impact of input changes on φ is the same as the impact of input changes on NPV does not heal the project ranking error, and, therefore, a high degree of correlation in the parameter ranking is useless.[13] Conversely, if the metric φ is weakly NPV-consistent but not strongly NPV-consistent, then it is important to assess its degree of (in)consistency with NPV.

In general, none of the three performance metrics (SLRR, average ROI, and IRR) is weakly NPV-consistent for project ranking (let alone strongly NPV-consistent). However, SLRR is strongly (even strictly) NPV-consistent if the competing projects have the same initial cash flows.

Proposition 2. *Suppose $F_0^j = F_0$ for every $j \in \{1, 2, \dots, N\}$. Then, the SLRR is strictly NPV-consistent for project ranking.*

Proof. Owing to Proposition 1, given a project j, the rank vector of φ^j is equal to the rank vector of NPV^j and the vectors of relevances coincide.

We then only have to show that $\varphi = \bar{\imath}^{sl}$ is NPV-consistent according to Definition 5. The overall average capital of project j is $C^{sl^j} = \sum_{t=1}^{p} b_0^j(1-(t-1)/p)(1+k)^{-(t-1)} = \sum_{t=1}^{p} -F_0(1-(t-1)/p)(1+k)^{-(t-1)} = C^{sl}$ and is constant for every $j \in \{1, 2, \dots, N\}$. If $F_0 < 0$, it results that $C^{sl^j} > 0$ for every $j \in \{1, 2, \dots, N\}$; therefore, every project is an investment project. If $F_0 > 0$, then $C^{sl^j} < 0$ for every $j \in \{1, 2, \dots, N\}$ and every project is a financing project. According to eq. (14), $\bar{\imath}^{sl^j} = k + \frac{\text{NPV}^j(\alpha)(1+k)}{C^{sl}}$ $\forall j \in \{1, 2, \dots, N\}$. This implies that the coefficients of the affine transformation $q = k$ and $m = (1+k)/C^{sl}$ are equal for all projects. If $F_0 < 0$, it results that $m > 0$ and, therefore, $\text{NPV}^j > \text{NPV}^h$ if and only if $\bar{\imath}^{sl^j} > \bar{\imath}^{sl^h}$; if $F_0 > 0$, it derives that $m < 0$ and, therefore, $\text{NPV}^j > \text{NPV}^h$ if and only if $\bar{\imath}^{sl^j} < \bar{\imath}^{sl^h}$. □

The proposition says that, whenever the firm has a given amount of capital $b_0 = -F_0$ to be invested, then the SLRR may be employed as a substitute for NPV (or be used in conjunction with it) for selecting the preferred alternative.

In contrast, if initial outlays F_0^j differ across the investments, SLRR and NPV are not consistent for project ranking and the selection of

[13] Indeed, the degree of NPV-(in)consistency if the metric is not weakly consistent is hardly interpretable in one sense or another.

the adequate valuation metric may depend on the presence of capital budget constraints: In case of capital rationing, decision makers may choose the SLRR in place of the NPV, whereas NPV is appropriate if no budget constraints exist and if absolute increase in wealth is set as the objective function instead of financial efficiency.

We now illustrate two simple numerical applications with $N = 2$. They serve as counter-examples for proving that the average ROI and the IRR are not strongly NPV-consistent for project ranking. We use Total Order FCSI to assess degrees of NPV-inconsistency. In the first example, both average ROI and IRR are weakly NPV-consistent for project ranking but not strongly NPV-consistent. In the second example, both the average ROI and the IRR are not even weakly NPV-consistent for project ranking.[14]

Example 11 (*Weak NPV-consistency for Project Ranking*)**.** Consider projects A and B with equal initial fixed assets, $\mathrm{NFA}_0 = 500$, and equal initial working capital, $\mathrm{WC}_0 = 250$. We assume that the book values of fixed assets are different, such that $\mathrm{NFA}_1^A = 300$, $\mathrm{NFA}_2^A = 100$, $\mathrm{NFA}_3^A = 50$ and $\mathrm{NFA}_1^B = 450$, $\mathrm{NFA}_2^B = 350$, $\mathrm{NFA}_3^B = 150$. The working capital of the two projects is assumed to amount to 20% of revenues, $\mathrm{WC}_t^j = 20\% \cdot \mathrm{Rev}_t^j$, $j = A, B$, for $t = 1, 2, 3$ (and $\mathrm{WC}_4 = 0$ for working capital is recovered at the end of the project).

On the basis of the input data, reported in Tables 17–18, the book values are calculated in the two scenarios: $\mathbf{b}^A(\alpha^0) = (750, 380, 184, 140, 0) \neq \mathbf{b}^B(\alpha^0) = (750, 520, 426, 220, 0)$ and $\mathbf{b}^A(\alpha^1) = (750, 382, 186.122, 142.246, 0) \neq \mathbf{b}^B(\alpha^1) = (750, 522, 428.122, 222.248, 0)$. Assuming $k = 6\%$, the overall book value capitals of A are $b^A(\alpha^0) = 1,389.80$ and $b^A(\alpha^1) = 1,395.46$ and the overall book value capitals of B are $b^B(\alpha^0) = 1,804.42$ and $b^B(\alpha^1) = 1,810.08$. The initial invested capital is $C_0 = \mathrm{NFA}_0 + \mathrm{WC}_0 = 500 + 250 = 750$, implying that the average capital vectors of A and B coincide: $\boldsymbol{C}^{sl} = (750, 562.5, 375, 187.5, 0)$ such that $C^{sl} = 1,771.84$. The performance metrics are collected in Tables 17–18 (the bold typeface represents the higher value of each performance metric). Project A is preferred to B, since $\mathrm{NPV}^A > \mathrm{NPV}^B$. All the three relative criteria average ROI, IRR, and SLRR satisfy the weak NPV-consistency for project ranking, since $\bar{\imath}^A(b) > \bar{\imath}^B(b)$, $\bar{\imath}^{sl^A} > \bar{\imath}^{sl^B}$, and $x^A > x^B$. However, the parameter ranking of average ROI and IRR is different from the NPV's parameter ranking. In particular, the degrees of NPV-consistency of average ROI for project A are $\rho^A_{\mathrm{roi,npv}} = 0.857$ and $\rho^A_{S^{\mathrm{roi}},S^{\mathrm{npv}}} = 0.553$ and, for project B, are $\rho^B_{\mathrm{roi,npv}} = 0.857$ and $\rho^B_{S^{\mathrm{roi}},S^{\mathrm{npv}}} = 0.553$. The degrees of NPV-inconsistency for IRR are very high. Specifically, $\rho^A_{\mathrm{irr,npv}} = 0.571$ and $\rho^A_{S^{\mathrm{irr}},S^{\mathrm{npv}}} = 0.483$ for project A; $\rho^B_{\mathrm{irr,npv}} = 0.048$ and $\rho^B_{S^{\mathrm{irr}},S^{\mathrm{npv}}} = 0.126$ for B. Therefore, while weakly NPV-consistent, average ROI and IRR are not strongly NPV-consistent for project ranking and their degree of NPV-inconsistency (especially, the IRR's) is remarkable. ⋄

Example 12 (*NPV-inconsistency for Project Ranking*)**.** Suppose, again, that projects A and B have the same initial fixed assets and same initial working capital: $\mathrm{NFA}_0 = 500$ and $\mathrm{WC}_0 = 250$. We assume that the two projects have different book values of fixed assets: $\mathrm{NFA}_1^A = 250$, $\mathrm{NFA}_2^A = 150$, $\mathrm{NFA}_3^A = 50$ and $\mathrm{NFA}_1^B = 40$, $\mathrm{NFA}_2^B = 20$, $\mathrm{NFA}_3^B = 10$. Tables 19–20 describe the input values in base case and perturbed case, respectively. We assume that the working capital of the two projects is endogenously determined: Specifically, it is equal to 20% of revenues in every period, $\mathrm{WC}_t^j = 20\% \cdot \mathrm{Rev}_t^j$, where $j = A, B$ and $t = 1, 2, 3$ ($\mathrm{WC}_4^j = 0$). The vectors of book value capitals are different for both cases: In the base case, $\mathbf{b}^A(\alpha^0) = (750, 334, 242, 146, 0) \neq \mathbf{b}^B(\alpha^0) = (750, 120, 88, 90, 0)$ and in the perturbed case $\mathbf{b}^A(\alpha^1) = (750, 340, 235.6, 152.6, 0) \neq \mathbf{b}^B(\alpha^1) = (750, 114, 94.34, 84, 0)$. Assuming $k = 6\%$, the overall book value capitals of A are $b^A(\alpha^0) = 1,403.06$ and $b^A(\alpha^1) = 1,408.56$; the overall book value capitals of B are $b^B(\alpha^0) = 1,017.09$ and $b^B(\alpha^1) = 1,012.04$. Given the input data, the initial invested capital is the same for A and B, $C_0 = \mathrm{NFA}_0 + \mathrm{WC}_0 = 500 + 250 = 750$; therefore, the vectors of average capital are the same, $\boldsymbol{C}^{sl} = (750, 562.5, 375, 187.5, 0)$. The overall SL capital is the same for the two projects and does not depend on the state: $C^{sl} = 1,771.84$, regardless of the scenario considered. The valuation metrics in the two cases are reported in Tables 19–20, respectively. Project A creates more value than B, since $\mathrm{NPV}^A > \mathrm{NPV}^B$. The SLRR provides the same answer as the NPV, since $\bar{\imath}^{sl^A} > \bar{\imath}^{sl^B}$. Also, considering Total Order FCSI, the parameters' relevances of NPV and SLRR are equal, implying that the SLRR is strictly NPV-consistent for project ranking. Conversely, the average ROI and the IRR provide an error in ranking projects, since $\bar{\imath}^A(b) < \bar{\imath}^B(b)$ and $x^A < x^B$, so they are not even weakly NPV-consistent.[15] ⋄

Table 17
Weak NPV-consistency for project ranking in α^0.

	A	B
Rev_1	400	350
Rev_2	420	380
Rev_3	450	350
Rev_4	500	350
OpC_1	300	220
OpC_2	290	210
OpC_3	280	195
OpC_4	260	190
Valuation	A	B
NPV	**15.95**	5.77
$\bar{\imath}$ (b)	**7.22%**	6.34%
$\bar{\imath}^{sl}$	**6.95%**	6.35%
x	**6.89%**	6.36%

Table 18
Weak NPV-consistency for project ranking in α^1.

	A	B
Rev_1	410	360
Rev_2	430.61	390.61
Rev_3	461.23	361.24
Rev_4	511.67	361.76
OpC_1	290.24	210.26
OpC_2	279.66	199.67
OpC_3	269.05	184.05
OpC_4	248.39	178.39
Valuation	A	B
NPV	**89.97**	79.85
$\bar{\imath}$ (b)	**12.83%**	10.68%
$\bar{\imath}^{sl}$	**11.38%**	10.78%
x	**10.92%**	10.83%

8. Concluding remarks

This paper builds upon three strands of literature, namely, (i) a methodological one, dealing with the NPV-consistency of measures of financial efficiency, (ii) a managerial one, dealing with management of uncertainty and sensitivity-analysis application to project appraisal, and (iii) an accounting one, dealing with the impact of working capital on financial performance. We introduce a new performance metric

[14] It is worthy of attention that, if working capital is zero or exogenous *and* if every project shares the same capital depreciation schedule, $\mathbf{b}^j = \mathbf{b}$, $\forall j \in \{1, 2, \dots, N\}$, then average ROI is indeed an affine transformation of NPV, $\bar{\imath}^j(b) = k + \mathrm{NPV}^j(\alpha)(1+k)/b$ with coefficients $q = k$ and $m = (1+k)/b$ equal for every project $j \in \{1, 2, \dots, N\}$; therefore, under these assumptions, average ROI is strictly NPV-consistent for project ranking.

[15] The correlation coefficients of average ROI for project A are $\rho^A_{\mathrm{roi,npv}} = 0.952$ and $\rho^A_{S^{\mathrm{roi}},S^{\mathrm{npv}}} = 0.799$ and, for project B, are $\rho^B_{\mathrm{roi,npv}} = 0.976$ and $\rho^B_{S^{\mathrm{roi}},S^{\mathrm{npv}}} = 0.953$. IRR's correlation coefficients are, for project A, $\rho^A_{\mathrm{irr,npv}} = 0.714$ and $\rho^A_{S^{\mathrm{irr}},S^{\mathrm{npv}}} = 0.354$ and, for project B, $\rho^B_{\mathrm{irr,npv}} = 0.976$ and $\rho^B_{S^{\mathrm{irr}},S^{\mathrm{npv}}} = 0.995$. However, these degrees are not relevant, given the error in project ranking.

Table 19
NPV-inconsistency for project ranking in α^0.

	A	B
Rev_1	420	400
Rev_2	460	340
Rev_3	480	400
Rev_4	520	450
OpC_1	300	200
OpC_2	290	200
OpC_3	280	352
OpC_4	260	100
Valuation	A	B
NPV	**110.54**	105.16
$\bar{\imath}$ (b)	14.35%	**16.96%**
$\bar{\imath}^{sl}$	**12.61%**	12.29%
x	12.02%	**12.03%**

Table 20
NPV-inconsistency for project ranking in α^1.

	A	B
Rev_1	450	370
Rev_2	428	371.7
Rev_3	513	370
Rev_4	487	370
OpC_1	329	190
OpC_2	321	189.3
OpC_3	249	347
OpC_4	292	80
Valuation	A	B
NPV	**57.35**	55.80
$\bar{\imath}$ (b)	10.32%	**11.84%**
$\bar{\imath}^{sl}$	**9.43%**	9.34%
x	9.18%	**9.37%**

for project appraisal, the straight-line rate of return (SLRR), which takes into explicit consideration the presence of (uncertain) working capital. We measure its NPV-consistency in both accept–reject decisions and project ranking and compare it with the average ROI introduced in Marchioni and Magni (2018) and the traditional Internal rate of Return (IRR). To this end, we analyze the impact on them of changes (perturbations) in the input data, also known as *value drivers* or *key parameters* (i.e., project's revenues and costs).

We find that the average ROI is not strongly NPV-consistent whenever working capital (WC) is present, uncertain, and endogenously dependent on the value drivers. We use the notion of Chisini mean to search for a measure which possesses strong NPV-consistency, thereby improving upon the average ROI. Two candidates arise: The well-known IRR and the newly-introduced SLRR, based on the (linear) average rate of change of the invested capital.

We find that the IRR is problematic, for its existence and uniqueness may depend on the project's key assumptions, and its financial nature may turn out to be ambiguous. In other words, a change in the value drivers may turn an investment IRR to a financing IRR (or vice versa) or generate multiple IRRs or make the IRR nonexistent. Further, even in favorable cases (as already displayed in Borgonovo and Peccati, 2004, 2006; Percoco and Borgonovo, 2012) the IRR is not strongly NPV-consistent for accept–reject decisions. For project ranking, we show that it is not NPV-consistent, not even in a weak sense.

In contrast, the SLRR is strongly NPV-consistent in a strict form for accept–reject decisions, regardless of whether the working capital is zero or not and regardless of whether it is endogenous or exogenous. Furthermore, its existence and uniqueness is guaranteed in every case. Moreover, the SLRR also enjoys strict NPV-consistency in project ranking if the initial cash flows of the competing projects are equal.

To wrap things up, as compared to the strand of literature about sensitivity analysis and project valuation, we make different and incremental findings:

– we show that a necessary condition for the average ROI to be strongly NPV-consistent in accept–reject decisions is that no use of WC is made in the operations (e.g., no inventory, and sales and purchases are made on a cash-only basis) or that the nonzero WC is managed by the firm's managers in such a way that it is unaffected by the value drivers (sales revenues and costs). In all other cases, the average ROI is not strongly consistent
– we introduce the SLRR (associated with the *average* invested capital) and show that it is strongly NPV-consistent, regardless of whether WC is present or not
– we compare the SLRR, the IRR, and the average ROI and measure the degree of NPV-inconsistency of IRR and average ROI
– we extend the study to project ranking and show, that, contrary to average ROI and IRR, the SLRR is (not only strongly but also) strictly NPV-consistent if the competing projects have the same initial outflow.

We illustrate these results by taking into account two sensitivity analysis techniques: FCSI (Borgonovo, 2010a) and DIM (Borgonovo and Apostolakis, 2001; Borgonovo and Peccati, 2004), and assess the degree of NPV-inconsistency of average ROI and IRR via Spearman's (1904) correlation coefficient and Iman and Conover's (1987) top-down coefficient and find that the degree of inconsistency of IRR and average ROI may vary case by case and may be very high.

The properties of average ROI, SLRR, and IRR are summarized in the following table.

Property	**Average ROI**	**SLRR**	**IRR**
Existence guaranteed	no	yes	no
Uniqueness guaranteed	yes	yes	no
Unambiguous financial nature	yes	yes	no
Accept–reject decisions			
Weak NPV-consistency	yes	yes	yes
Strong NPV-consistency			
with exogenous WC	yes	yes	no
with endogenous WC	no	yes	no
Project ranking			
Weak NPV-consistency (if $F_0^j = F_0 \ \ \forall j$)	no	yes	no
Strong NPV-consistency (if $F_0^j = F_0 \ \ \forall j$)	no	yes	no

These findings show that

- the IRR meets new, previously unknown difficulties in several respects
- the average ROI is more reliable than IRR, but it may incur NPV-inconsistency for both accept–reject decisions and project ranking as well as possible nonexistence
- the SLRR, based on the average rate of change, is reliable and robust and is an appropriate candidate for economic analysis in accept–reject decisions. It is also sound for project ranking if the initial cash flows of the competing projects are equal.

Appendix. Finite change sensitivity index and differential important measure

Finite Change Sensitivity Indices.

The Finite Change Sensitivity Indices (FCSIs) study the effect of a finite change in the inputs on the model output (Borgonovo, 2010a,b). Two versions of FCSIs are defined: First Order FCSI and Total Order FCSI. The First Order FCSIs measure the individual effects of the parameters on f, whereas the Total Order FCSIs consider both the individual contributions and the interactions between parameters. The parameters change from the base value α^0 to $\alpha^1 = \left(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1\right) \in$

A. The corresponding output variation is $\Delta f = f(\alpha^1) - f(\alpha^0)$. The individual effect of α_i on Δf is

$$\Delta_i f = f(\alpha_i^1, \alpha_{(-i)}^0) - f(\alpha^0)$$

where $(\alpha_i^1, \alpha_{(-i)}^0) = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{i-1}^0, \alpha_i^1, \alpha_{i+1}^0, \ldots, \alpha_n^0)$ is obtained by varying the parameter α_i to the new value α_i^1, while the remaining $n-1$ parameters are fixed at α^0. The First Order FCSI of α_i, denoted as $\Phi_i^{1,f}$, is

$$\Phi_i^{1,f} = \frac{\Delta_i f}{\Delta f} \tag{17}$$

(Borgonovo, 2010a). The total effect of the parameter α_i, denoted as $\Delta_i^T f$, is

$$\Delta_i^T f = f(\alpha^1) - f(\alpha_i^0, \alpha_{(-i)}^1), \ \forall i = 1, 2, \ldots, n,$$

(Borgonovo, 2010a, Proposition 1) where $(\alpha_i^0, \alpha_{(-i)}^1)$ is the point with all the parameters equal to the new value α^1, except the parameter α_i, which is equal to α_i^0. The Total Order FCSI of the parameter α_i, denoted as $\Phi_i^{T,f}$, is (Borgonovo, 2010a):

$$\Phi_i^{T,f} = \frac{\Delta_i^T f}{\Delta f} = \frac{f(\alpha^1) - f(\alpha_i^0, \alpha_{(-i)}^1)}{\Delta f}. \tag{18}$$

Differential Importance Measure. The Differential Importance Measure (DIM) of parameter α_i is the ratio of the partial differential of f with respect to α_i to the total differential of f (Borgonovo and Apostolakis, 2001; Borgonovo and Peccati, 2004):

$$DIM_i^f(\alpha^0, \mathrm{d}\alpha) = \frac{\mathrm{d}f_{\alpha_i}}{\mathrm{d}f} = \frac{\frac{\partial f}{\partial \alpha_i}(\alpha^0) \cdot \mathrm{d}\alpha_i}{\sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\alpha^0) \cdot \mathrm{d}\alpha_j}. \tag{19}$$

Two versions of DIM are defined, according to the assumption made upon the variation structure of parameters: Uniform variation assumption (H1) or proportional variation assumption (H2).

H1 implies $\mathrm{d}\alpha_i = \mathrm{d}\alpha_j$, $\forall \alpha_i, \alpha_j$; the resulting DIM is

$$DIM1_i^f(\alpha^0) = \frac{\frac{\partial f}{\partial \alpha_i}(\alpha^0) \cdot \mathrm{d}\alpha_i}{\sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\alpha^0) \cdot \mathrm{d}\alpha_j} = \frac{\frac{\partial f}{\partial \alpha_i}(\alpha^0)}{\sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\alpha^0)}. \tag{20}$$

H2 implies $\mathrm{d}\alpha_i = \xi \cdot \alpha_i^0$ for some $\xi \neq 0$; the resulting DIM is

$$DIM2_i^f(\alpha^0) = \frac{\frac{\partial f}{\partial \alpha_i}(\alpha^0) \cdot \xi \cdot \alpha_i^0}{\sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\alpha^0) \cdot \xi \cdot \alpha_j^0} = \frac{\frac{\partial f}{\partial \alpha_i}(\alpha^0) \cdot \alpha_i^0}{\sum_{j=1}^n \frac{\partial f}{\partial \alpha_j}(\alpha^0) \cdot \alpha_j^0}. \tag{21}$$

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Coherent Portfolio Performance Ratios

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Coherent Portfolio Performance Ratios

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ABSTRACT

In Quantitative Finance 2016, Chen, Hu and Lin (CHL) claimed the following: '…there is yet no coherent risk measure related to investment performance.' (p. 682). Our paper suggests and analyzes four coherence axioms that portfolio performance ratios should satisfy.

Our Portfolio Riskless Translation Invariance axiom must be satisfied to assure separation of the objective decision to optimize a portfolio's risky composition from the subjective decision to optimize the weight of the portfolio's level of risk-free asset. Performance ratios with fixed thresholds other than the risk-free rate do not satisfy this axiom, allowing portfolio managers to affect an ex-ante performance ratio merely by changing the proportion of the risk-free asset in the portfolio rather than by improving the composition of the portfolio's risky components. The magnitude of this potential drawback is examined using the S&P-500 stock index data.

Replacing the fixed threshold, T, with a threshold $T(\gamma, \alpha)$ that equals γ times the portfolio's risk premium plus (1- γ) times the risk-free rate, eliminates the above shortcoming for any selected γ. In addition, using performance ratios with the threshold $T(\gamma, \alpha)$ rather than the fixed T, assures consistency of the performance ratios with the effective stochastic dominance with the risk-free asset rules.

Keywords. Coherent performance ratios, Coherent risk measures, Downside risk measures, Lower partial moments, Sortino ratio, Omega ratio, Kappa ratio, stochastic dominance, FSDR and SSDR rules.

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I. INTRODUCTION

A performance ratio is typically defined as the ratio of a portfolio's excess return to a quantitative risk measure. The most commonly used performance ratio is the Sharpe ratio (Sharpe 1966), which uses the standard deviation (StD) as the risk measure. However, the StD of returns is a proper measure of risk only in the limited case of normal return distributions (or where the utility function is quadratic). For all other distributions, the preference by the mean-variance criterion (MVC) is neither a necessary nor sufficient condition for preference by all expected utility investors[1]. Indeed, the StD as a measure of risk has been heavily criticized, even by its originator, Markowitz 1959 (pp. 286-288), and by many other researchers who suggested downside risk measures that consider only deviations below a minimum acceptable threshold[2].

In an attempt to rectify the shortcomings of the standard deviation as a risk measure, several alternative risk-adjusted performance measures were suggested (Carles 2015). A general expression of downside-risk-adjusted performance ratios is the *n*-th degree Kappa ratio (Kaplan and Knowles 2004), which uses the *n*-th root of the lower partial moment (LPM_n) as the measure of risk, and is defined as follows:

(1) $$K_n(\tilde{R},T) = \frac{E(\tilde{R})-T}{[LPM_n(\tilde{R},T)]^{\frac{1}{n}}}$$

where $\tilde{R}$ is the (random) rate of return on the investment. The numerator of the Kappa ratio equals the expected risk premium over the investor's (subjective) threshold, T, where all returns lower than T are considered to be in the "loss" range. The threshold T could be equal to the risk-free rate, R_f, or different from it. The risk measure is the *n*-th root of the lower partial moment of the *n*-th degree, $LPM_n(\tilde{R},T)$, which is measured as follows:

(2) $$LPM_n(\tilde{R},T) = \int_{-\infty}^{T} \left|\tilde{R}-T\right|^n f(\tilde{R})d\tilde{R}.$$

The Kappa ratio incorporates other downside-risk-based performance ratios. Kazemi, Schneeweis, and Gupta 2004 show that the Kappa ratio of the first degree (n = 1), which they call the Omega-Sharpe ratio, provides exactly the same information that Omega provides and always leads to the same ranking as Omega. The Kappa ratio of the second degree (n = 2) is, in fact, identical to the Sortino ratio (Sortino and Van del Meer 1991).

About two decades ago, Artzner, Dalbaen and Eber 1999 (ADE) introduced four axioms that a risk measure must satisfy to qualify as a coherent measure of risk. Their study was followed by many who extended their work in different directions. Recently, Koumou and Dionne 2019 (KD) presented an axiomatic foundation for coherent portfolio diversification measures of correlation,Φ, as functions of the portfolio's weights, w, and the assets' returns, $\tilde{R}$. KD's work compliments the work of ADE on coherent risk measures, integrating the requirements of coherent correlation and coherent risk measures to define axioms for coherent diversification. KD affirms that their axioms for coherent diversification measures are only a first step toward a rigorous theory of correlation diversification measures. The current paper defines axioms that

[1] The stochastic dominance rules for all rational investors (First degree Stochastic Dominance rule - FSD) and for all rational risk averse investors (Second degree Stochastic Dominance rule - SSD), respectively, provide the necessary and sufficient (optimal) efficiency rules for preferences. However, the practical application of these rules for constructing optimal portfolios and obtaining market equilibrium conditions is quite limited.

[2] For a review of the history of downside risk measures, see Nawrocki 1999.

must be satisfied by a performance ratio to qualify as a coherent performance ratio. It examines the coherence (or the lack of it) of existing portfolio performance ratios (PPRs) and suggests a modification of the threshold that guarantees the coherence of the PPRs for a wide range of selected thresholds.

As an element of our original set of axioms, the Portfolio Riskless Translation Invariance (PRTI) axiom implies that a coherent performance ratio must be invariant with respect to the (subjectively selected) proportion of the risk-free asset in the portfolio. We show that PPRs that use $T \neq R_f$ are coherent according to PRTI if and only if the selected threshold is a specific combination of the risk-free rate and the portfolio's risk premium. Note that if a performance ratio does not satisfy this PRTI axiom, portfolio managers may increase the ratio simply by changing the portfolio's proportion of the risk-free asset rather than by selecting a better composition of the portfolio's risky assets. In other words, if the threshold is different from the risk-free rate, increasing or decreasing the portfolio's leverage using the risk-free asset affects the performance measure for any given composition of the portfolio's risky assets component.

In terms of applied financial value, the adoption of coherent performance metrics will drive portfolio managers to maximize the ex-ante ratios of an investment portfolio by optimizing the composition of the portfolio's risky component, independent of the portfolio's proportion of the riskless asset, focusing on their technical skills of asset allocation and selection under risk while avoiding value-neutral decisions regarding the employment of the riskless asset that can be easily taken by their clients.

Our paper is organized as follows. The next section presents the major relevant literature on coherent risk and coherent diversification measurement. Section three presents our axioms for the coherence of portfolio performance ratios. In section four, we examine whether some well-known performance ratios satisfy our required axioms. It is shown that the original Sharpe ratio and some of its variants are not coherent. They do not satisfy either the monotonicity axiom or the PRTI axiom. In addition, portfolio performance ratios that use downside risk measures with $T \neq R_f$ do not satisfy the PRTI axiom. We present empirical evidence demonstrating the effect of the equity level on the performance ratio indicating a lack of coherence of these ratios. In section five, we suggest a replacement for the constant threshold using a more economically logical threshold, $T(\gamma,\alpha)$, which is responsive to the portfolio's risk level. This suggested threshold guarantees that all the performance ratios examined in this paper satisfy the set of required axioms and qualify as coherence ratios. Section six shows how performance ratios that use downside risk measures and employ our responsive threshold, can be maximized by minimizing the coefficient of risk, and demonstrates the maximization process using the Sortino ratio and the Reward-to-VaR ratio. In section seven, we prove that dominance by first degree stochastic dominance with the riskless asset rule (FSDR) and by second degree stochastic dominance with the riskless asset rule (SSDR) of Levy and Kroll 1976, imply dominance by performance ratios that employ our $T(\gamma,\alpha)$ threshold. In the final section, we present a summary and some conclusions.

II. PRELIMINARIES: AXIOMS FOR COHERENT RISK AND CORRELATION DIVERSIFICATION MEASURES

In this section, we present previously suggested sets of axioms for coherent risk and coherent diversification measures, some of which we modify and adapt for our set of axioms for coherent performance ratios.

The seminal study by ADE (1999) presents the following four axioms that a risk measure ρ should satisfy in order to be considered a *coherent risk measure* for ranking two random loss variants $\tilde{R}_x$ and $\tilde{R}_y$:

1. ***Monotonicity***: If $\tilde{R}_x \geq \tilde{R}_y$, then $\rho(\tilde{R}_x) \leq \rho(\tilde{R}_y)$;
2. ***Positive homogeneity***: $\rho(\lambda\tilde{R}_x) = \lambda\rho(\tilde{R}_x) \;\; for\ \lambda > 0$;
3. ***Riskless translation invariance***: $\rho(\tilde{R}_X + M) = \rho(\tilde{R}_X) - M$; and
4. ***Risk subadditivity***: $\rho(\tilde{R}_x + \tilde{R}_y) \leq \rho(\tilde{R}_x) + \rho(\tilde{R}_y)$.

Föllmer and Schied (2002) proposed an extension of the notion of a coherent risk measure by introducing the definition of a convex measure of risk. They augment axioms 2 and 4 above, and account for the fact that large positions may introduce liquidity risk by defining the following convexity axiom:

5. ***Convexity***: $\rho[\alpha\tilde{R}_x + (1-\alpha)\tilde{R}_y] \leq \rho(\alpha\tilde{R}_x) + \rho[(1-\alpha)\tilde{R}_y]$.

The above axioms are reasonable for rational investors who prefer more money to less, although the monotonicity and subadditivity axioms can be weakened without losing any practical importance by employing stochastic dominance (hereafter, SD) rules. We denote dominance of $\tilde{R}_x$ over $\tilde{R}_y$ by first degree stochastic dominance (FSD) or by second degree stochastic dominance (SSD), respectively, as follows: $\tilde{R}_x \underset{FSD}{D} \tilde{R}_y$ or $\tilde{R}_x \underset{SSD}{D} \tilde{R}_y$. Then, the monotonicity axiom may thus be written as follows.

Monotonicity for all rational risk averse expected utility maximizers*:* If $\tilde{R}_x \underset{FSD}{D} \tilde{R}_y$ or only $\tilde{R}_x \underset{SSD}{D} \tilde{R}_y$, then $\rho(\tilde{R}_x) \geq \rho(\tilde{R}_y)$.

We thus replace ADE's monotonicity axiom with a weaker but perhaps more applicable monotonicity axiom, by which a performance ratio is higher for a return distribution $\tilde{R}_x$ than for a return distribution $\tilde{R}_y$ if $\tilde{R}_x$ dominates $\tilde{R}_y$ by either FSD or only by SSD.

This two-tiered monotonicity requirement yields a two-tiered coherence of risk measures that depend on the investor's utility. Risk measures that are coherent for all rational expected utility maximizers are coherent for all rational risk averse expected utility maximizers. On the other hand, risk measures that are coherent for all rational risk averse expected utility maximizers are not necessarily coherent for all rational expected utility maximizers. If the vast majority of investors are risk averse, the weaker requirement, $\tilde{R}_x \underset{SSD}{D} \tilde{R}_y$, is relevant to the vast majority of investors, and, in general, it potentially produces a larger group of coherent risk measures.
Several different contributions have enriched the knowledge in the field of coherent risk measures. Among others, Denault 2001 introduced an axiomatic foundation of coherence for risk capital allocation, Delbaen 2002 extended the notion of coherent risk measures in ADE to general probability spaces and related the theory of coherent risk measures to game theory, Föllmer and Schied 2002 proposed an extension of the notion of a coherent risk measure by introducing the definition of a convex measure of risk (as mentioned above), Artzner et al 2007 defined coherence in risk management in a multiperiods setting with intermediate stopping times, and Chen and Wang 2008 constructed a class of two-sided coherent risk measures for controlling the asymmetry and fat-tail characteristics. The development of coherent performance measures, to the best of our knowledge, includes a limited series of academic contributions. Among these, Cherny and Madan 2009 proposed an axiomatic foundation of coherent *trading* performance in a static one-time period setup, linked to positive expectations

resulting from a stressed sampling of the cash-flow distribution, based on the eight axioms of quasi-concavity, monotonicity, scale invariance, Fatou's property, law invariance, consistency with second order stochastic dominance, arbitrage consistency, and expectations consistency. Then, Bielecki, Cialenco, and Zhang 2014 built upon Cherny and Madan 2009, changing the mathematical framework to a dynamical multiperiod setup, where cash flows are treated as random processes, and considering the cumulative cash flows at each intermediate time. Furthermore, it is also worth noting the work of CHL 2016. Although they did not propose an axiomatic foundation for coherent performance measurement, they developed and analyzed the performance ratios based on coherent risk measures and obtained a portfolio selection model that, considering transaction costs, empirically performed much better than the corresponding alternative optimal portfolio.

As noted above, KD presented an axiomatic foundation for coherent portfolio diversification measures of correlation, as functions of the portfolio's weights w and the assets' returns$\tilde{R}$, that compliments the work of ADE on coherent risk measures. Below we list KD's axioms for *coherent portfolio diversification measures of correlation* Φ.

i. ***Concavity***: The Φ of a portfolio is not less than the sum of the weighted average of the Φs of the portfolio's single assets.
ii. ***Size degeneracy*****:** The Φs of all single-assets portfolios are minimal and equal $\underline{\Phi}$.
iii. ***Risk degeneracy*****:** In case all the individual assets have the same distribution, diversification has no benefit and the Φ of any portfolio is equal to $\underline{\Phi}$ **.**
iv. ***Reverse risk degeneracy*****:** If diversification has no benefit, then for any portfolio composition, the diversification measure is equal to $\underline{\Phi}$.
v. ***Duplication invariance*****:** If some assets in the portfolio have identical distributions, the optimal weights of the other assets should consider the total weights of the identical assets as if they were one**.**
vi. ***Size Monotonicity*****:** Increasing the size of the portfolio does not decreaseΦ.
vii. ***Translation Invariance* (*TI*):** Adding a given amount, a, to returns R_A does not change$\boldsymbol{\Phi}$**.** Namely, $\Phi(w|R_{A+a}) = \Phi(w|R_A)$.
viii. ***Homogeneity*****:** Multiplying returns $\tilde{R}_A$ by a positive constant, b, does not change the optimal diversification: $\Phi(w|R_{bA}) = b^k\Phi(w|R_A)$.
ix. ***Symmetry*****:** A portfolio diversification measure must be symmetric with respect to the exchangeable variates of w**.**

KD correctly asserted that their nine axioms are only related to the measurement of the isolated impact of correlations through diversification where the overall value of portfolio diversification is generated from other sources such as the law of large numbers and the beta with the market portfolio.

III. THE AXIOM SET FOR COHERENT PERFORMANCE RATIOS, $\boldsymbol{\Psi}$.

In this section, we adapt and modify the relevant axioms proposed by ADE and KD in an attempt to put forth a set of axioms for coherent portfolio performance ratios. We then show that the existing performance measures, even when based on coherent risk measures with thresholds different from the risk-free rate, violate the Portfolio Riskless Translation Invariance axiom (PRTI), which is an essential requirement for the separation between the optimal composition of the portfolio's risky component[3] and the subjective choice of splitting the portfolio into its risk-free and the risky components.

[3] We interchangeably refer to the "risky component" as the "equity component" or the "equity level".

Recall that ADE's coherence axioms are relevant for risk measures and not for performance ratio measures. Similarly, KD's coherence axioms are relevant for the partial mutual impact of diversification and correlation on portfolio performance but they are not intended to guarantee the coherence of the overall performance of a portfolio. Consequently, some of their axioms may be irrelevant for the axiomatic base of coherent portfolio performance ratios and the relevant axioms may be inadequate for guaranteeing the coherence of performance ratios.

We now introduce our axioms for coherent portfolio performance ratios, Ψ. Assume two portfolio return distributions $\tilde{R}_x$ and $\tilde{R}_y$. Coherent performance ratios must satisfy the following four axioms.

A. ***Monotonicity*****:** If $\tilde{R}_x \underset{SD(n)}{D} \tilde{R}_y$, then $\Psi(\tilde{R}_x) \underset{(n)}{\geq} \Psi(\tilde{R}_y)$.

Here, *n* stands for the degree of stochastic dominance. The essence of this axiom is to create consistency between the rankings of investment portfolios on the one hand, and the investors' expected utility on the other hand. A SD rule divides the potential distributions into an "efficient set" and an "inefficient" set. It follows that $\tilde{R}_x \underset{SD(n)}{D} \tilde{R}_y$ indicates dominance of $\tilde{R}_x$ over $\tilde{R}_y$ according to the SD rule of the *n*-th degree and $\Psi(\tilde{R}_x) \underset{(n)}{\geq} \Psi(\tilde{R}_y)$ indicates that $\tilde{R}_x$ is ranked higher than (or equal to) $\tilde{R}_y$ by the relevant performance ratio, and the ranking applies to distributions that belong to the "efficient" set as defined by the relevant SD rule.

The need for screening potential return distributions using SD rules stems from the fact that performance ratios may potentially rank an "inefficient" return distribution higher than an "efficient" return distribution, resulting in an erroneous ranking in the sense that, for the relevant utility group (rational expected utility maximizers, rational as well as risk averse expected utility maximizers and so on), there is at least one distribution in the "efficient set" which provides a higher expected utility than all the return distributions in the "inefficient set" for all the relevant utilities belonging to the group.

In principle, the application of SD rules and coherent performance ratios represents two types of screening tests in the performance evaluation process. The SD tests, defining the first kind of controls, apply to the potential distributions from which an investment is to be selected. These tests do not provide a complete ranking, but they rather identify the set of dominated distributions that need not be evaluated by a performance ratio since they are clearly "inefficient" for the defined group of investors. A performance ratio that satisfies the *monotonicity* axiom, is capable of ranking alternative return distributions, knowing that a distribution from the "inefficient set" will not be preferred to a distribution from the "efficient set".

With respect to the SD of *n* degree, recall that when a given number of return distributions are evaluated, the number of distributions in the "efficient set" is generally decreasing with *n*. Therefore, on the one hand, as *n* increases there are likely to be less potential conflicts between the SD ranking and performance ratios ranking, but on the other hand, as *n* increases, the analysis is relevant for a smaller group of utilities due to the additional constraints on the utility function of the investors belonging to that group[4].

[4] FSD applies to investors whose utility functions are not decreasing with wealth (positive first derivative). SSD applies to the utility functions of risk avert investors (positive first derivative and negative second derivative). In general, SD rules of the *n*-th degree apply to utilities with positive odd derivatives and negative even derivatives. (For more details, see Levy 2006, Theorem 3.5, page 131.) In Section VII of this paper we show that when borrowing and lending at the riskless rate is allowed, the monotonicity requirement is less restrictive

Finally note that our *monotonicity* axiom is much weaker than ADE's monotonicity axiom, which may be interpreted as requiring that the returns of one distribution are higher than the returns of an alternative distribution under *all* states of nature. Although our *monotonicity* axiom is not part of KD's axioms, their size monotonicity is directly relevant to our set of axioms for coherent PPRs.

B. *Size Monotonicity*: $\Psi\big(\lambda(1+\tilde{R}_x)\big) = \Psi(1+\tilde{R}_x) = \Psi(\tilde{R}_x) \quad \lambda > 0$
Our *size monotonicity* axiom implies that the performance ratio per unit of invested capital must remain invariant with respect to the invested amount. The positive constant, λ, is thus interpreted as a wealth multiplier. With respect to the Sharpe ratio, for example, we get:

$$(3) \quad SR\big(\lambda(1+\tilde{R}_x)\big) = \frac{E(\lambda(1+\tilde{R}_x)) - \lambda(1+R_f)}{StD(\lambda(1+\tilde{R}_x))} = $$
$$= \frac{E(\lambda\tilde{R}_x) - \lambda R_f}{StD(\lambda\tilde{R}_x)} = \frac{E(\tilde{R}_x) - R_f}{StD(\tilde{R}_x)} = SR(\tilde{R}_x)$$

It follows that the Sharpe performance ratio is invariant with respect to the wealth level. In fact, as detailed in Table 1, all the other performance ratio that are analyzed in this paper (as well as perhaps all other common performance ratios), satisfy this *size monotonicity* axiom. Note that this axiom is obviously entirely different from the corresponding positive homogeneity axiom put forth by ADE, according to which, as the size of the portfolio increases, the risk of the invested amount increases proportionally. The difference is rooted in the fact that while ADE consider the investment amount, performance ratios focus on performance per unit of invested capital.

C. *Portfolio Riskless Translation Invariance* (*PRTI*): $\Psi(\alpha\tilde{R}_x + (1-\alpha)R_f) = \Psi(\tilde{R}_x)$, where $0 < \alpha \leq 1$ is the proportion of the risky assets in the portfolio and R_f is the risk-free rate[5].

This PRTI axiom is different from ADE's RTI axiom. Their axiom correctly asserts that the additional riskless amount, *M*, reduces risk by *M*. The interpretation of KD's TI axiom is also different from our interpretation. KD asserts that adding a riskless amount of money to a portfolio does not change the advantage of diversification through correlation. Our PRTI axiom is based on the following assumptions.

1. A portfolio manager cannot generate value for investors using a strategic long term holding of a risk-free asset since there is no required professional expertise for this holding. Investment in the risk-free asset is a trivial investment, readily accessible to the ultimate investor, and yields a known return. Hence, the performance evaluation ratio should be limited solely to the risky component of the portfolio[6].
2. The selection of the level of risk is a subjective decision of the investor. In principle, any choice of the proportion of the risk-free vs. the risky component by the professional

than would seem at first glance, since the inefficient set tends to increase with the use of Stochastic Dominance with Riskless Asset Rules.

[5] The case where $\alpha = 0$ is redundant since in this case the entire portfolio's capital is invested in the riskless asset, there is no risk and a ratio of reward to variability is undefined.

[6] This axiom is related only to the expected strategic level of the riskless asset in the portfolio. It does not preclude the potential gains and a higher performance ratio due to successful timing in entering or exiting the risky market, as well as selecting a high beta portfolio before a bullish market, even though several empirical studies indicate that investment professionals lack a return timing ability (see: Sherman, O'Sullivan and Gao 2017, Bodson, Cavenaile and Sougné 2013, Cuthbertson, Nitzsche and O'Sullivan 2010, Friesen and Sapp 2007).

portfolio manager can be offset by the ultimate investor. Hence, the axiom postulates a separation between the optimal composition of the portfolio's risky component (usually determined by a professional portfolio manager) and the overall portfolio split between the risky component and the risk-free component (usually determined by the ultimate investor)[7].

The above PRTI requirement is extremely important since in its absence the ex-ante expected performance ratio may be increased (or perhaps manipulated) by merely changing the portfolio's proportion of the risk-free asset while it should only reflect the performance of the portfolio's risky component.

D. *Concavity:* $\Psi\left[\alpha\tilde{R}_x + (1-\alpha)\tilde{R}_y\right] \geq \alpha\Psi(\tilde{R}_x) + (1-\alpha)\Psi(\tilde{R}_y)$
This axiom reflects the potential advantage of diversification due to correlation. Equality in the concavity relationship is a corner situation where $\tilde{R}_x$ and $\tilde{R}_y$ are identically distributed and perfectly positively correlated. It is trivially tantamount to holding two shares of the same company. Note that the axiom is equivalent to KD's concavity axiom and to ADE's convexity axiom (Föllmer and Schied 2002).

IV. THE COHERENCE OF SOME KNOWN PERFORMANCE RATIOS

In this section, we examine whether some well-known performance ratios satisfy the above coherence axioms and especially the PRTI and *monotonicity* axioms by focusing on a portfolio that consists of a risk-free asset and a (portfolio of) risky asset(s). The return of the overall portfolio and the return of the portfolio's risky (equity) component are denoted $\tilde{R}_P$ and $\tilde{R}_e$, respectively. The portfolio's weights are α for the proportion invested in the risky equities and $(1-\alpha)$ for the proportion invested in the risk-free asset. The ratios we examine are Sharpe, Kappa, Omega, Reward-to-VaR, and Reward-to-CVaR as listed below.

Sharpe ratio. The portfolio's Sharpe ratio is defined as follows:

(4) $$SR(\tilde{R}_P) \equiv \frac{E(\tilde{R}_P) - R_f}{StD(\tilde{R}_P)}$$

Kappa ratio. From Eq. (1), the generalized n^{th} degree Kappa portfolio performance ratio (for $n \geq 1$) is as follows:

(5) $$K_n(\tilde{R}_P, T) \equiv \frac{E(\tilde{R}_P) - T}{[LPM_n(\tilde{R}_P, T)]^{1/n}}$$

where the lower partial moment is defined by Eq. (2). Note that the Kappa ratio of the second degree is, in fact, the Sortino ratio:

(6) $$\text{SOR}(\tilde{R}_p, T) = \frac{E(\tilde{R}_p) - T}{\left[\int_{-\infty}^{T} (T - \tilde{R}_p)^2 f(\tilde{R}_p) d(\tilde{R}_P)\right]^{\frac{1}{2}}} = K_2(\tilde{R}_p, T)$$

Omega ratio. The first degree Omega is given by the following:

[7] The monetary separation theorem is the basis of the Sharpe-Lintner-Mossin Capital Asset Pricing Model (CAPM).

(7) $\Omega_1(\tilde{R}_p,T) \equiv \frac{\int_T^{\infty}[1-F(\tilde{R}_p)]d\tilde{R}_p}{\int_{-\infty}^{T}F(\tilde{R}_p)d\tilde{R}_p} = \frac{E[\max(\tilde{R}_P-T,0)]}{E[\max(T-\tilde{R}_P,0)]}$,

Where $F(\tilde{R}_P)$ represents the cumulative distribution function of the portfolio return $\tilde{R}_P$. The Omega ratio of the n^{th} degree is given by the following:

(8) $\Omega_n(\tilde{R}_p,T) = \frac{\{E[\max(\tilde{R}_P-T,0)^n]\}^{\frac{1}{n}}}{\{E[\max(T-\tilde{R}_P,0)^n]\}^{\frac{1}{n}}}$.

Kaplan and Knowles 2004 proved that $\Omega_1(\tilde{R}_P,T)$ belongs to the Kappa group such that the following holds:

(9) $\Omega_1(\tilde{R}_p,T) = \frac{E(\tilde{R}_p)-T}{LPM_1(\tilde{R}_p,T)} + 1 = K_1(\tilde{R}_p,T) + 1$

Reward-to-VaR. The next ratio to be examined is the Reward-to-VaR ratio:

(10) $\text{RVaR}(\tilde{R}_p,q,T) = \frac{E(\tilde{R}_p)-T}{VaR(\tilde{R}_P,q,T)}$

$VaR(\tilde{R}_P,q,T)$ is given by Equation (11), where $Q_P(q)$ is the portfolio's q quantile, such that the probability that the portfolio's return will be less than or equal to $Q_P(q)$ is equal to q.

(11) $VaR(\tilde{R}_P,q,T) = T - Q_P(q)$

Reward-to-CVaR. The last ratio we consider is the Reward to CVaR ratio:

(12) $RCVaR(\tilde{R}_P,T) \equiv \frac{E(\tilde{R}_P)-T}{T-\int_0^{F_P(T)} Q_P(q)dq}$

Where $F_P(T)$ is the cumulated probability of the portfolio's return up to the threshold T, and the CVaR is given by the denominator of Eq. (12): $CVaR(\tilde{R}_P,T) = T - \int_0^{F_P(T)} Q_P(q)dq$.

We now test the coherence of the five ratios with respect to the four axioms. Beginning with the Sharpe ratio, we note that it clearly complies with our PRTI axiom since the following holds:

(13) $SR(\tilde{R}_P) \equiv \frac{E(\tilde{R}_P)-R_f}{StD(\tilde{R}_P)} = \frac{E[\alpha\tilde{R}_e+(1-\alpha)R_f]-R_f}{StD(\alpha\tilde{R}_e)} = \frac{E(\tilde{R}_e-R_f)}{StD(\tilde{R}_e)} = SR(\tilde{R}_e)$

Indeed, this ratio obeys the separation between the decision to include the risk-free asset as part of the investment portfolio and the decision with respect to the composition of the portfolio's risky component, a separation that is a central premise of the CAPM. On the other hand, it is well known that the Sharpe ratio *does not* comply with the *monotonicity* axiom, even at the basic level of the FSD rule, since it is possible that $\tilde{R}_x$ dominates $\tilde{R}_y$ according to FSD so that all rational expected utility investors prefer $\tilde{R}_x$ to $\tilde{R}_y$, but yet the Sharpe ratio falsely ranks $\tilde{R}_y$ as a better performing distribution: $SR(\tilde{R}_y) > SR(\tilde{R}_x)$[8]. The four performance ratios other than the Sharpe ratio may fail to satisfy the PRTI axiom unless $T = R_f$, as stated in Proposition 1.

[8] For example, assume two uniform distributions as follows: $\tilde{R}_y \sim U(0.05,0.07)$ and $\tilde{R}_x \sim U(0.10,0.20)$ and $R_f = 0$. Clearly, x dominates y according to FSD since even the lowest return of x is higher than the highest

Proposition 1. The four portfolio performance ratios, i.e., the Kappa ratio $K_n(\tilde{R}_P)$, the Omega ratio $\Omega_n(\tilde{R}_p, T)$, the Reward-to-VaR ratio $RVaR(\tilde{R}_p, q, T)$ and the Reward-to-CVaR ratio $RCVaR(\tilde{R}_p, q, T)$ are coherent with the PRTI axiom (invariant with respect to α changes) if and only if $T = R_f$. Furthermore, they increase (decrease) with α if and only if $T > R_f$ ($T < R_f$).

The proofs are provided in Appendix A, where we furthermore analyze these ratios for their compliance with the *size monotonicity* and *concavity* axioms, showing that the ratios considered do, in general, satisfy these axioms. The compliance with the *monotonicity* axiom is analyzed in Section VII of the paper.

Figure 1 presents an example of the empirical consequence of Proposition 1. The significant effect of a portfolio's α level on performance ratios is demonstrated using the S&P-500 index monthly returns over the 240 months during the period from 02.2000 to 01.2020. The risk-free rate was estimated to equal 0.1% per month. Figure 1 contains three panels. Panel A exhibits the ratios where the threshold is set at $T = 0.5 \times R_f = 0.05\%$ per month. The ratios shown in Panel B use the threshold $T = 1.5 \times R_f = 0.15\%$ per month and in Panel C the threshold is set to be $T = R_f = 0.1\%$ per month. It is evident that the α level has a significant and even dramatic effect on the ratios, particularly at relatively low levels of α[9]. In line with Proposition 1, only in the case where the threshold is set equal to the risk-free rate, i.e., $T = R_f$ (Panel C), do we see that the weight of the risky asset, α, has no effect on the performance ratio. Given these results, one may assume that where $T < R_f$, portfolio managers who wish to maximize any of these performance ratios have a significant incentive to decrease the portfolio's weight of the risky asset, an incentive that is evidently stronger over the lower α range. The opposite holds where $T > R_f$. As exhibited in Panel B, the ratios increase significantly with the level of α such that a portfolio manager might want to increase the weight of the risky asset if they wish to increase their portfolio's performance ratio.

V. THE MODIFIED PORTFOLIO PERFORMANCE RATIOS

Rather than using fixed thresholds, we now modify the ratios and employ thresholds that are responsive to the portfolio's α. Specifically, we use the threshold $T(\gamma, \alpha)$ which is set equal to the weighted average of the expected portfolio return and the risk-free return, as follows:

(14) $T(\gamma, \alpha) = \gamma E(\tilde{R}_P) + (1 - \gamma)R_f = \gamma\left[E(\tilde{R}_P) - R_f\right] + R_f$

where $\gamma \in \mathbb{R}$. Eq. (14) may be rewritten also as follows:

(15) $T(\gamma, \alpha) = \gamma\alpha\left(E(\tilde{R}_e) - R_f\right) + R_f$

return of y, leading to the preference of x over y by every rational investor. However, calculating the Sharpe ratios, we obtain the following: $SR(\tilde{R}_y) = \frac{0.06}{0.01} = 6 > 3 = \frac{0.15}{0.05} = SR(\tilde{R}_x)$. Namely, according to the Sharpe ratio, $\tilde{R}_y$ is a better performing investment than $\tilde{R}_x$.

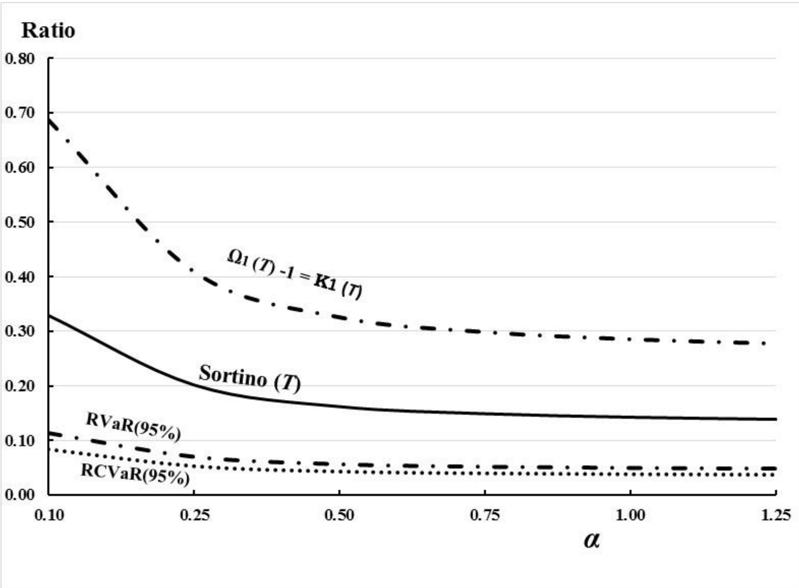
[9] Note that as α decreases, the portfolio's risk-free asset weight increases and the portfolio's return gets closer to the risk-free rate. If $T > R_f$, the ratios are zero at some low level of α and negative at even lower α levels.

Figure 1

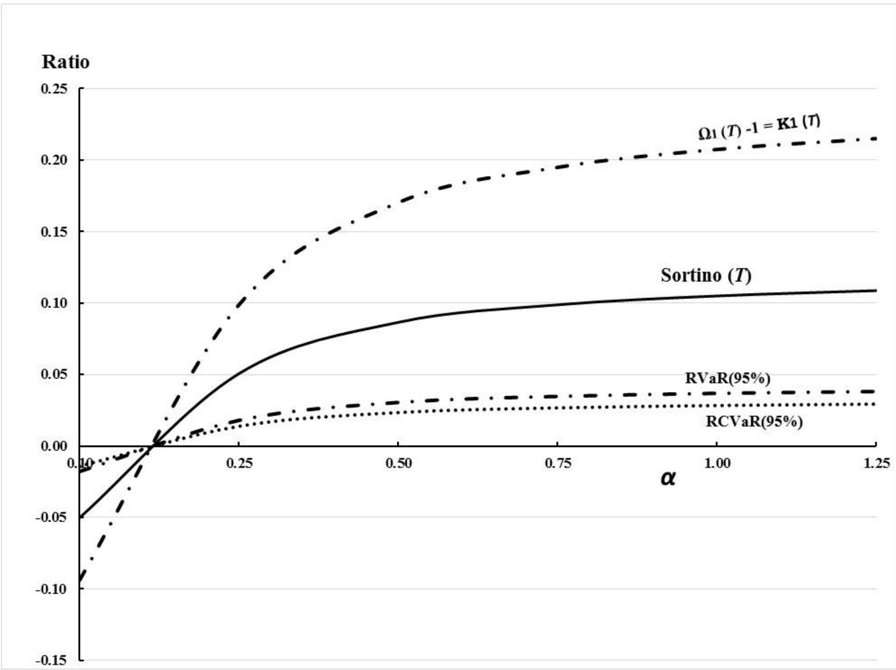
Portfolio Performance Ratios and the proportion of the portfolio's risky asset, α.

The ratios were calculated using 240 monthly returns
of the S&P-500 index from 02.2000 to 01.2020.
The risk-free rate was estimated as a constant 0.1% per month.

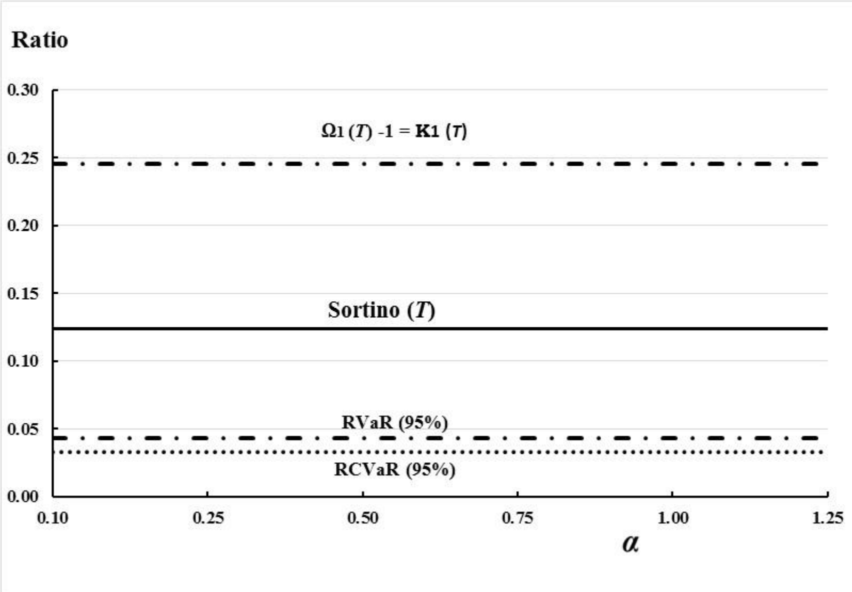
Panel A: $\boldsymbol{T = 0.5 \times R_f = 0.05\%}$



Panel B: $\boldsymbol{T = 1.5 \times R_f = 0.15\%}$



Panel C: $T = R_f = 0.1\%$



where $E(\tilde{R}_e)$ is the expected return of the portfolio's risky component. The economic logic for choosing $T(\gamma, \alpha)$ as a threshold rate is that the threshold for measuring the downside risk of a portfolio should adjust in response to the portfolio's risk premium. This is because it is likely that as the selected overall expected volatility of the portfolio increases, the investor's propensity to absorb losses increases as well. Of course, this is not a necessary behavioral attitude of all investors. However, we will prove in Proposition 2 below that it is necessary for measuring the portfolio's performance when one subjectively sets a threshold that differs from the risk-free rate since otherwise the PRTI axiom is not satisfied. Additionally, note that the responsive threshold, as defined in Equation (14) or (15), is a weighted average between the portfolio's risky component fixed threshold $T(\gamma, \alpha = 1)$,and the risk-free return:

(16) $$T(\gamma, \alpha) = \alpha[T(\gamma, \alpha = 1)] + (1 - \alpha)R_f = \\ = \alpha\left[\gamma\left(E(\tilde{R}_e) - R_f\right) + R_f\right] + (1 - \alpha)R_f = \gamma\alpha\left(E(\tilde{R}_e) - R_f\right) + R_f$$

Equation (16) implies that as the portfolio's threshold changes, the threshold for the risky component remains unchanged at $T(\gamma, \alpha = 1) = \gamma\left(E(\tilde{R}_e) - R_f\right) + R_f$.

Proposition 2. For every γ, PPRs that use the responsive threshold $T(\gamma, \alpha)$ are invariant with respect to the portfolio's equity level, α, and hence they satisfy the PRTI axiom.

The proofs are presented in Appendix B. Note that γ is a subjective loss aversion factor. Assuming a positive expected equity risk premium, a higher γ leads to a higher threshold for any given proportion α of the portfolio's risky equity investment. Using the modified performance ratios in the financial practice, entails the need to estimate the subjective loss aversion of the client, γ, <u>instead</u> of the fixed threshold T. The other parameters, $E(\tilde{R}_e)$, R_f and α, are generally available and required for measuring the traditional performance ratios. It is worth noting that non-positive thresholds exist for the following γ values:

(17) $$\gamma \leq -\frac{R_f}{\alpha[E(\tilde{R}_P) - R_f]}$$

Selecting a negative γ while the expected risk premium $E(\tilde{R}_e) - R_f$ is positive and α is positive, implies a threshold below the risk-free rate.

VI. THE PORTFOLIO'S OPTIMAL RISKY COMPONENT FOR A GIVEN γ

As claimed by Proposition 2, performance ratios that use the responsive threshold $T(\gamma, \alpha)$ are invariant with respect to α. Therefore, they sustain the important separation between the composition of the portfolio's risky component and the weight given to the risk-free asset without affecting the performance ratios. As a result, one is able to maximize the ex-ante ratios only by optimizing the composition of the portfolio's risky component.

In all our PPRs that uses $T(\gamma, \alpha)$ the numerator is 1-γ times the expected equity risk premium in excess of the risk-free rate$\left(E(\tilde{R}_e) - R_f\right)$ and the denominator is the expected risk measure of the relevant performance ratio. Denote the ratio of this expected portfolio risk measure to the expected equity risk premium by Coefficient of Risk (CR)[10]. Thus, maximization of the expected PPR can be expressed by the following proposition.

Proposition 3. The five PPRs examined in this paper that use the responsive threshold$T(\gamma, \alpha)$, can be maximized by minimizing the expected relative risk measure of the equity, CR.

The proof is rooted in the fact that the responsive threshold, $T(\gamma, \alpha)$, creates a separation between the composition of the risky portion of a portfolio and the extent to which the riskless asset is employed in the portfolio. Let us demonstrate the application of such an optimization process using, as an example, the Sortino ratio, $\mathrm{SOR}\left(\tilde{R}_P, T(\gamma, \alpha)\right)$. A similar optimization process may be applied to the other performance ratios presented here since they all satisfy the PRTI axiom when using the responsive threshold $T(\gamma, \alpha)$. Additional demonstration of the maximization of the PPRs by minimizing the relevant coefficient of risk, is provided in the proof of proposition 5 below, which focuses on the Reward-to-VaR ratio.

The modified Sortino ratio $\mathrm{SOR}\left(\tilde{R}_P, T(\gamma, \alpha)\right)$ is as follows:

(18) $$\mathrm{SOR}\left(\tilde{R}_P, T(\gamma, \alpha)\right) = \frac{E(\tilde{R}_P) - T(\gamma,\alpha)}{\{E\ [Max(T(\gamma,\alpha) - \tilde{R}_P, 0)]^2\}^{\frac{1}{2}}}.$$

This is the same as the following:

(19) $$\mathrm{SOR}(\tilde{R}_P, T(\gamma, \alpha)) = \frac{\alpha\left(E(\tilde{R}_e) - R_f\right) - \gamma\alpha\left(E(\tilde{R}_e) - R_f\right)}{\left\langle E\left\{\max\left[\gamma\alpha\left(E(\tilde{R}_e) - R_f\right) - \alpha\left(\tilde{R}_e - R_f\right), 0\right]\ \right\}^2\right\rangle^{\frac{1}{2}}} = $$
$$= \frac{(1-\gamma)\left[E(\tilde{R}_e) - R_f\right]}{\left\langle E\left\{\max\left[\gamma\left(E(\tilde{R}_e) - R_f\right) - \left(\tilde{R}_e - R_f\right), 0\right]\right\}^2\right\rangle^{\frac{1}{2}}}$$

We define the "risk premium ratio" of the risky component as the ratio of the (random) risky component's risk premium to its expected value, and denote it as $\widetilde{rpr_e}$:

(20) $$\widetilde{rpr_e} = \frac{\tilde{R}_e - R_f}{E(\tilde{R}_e) - R_f}$$

[10] We use this notation as a homage to the well-known "Coefficient of Variation" risk measure which is defined as a variable's standard deviation divided by its expected return.

Proposition 4. For a given γ, a performance ratio, including the Sortino ratio that uses the threshold $T(\gamma, \alpha)$, is maximized by minimizing the expected downside square deviations of the "risk premium ratio" from γ, namely, the following:

$$(21) \qquad \min_{\underset{\sim}{q}}\{E[\max(\gamma - \widetilde{rpr}_e, 0)]^2\}$$

where $\underset{\sim}{q}$ is the vector of the proportions invested in the individual risky securities.

The proof is based on Eq. (19) that can be rewritten (for the Sortino ratio example) as follows:

$$(22) \qquad \mathrm{SOR}(\tilde{R}_P, T(\gamma, \alpha)) = \frac{1-\gamma}{\{E[\max(\gamma - \widetilde{rpr}_e, 0)]^2\}^{\frac{1}{2}}}$$

As argued, the conventional Sortino ratio is not invariant with respect to α (except for $T = R_f$) and, therefore, its reward vs. downside risk frontier changes with α as well. In contrast, $\mathrm{SOR}(\tilde{R}_P, T(\gamma, \alpha))$ is invariant with respect to α; thus, one may apply Eq. (21) subject to any given expected return and obtain the minimum downside risk for each expected return, thereby creating the efficient mean-downside risky frontier of the risky portion of the portfolio for the chosen γ. Consequently, for any $T(\gamma, \alpha)$, one can use the minimization process of Eq. (21) to find the optimal composition of the risky component of the portfolio. The portfolio's optimal split between the risk-free asset and the optimal risky component is determined subjectively by the investor.

Figure 2 depicts the result of such an optimization process. It shows the trade-off between the expected return and the downside risk for a given positive γ. The portfolio's optimal equity-only component has an expected return of $E(\tilde{R}_e^*)$. This optimal portfolio is determined objectively and is applicable only for investors who select a specific γ. The overall optimal subjective combination of the risky asset component vs. the risk-free asset for the investor who selected the said γ has an expected rate of return $E(\tilde{R}_P^*)$ and a downside risk that is found at the tangency point between the investor's relevant indifference curve and the tangent line that runs from R_f toward the tangency point with the efficient risky frontier at point O.

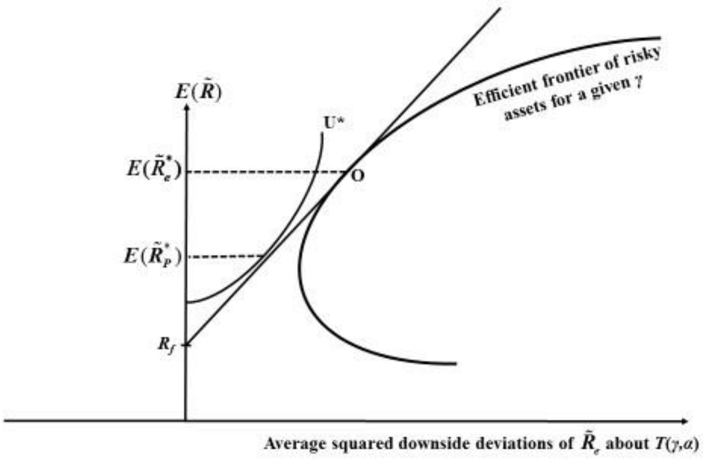
Proposition 5 below presents a second example of optimization process, applying it to the RVaR ratio which uses the responsive threshold$T(\gamma, \alpha)$.

Proposition 5. For a given γ, The $\mathrm{RVaR}(\tilde{R}_P, q, T(\gamma, \alpha))$ performance ratio, is maximized by minimizing the coefficient of risk in Eq. 23:

$$(23) \qquad \frac{R_f - Q_e(q)}{E(\tilde{R}_e) - R_f} = \frac{VaR_e(q)}{E(\tilde{R}_e) - R_f}$$

The numerator of Eq. 23 is again a measure of risk in terms of VaR of only the equity portion of the portfolio where the VaR is the difference between the riskless return and the q order quantile $Q_e(q)$ of the equity risky component. Namely, there is a probability of q that the equity component's return will be below R_f by more than $R_f - Q_e(q)$. Minimizing the equity Coefficient of Risk (i.e., expected equity risk over expected equity premium), leads to the maximization of the PPR. To prove this, note that in Appendix B we show that $\mathrm{RVaR}(\tilde{R}_P, T(\gamma, \alpha))$ can be written as:

Figure 2
The efficient risky asset frontier; the optimal expected rate of return of the portfolio's risky component, $E(\tilde{R}_e^*)$; and the optimal expected rate of return of the overall portfolio $E(\tilde{R}_p^*)$ for a chosen γ



(24) $\quad \mathrm{RVaR}\big(\tilde{R}_P, q, T(\gamma,\alpha)\big) \equiv \frac{E(\tilde{R}_P)-T(\gamma,\alpha)}{VaR(Q_P(q))} = \frac{(1-\gamma)\left(E(\tilde{R}_e)-R_f\right)}{\gamma\left(E(\tilde{R}_e)-R_f\right)-\left(Q_e(q)-R_f\right)}$

Eq. 23 is directly developed from the right hand side of Eq. 24. Using the expressions presented in Appendix B for the other PPRs, it is easy to show that minimizing the relevant coefficient of risk, leads to the maximization of the PPRs provided they apply our responsive threshold.

VII. CONSISTENCY OF OUR PORTFOLIO PERFORMANCE RATIOS THAT USE $T(\gamma,\alpha)$ WITH STOCHASTIC DOMINANCE WITH A RISKLESS ASSET (SDR) RULES.

To simplify the notation, let *X* and *Y* be the names and returns of two alternative portfolios with returns that formerly were denoted by $\tilde{R}_x$ and $\tilde{R}_Y$, respectively, whose performance are to be evaluated by applying PPRs. We denote the q order quantile of *X* and *Y* as *X*(*q*) and *Y*(*q*), respectively.[11] Let the preference of *X* over *Y* according to a performance ratio that employs a fixed threshold T be denoted as $X \underset{\Psi(T)}{D} Y$ and let the same preference according to a performance ratio that uses $T(\gamma,\alpha)$ as a threshold be denoted as $X \underset{\Psi(T(\gamma,\alpha))}{D} Y$. These preferences present a complete ordering that, a priori, may be inconsistent with SD rules. Thus, in general, a performance ratio's ordering may not be sufficient for dominance according to SD rules[12]. Levy and Kroll (1976) extended the SD rules to portfolios of risky assets that could be diversified with the risk-free asset and denoted them as SDR rules (i.e., Stochastic Dominance with Riskless asset rules). The First and Second degree SDR rules are denoted as FSDR and

[11] The quantile function is the inverse of the cumulative distribution function (CDF) and the q order quantile $X(q)$ of a random variate $\tilde{X}$ satisfies the following probability condition: $Pr\big(\tilde{X} \le X(q)\big) = q$.

[12] With respect to SOR(*T*), Balder and Schweizer 2017 showed that if $X \underset{SSD}{D} Y$ and $E(X) \ge T \ge E(Y)$, then $X \underset{\mathrm{SOR}(T)}{D} Y$.

SSDR rules, respectively. They proved that if a portfolio consists of proportions α and (1-α) invested in X and in the riskless asset, respectively, such that this portfolio dominates Yaccording to FSD or SSD, then for any combination of Y with the riskless asset, there is at least one other combination of X with the riskless asset that dominates it according to the relevant FSD or SSD rules, respectively.

It should be noted that the partial ordering according to SDR rules is potentially more effective than the ordering according to SD rules. For example, assume that X and Y are uniformly distributed returns such that: $X \sim U(0.0{,}0.2)$ and$Y \sim U(0.05{,}0.10)$. In this example, there is no FSD or SSD dominance relationships between X and Y. The expected return of X is greater than that of Y (0.10 >0.075); hence, Y clearly does not dominate X. In addition, X's lowest outcome is smaller than that of Y's (0.00 < 0.05) and thus X does not dominate Y. However, if each of the risky assets could be diversified with a risk-free asset whose return is 7.2%, then, for example, a portfolio of 30% X and 70% R_f is also distributed uniformly, $X_{\alpha=30\%} \sim U(0.0504{,}0.1104)$, and it dominates Y according to FSD. Likewise, according to SDR rules, for any combination of Y and R_f, one can find *at least* one combination of X with R_f that dominates it.

This example shows that by considering the diversification opportunities of the risky and risk-free alternatives, a lack of dominance according to the FSD or SSD rules between two distributions may nevertheless exhibit a dominance relationship according to the FSDR or SSDR rules, respectively.

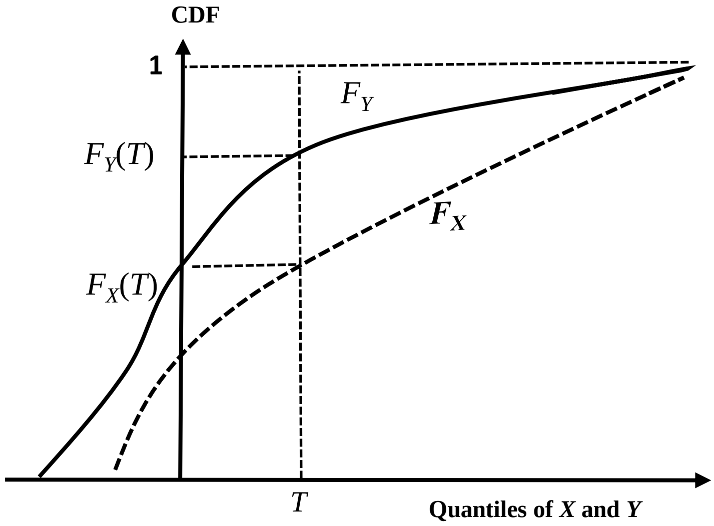
Note that dominance by SD rules (FSD, SSD, FSDR and FSDR rules) guarantees that all expected utility investors who fulfil the appropriate general utility assumptions (U'>0 or also U"<0 for risk averters), will select the dominating alternative. A more effective SD rule tends to generate more cases of dominance out of the feasible set of alternatives. Thus if dominance by the SD rules implies also dominance by the PPR rule there will be also less conflicts between the PPR rule and the SD rules as conflicts can be accrue only between the PPR and SD dominance for the smaller group of alternatives which belong to the efficient set.

Proposition 6. If $X \underset{FSD}{D} Y$ and there are no short sales of either X or Y, then $X \underset{\Psi(T)}{D} Y$ for every T and $X \underset{\Psi(T(\gamma,\alpha))}{D} Y$ for every $\gamma < 1$, where the set ψ includes the Kappa ratios of all degrees (and the Sortino ratio as a special case), the Omega ratios of all degrees, RVaR and RCVaR[13].

Proof: The proof is almost immediate, as is evident from Figure 3 that graphically depicts a case where $X \underset{FSD}{D} Y$. Such dominance implies that for each quantile of order $0 \leq q \leq 1$, $X(q) \geq Y(q)$. Thus, for each constant T or $T(\gamma,\alpha)$, we have $T - X(q) \leq T - Y(q)$. We denote $Pr\big(\tilde{X} \leq X(q) = T\big)$ and $Pr\big(\tilde{Y} \leq Y(q) = T\big)$ as the q order probabilities that lead to the T value under X and Y, respectively (see Figure 3). Due to the FSD assumption, $Pr\big(\tilde{X} \leq T\big) \leq Pr\big(\tilde{Y} \leq T\big)$; hence, the lower partial moment of any degree n under X with respect to T , is smaller than the respective lower partial moment of degree n under Y.

[13] Note that propositions 4 through 7 hold for the Kappa ratio of all $n \geq 1$ degrees. Recall that the Omega ratio of the first degree is identical to 1 plus the Kappa ratio of the first degree and the Sortino ratio is identical to the Kappa ratio of the second degree.

Figure 3
Presentation of the FSD of *X* and *Y*



In general, dominance according to FSD is relatively scarce among competing return distributions in competitive markets. Thus, it is expected that there will be many cases of conflicts between the complete rankings of the ratios that are included in ψ and many distributions that belong to the "efficient set" according to the FSD rule. These cases of conflicts can be reduced by including the opportunity to diversify the portfolio's risky component with the risk-free asset[14] and by using higher moment stochastic dominance rules such as SSD and TSD.

In the following three propositions, we extend proposition 6 to allow the opportunity to diversify the portfolio with a risk-free asset and by employing FSDR, SSD and SSDR rules.

Proposition 7. If $X \underset{FSDR}{D} Y$, then $X \underset{\Psi(T(\gamma,\alpha))}{D} Y$ for all $\gamma < 1$, where the set ψ includes the Kappa ratios of all degrees (and the Sortino ratio as a special case), the Omega ratios of all degrees, the RVaR and the RCVaR.

Proof. If there is FSDR of X over Y, then there is a combination of X and the risk-free asset that dominates a given combination of Y with the risk-free asset; thus, we are back in a situation that is presented in Proposition 6. FSDR guarantees that for any other combination of Y with the risk-free asset, there is *at least* one other dominating combination of X with the risk-free asset, and the conditions of Proposition 6 hold again.

Proposition 8. If $X \underset{SSD}{D} Y$ and there are no short sales of either X or Y, then $X \underset{\Psi(T)}{D} Y$ for every T and $X \underset{\Psi(T(\gamma,\alpha))}{D} Y$ for every $\gamma < 1$, where the set ψ includes the Kappa ratios of all degrees (and the Sortino ratio as a special case), the Omega ratios of all degrees and the RCVaR but not the RVaR.

[14] Additionally, the diversification of the risky assets with other risky assets can reduce the number of conflicts, but the analysis of these diversification possibilities is related more to the determination of the parametric optimal portfolio.

Proof: For every quantile of order $0 \leq \hat{q} \leq 1$, the relationship $X \underset{SSD}{D} Y$is equivalent to the following:

(25) $\int_0^{\hat{q}} X(q)\, dq \;\geq\; \int_0^{\hat{q}} Y(q)\, dq$ for all $0 \leq \hat{q} \leq 1$

Thus, for every $\hat{q}$ and *T*, we have the following:

(26) $\int_0^{\hat{q}} [T - X(q)]dq \leq \int_0^{\hat{q}} [T - Y(q)]dq$

Eq. (26) holds since the variables $T - X(q)$ and $T - Y(q)$ are integrated only over their respective positive domains, and the SSD of *X* over *Y* implies that the integral from zero to *q* under *X*(*q*) is greater (in the weak sense) than the respective integral under *Y*(*q*). Thus, the q_X^* and q_Y^* for which $X(q^*) = T$ and $Y(q^*) = T$ must satisfy the relationship $q_X^* \leq q_y^*$. It follows that the average lower partial moment of degree *n* of X with respect to *T* for all $X \leq T$, is smaller than the respective average lower partial moment of degree *n* of Y with respect to *T*:

(27) $\left\{\int_0^{\hat{q}_X^*} [T - X(q)]^n dq\right\}^{\frac{1}{n}} \leq \left\{\int_0^{\hat{q}_Y^*} [T - Y(q)]^n dq\right\}^{\frac{1}{n}}$

Note that Proposition 8 does not hold for the RVaR performance ratio since the VaR measures the value of the CDF at *T* without integrating the CDF below *T*. It is therefore possible that at a specific probability level q^*, $X(q^*) < Y(q^*)$ so that at that point $VaR_X(q^*) > VaR_Y(q^*)$, even though X dominates Y according to SSD, and Eqs. (25) and (26) hold. Additionally, note that the non-coherence of the RVaR performance ratio with respect to the SSD rule may or may not cause conflicting SSD and RVaR ranking. For example, if X dominates Y according to SSD, then necessarily $E(X) \geq E(Y)$. However, a conflict between SSD dominance and RVaR ranking is possible only if the following holds:

(28) $\frac{E(X)-T}{VaR_X(q*)} < \frac{E(Y)-T}{VaR_Y(q*)}$

Namely, the ratio of the expected risk premium of X over its appropriate VaR is lower than the respective ratio under Y. The incoherency of RVaR as a performance ratio is analogous to the incoherency of VaR as a risk measure.

Proposition 9. If $X \underset{SSDR}{D} Y$ and there are no short sales of either *X* or *Y*, then $X \underset{\psi(T(\gamma,\alpha))}{D} Y$ for every $\gamma < 1$, where the set ψ includes the Kappa ratios of all degrees (and the Sortino ratio as a special case), the Omega ratios of all degrees, and the RCVaR but not the RVaR.

Proof. If there is SSDR of *X* over *Y*, then there is a combination of *X* and the risk-free asset that dominates a given combination of Y with the risk-free asset, and thus we are back in the situation that is presented in Proposition 8. It is guaranteed that for any other combination of *Y* with the risk-free asset, there is *at least* one other dominating combination of *X* with the risk-free asset, and the conditions of Proposition 8 hold again.

Table 1 below summarizes the relationships between the five PPRs examined in this paper and our coherence axioms.

Table 1

Analysis of five common Portfolio Performance Ratios (PPRs) given a set of axioms that renders coherence

("Yes" indicates that the axiom is satisfied; "No" indicates that the axiom is not satisfied)

Panel A: Portfolio Performance Ratios with a fixed threshold, T

	Reward-to-Standard Deviation	**Kappa Ratio of the nth degree ($n \geq 1$)**	**Omega Ratio of the nth degree ($n \geq 1$)**	**Reward-to-VaR**	**Reward-to-CVaR**
Monotonicity with respect to FSD and SSD*	No	Yes	Yes	No	Yes.
Monotonicity with respect to FSDR and SSDR*	No	No, if $T \neq R_f$	No, if $T \neq R_f$	No	No, if $T \neq R_f$
Size monotonicity	Yes	Yes	Yes	Yes	Yes
PRTI	No, except if $T = R_f$	No, except if $T = R_f$	No, except if $T = R_f$	No, except if$T = R_f$	No, except if $T = R_f$
Concavity	Yes	Yes	Yes, if n = 1	Yes	Yes

Panel B: Portfolio Performance Ratio with a responsive threshold, $T(\gamma, \alpha)$

	Reward-to-Standard Deviation	**Kappa Ratio of the nth degree ($n \geq 1$)**	**Omega Ratio of the nth degree ($n \geq 1$)**	**Reward-to-VaR**	**Reward-to-CVaR**
Monotonicity with respect to FSD and SSD*	No	Yes	Yes	No	Yes
Monotonicity with respect to FSDR and SSDR*	No	Yes	Yes	No	Yes
Size monotonicity	Yes	Yes	Yes	Yes	Yes
PRTI	Yes	Yes	Yes	Yes	Yes
Concavity	Yes	Yes	Yes, if n = 1	Yes	Yes

* Monotonicity of a PPR with respect to a given SD rule means that if $\tilde{R}_x$ dominates $\tilde{R}_y$ by that SD rule, the PPR of $\tilde{R}_x$ is surely not less than the PPR of $\tilde{R}_y$.

Table 1 exposes the two main contributions of our portfolio performance ratios with adjustable thresholds. First, the PRTI axiom is satisfied by all the five PPRs examined, when using the responsive threshold $T(\gamma, \alpha)$. This is not so when the ratios employ a fixed threshold, $T \neq R_f$. This change is due to the fact that the responsive threshold restores the separation between the decision concerning the composition of the portfolio's risky component and the extent to which the riskless asset is used. The second main benefit is apparent with respect to the *monotonicity* axiom when riskless borrowing and lending is available. Two ratios do not satisfy the *monotonicity* axiom with respect to FSDR and SSDR for any fixed threshold T, and three ratios (Kappa, Omega and RCVaR) satisfy the axiom only for $T = R_f$. On the other hand, these latter ratios satisfy the *monotonicity* axiom with respect to FSDR and SSDR, given that the ratios employ the responsive threshold $T(\gamma, \alpha)$.

VIII. SUMMARY AND CONCLUSIONS

This paper suggests four axioms for coherent PPRs and examines whether some well-known PPRs are coherent with respect to these axioms. In addition, the paper proposes the use of a responsive threshold, $T(\gamma, \alpha)$, which equals γ times the expected return of the portfolio plus (1-γ) times the risk-free rate. As shown, the responsive threshold for the portfolio ensures that the threshold for the equity (risky) component of the portfolio remains unchanged when the portfolio's equity level changes. The use of the responsive threshold ensures the coherence of most of the PPRs that are analyzed in this paper reflecting the fact that, unlike the fixed threshold, the responsive threshold maintains the desired premise of separation between the objective optimal composition of the portfolio's risky component and the subjective optimal portfolio's equity level.

The *Portfolio Riskless Translation Invariance* (*PRTI*) axiom, which is one of our four axioms, requires that the degree to which the risk-free asset is used in a portfolio will not have an effect on the performance measure. We prove theoretically and demonstrate empirically that common performance ratios that employ fixed thresholds other than the risk-free rate, do not satisfy this necessary axiom. The severity of the above PRTI incoherence is demonstrated by examining the performance ratios of portfolios of treasury bills and the S&P-500 index returns that span over 240 months, from 02.2000 to 01.2020. Four different ratios, that often use fixed thresholds other than the risk-free rate, are examined: the Kappa ratios of all degrees (including the Sortino ratio as a special case), the Omega ratios of all degrees, the Reward-to-VaR and the Reward-to-CVaR ratios. If the fixed threshold T is set to be higher (lower) than the risk-free rate, all the examined ratios increase (decrease) substantially with the portfolio's equity level. This undesirable shortcoming enables funds managers to increase the portfolio performance ratio by simply changing the portfolio's degree of leverage rather than by better selection of the risky assets. Thus, we suggest and rationalize a remedy that renders the above performance measures coherence by replacing the fixed portfolio's threshold with a threshold that responds to the portfolio's equity level. We recall that the required informational content is the same as for determining the traditional performance ratio with fixed threshold T, except for the subjective loss aversion γ in place of the assumption on T.

It should be noted that using the risk-free rate as a fixed threshold, is a specific case, in which $T(\gamma = 0, \alpha) = R_f$. Given that the expected ex-ante return of a risky portfolio is higher than the risk-free return, the threshold $T(\gamma, \alpha)$ which reflects the investor's sensitivity to loss, increases with γ.

In contrast to conventional ratios, our modified performance ratios are invariant with respect to the portfolio's equity level, α, and depend only on the selected subjective "loss benchmark" γ. Hence, they satisfy the PRTI axiom and thus cannot be changed by merely changing the portfolio's weight of the risk-free asset.

Stochastic dominance with the riskless asset rules (SDR) are more effective (i.e., generally generate smaller efficient sets) relatively to SD rules that do not consider diversification between the risky and the risk-free assets. With the exception of RVaR(γ), dominance by SDR rules implies dominance by PPRs that use $T(\gamma, \alpha)$, but not necessarily by PPRs that use fixed thresholds. Thus, conflicts between the preferences of expected utility investors and our PPRs that employ the responsive thresholds can occur only among return distributions belonging to the smaller SDR efficient set.

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APPENDIX A

THE COHERENCE AXIOMS AND SOME COMMON PORTFOLIO PERFORMANCE RATIOS WITH FIXED THRESHOLDS

In this appendix, we examine the coherence of five common portfolio performance ratios with respect to our axiomatic foundation. We show that they do not satisfy the PRTI axiom except for the case where $T = R_f$. They do satisfy the size *monotonicity* and *concavity* axioms. Monotonicity with respect to FSD, SSD, FDSR and SSDR was analyzed in the text. The ratios examined are the following:

1. Reward to standard deviation ratios,
2. The Kappa family ratios,
3. The Omega family ratios,
4. The Reward-to-VaR ratio, and
5. The Reward-to-CVaR ratio.

Part 1: Reward to standard deviation ratios

This part of the appendix examines whether ratios that use the StD as its risk measure satisfy our PRTI axiom. We denote the standard deviation of the portfolio's rate of return as $StD(\tilde{R}_P)$ and the standard deviation of the risky component's rate of return as $StD(\tilde{R}_e)$. A general expression for a Sharpe-like ratio, denoted $S(\tilde{R}_P, T)$, that uses a fixed threshold $T = R_f + \Delta$ is as follows:

(A1) $$S(\tilde{R}_P, T) = \frac{E(\tilde{R}_P) - T}{StD(\tilde{R}_P)} = \frac{E(\tilde{R}_P) - (R_f + \Delta)}{StD(\tilde{R}_P)} =$$
$$= \frac{\left[\alpha E(\tilde{R}_e) + (1-\alpha)R_f\right] - (R_f + \Delta)}{\alpha StD(\tilde{R}_e)} =$$
$$= \frac{\alpha\left[E(\tilde{R}_e) - R_f\right] - \Delta}{\alpha StD(\tilde{R}_e)} = \frac{E(\tilde{R}_e) - R_f - \dfrac{\Delta}{\alpha}}{StD(\tilde{R}_e)} =$$
$$= \frac{E(\tilde{R}_e) - R_f}{StD(\tilde{R}_e)} - \frac{\Delta}{\alpha StD(\tilde{R}_e)}$$

Equation (A1) shows that if $\Delta = 0 \Rightarrow T = R_f$, the ratio $S(\tilde{R}_P, T)$ is the familiar Sharpe ratio, $SR(\tilde{R}_e)$, and it is invariant with respect to α. However, if $\Delta \neq 0 \ \Rightarrow T \neq R_f$, the ratio is not invariant with respect to changes in α and therefore it does not satisfy the PRTI axiom.
For $\alpha > 0$, and a positive Δ, the Sharpe ratio increases with α, and the opposite holds for a negative Δ.

Size monotonicity is satisfied by the Sharpe-like ratios as is easily verified by Eq. (A2):

(A2) $S\big(\lambda(1+\tilde{R}_P),T\big) = \frac{E(\lambda(1+\tilde{R}_P))-\lambda(1+T)}{StD(\lambda\tilde{R}_P)} = \frac{E(1+\tilde{R}_P)-(1+T)}{StD(\tilde{R}_P)} = \frac{E(\tilde{R}_P)-T}{StD(\tilde{R}_P)} = S(\tilde{R}_P,T)$

Finally, *concavity* is satisfied since the numerator is a weighted average of the respective numerators of the ratios of the individual securities while the denominator is less than (or equal to) the weighted average of the standard deviations of the individual securities due to diversification.

Part 2: The Kappa family ratios

This part of the appendix examines whether the Kappa ratios satisfy the PRTI axiom, the *size monotonicity* and the *concavity* axioms, starting with the PRTI axiom.

Recall that the Kappa ratio of the first degree relates to the Omega ratio of the first degree so that $\Omega_1(\tilde{R}_P,T) = K_1(\tilde{R}_P,T)+1$, and the Kappa ratio of the second degree is identical to the Sortino ratio. It is reasonable to assume $T = R_f + \Delta < E(\tilde{R}_P)$ since otherwise the expected reward to risk is negative. The n^{th} degree Kappa ratio is given by the following:

(A3) $$K_n(\tilde{R}_P,T) = \frac{E(\tilde{R}_P)-T}{\{E[\max(T-\tilde{R}_P,0)]^n\}^{\frac{1}{n}}} = \\ = \frac{E[\alpha\tilde{R}_e + (1-\alpha)R_f] - (R_f+\Delta)}{\left\langle E\big\{\max\big[(R_f+\Delta) - \big(\alpha\tilde{R}_e + (1-\alpha)R_f\big),0\big]\big\}^n\right\rangle^{\frac{1}{n}}}$$

For $\alpha > 0$, we may write the following:

(A4) $$K_n(\tilde{R}_P,T) = \frac{E(\tilde{R}_e - R_f - \frac{\Delta}{\alpha})}{\left\{E\left[\max\left(\frac{\Delta}{\alpha}-\tilde{R}_e+R_f,0\right)\right]^n\right\}^{\frac{1}{n}}}$$

Denoting $\tilde{u}_\alpha = \tilde{R}_e - R_f - \frac{\Delta}{\alpha}$, we rewrite the ratio as follows:

(A5) $$K_n(\tilde{R}_P,T) = \frac{E(\tilde{u}_\alpha)}{\{E[\max(-\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}}}$$

Since $E\big(\tilde{R}_p\big) > T$, for $\alpha > 0$, we get $E(\tilde{u}_\alpha) > 0$:

(A6) $$E(\tilde{R}_P) > T \quad \Rightarrow \alpha E(\tilde{R}_e) + (1-\alpha)R_f > R_f + \Delta \quad \Rightarrow \\ \Rightarrow \alpha\big(E(\tilde{R}_e) - R_f\big) - \Delta > 0 \\ \Rightarrow E(\tilde{R}_e) - R_f - \frac{\Delta}{\alpha} > 0 \quad \Rightarrow E(\tilde{u}_\alpha) > 0$$

In addition, we note that $\frac{\partial E(\tilde{u}_\alpha)}{\partial\alpha} = \frac{\partial\tilde{u}_\alpha}{\partial\alpha} = \frac{\Delta}{\alpha^2}$. For $\Delta = 0$, we have $T = R_f$, and from Eq. (A4), the ratio is invariant with respect to α:

(A7) $$K_n(\tilde{R}_P,T) = \frac{E(\tilde{R}_e - R_f)}{\left\{E\left[\max\left(R_f - \tilde{R}_e,0\right)\right]^n\right\}^{\frac{1}{n}}} \quad \Rightarrow \frac{\partial K_n(\tilde{R}_P,T)}{\partial\alpha} = 0$$

For $\Delta \neq 0$, from Eq. (A5), we obtain the following:

(A8) $\frac{\partial K_n(\tilde{R}_P,T)}{\partial \alpha} =$

$$= \frac{\frac{\Delta}{\alpha^2}\left\{\{E[\max(-\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}} + E(\tilde{u}_\alpha) \times \left(\frac{1}{n}\right) \times \{E[\max(-\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}-1} \quad \times n \times E[\max(-\tilde{u}_\alpha,0)]^{n-1}\right\}}{\left\{\{E[\max(-\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}}\right\}^2}$$

The denominator of the derivative is clearly positive (in the weak sense – here and in what follows). The expression in the numerator's curly brackets is positive as well since $E(u_\alpha) > 0$, as noted above. It follows that the sign of the derivative, $\frac{\partial K_n(\tilde{R}_P,T)}{\partial \alpha}$, is determined by the sign of Δ. A positive Δ indicates a threshold higher than the risk-free rate, in which case the derivative is positive; meanwhile, a negative Δ indicates a threshold lower than the risk-free rate, in which case the derivative is negative. It follows that the Kappa ratios do not satisfy the PRTI axiom for all $T \neq R_f$.

The *size monotonicity* axiom is satisfied by the Kappa ratio, since it is expressed in terms of rates of return, and indeed, $K_n(\lambda\tilde{R}_P,T) = \frac{E(\lambda\tilde{R}_P)-\lambda T}{[E[\max(\lambda T-\lambda\tilde{R}_P,0]^n]^{\frac{1}{n}}} = K_n(\tilde{R}_P,T)$.

The *concavity* axiom is satisfied by the Kappa ratio since the ratio's numerator is a weighted average of the individual securities in the portfolios while the denominator is less than (or equal to) the weighted average of the downside deviations of the individual securities due to diversification.

Part 3: The Omega family ratios

We first analyze the compliance of the *n*-th degree Omega ratio as in Eq. (A9), with the PRTI axiom:

(A9) $\Omega_n(\tilde{R}_P,T) = \frac{\{E[\max(\tilde{R}_P-T,0)]^n\}^{\frac{1}{n}}}{\{E[\max(T-\tilde{R}_P,0)]^n\}^{\frac{1}{n}}}$

As T increases, the Omega ratio's numerator decreases (in the weak sense, here and in what follows) and its denominator increases. Therefore, clearly, the ratio decreases. Hence, if $T > R_f$, then $\Omega_n(\tilde{R}_P,T) < \Omega_n(\tilde{R}_P,R_f)$; and if $T < R_f$, then $\Omega_n(\tilde{R}_P,T) > \Omega_n(\tilde{R}_P,R_f)$.

To determine if the Omega ratio increases or decreases with α, we take the ratio's first derivative with respect to α. We first rewrite the ratio as follows:

$$\text{(A10)} \quad \Omega_P^n(T) = \frac{\{E[\max(\tilde{R}_P-T,0)]^n\}^{\frac{1}{n}}}{\{E[\max(T-\tilde{R}_P,0)]^n\}^{\frac{1}{n}}} = \frac{\left\{E\left[\max\left(\alpha\tilde{R}_e+(1-\alpha)R_f-T,0\right)\right]^n\right\}^{\frac{1}{n}}}{\left\{E\left[\max\left(T-\alpha\tilde{R}_e-(1-\alpha)R_f,0\right)\right]^n\right\}^{\frac{1}{n}}}$$

For $\alpha > 0$ and using $T = R_f + \Delta$, we may write the following:

(A11) $$\Omega_n(\tilde{R}_P,T) = \frac{\left\{E\left[\max\left(\tilde{R}_e - R_f - \frac{\Delta}{\alpha},0\right)\right]^n\right\}^{\frac{1}{n}}}{\left\{E\left[\max\left(\frac{\Delta}{\alpha} - \tilde{R}_e + R_f,0\right)\right]^n\right\}^{\frac{1}{n}}}$$

Using $\tilde{u}_\alpha = \tilde{R}_e - R_f - \frac{\Delta}{\alpha}$ again, we rewrite the ratio in a more compact way as follows:

(A12) $$\Omega_n(\tilde{R}_P,T) = \frac{\{E[\max(\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}}}{\{E[\max(-\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}}}$$

For $\Delta = 0$, we have $T = R_f$; and from Eq. (A11), the ratio is invariant with respect to α:

(A13) $$\Omega_n(\tilde{R}_P,T) = \frac{\left\{E\left[\max\left(\tilde{R}_e - R_f,0\right)\right]^n\right\}^{\frac{1}{n}}}{\left\{E\left[\max\left(R_f - \tilde{R}_e,0\right)\right]^n\right\}^{\frac{1}{n}}} \quad \Rightarrow \frac{\partial \Omega_n(\tilde{R}_P,T)}{\partial \alpha} = 0$$

We now take the derivative of the Omega ratio with respect to αassuming $\Delta \neq 0$. From Eq. (A12), we obtain the following:

(A14) $$\frac{\partial \Omega_n(\tilde{R}_P,T)}{\partial \alpha} = \frac{\left\{\left(\frac{1}{n}\right)\left[E\left(\max(\tilde{u}_\alpha,0)\right)^n\right]^{\frac{1}{n}-1}(n)[E(\max(\tilde{u}_\alpha,0))^{n-1}]\left(\frac{\Delta}{\alpha^2}\right)\right\}\left\{E[(\max(-\tilde{u}_\alpha,0))^n]^{\frac{1}{n}}\right\}}{\left\{[E(Max(-\tilde{u}_\alpha,0))^n]^{\frac{1}{n}}\right\}^2}$$
$$-\frac{\left\{\left(\frac{1}{n}\right)\left\{E[(\max(-\tilde{u}_\alpha,0))^n]^{\frac{1}{n}-1}\right\} \ (n)[E(\max(-\tilde{u}_\alpha,0))]^{n-1}\left(-\frac{\Delta}{\alpha^2}\right)\right\}\{E[\max(\tilde{u}_\alpha,0)]^n\}^{\frac{1}{n}}}{\left\{[E(\max(-\tilde{u}_\alpha,0))^n]^{\frac{1}{n}}\right\}^2}$$

Clearly, the sign of the derivative is determined by the sign of Δ. For $\Delta > 0$, the sign of the derivative is positive; and for $\Delta < 0$, the sign is negative.

The *size monotonicity* axiom is satisfied by the Omega family ratios, since from eq. (A9), it follows that the ratio is unaffected by the size in the investment.

The *concavity* axiom is satisfied by the Omega ratio provided *n* = 1 since the ratio's numerator is a weighted average of the individual securities in the portfolios while the denominator is less than (or equal to) the weighted average weighted average of the downside deviations of the individual securities due to diversification.

Part 4: The Reward-to-VaR ratio

This part of the appendix shows that the Reward-to-VaR ratio that uses a fixed threshold does not satisfy our PRTI axiom. However, the *size monotonicity* and the *concavity* axioms are satisfied by the ratio. As noted in Eq. (11), a portfolio's value-at-risk is given by the following: $VaR(\tilde{R}_P,q,T) = T - Q_P(q)$, where $Q_P(q)$ is the portfolio's q quantile such that the probability that the portfolio's return will be less than or equal to $Q_P(q)$ is equal to q. It follows that

(A15) $\Pr[\tilde{R}_P \leq Q_P(q)] = q$

and the value-at-risk of an all-equity portfolio is as follows:

(A16) $\Pr[\tilde{R}_e \leq Q_e(q)] = q$.

Since $\tilde{R}_P = \alpha\tilde{R}_e + (1-\alpha)R_f$ when $\alpha > 0$ is a positively monotone transformation of $\tilde{R}_e$, the portfolio's q quantile corresponds to the risky component's q quantile such that

(A17) $Q_P(q) = \alpha Q_e(q) + (1-\alpha)R_f$.

From here and Eq. (10), we obtain that

(A18) $$\mathrm{RVaR}(\tilde{R}_P, q, T) = \frac{E(\tilde{R}_P)-T}{VaR(\tilde{R}_P,q,T)} =$$
$$= \frac{E(\tilde{R}_P)-(R_f+\Delta)}{(R_f+\Delta)-Q_P(q)} = \frac{\left[\alpha E(\tilde{R}_e)+(1-\alpha)R_f\right]-(R_f+\Delta)}{(R_f+\Delta)-\left[\alpha Q_e(q)+(1-\alpha)R_f\right]} =$$
$$= \frac{\alpha\left[E(\tilde{R}_e)-R_f\right]-\Delta}{\Delta-\alpha\left(Q_e(q)-R_f\right)} = \frac{E(\tilde{R}_e)-R_f-\dfrac{\Delta}{\alpha}}{\dfrac{\Delta}{\alpha}+R_f-Q_e(q)} \Rightarrow$$
$$\Rightarrow \begin{Bmatrix} if\Delta > 0 & \mathrm{RVaR}(\tilde{R}_P,q,T) < \mathrm{RVaR}(\tilde{R}_e,q,T) \\ if\Delta < 0 & \mathrm{RVaR}(\tilde{R}_P,q,T) > \mathrm{RVaR}(\tilde{R}_e,q,T) \end{Bmatrix}$$

Note that the denominator of (A18) is positive since $Q_e(q)$ is typically negative and clearly $Q_e(q) - R_f$ is more negative.

Equation (A18) shows that if $\Delta = 0$, the Reward-to-VAR ratio is invariant with respect to changes in α, in which case it satisfies the PRTI axiom. However, when $\Delta > 0$, the term Δ/α decreases as α increases; therefore, the numerator of $\mathrm{RVaR}(\tilde{R}_P, q, T)$ increases and its denominator decreases so that $\mathrm{RVaR}(\tilde{R}_P, q, T)$ increases as α increases. Conversely, when $\Delta < 0$, the term Δ/α increases (i.e., becomes less negative) as α increases; therefore, the numerator of $\mathrm{RVaR}(\tilde{R}_P, q, T)$ decreases and its denominator increases so that $\mathrm{VaR}\left(\tilde{R}_p, q, T\right)$ decreases as α increases. Hence, if $\Delta \neq 0$, the ratio is not invariant with respect to changes in $\alpha$and therefore it does not satisfy the PRTI axiom.

The *size monotonicity* axiom is satisfied by the RVaR ratio, since it is expressed in terms of return and is unaffected by the size in the investment.

The *concavity* axiom is satisfied by the RVaR ratio since the numerator of the ratio is a weighted average of the similar numerators for the individual securities in the portfolios while the denominator is equal to the weighted average of the similar denominators of the individual securities.

Part 5: The Reward-to-CVaR ratio

The $\mathrm{CVaR}(\tilde{R}_P, T)$ is the cumulative difference between T and the quantiles below it. Namely, the CVaR measure the average downside loss that has a probability of p.

We now show that the Reward-to-CVaR ratio with a fixed threshold does not satisfy the PRTI axiom when $T \neq R_F$.

A(19) $\text{RCVaR}(\tilde{R}_P, T) = \frac{E(\tilde{R}_P) - T}{CVaR(\tilde{R}_P, T)} =$

$$= \frac{E(\tilde{R}_P) - (R_f + \Delta)}{E[\max(R_f + \Delta - Q_P(q), 0)]} = \frac{\left[\alpha E(\tilde{R}_e) + (1-\alpha)R_f\right] - (R_f + \Delta)}{E[\max(R_f + \Delta - \alpha Q_e(q) - (1-\alpha)R_f, 0)]} =$$

$$= \frac{E(R_e) - R_f - \frac{\Delta}{\alpha}}{E[\max(\frac{\Delta}{\alpha} + R_f - Q_e(q), 0)]} =$$

Equation (A19) shows that only in the case where $\Delta = 0$ do we get the result that the ratio is invariant with respect to α, in which case it satisfies the PRTI axiom. If $\Delta \neq 0$, the ratio is not invariant with respect to α and therefore the PRTI axiom is not satisfied. Specifically, assuming $\alpha > 0$, if $T > R_f \Rightarrow \Delta > 0$, then $\text{RCVaR}(\tilde{R}_p, T)$ is smaller than $\text{RCVaR}(\tilde{R}_e, T)$. If $T < R_f \Rightarrow \Delta < 0$, then $\text{RCVaR}(\tilde{R}_p, T)$ is higher than $\text{RCVaR}(\tilde{R}_e, T)$.

The *size monotonicity* axiom is satisfied by the RCVaR ratio, since it is expressed in terms of of return and is unaffected by the size in the investment.

The *concavity* axiom is satisfied by the RCVaR ratio since the numerator of the ratio is a weighted average of the similar numerators for the individual securities in the portfolios while the denominator is equal to the weighted average of the similar denominators of the individual securities.

APPENDIX B

THE COHERENCE OF SOME COMMON PORTFOLIO PERFORMANCE RATIOS WITH A RESPONSIVE THRESHOLD

In this appendix, we show that the portfolio performance ratios that use our responsive threshold satisfy our PRTI axiom. The ratios examined are the following:

1. Reward to standard deviation ratios,
2. The Kappa family ratios,
3. The Omega family ratios,
4. The Reward-to-VaR ratio,
5. The Reward-to-CVaR ratio.

Part 1: Reward to standard deviation ratio

This part of the appendix proves that Sharpe-like ratios that use the STD for a risk measure in conjunction with our responsive threshold, $T(\gamma,\alpha)$, as defined in Eqs. (13) and (14), satisfy our PRTI axiom. To see this, note that the Sharpe ratio of an all-equity portfolio with $\gamma = 0$ is given by the following:

(B1) $$\mathrm{SR}\big(\tilde{R}_e, T(\gamma = 0, \alpha = 1)\big) = \frac{E(\tilde{R}_e) - R_f}{StD(\tilde{R}_e)}$$

In the general case where $\gamma \neq 0$ and/or $\alpha \neq 1$, from Eq. (14), the Sharpe-like ratio is as follows:

(B2) $$\begin{aligned} S\big(\tilde{R}_P, T(\gamma,\alpha)\big) &\equiv \frac{E(\tilde{R}_P) - T(\gamma,\alpha)}{StD(\tilde{R}_p)} = \\ &= \frac{\{\alpha E(\tilde{R}_e) + (1-\alpha)R_f\} - \{\gamma\alpha[E(\tilde{R}_e) - R_f] + R_f\}}{\alpha StD(\tilde{R}_e)} = \\ &= \frac{\alpha[E(\tilde{R}_e) - R_f] - \gamma\alpha[E(\tilde{R}_e) - R_f]}{\alpha StD(\tilde{R}_e)} = \frac{(1-\gamma)\big(E(\tilde{R}_e) - R_f\big)}{StD(\tilde{R}_e)} = \\ &= (1-\gamma)\mathrm{SR}\big(\tilde{R}_e, T(\gamma = 0, \alpha = 1)\big) \end{aligned}$$

This relationship demonstrates that for any chosen $0 \leq \gamma < 1$, the Sharpe-like ratio is invariant with respect to changes in α and thus it satisfies the PRTI axiom.

Part 2: The Kappa family ratios with $\boldsymbol{T(\gamma,\alpha)}$

We now examine the Kappa ratios where the threshold is equal to $T(\gamma,\alpha)$ of Equation (13) and demonstrate that the PRTI axiom is satisfied.

(B3) $$K_n\big(\tilde{R}_P, T(\gamma,\alpha)\big) = \frac{E(\tilde{R}_P) - T(\gamma,\alpha)}{\{E[\max(T(\gamma,\alpha) - \tilde{R}_P, 0)]^n\}^{\frac{1}{n}}}$$

Eq. (B3) can be specified as follows:

(B4) $K_n\big(\tilde{R}_P, T(\gamma,\alpha)\big) = \dfrac{\alpha\big(E(\tilde{R}_e)-R_f\big)-\gamma\alpha\big(E(\tilde{R}_e)-R_f\big)}{\left\langle E\left\{\max[\gamma\alpha\big(E(\tilde{R}_e)-R_f\big)-\alpha(\tilde{R}_e-R_f),0]\ \right\}^n\right\rangle^{\frac{1}{n}}}$

This is the same as the following:

(B5) $K_n\big(\tilde{R}_P, T(\gamma,\alpha)\big) = \dfrac{(1-\gamma)[E(\tilde{R}_e)-R_f]}{\left\langle E\left\{\max[\gamma\big(E(\tilde{R}_e)-R_f\big)-(\tilde{R}_e-R_f),0]\ \right\}^n\right\rangle^{\frac{1}{n}}}$

The last formulation of $K_n\big(\tilde{R}_P, T(\gamma,\alpha)\big)$ is invariant with respect to α.

Part 3: The Omega family ratios with $T(\gamma,\alpha)$.

The Omega ratio of the n^{th} degree with our responsive threshold is given by the following:

(B6) $\Omega_n(\tilde{R}_P, T(\gamma,\alpha)) = \dfrac{\{E[\max(\tilde{R}_P-T(\gamma,\alpha),0)]^n\}^{\frac{1}{n}}}{\{E[\max(T(\gamma,\alpha)-\tilde{R}_P,0)]^n\}^{\frac{1}{n}}} =$

$$= \frac{\left\langle E\left\{\max\left[\left(\alpha\tilde{R}_e+(1-\alpha)R_f\right)-\gamma\alpha\big(E(\tilde{R}_e)-R_f\big)-R_f,0\right]\right\}^n\right\rangle^{\frac{1}{n}}}{\left\langle E\left\{\max\left[\gamma\alpha\big(E(\tilde{R}_e)-R_f\big)+R_f-\left(\alpha\tilde{R}_e+(1-\alpha)R_f\right),0\right]\right\}^n\right\rangle^{\frac{1}{n}}} = $$

$$= \frac{\left\langle E\left\{\max\left[\big(\tilde{R}_e-R_f\big)-\gamma\big(E(\tilde{R}_e)-R_f\big),0\right]\right\}^n\right\rangle^{\frac{1}{n}}}{\left\langle E\left\{\max\left[\gamma\big(E(\tilde{R}_e)-R_f\big)-\big(\tilde{R}_e-R_f\big),0\right]\right\}^n\right\rangle^{\frac{1}{n}}}$$

As seen, the resulting ratio is invariant with respect to α.

Part 4: The Reward-to-VaR ratio with $T(\gamma,\alpha)$.

This part of the appendix shows that the RVaR ratio that uses the threshold $T(\gamma,\alpha)$ satisfies our PRTI axiom. Using $T(\gamma,\alpha)$, we have the following:

(B7) $\mathrm{RVaR}\big(\tilde{R}_P, q, T(\gamma,\alpha)\big) = \dfrac{E(\tilde{R}_P)-T(\gamma,\alpha)}{VaR(Q_P(q))} =$

$$= \frac{E\left[\left(\alpha E(\tilde{R}_e)+(1-\alpha)R_f\right)\right]-\left[\gamma\alpha\big(E(\tilde{R}_e)-R_f\big)+R_f\right]}{\gamma\alpha\big(E(\tilde{R}_e)-R_f\big)+R_f-\left[\alpha Q_e(q)+(1-\alpha)R_f\right]} = $$

$$= \frac{(1-\gamma)\big(E(\tilde{R}_e)-R_f\big)}{\gamma\big(E(\tilde{R}_e)-R_f\big)-\big(Q_e(q)-R_f\big)}$$

Clearly, $\mathrm{RVaR}\big(\tilde{R}_P, q, T(\gamma,\alpha)\big)$ is invariant with respect to α and satisfies our PRTI axiom. As noted in the text, the RVaR ratio may be inconsistent with the claim of Proposition 6.

Part 5: The RCVaR ratio with $T(\gamma,\alpha)$.

This part of the appendix shows that $\mathrm{RCVaR}\big(\tilde{R}_P, q, T(\gamma,\alpha)\big)$ is invariant with respect to α and thus satisfies our PRTI axiom.

(B8) $\quad \text{RCVaR}\big(\tilde{R}_P, T(\gamma,\alpha)\big) = \frac{E(\tilde{R}_P)-T(\gamma,\alpha)}{\int_0^q VaR(Q_P(q))} = \frac{E\left[\left(\alpha E(\tilde{R}_e)+(1-\alpha)R_f\right)\right]-\left[\gamma\alpha\left(E(\tilde{R}_e)-R_f\right)+R_f\right]}{\int_0^q\{\gamma\alpha\left(E(\tilde{R}_e)-R_f\right)+R_f-[\alpha Q_e(q)+(1-\alpha)R_f]\}dq} = \frac{(1-\gamma)\left(E(\tilde{R}_e)-R_f\right)}{\gamma\int_0^q\{\left[\left(E(\tilde{R}_e)-R_f\right)-\left(Q_e(q)-R_f\right)\right]\}dq}.$

This results in the following:

(B9) $\quad \text{RCVaR}\big(\tilde{R}_P, T(\gamma,\alpha)\big) = \frac{(1-\gamma)}{\gamma}\text{RCVaR}\big(\tilde{R}_P, T(1{,}0)\big).$

Equation (B9) shows that for any chosen γ, the ratio is invariant with respect to α and therefore the ratio satisfies the PRTI axiom. Thus, maximizing the relative risk ratio (coefficient of risk of equity $\frac{\left(Q_e(q)-R_f\right)}{\left(E(\tilde{R}_e)-R_f\right)}$) is equivalent to maximizing $\text{RCVaR}\big(\tilde{R}_P, T(\gamma,\alpha)\big)$.

Value-based performance measurement with the Attribution Matrix and the Finite Change Sensitivity Index

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Value-based performance measurement with the Attribution Matrix and the Finite Change Sensitivity Index

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Abstract

We present a model of performance measurement and attribution for delegated investments that summarizes the *manager effect* and the *client effect* on value creation. In particular, we introduce an innovative two-dimensional approach that, on one hand, detects the (manager and client) *decision effects*, measuring the impact of manager/investor choices on the overall investment performance and, on the other hand, detects the (manager and client) *period effects*, measuring the impact of all the (manager and client) decisions on the investment performance in a given assessment interval. As for the decision effects, the value added of an active investment portfolio is broken down in terms of the value generated by the decisions made by the manager (manager decision effect) and the value generated by the client/investor (client decision effect). As for the period effects, we quantify the impact of all the decisions made in the assessment interval on the value creation generated in a single period by the manager (manager period effect) and the client (client period effect), so that the sum of the periods' attribution values is the investment's value added.

In order to accomplish the task, we employ the Finite Change Sensitivity Index (FCSI), which enables one to quantify the impact of the most influential decisions made by manager and investor, and a truncation approach which is equivalent to the well-known residual-income approach. We then combine the two attribution dimensions into an Attribution Matrix (AM). Each element of the AM provides the amount of value added generated in a given period by the decisions made by the manager or by the investor in (the same or) another period.

Keywords. Value added, performance measurement, attribution matrix, sensitivity analysis, FCSI, manager effect, client effect.

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1 Introduction

A number of metrics are used in practice for measuring the performance of a financial investment and a substantial amount of contributions have recently dealt with pros and cons of various metrics from several points of view, all of which taking into account the role of a benchmark return in assessing the investment's value added. Most of these measures are return-based, that is, expressed as relative measures of worth (see Long and Nickels 199; Gredil et al. 2014; Magni 2014; Altshuler and Magni 2015; Jiang 2017; Cuthbert and Magni 2018).

Investment performance of a delegated portfolio depends on two sets of decisions: Investment decisions made by the manager and cash-flow decisions made by the client/investor. Notwithstanding this double dependence and the prosperous development of mathematical techniques for the optimization of portfolio allocation and selection (Jin and Yu Zhou 2008; Lim et al 2011; Low et al 2012; Wang and Yu Zhou 2020; Cerny 2020), the role of client's performance has been somewhat disregarded, and a long-standing tradition in academic literature has been mainly focused on the empirical measurement of managerial skills such as the ability to invest/disinvest in undervalued/overvalued securities (asset allocation and selection policy) or the capability to anticipate market behaviour and vary the levels of risk exposures in upward and downward markets accordingly (market timing) (Angelidis, Giamouridis, and Tessaromatis 2013; Spaulding 2014, Andreu Sánchez, Matallí-Sáez, and Sarto Marzal 2018, Crane and Crotty 2018, Elton and Gruber 2020, Bali et al 2020. See also Banker, Chen, and Klumpes 2016 for the asymmetric ability of managers in buying and selling), as well as the ability to control expenses and transaction costs (Andreu, Serrano, and Vicente 2019, Galagedera et al. 2020). With a similar attitude, Levy (1968) segregated (more precisely, cleaned) the manager's results from the client's contribution and distribution decisions. Considering the client's perspective, strong empirical evidence has been found about the relation between past investment performance and investors' contribution-and-distribution decisions (Del Guercio and Tkac 2002; Ippolito 1992; Chevalier and Ellison 1997; Bollen 2007, Goyal, and Wahal 2008; Goriaev, Nijman, and Werkel 2008). Futrhermore, Rakowski (2010) analysed the effect of daily mutual fund flow volatility on fund performance, Jones and Martinez (2017) studied the impact on asset allocation decisions of the investors' expectations about the fund's future performance, and Kostovetsky and Warner (2015) found evidence on how past fund investment perfomance and past cash contributions and distributions predict managerial turnover.

Bagot and Armitage (2004) moved a step forward, from the analysis of managerial skills to the contributions to value creation, however still concentrating just on managerial performance. They noted that return-based methods of performance measurement and attribution, such as the time-weighted return (TWR), do not answer the question about 'What has the manager done for me, given my initial investment and the cash inflows and outflows by me along the way?', since these metrics assess the manager's skills but do not measure the manager's contribution to the investment's value added. Bagot and Armitage (2004) and Armitage and Bagot (2009) endorsed a value-based method to answer this question about managerial contribution, since multiperiod attribution analysis is easier using values than using returns. Furthermore, in a multiperiod relationship between a fund manager and a client, Heinkel and Stoughton (1994) studied the contracts and the client's retention policies that most motivate the manager to acquire valuable information.

Despite the considerable attention drawn on the appropriateness of a performance criterion and on the role of the manager in affecting the performance, financial models explicitly measuring the impact of the investor's decisions and the interaction between the two kinds of decisions are lacking in literature. This paper aims to fill the gap. We elaborate on the question by Bagot and Armitage (2004) and further ask, 'What has the client done for himself, with his own decisions on the intermediate cash deposits and withdrawals into the investment portfolio, given the previous and future realized returns derived from the investment policies of the managers?'.

More precisely, our paper measures the *manager effect* and the *client effect*, respectively defined as the impact of the decisions made by the manager and by the investor on the value added by an actively managed investment as opposed to a passively-managed investment in the benchmark over a pre-selected assessment interval $[0, n]$.

We identify, for each period and for each decision maker (manager and investor), the decisions and the group of decisions that have been the most influential performance drivers; we also attribute a specific value for each decision made in every period and rank the decisions according to their impact on the investment's performance.

Since the decisions by the fund manager about selection and allocation of assets in a given period generate a well-determined holding period rate and the decisions of the investor gives rise to a cash flow (into or out of the investment), the problem of measuring the impact of decisions boils down to measuring the impact of the holding period rates and the intermediate cash flows on the investment's value added. Holding period rates and interim cash flows will then give rise to the set of input parameters of the model, the output being the investment's value added. The analysis is then refined so as to assess the impact of the decisions on each period performance. As a result, we propose a two-dimensional model which enables one to understand in which periods and by whom the most important (and less important) decisions have been made. The first dimension of the analysis addresses the problem of assessing the impact of the investment decisions and contribution-and-distribution decisions made in a given period onto the investment's value added in the assessment interval $[0, n]$. To accomplish this objective, the active investment derived from the decisions of manager and client is compared with a passive investment in a benchmark portfolio with no intermediate cash flows. We make use of a recently-conceived technique of sensitivity analysis, which apportions a discrete change in a model output to the discrete changes in the model inputs, the Finite Change Sensitivity Index (FCSI), introduced in Borgonovo (2010a, 2010b). We suitably supplement this technique with the fine-tuning of the FCSI procedure introduced in Magni et al. (2020) which allows a perfect (i.e., 100%) decomposition of the value added. In the following step, we address the second analytical dimension, namely, the determination of the effects of the decisions made in the assessment interval $[0, n]$ onto the value created in one single period. To accomplish this objective, we use a *truncation approach* whereby truncated projects are generated under the assumption of liquidation of the investment at the various dates of the assessment interval. We show that this truncation approach is equivalent to the residual-income approach, well-known in finance and accounting (Lundholm and O'Keefe 2001; Magni 2009). Finally, combining the two attribution analyses we obtain an Attribution Matrix (AM) whose cells are the *attribution values*; an attribution value measures the value added in a period t by the decisions made by the manager or the investor in a (same or other) period. The sum of the elements of a row of the AM attributable to the manager is the *manager decision effect*, whereas the sum of the elements of a row attributable to the client is the *client decision effect*. The sum of the manager decision effect and the client decision effect referred to the same period is the *joint decision effect*. Furthermore, the sum of the elements of a column is the *period effect*, which may be partitioned into *manager period effect* and *client period effect*.

We prove that the investment's value added is equal to the sum of the decision effects, which is also equal to the sum of the period effects. As a result, the model generates a twofold decomposition of the investment's value added in terms of decisions and in terms of periods.

The remainder of the paper is structured as follows. Section 2 presents the setting and Section 3 defines the investment's performance in terms of finite changes. This enables us to apply the (Clean) Finite Change Sensitivity Index (FCSI), which is described in Section 4. Section 5 uses the Clean FCSI technique for finding what we call the (manager and client) *decision effects*, that is, the impact of the investment decisions and the contribution-and-distribution decisions made in a given period onto the overall performance of an actively-managed investment. Section 6 presents a truncation approach to find what we call the *period effects*, that is, the impact of the investment decisions and contribution-and-distribution decisions made in the overall assessment interval onto the value created in a single period; we show that such an impact is equal to the investment's residual income. Section 7 illustrates the procedure with a numerical example for an eight-period investment. Section 8 builds the Attribution Matrix (AM) which combines the two dimensions decomposing the decisions effects and the period effects and giving rise to the aforementioned attribution values. Section 9 continues the example previously introduced and completes it by building and commenting its AM. Some remarks conclude the paper.

2 Economic setting

We analyze an investment (a fund or a portfolio of assets), starting at time $t = 0$ and liquidating at time $t = n$, involving a client/investor who endows the fund manager a monetary amount for actively managing the investment. Cash flows into and out of the fund at time t are denoted as F_t, where $F_t < 0$ represents a net contribution into the fund (outflow for the investor) and $F_t > 0$ represents a distribution from the fund (inflow for the investor), with $t = 0, 1, 2, \ldots, n$. While the investor makes the periodic decisions on contributions and distributions, the fund manager makes periodic decisions on the selection and allocation of the amount that remains invested in the fund. These decisions affect the beginning-of-period capital invested (investor's decisions) and the single-period rate of return of the fund managed (manager's decisions). Consider period t (i.e., the interval $[t-1, t]$) and let E_t be the end-of-period portfolio value at time t (i.e., before cash movement) and B_t denote the beginning-of-period portfolio value at time t (i.e., after cash movement). The rate of return is calculated as

$$i_t = \frac{E_t}{B_{t-1}} - 1. \tag{1}$$

This relation says that the investment's holding period rate i_t represents the relative increase in the investment value. For example, if $B_{t-1} = 100$, E_t=110, then the increase in value is $i_t = 110/100 - 1 = 10\%$. Therefore, one may also write

$$E_t = B_{t-1} \cdot (1 + i_t) \tag{2}$$

which says that the end-of-period value is equal to the beginning value marked up by the return rate i_t.

The beginning-of-period value at time t (i.e., at the beginning of period $t+1$) may be obtained from the end-of-period value by deducting the cash withdrawal from the fund or adding the capital injections into the fund. This may be expressed formally as

$$\begin{aligned} B_t &= E_t - F_t \\ B_t &= B_{t-1}(1 + i_t) - F_t. \end{aligned} \tag{3}$$

Completing the numerical example, if $F_t = 20$, the portfolio value at the beginning of period $t+1$ will be $B_t = 110 - 20 = 90$ or, equivalently, $100(1 + 10\%) - 20 = 90$.

The above relation formally describes the change in portfolio value caused by both the fund manager and the client/investor. More precisely, eq. (3) depends on both the manager's decisions, which affect i_t via the allocation and selection choices, and the client's decisions, which determine F_t via the contribution and distribution choices. The two effects are intertwined, since B_{t-1} is determined by past decisions of both manager and investor. This means that the manager's decisions and the investor's decisions interact in each period to determine the next-period investment value (see also Table 1).

Table 1: Breakdown of beginning-of-period investment value, B_t

decisions made by the manager and the investor in the interval $[0, t-1]$	$\Longrightarrow B_{t-1}$
decisions made by the manager in period t, i.e. in the interval $[t-1, t]$	$\Longrightarrow i_t$
decisions made by the investor at the end of period t, i.e. in date t	$\Longrightarrow F_t$

At time 0, the beginning-of-period value is $B_0 = -F_0 > 0$, and the ending value of the portfolio at the liquidation time n, denoted as E_n, is entirely distributed to the investor (i.e., $F_n = E_n$), so that the cash-flow stream for the investor is $(F_0, F_1, \ldots, E_n)$. We denote as $F = (F_1, F_2, \ldots, F_{n-1}) \in \mathbb{R}^{n-1}$ the vector collecting the intermediate cash flows, from $t = 1$ to $t = n-1$, while $i = (i_1, i_2, \ldots, i_n) \in \mathbb{R}^n$ denotes the vector collecting the single-period rates of return from $t = 1$ until the liquidation date $t = n$.

Focusing on the terminal date n, and using (1)-(3), one can express the net terminal value E_n as a function of the return rates and the cash flows prior to n, collected in vectors i and F, respectively:

$$E_n = E_n(i, F) = -\sum_{t=0}^{n-1}(1 + i_{t+1})(1 + i_{t+2}) \dots (1 + i_n) \cdot F_t. \tag{4}$$

The above relation tells us that the portfolio's terminal value E_n is the result of the previous decisions made by both the manager (who affects i_t) and the client (who affects F_t), and it is formally equal to the difference between the future value of the contributions and the future value of the distributions.

Consider now a benchmark index traded on the financial market and let $i^* = (i_1^*, i_2^*, \dots, i_n^*)$ be the vector collecting the benchmark single-period returns. The benchmark is used as a reference index, and the single-period benchmark returns i_t^* are used to capitalize, to a given point in time, the interim contributions and distributions as well as the portfolio's net terminal value. If the point in time is $t = 0$, the discounting process leads to the investor's Net Present Value (NPV):

$$\text{NPV} = \sum_{t=0}^{n} \frac{F_t}{(1 + i_1^*)(1 + i_2^*) \dots (1 + i_t^*)}. \tag{5}$$

If the point in time is $t = n$, the compounding process leads to the investor's Net Future Value, also known as Value Added (VA):

$$\text{VA} = \sum_{t=0}^{n}(1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*) \cdot F_t = \text{NPV} \cdot (1 + i_1^*)(1 + i_2^*) \dots (1 + i_n^*). \tag{6}$$

Since we aim at measuring the economic value created by the investment *ex post*, we will focus on the latter. The investment creates value for the client if and only if the value added is positive: VA > 0.

3 Value added: Active vs. passive investment

For a given vector of benchmark rates $i^* = (i_1^*, i_2^*, \dots, i_n^*)$ and a given initial contribution F_0, (6) may be reframed in terms of i and F as follows:

$$\begin{aligned} \text{VA} = f(i, F) &= \Big(\sum_{t=0}^{n-1}(1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*) \cdot F_t\Big) + E_n = \\ &= \Big(\sum_{t=0}^{n-1}(1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*) \cdot F_t\Big) + \Big(-\sum_{t=0}^{n-1}(1 + i_{t+1})(1 + i_{t+2}) \dots (1 + i_n) \cdot F_t\Big) \\ &= \sum_{t=0}^{n-1}\Big((1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*) - (1 + i_{t+1})(1 + i_{t+2}) \dots (1 + i_n)\Big) \cdot F_t. \end{aligned} \tag{7}$$

Abusing notation, we denote as $0 \in \mathbb{R}^{n-1}$ the null vector, whose components are all equal to zero. It is then worth noting that $f(i^*, 0)$ denotes the value added of a passive investment whereby an investor invests in the benchmark index and does not make any contribution nor distribution between $t = 1$ and $t = n$. Replacing i with i^* and F with 0 in (7) one finds that the value added by such a passive investment is zero (as expected):

$$f(i^*, 0) = \Big((1 + i_1^*)(1 + i_2^*) \dots (1 + i_n^*) - (1 + i_1^*)(1 + i_2^*) \dots (1 + i_n^*)\Big) F_0 = 0. \tag{8}$$

In other words, the passive investment is value neutral. This implies that the value added by the investment under consideration, VA, may be viewed as the result of switching from a passive investment where the single-period rate is i_t^* and the interim cash flows are zero to an active investment where the single-period rate is i_t and the interim cash flows are equal to F. Switching from $(i^*, 0)$ to (i, F) means to switch from a passive investment in the benchmark (with no interim contributions nor distributions) to an active investment where

(i) the fund manager selects assets and allocates the endowed amounts to the various assets

(ii) the client selects the time and the size of contributions and distributions.

As a result, the value added changes from $f(i^*, 0)$ to $f(i, F)$. Since $f(i^*, 0) = 0$, eq. (6) may be rewritten as

$$\text{VA} = \overbrace{f(i, F)}^{\substack{\text{value added} \\ \text{by the active investment}}} - \overbrace{f(i^*, 0).}^{\substack{\text{value added} \\ \text{by the passive investment}}} \tag{9}$$

We now analyze and interpret (9) in some detail.

Given a generic initial outflow y_0 and a vector of benchmark returns i^*, consider an asset with a set of single-period rates $x = (x_1, x_2, \dots, x_n)$ and a set of interim contributions and distributions $y = (y_1, y_2, \dots, y_{n-1})$. The terminal asset value, denoted as $E_n(x, y)$, is

$$E_n(x, y) = -\sum_{t=0}^{n-1} (1 + x_{t+1})(1 + x_{t+2}) \dots (1 + x_n) \cdot y_t \tag{10}$$

while the value added is

$$f(x, y) = \sum_{t=0}^{n-1} \Big((1 + i_{t+1}^*)(1 + i_{t+2}^*) \cdots (1 + i_n^*) - (1 + x_{t+1})(1 + x_{t+2}) \cdots (1 + x_n)\Big) \cdot y_t. \tag{11}$$

Therefore, (9) expresses the difference between the function f evaluated at the point $(x^1, y^1) = (i, F)$ and the same function evaluated at the point $(x^0, y^0) = (i^*, 0)$, assuming $y_0 = F_0$. Hence, (9) tells us that the economic value created by any investment is the *change* in the value added obtained by turning from a passive strategy to an active strategy, which shifts the value added from $f(i^*, 0)$ to $f(i, F)$.

From now on, we will use eq. (9), not eq. (6). The reason is that, analytically, eq. (9) is more useful for our ends, because it represents a finite change: The change of $f(x, y)$ when the independent variables shift from the point $(x^0, y^0) = (i^*, 0)$ to the point $(x^1, y^1) =$

(i, F). This fact enables us to apply a most recent technique of sensitivity analysis to $f(x, y)$ so as to measure the effects of the decisions made by the manager and the investor on VA (i.e., the manager decision effects and the client decision effects). In Section 4, we describe the technique, so-called Finite Change Sensitivity Index and, then, in Section 5, we show how to derive the (manager and client) decision effects.

4 Finite Change Sensitivity Indices

Sensitivity analysis (SA) is the study of how the variance of the output of a model (numerical or otherwise) can be apportioned to different input key parameters (Saltelli et al. 2004). As such, it aims at quantifying how much of an output change is attributed to a given parameter or a set of parameters. It is widely employed in finance and management (Huefner 1972), for instance in analyzing the value creation of industrial projects (Borgonovo and Peccati 2004, 2006; Borgonovo, Gatti, and Peccati 2010; Percoco and Borgonovo 2012; Marchioni and Magni 2018; Magni and Marchioni 2020), the composition of optimal financial portfolios (Luo, Seco and Wu 2015), and the effects of corporate debt (Donders, Jara and Wagner 2018; Délèze and Korkeamäki 2018).

There exist several SA techniques defined in the literature (see Borgonovo and Plischke 2016; Pianosi et al. 2016; Saltelli et al. 2008, 2004 for reviews of SA methods). Among others, the Finite Change Sensitivity Indices (FCSIs) have been recently introduced in Borgonovo (2010a, 2010b) for analyzing the impact of a finite change in the model inputs on the model output and apportioning the influence of each input on the output change. Formally, let f be the objective function, which maps the vector of inputs (also called parameters, or key drivers) $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p$ onto the model output $f(\alpha) \in \mathbb{R}$. Let the inputs vary from $\alpha^0 = (\alpha_1^0, \ldots, \alpha_p^0)$, the so-called *base value*, to $\alpha^1 = \left(\alpha_1^1, \alpha_2^1, \ldots, \alpha_p^1\right)$, the *realized value*. The corresponding model outputs are $f(\alpha^0)$ and $f(\alpha^1)$, so that the output variation is

$$\Delta f = f(\alpha^1) - f(\alpha^0). \tag{12}$$

Let $(\alpha_j^1, \alpha_{(-j)}^0) = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{j-1}^0, \alpha_j^1, \alpha_{j+1}^0, \ldots, \alpha_p^0)$ be the vector consisting of all the inputs set at their base value α^0, except parameter α_j which is given the realized value α_j^1. Analogously, let

$$(\alpha_j^1, \alpha_k^1, \alpha_{(-j,k)}^0) = (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{j-1}^0, \alpha_j^1, \alpha_{j+1}^0, \ldots, \alpha_{k-1}^0, \alpha_k^1, \alpha_{k+1}^0, \ldots, \alpha_p^0)$$

be the input vector where α_j and α_k are set to the realized values, while the remaining $p-2$ parameters are set at their base value, and so forth for all s-tuples of inputs, $s = 1, 2, \ldots, p$.

Borgonovo (2010a, 2010b) defines two versions of FCSIs: First Order FCSI and Total Order FCSI. The First Order FCSI of parameter α_j measures the individual effect of α_j (Borgonovo 2010a) on the output change and is obtained as

$$\Delta_j^1 f = f(\alpha_j^1, \alpha_{(-j)}^0) - f(\alpha^0) \tag{13}$$

or, in normalized version, $\Phi_j^1 f = \frac{\Delta_j^1 f}{\Delta f}$. The Total Order FCSI quantifies the total effect of

α_j, including both its individual contribution and its interactions with the other parameters. Before giving the definition of the Total Order FCSI, we need to understand the interaction effects. Let $\Delta_{j,k}f$ be the interaction between α_j and α_k, that is, the portion of $f(\alpha_j^1, \alpha_k^1, \alpha_{(-j,k)}^0) - f(\alpha^0)$ which is not explained by the individual effects $\Delta_j^1 f$ and $\Delta_k^1 f$. Specifically,

$$\overbrace{f(\alpha_j^1, \alpha_k^1, \alpha_{(-j,k)}^0) - f(\alpha^0)}^{\text{change in } f \text{ caused by } \alpha_j \text{ and } \alpha_k} = \overbrace{\Delta_j^1 f + \Delta_k^1 f}^{\text{individual contributions of } \alpha_j \text{ and } \alpha_k} + \overbrace{\Delta_{j,k} f}^{\text{interaction effect of } \alpha_j \text{ and } \alpha_k}$$

whence the interaction effect can be calculated as

$$\Delta_{j,k}f = f(\alpha_j^1, \alpha_k^1, \alpha_{(-j,k)}^0) - f(\alpha^0) - \Delta_j^1 f - \Delta_k^1 f.$$

Similarly, let $\Delta_{j,k,h}f$ be the interaction among the inputs α_j, α_k and α_h, which is the portion of $f(\alpha_j^1, \alpha_k^1, \alpha_h^1, \alpha_{(-j,k,h)}^0) - f(\alpha^0)$ not explained by the individual effects and by the interactions between any pair:

$$\begin{aligned} \overbrace{f(\alpha_j^1, \alpha_k^1, \alpha_h^1, \alpha_{(-j,k,h)}^0) - f(\alpha^0)}^{\text{change in } f \text{ caused by } \alpha_j,\ \alpha_k, \text{ and } \alpha_h} &= \overbrace{\Delta_j^1 f + \Delta_k^1 f + \Delta_h^1 f}^{\substack{\text{individual contributions} \\ \text{of } \alpha_j,\ \alpha_k, \text{ and } \alpha_h}} \\ &+ \overbrace{\Delta_{j,k}f + \Delta_{j,h}f + \Delta_{k,h}f}^{\substack{\text{pairwise interaction effect} \\ \text{of } \alpha_j,\ \alpha_k, \text{ and } \alpha_h}} + \overbrace{\Delta_{j,k,h}f}^{\substack{\text{threewise interaction effect} \\ \text{of } \alpha_j,\ \alpha_k, \text{ and } \alpha_h}} \end{aligned} \tag{14}$$

whence

$$\Delta_{j,k,h}f = f(\alpha_j^1, \alpha_k^1, \alpha_h^1, \alpha_{(-j,k,h)}^0) - f(\alpha^0) - \Delta_j^1 f - \Delta_k^1 f - \Delta_h^1 f - \Delta_{j,k}f - \Delta_{j,h}f - \Delta_{k,h}f$$

(analogously for a s-tuple, with $s > 3$). Switching from α^0 to α^1, the output change is equal to the sum of all the individual effects and all the s-wise interactions, $s = 1, 2, \ldots, p$ between parameters:

$$\begin{aligned} \Delta f = &\overbrace{\sum_{i=j}^{p} \Delta_j^1 f}^{\text{individual contributions}} + \\ &\underbrace{\overbrace{\sum_{j_1<j_2} \Delta_{j_1,j_2} f}^{\text{pairs}} + \overbrace{\sum_{j_1<j_2<j_3} \Delta_{j_1,j_2,j_3} f}^{\text{triplets}} + \cdots + \overbrace{\sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f}^{s\text{-tuples}} + \ldots + \overbrace{\Delta_{j_1,j_2,\ldots,j_p} f}^{p\text{-tuple}}}_{\text{overall interaction effects}}, \end{aligned}$$

where $\sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f$ is the sum of the interactions between s-tuples.

Borgonovo (2010a) defines the Total Order FCSI of α_j, denoted as $\Delta_j^{\mathcal{T}} f$, as the sum of First Order FCSI of α_j, $\Delta_j^1 f$, and the interaction effect of α_j, denoted as $\Delta_j^{\mathcal{I}} f$ and called

Interaction FCSI, which is the sum of every interaction involving α_j:

$$\Delta_j^{\mathcal{I}} f = \sum_{\substack{j_1<j_2 \\ j\in\{j_1,j_2\}}} \Delta_{j_1,j_2} f + \ldots + \sum_{\substack{j_1<j_2\ldots<j_s \\ j\in\{j_1,j_2,\ldots,j_s\}}} \Delta_{j_1,j_2,\ldots,j_s} f + \ldots + \Delta_{j_1,j_2,\ldots,j_p} f.$$

Therefore,

$$\Delta_j^{\mathcal{T}} f = \Delta_j^1 f + \Delta_j^{\mathcal{I}} f = \Delta_j^1 f + \sum_{\substack{j_1<j_2 \\ j\in\{j_1,j_2\}}} \Delta_{j_1,j_2} f + \ldots + \sum_{\substack{j_1<j_2\cdots<j_s \\ j\in\{j_1,j_2,\ldots,j_s\}}} \Delta_{j_1,j_2,\ldots,j_s} f + \ldots + \Delta_{j_1,j_2,\ldots,j_p} f \tag{15}$$

and, in normalized version, $\Phi_j^{\mathcal{T}} f = \frac{\Delta_j^{\mathcal{T}} f}{\Delta f}$.

Computationally, the calculation of the Interaction FCSIs (and, therefore, the Total Order FCSIs) may be extremely burdensome if the model does not contain a very small number of inputs.[1] However, Borgonovo (2010a, Proposition 1) provides a useful result for reducing the number of calculations:

$$\Delta_j^{\mathcal{T}} f = f(\alpha^1) - f(\alpha_j^0, \alpha_{(-j)}^1),\ \forall j = 1, 2, \ldots, p, \tag{16}$$

where $(\alpha_j^0, \alpha_{(-j)}^1)$ denotes the vector with each input equal to the realized value α^1, except for α_j which is set equal to α_j^0. This enables computing the total FCSI of α_j without calculating the Interaction FCSI of α_j.

Unfortunately, the Total Order FCSI has an unpleasant feature: It does not provide a complete decomposition of the output change. That is,

$$\sum_{l=1}^{p} \Delta_l^{\mathcal{T}} f \neq \Delta f = f(\alpha^1) - f(\alpha^0) \quad \text{or, equivalently,} \sum_{l=1}^{p} \Phi_l^{\mathcal{T}} f \neq 1.$$

In other words, the sum of Total FCSIs explains less (or more) than 100% of the output change.[2] Recently, Magni et al. (2020) introduced a duplication-clearing factor which eliminates the redundant, multiple interactions and allows a complete and exact decomposition of the output change. The *Clean* Interaction FCSI of α_j, here denoted as $\Delta_j^I f$, is defined as the product of the Interaction FCSI $\Delta_j^{\mathcal{I}} f$ and a suitable correction factor, defined as the ratio of the overall interaction effects over the sum of Interaction FCSIs

[1]The number of individual contributions is p and the number of the interactions between parameters and groups of parameters is equal to $2^p - p - 1$.

[2]To understand why this happens, consider that, in the sum of the Interaction FCSIs, $\sum_{l=1}^{p} \Delta_l^{\mathcal{I}} f$, the pairwise interactions of α_j and α_k, appear twice (in $\Delta_j^{\mathcal{I}} f$ and in $\Delta_k^{\mathcal{I}} f$); the three-wise interactions of α_j, α_k, and α_h appear three times (in $\Delta_j^{\mathcal{I}} f$, in $\Delta_k^{\mathcal{I}} f$, and in $\Delta_h^{\mathcal{I}} f$); and so on for all the s-wise interactions, $s = 2, 3, \ldots, p$. This implies that the sum of Interaction FCSIs does not equate the overall interaction effects:

$$\underbrace{\sum_{l=1}^{p} \Delta_l^{\mathcal{I}} f}_{\text{sum of Interaction FCSIs}} \neq \underbrace{\overbrace{\sum_{j_1<j_2} \Delta_{j_1,j_2} f}^{\text{pairs}} + \overbrace{\sum_{j_1<j_2<j_3} \Delta_{j_1,j_2,j_3} f}^{\text{triplets}} + \cdots + \overbrace{\sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f}^{s\text{-tuples}} + \ldots + \overbrace{\Delta_{j_1,j_2,\ldots,j_p} f}^{p\text{-tuple}}}_{\text{overall interaction effects}}$$

and, therefore, $\sum_{l=1}^{p} \Delta_l^{\mathcal{T}} f \neq \Delta f$.

(Magni et al. 2020):

$$\Delta_j^I f = \Delta_j^{\mathcal{I}} f \cdot \frac{\overbrace{\sum_{j_1<j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_p} f}^{\text{overall interaction effects}}}{\underbrace{\sum_{l=1}^{p} \Delta_l^{\mathcal{I}} f}_{\text{sum of Interaction FCSIs}}}. \tag{17}$$

Considering that $\Delta_j^{\mathcal{I}} f = \Delta_j^{\mathcal{T}} f - \Delta_j^1 f$ and

$$\sum_{j_1<j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_p} f = \Delta f - \sum_{j=1}^{p} \Delta_j^1 f,$$

one may reframe (17) as

$$\Delta_j^I f = \frac{\Delta_j^{\mathcal{T}} f - \Delta_j^1 f}{\sum_{l=1}^{p} (\Delta_l^{\mathcal{T}} f - \Delta_l^1 f)} \cdot \big(\Delta f - \sum_{l=1}^{p} \Delta_l^1 f\big). \tag{18}$$

In other words, the Clean Interaction FCSI is computed by imputing a share of the overall true interaction effect $(\Delta f - \sum_{l=1}^{p} \Delta_l^1 f)$ to parameter α_j. This share is obtained as the ratio of the Interaction FCSI of α_j and the sum of all Interaction FCSIs (Magni et al. 2020).

The Clean Total Order FCSI of parameter α_j, denoted as $\Delta_j^T f$, is defined as the sum of individual contribution and Clean Interaction FCSI of α_j (Magni et al. 2020):

$$\Delta_j^T f = \Delta_j^1 f + \Delta_j^I f \tag{19}$$

and, in normalized version, $\Phi_j^T f = \frac{\Delta_j^T f}{\Delta f}$. It is easy to see that the Clean Total FCSIs completely explain the output variation:

$$\sum_{l=1}^{p} \Delta_l^T f = \Delta f, \tag{20}$$

and, in normalized version, $\sum_{l=1}^{p} \Phi_l^T f = 1$.

The sign of a Clean Total FCSI, $\Delta_j^T f$, signals the directional effect of an input change onto the output change: A positive (negative) index signals that the change in the input has the effect of increasing (decreasing) the output. The absolute value of the Clean Total FCSI quantifies the magnitude of the effect; one may then rank the input factors according to their influence on the change in the objective function: Input α_j has higher rank than α_k if and only if $|\Delta_j^T f| > |\Delta_k^T f|$. We denote the rank of parameter α_j as R_j. The rank vector is $R = (R_1, R_2, \ldots, R_p)$.

5 Decision effects

Consider the vector of input factors $\alpha = (x, y) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{2n-1}$ where x_t denotes a rate of return and y_t denotes a cash flow. For a given initial outflow $y_0 < 0$ (investor's initial contribution) and a given vector of benchmark returns i^*, the asset allocation-and-selection policy followed by the manager in the various periods has the effect of shifting the rates from $x^0 = i^*$ to $x^1 = i$ and the client's decisions about contributions and distributions shift the cash flows from $y^0 = 0$ to $y^1 = F$. As already seen, the change from

$$\alpha^0 = (i^*, 0) = (i_1^*, i_2^*, \ldots, i_n^*, 0, 0, \ldots, 0)$$

to

$$\alpha^1 = (i, F) = (i_1, i_2, \ldots, i_n, F_1, F_2, \ldots, F_{n-1})$$

expresses the change from a passive investment policy to an active investment policy, which makes the value added change from $f(\alpha^0) = f(i^*, 0)$ to $f(\alpha^1) = f(i, F)$ (see Table 2). The output change is

$$f(\alpha^1) - f(\alpha^0) = f(i, F) - f(i^*, 0)$$

which is (9). Therefore, one may apply the Clean FCSI technique illustrated in Section 4 for decomposing VA in terms of period return rates and interim cash flows.

Table 2: Passive vs. active investment: Inputs, terminal value and value added

Inputs	**Passive**	**Active**
$\alpha = (x, y)$	α^0	α^1
$\alpha_1 = x_1$	i_1^*	i_1
$\alpha_2 = x_2$	i_2^*	i_2
$\alpha_3 = x_3$	i_3^*	i_3
$\vdots$	$\vdots$	$\vdots$
$\alpha_{n-1} = x_{n-1}$	i_{n-1}^*	i_{n-1}
$\alpha_n = x_n$	i_n^*	i_n
$\alpha_{n+1} = y_1$	0	F_1
$\alpha_{n+2} = y_2$	0	F_2
$\alpha_{n+3} = y_3$	0	F_3
$\vdots$	$\vdots$	$\vdots$
$\alpha_{2n-1} = y_{n-1}$	0	F_{n-1}
Terminal value and value added		
$E_n(\alpha) = E_n(x, y)$	$-\prod_{l=1}^{n}(1 + i_l^*) \cdot F_0$	$-\sum_{t=0}^{n-1} \prod_{l=t+1}^{n}(1 + i_l) \cdot F_t$
$f(\alpha) = f(x, y)$	$f(\alpha^0)$	$f(\alpha^1)$

It is then possible to identify the investment choices made by the manager and the contributions/distributions decisions made by the client which have most affected the overall investment's performance. In particular, the value added may be considered as the sum of all the effects of the active selection and allocation choices made in the various periods and the contribution-and-distribution decisions, as opposed to the passive strategy consisting in investing in the benchmark portfolio with no contributions nor distributions.

The Clean Total FCSI, $\Delta_j^T f$, provides the amount of value added that is determined by the decision made in a period by the manager or the client. We call $\Delta_j^T f$ the *decision effect* of parameter α_j. It is worth noting that the piece of information provided by $\Delta_j^T f$ is not whether and how much the investment outperforms or underperforms the benchmark in a given period, but whether the decisions made by the manager or the client in a given period have contributed, overall, to outperform or underperform the passive benchmark investment in the time interval $[0, n]$ and how much of the value added is attributable to them. This piece of information necessarily takes account of the interactions with the decisions made in the other periods. Indeed, the manager's investment decisions made in period t determine i_t, which measures the relative period growth in the investment's value and, therefore, affect the magnitude of the value added (not only in period t, but also) in the following periods $t+1$, $t+2$, …, n. Analogously, the client's choices about contributions and distributions made by the client in period t determine F_t, which affects the beginning-of-period capital B_t, and, therefore, the magnitude of the value added in period $t+1$, and also in the following periods $t+2$, $t+3$, …, n. Overall, there are $p = 2n-1$ decision effects attributable to the decisions of manager and client: The first n effects are attributable to the manager's decisions and are called *manager decision effects*. The remaining $n-1$ effects are attributable to the investor's decisions and are called *client decision effects*. Finally, we define the *joint decision effect* as the sum of the manager decision effect and the client decision effect related to the decisions made in the same period:

$$\text{joint decision effect in period } j = \Delta_j^T f + \Delta_{n+j}^T f, \qquad \text{for } j = 1, \dots, n \tag{21}$$

with $\Delta_{2n}^T f = 0$. The value added is equal to the sum of all the joint decision effects:

$$\text{VA} = \sum_{j=1}^{n} (\Delta_j^T f + \Delta_{n+j}^T f) \qquad \text{for } j = 1, \dots, n. \tag{22}$$

For summarizing the role of the two decision makers on value creation, we define the *manager effect* as the sum of the n manager decision effects, $\sum_{j=1}^{n} \Delta_j^T f$ (and, in normalized version, $\sum_{j=1}^{n} \Phi_j^T f$) and the *client effect* as the sum of the $n-1$ client decision effects, $\sum_{j=n+1}^{2n-1} \Delta_j^T f$ (and, in normalized version, $\sum_{j=n+1}^{2n-1} \Phi_j^T f$), such that the value added is equal to the addition of manager effect and client effect:

$$\text{VA} = \overbrace{\sum_{j=1}^{n} \Delta_j^T f}^{\text{manager effect}} + \overbrace{\sum_{j=n+1}^{2n-1} \Delta_j^T f}^{\text{client effect}}. \tag{23}$$

6 Period effects

As seen, the Clean Total FCSI $\Delta_j^T f$ represents the decision effect and measures the global effect of a given *decision* onto VA. In this section, we want to capture the period effect, that is, the global effect of a given *period* onto VA. In order to do so, we use what we call a truncation approach: We assume that the investment is fully liquidated at the date m such that $0 \leq m \leq n$. This implies that the cash-flow stream of the investment truncated at time m is $F^{(m)} = (F_0, F_1, \ldots, F_{m-1}, E_m, 0, 0, \ldots, 0)$ where

$$E_m = E_m(x, y) = -\sum_{t=0}^{m-1}(1 + x_{t+1})(1 + x_{t+2}) \ldots (1 + x_m) \cdot y_t. \tag{24}$$

From (5), the NPV of such a truncated project, denoted as $\text{NPV}^{(m)}$, is

$$\text{NPV}^{(m)} = \sum_{t=0}^{m-1} \frac{F_t}{(1 + i_1^*)(1 + i_2^*) \ldots (1 + i_t^*)} + \frac{E_m}{(1 + i_1^*)(1 + i_2^*) \ldots (1 + i_m^*)}$$

for $m = 1, 2, \ldots, n$. We denote as $f^{(m)}$ the value added (at time $t = n$) by the project truncated at time $t = m$. Using (6),

$$\begin{aligned} f^{(m)} &= \text{NPV}^{(m)} \cdot (1 + i_1^*)(1 + i_2^*) \ldots (1 + i_n^*) \\ &= \sum_{t=0}^{m-1} F_t \cdot (1 + i_{t+1}^*)(1 + i_{t+2}^*) \ldots (1 + i_n^*) + E_m \cdot (1 + i_{m+1}^*)(1 + i_{m+2}^*) \ldots (1 + i_n^*) \end{aligned} \tag{25}$$

with $f^{(0)} = 0$. Consider now two consecutive truncated projects: The difference $f^{(m)} - f^{(m-1)}$ represents that part of the investment's VA generated in period m (i.e., the interval $[m-1, m]$). We denote it as $\Delta^T f_m$ and call it *period effect*:

$$\Delta^T f_m = f^{(m)} - f^{(m-1)}; \tag{26}$$

its normalized version is denoted as $\Phi^T f_m = \frac{\Delta^T f_m}{\text{VA}}$.

The period effect is the effect on the value created in period m by the decisions made by the manager and the investor in the various periods .[3] It is easy to check that the sum of the period effects is exactly equal to VA:

$$\begin{aligned} \sum_{m=1}^{n} \Delta^T f_m &= \sum_{m=1}^{n} (f^{(m)} - f^{(m-1)}) \\ &= (f^{(1)} - f^{(0)}) + (f^{(2)} - f^{(1)}) + \ldots (f^{(n-1)} - f^{(n-2)}) + (f^{(n)} - f^{(n-1)}) \\ &= f^{(n)} = f(i, F) = \text{VA}. \end{aligned}$$

We have then generated two attribution groups: The group of the manager and client decision effects (the clean FCSIs) and the group of the period effects (the change in value

[3]Notably, only the decisions made up to time m (may) have a nonzero impact on the VA generated in period m and following periods, whereas any decision made after time m has no effect whatsoever on period m and previous periods (see Remark 2).

added of the truncated projects), which perfectly decompose the investment's VA:

$$\sum_{j=1}^{p} \Delta_j^T f = \text{VA} = \sum_{t=1}^{n} \Delta^T f_t. \tag{27}$$

In other words, we have two dimensions of analysis and two vectors, $(\Delta_1^T f, \Delta_2^T f, \ldots, \Delta_p^T f) \in \mathbb{R}^p$ and $(\Delta^T f_1, \Delta^T f_2, \ldots, \Delta^T f_n) \in \mathbb{R}^n$, both accomplishing a perfect breakdown of the VA. (In the next section, we combine the decision effects and the period effects and flesh out the contribution to VA of a given parameter α_j in a given period t.)

Table 3: Period effect and residual income

Time	$F^{(m-1)}$	$F^{(m)}$	$\Delta F^{(m)}$
0	F_0	F_0	0
1	F_1	F_1	0
2	F_2	F_2	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-2$	F_{m-2}	F_{m-2}	0
$m-1$	E_{m-1}	F_{m-1}	$-B_{m-1}$
m	0	E_m	E_m
$m+1$	0	0	0
$m+2$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	0	0

Remark 1. It is worth noting that the information supplied by the period effect is logically equivalent to the information provided by the well-known notion of *residual income* (Lundholm and O'Keefe 2001; Magni 2009). The latter expresses the value created by an investment in a given period $[m-1, m]$. It is defined as $B_{m-1}(i_m - i_m^*)$ and, as such, it measures the return over and above the normal return that would be generated by investing the same beginning-of-period capital B_{m-1} in the passive benchmark portfolio. Using (25) and (26), one gets

$$\begin{aligned}
\Delta^T f_m &= \Big((1+i_{m+1}^*)\ldots(1+i_n^*)\Big)\cdot\Big(E_m - (E_{m-1} - F_{m-1})(1+i_m^*)\Big) \\
&= \Big((1+i_{m+1}^*)\ldots(1+i_n^*)\Big)\cdot\Big(B_{m-1}(1+i_m) - B_{m-1}(1+i_m^*)\Big) \\
&= \Big((1+i_{m+1}^*)\ldots(1+i_n^*)\Big)\cdot B_{m-1}(i_m - i_m^*)
\end{aligned}$$

or, equivalently,

$$B_{m-1}(i_m - i_m^*) = \frac{\Delta^T f_m}{(1+i_{m+1}^*)\cdots(1+i_n^*)}.$$

Therefore, the period-m effect, $\Delta^T f_m$, is the value, at time n, of the residual income of period m or, equivalently, the residual income of period m, $B_{m-1}(i_m - i^*_m)$, is the value of the period-m effect discounted to time m. Conceptually, the equivalence may be best understood by considering two consecutive truncated projects. Continuing the investment from $m-1$ to m, and considering that $E_{m-1} - F_{m-1} = B_{m-1}$, the incremental cash-flow stream is $\Delta F^{(m)} = F^{(m)} - F^{(m-1)} = (0, 0, \ldots, -B_{m-1}, E_m, 0, 0, \ldots, 0)$ (see Table 3). This incremental cash-flow stream tells us that, by continuing the investment from $m-1$ to m the investor gives up B_{m-1} at time $m-1$ but receives an additional E_m at time m. Whether the continuing value is greater or smaller of the truncation value depends on whether the manager's investment decisions are value-creating or not. Specifically, if B_{m-1} were invested in the benchmark, the end-of-period market value would be $B_{m-1}(1 + i^*_m)$. The difference $E_m - B_{m-1}(1 + i^*_m)$ is then the incremental value generated in period m; owing to (2), it may be reframed as the residual income of period m: $E_m - B_{m-1}(1+i^*_m) = B_{m-1}(i_m - i^*_m)$. The value of this residual income at time n is precisely the m-period effect, $\Delta^T f_m$.

Remark 2. Inspecting (24) and (25), it should be clear that the return rates x_t of periods $t > m$ and the cash flows y_t of periods $t \geq m$ do not affect $E_m(x, y)$ and $f^{(m)}(x, y)$, because these decisions only intervene after the liquidation date m. Therefore,

$$\Delta^T_j f^{(m)} = 0 \quad \text{for the inputs } \alpha_j = x_t \text{ with } t > m \text{ or } \alpha_j = y_t \text{ with } t \geq m. \tag{28}$$

This implies that the effect of α_j is null on the periods preceding period t, that is,

$$\Delta^T_j f_m = 0 \text{ for the inputs } \alpha_j = x_t \text{ with } t > m \text{ or } \alpha_j = y_t \text{ with } t \geq m. \tag{29}$$

7 Worked example

In this section, we consider an investment management agreement whereby an investor endows a fund manager the capital amount $B_0 = -F_0 = 100$ (in thousands). The investment lasts $n = 8$ periods. The input data are described in Table 4: The first column describes the $15 (= 2 \cdot 8 - 1)$ variables of the model, distinguishing the rates (x_t) from the cash flows (y_t) (and, therefore, the manager's decisions from the investor's decisions). The second column expresses the benchmark (i.e., base) case and the third column describes the realized case.

Table 5 describes the beginning-of-period and end-of-period values of both passive investment and active investment, as well as the returns and the cash flows. From (8), the value added of the passive investment is 0 (as expected), and, from (7), the value added of the active investment is $f(i, F) = 2.466$ (see last row of the table).

Therefore, from (9), the increase in value added from the passive (value-neutral) investment policy to the active investment policy is $\text{VA} = f(i, F) - f(i^*, 0) = 2.466 - 0 = 2.466 > 0$, meaning that the active investment creates value.

Using Clean FCSIs, we now decompose the value added in terms of the influences of active investment choices and contribution/distribution decisions made in the various periods, by evaluating the effect on $f(\alpha)$ when the input vector is changed from the

Table 4: Worked example: Inputs

α	$\alpha^0 = (i^*, 0)$	$\alpha^1 = (i, F)$
$\alpha_1 = x_1$	$i_1^* = 3\%$	$i_1 = 4\%$
$\alpha_2 = x_2$	$i_2^* = 4\%$	$i_2 = 5\%$
$\alpha_3 = x_3$	$i_3^* = 3\%$	$i_3 = 2\%$
$\alpha_4 = x_4$	$i_4^* = 6\%$	$i_4 = 4\%$
$\alpha_5 = x_5$	$i_5^* = 1\%$	$i_5 = 3\%$
$\alpha_6 = x_6$	$i_6^* = 2\%$	$i_6 = 3\%$
$\alpha_7 = x_7$	$i_7^* = 2\%$	$i_7 = 5\%$
$\alpha_8 = x_8$	$i_8^* = 5\%$	$i_8 = 4\%$
$\alpha_9 = y_1$	0.00	30.00
$\alpha_{10} = y_2$	0.00	−20.00
$\alpha_{11} = y_3$	0.00	40.00
$\alpha_{12} = y_4$	0.00	10.00
$\alpha_{13} = y_5$	0.00	−30.00
$\alpha_{14} = y_6$	0.00	60.00
$\alpha_{15} = y_7$	0.00	20.00

Table 5: Passive vs. active investment: Cash flows, market values, and value added

	Passive Investment $(i^*, 0)$					**Active Investment** (i, F)					
t	Beginning value $(t-1)$	Rate of return	Ending value (t)	Cash flow	Beginning value (t)	Beginning value $(t-1)$	Rate of return	Ending value (t)	Cash flow	Beginning value (t)	t
0				−100.00	100.00				−100.00	100.00	0
1	100.00	3.00%	103.00	0.00	103.00	100.00	4.00%	104.00	30.00	74.00	1
2	103.00	4.00%	107.12	0.00	107.12	74.00	5.00%	77.70	−20.00	97.70	2
3	107.12	3.00%	110.33	0.00	110.33	97.70	2.00%	99.65	40.00	59.65	3
4	110.33	6.00%	116.95	0.00	116.95	59.65	4.00%	62.04	10.00	52.04	4
5	116.95	1.00%	118.12	0.00	118.12	52.04	3.00%	53.60	−30.00	83.60	5
6	118.12	2.00%	120.49	0.00	120.49	83.60	3.00%	86.11	60.00	26.11	6
7	120.49	2.00%	122.90	0.00	122.90	26.11	5.00%	27.41	20.00	7.41	7
8	122.90	5.00%	129.04	129.04		7.41	4.00%	7.71	7.71		8
$f(x,y)$				0.000					2.466		$f(x,y)$

benchmark vector $\alpha^0 = (i^*, 0)$ to the active-investment vector $\alpha^1 = (i, F)$.

Table 6: Decomposition of the value added: Decision effects ($\Delta_j^T f$)

$\alpha = (x, y)$	$\Delta_j^1 f$	$\Delta_j^I f$	$\Delta_j^T f$	$\Phi_j^T f$	R_j
Manager decision effects					
$\alpha_1 = x_1$	1.253	0.019	1.272	51.57%	4
$\alpha_2 = x_2$	1.241	−0.167	1.074	43.56%	7
$\alpha_3 = x_3$	−1.253	0.038	−1.215	−49.26%	5
$\alpha_4 = x_4$	−2.435	0.529	−1.905	−77.27%	2
$\alpha_5 = x_5$	2.555	−0.696	1.859	75.39%	3
$\alpha_6 = x_6$	1.265	−0.177	1.088	44.12%	6
$\alpha_7 = x_7$	3.795	−1.499	2.296	93.13%	1
$\alpha_8 = x_8$	−1.229	0.581	−0.648	−26.29%	9
Client decision effects					
$\alpha_9 = y_1$	0.000	−0.567	−0.567	−22.99%	11
$\alpha_{10} = y_2$	0.000	0.244	0.244	9.91%	14
$\alpha_{11} = y_3$	0.000	−0.710	−0.710	−28.79%	8
$\alpha_{12} = y_4$	0.000	−0.277	−0.277	−11.25%	13
$\alpha_{13} = y_5$	0.000	0.488	0.488	19.79%	12
$\alpha_{14} = y_6$	0.000	−0.634	−0.634	−25.70%	10
$\alpha_{15} = y_7$	0.000	0.101	0.101	4.08%	15
Total	5.193	−2.727	2.466	100%	

Table 6 collects the results of the sensitivity analysis: Column 1 presents the input parameters, column 2 supplies the individual contributions of α_j, calculated as in (13); column 3 reports the Clean Interaction FCSI, which is computed as in (18); column 4 (in gray) shows the Clean Total Order FCSI as defined in (19). They represent the manager decision effects (the first eight effects, whose sum is the manager effect) and the client decision effects (the following seven effects, whose sum is the client effect). As expected, they exactly decompose the value added, with $\sum_{j=1}^{15} \Delta_j^T f = f(i, F) - f(i^*, 0) = 2.466 =$ VA.[4] Column 5 reports the normalized decision effects $\Phi_j^T f$ and, finally, column 6 shows their ranking (see also the bar chart in Figure 1).

The most influential parameter on VA is the return rate in period $t = 7$, $\alpha_7 = x_7$, with $\Delta_7^T f = 2.296$, signifying that the investment decisions made by the manager in period 7, realizing the return rate $i_7 = 5\%$ (greater than the benchmark index of the same period $i_7^* = 2\%$), have overall contributed positively to the active-investment performance and have had the greatest impact on VA.

For the sake of interpretability, it is worth noting that the *individual* contribution of $\alpha_7 = x_7$ to the value added is obtained with the following argument: Suppose the client invests passively in the benchmark index from time $t = 0$ to time $t = 6$, then switches

[4]It can be shown that the standard Total FCSIs (Borgonovo 2010a, 2010b) do not accomplish a perfect decomposition. Specifically,

$$\sum_{j=1}^{15} \Delta_j^T f = -0.230 \neq 2.466 = \text{VA}.$$

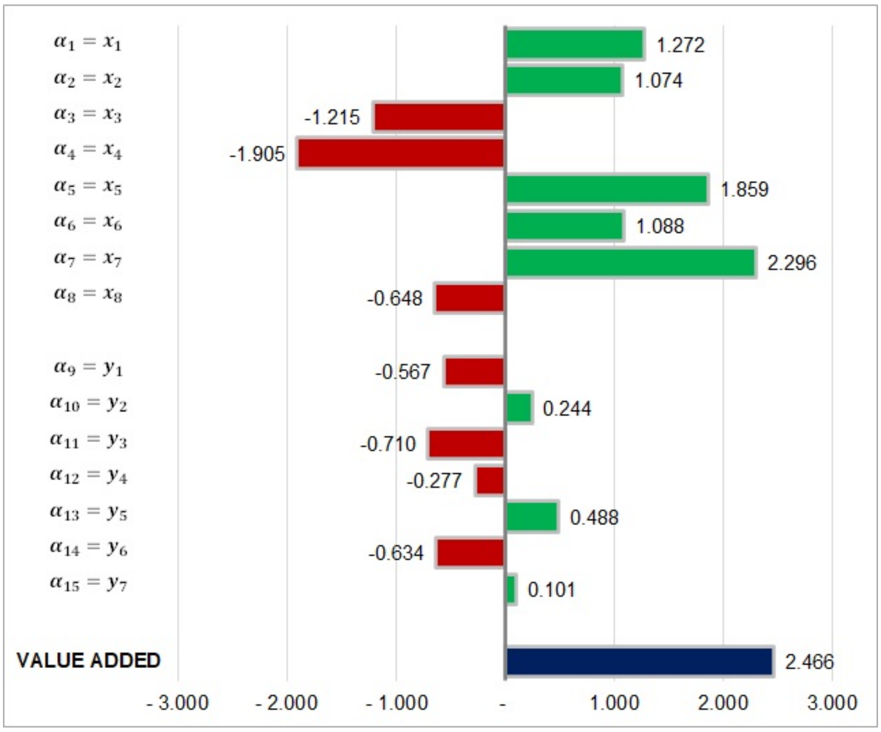

Figure 1: Attribution chart: Decomposition of value added into manager decision effects (first eight bars) and client decision effects (remaining seven bars).

Table 7: Individual contribution of the decisions made by the manager in period 7

$\alpha = (x, y)$	(α^0)	$(\alpha_7^0, \alpha_{(-7)}^0)$
$\alpha_1 = x_1$	$i_1^* = 3\%$	$i_1^* = 3\%$
$\alpha_2 = x_2$	$i_2^* = 4\%$	$i_2^* = 4\%$
$\alpha_3 = x_3$	$i_3^* = 3\%$	$i_3^* = 3\%$
$\alpha_4 = x_4$	$i_4^* = 6\%$	$i_4^* = 6\%$
$\alpha_5 = x_5$	$i_5^* = 1\%$	$i_5^* = 1\%$
$\alpha_6 = x_6$	$i_6^* = 2\%$	$i_6^* = 2\%$
$\alpha_7 = x_7$	$\boldsymbol{i_7^* = 2\%}$	$\boldsymbol{i_7 = 5\%}$
$\alpha_8 = x_8$	$i_8^* = 5\%$	$i_8^* = 5\%$
$\alpha_9 = y_1$	0	0
$\alpha_{10} = y_2$	0	0
$\alpha_{11} = y_3$	0	0
$\alpha_{12} = y_4$	0	0
$\alpha_{13} = y_5$	0	0
$\alpha_{14} = y_6$	0	0
$\alpha_{15} = y_7$	0	0

to the fund manager's active investment at time $t = 6$ and then switches back to the benchmark index at time $t = 7$, without intermediate contributions and distributions. This means that α shifts from α^0 to $(\alpha_7^1, \alpha_{(-7)}^0)$ (i.e., all parameters are unvaried at their base value while $\alpha_7 = x_7$ is changed from $\alpha_7^0 = x_7^0 = 2\%$ to $\alpha_7^1 = x_7^1 = 5\%$. From (11), and considering that $y_t = 0$ for $t = 1, 2, \ldots, 7$, the switching strategy leads to

$$\begin{aligned}
f(\alpha_7^1, \alpha_{(-7)}^0) &= f(0.03, 0.04, 0.03, 0.06, 0.01, 0.02, \mathbf{0.05}, 0.05, 0, 0, 0, 0, 0, 0, 0) \\
&= -100\Big((1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(1.02)(1.05)+ \\
&\qquad - (1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(\mathbf{1.05})(1.05)\Big) = 3.795
\end{aligned}$$

and the no-switching strategy is the passive investment which, owing to (8), leads to

$$\begin{aligned}
f(\alpha^0) &= f(0.03, 0.04, 0.03, 0.06, 0.01, 0.02, \mathbf{0.02}, 0.05, 0, 0, 0, 0, 0, 0, 0) = \\
&= -100\Big((1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(1.02)(1.05)+ \\
&\qquad - (1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(\mathbf{1.02})(1.05)\Big) = 0.
\end{aligned}$$

The difference

$$\Delta_7^1 f = f(\alpha_7^1, \alpha_{(-7)}^0) - f(\alpha^0) = 3.795 - 0 = 3.795$$

represents the individual contribution of $\alpha_7 = x_7$, calculated as in (13), that is the impact of the investment decisions made by the manager in period $t = 7$ on the value added, taken in isolation from the other inputs. The interaction effect is calculated as in eq. (18): $\Delta_7^I f = -1.499$. That is, the interaction shows a partial compensating effect. Overall, the manager contribution (i.e., the contribution to VA of the investment policy made by the manager) in period 7 is $\Delta_7^T f = 3.795 - 1.499 = 2.296$.

In terms of weight, the manager's contribution in period 7 explains VA almost entirely ($\Phi_7^T f = 93.13\%$). However, this does not mean that the impact of the individual decisions made in the other periods is small, because some of the parameters have had a strong *positive* impact and some other parameters have had a strong *negative* impact. For example, $\alpha_4 = x_4$ is the second most influential input and it contributes negatively ($i_4 = 4\%$ is lower than the passive index $i_4^* = 6\%$) with $\Delta_4^T f = -1.905$, corresponding to $\Phi_4^T f = -77.27\%$, which means that the manager has destroyed much value in that period. However, in the following period, the manager's decisions have created value ($i_5 = 3\% > 1\% = i_5^*$): The total contribution of $\alpha_5 = x_5$ is $\Delta_5^T f = 1.859$ which corresponds to $\Phi_5^T f = 75.39\%$ of VA, implying that this is the third most influential parameter and that it has almost entirely offset the poor performance of period 4.

At the opposite side of the parameters' ranking, the least influential input in the whole set is the client contribution in period 7, $y_7 = \alpha_{15}$. This means that the client's decision of withdrawing 20 from the investment at $t = 7$ is the lowest-impact decision. The contribution of y_7 is $\Delta_{15}^T f = 0.101$, corresponding to a 4.08% of the value added. The penultimate rank and the third-last rank are also determined by client's decisions, namely, $y_2 = \alpha_{10}$ and $y_4 = \alpha_{12}$, with $\Delta_{10}^T f = 0.244$ and $\Delta_{12}^T f = -0.277$.

As anticipated, for any fixed period t, a joint decision effect is obtained as the sum of

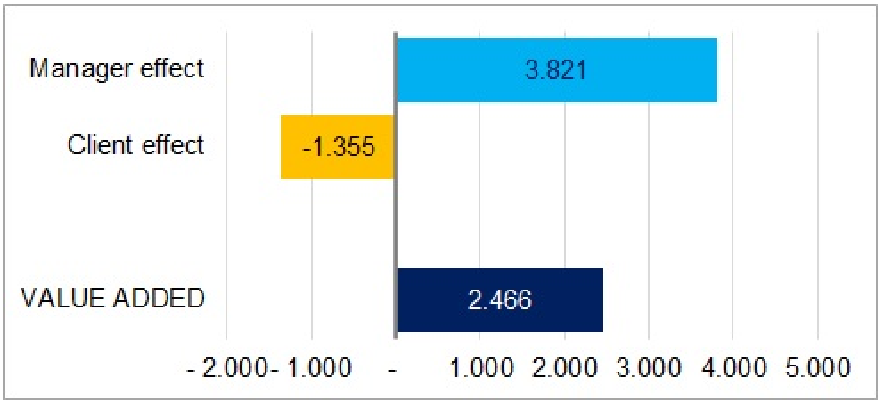

Figure 2: Attribution chart: Decomposition of value added into manager effect and client effect (see eq. (23))

the manager decision effect and the client decision effect of period t. Table 8 reports the joint decision effects as defined in eq. (21). The highest positive effect is in period $t = 7$, equal to 2.397, meaning that the decisions made in period 7 by manager and client jointly generate 2.397; the highest negative effect is in period $t = 4$ and amounts to -2.182.

Table 8: Decision effects as the sum of manager decision effects and client decision effects

Effect	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	Sum
Manager decision effect	1.272	1.074	-1.215	-1.905	1.859	1.088	2.296	-0.648	3.821
Client decision effect	-0.567	0.244	-0.710	-0.277	0.488	-0.634	0.101	0	-1.355
Joint decision effect	0.705	1.318	-1.925	-2.182	2.347	0.454	2.397	-0.648	2.466

The manager effect, determined by the group of parameters $x = \{x_1, x_2, \ldots, x_8\}$ and computed as $\sum_{j=1}^{8} \Delta_j^T f = 3.821$, is considerably more impactful than the client effect, determined by the group of parameters $y = \{y_1, y_2, \ldots, y_7\}$ and calculated as $\sum_{j=9}^{15} \Delta_j^T f = -1.355$. Moreover, the former is positive while the latter is negative. Therefore, the manager has, overall, performed positively and created value, thereby offsetting the value destruction caused by the investor's decisions regarding interim contributions and distributions (see also the bar chart in Figure 2).

Remark 3. The decisions of contributions and distributions of the investor, taken in isolation, have no effect on the value added: $\Delta_j^1 f = 0$ for all $j = 9, 10, \ldots, 15$ (see column 2 of Table 6). Indeed, if in a given period the investor funds are invested at a rate of return equal to the benchmark return, the amount of money which is deposited or withdrawn at the beginning of that period will neither increase the value added nor descrease it (the decisions will be neutral). The effects of deposits and withdrawals are indirect, mediated by the manager's performance. In other words, it is the interaction between rates (affected by manager's decisions) and cash flows (determined by the investor) that activates a nonzero effect of the cash flows on the investment's performance (see columns 3 and 4 of the table). Specifically, if the investor deposits (withdraws) money at the beginning of a value-creating period (i.e., $i_t > i_t^*$), then the investor's decision will amplify (reduce) the good manager's performance; if, instead, the investor deposits (withdraws) money at

Table 9: Decomposition of the value added: Period effects ($\Delta^T f_m$)

Cash-flow dates (t)	**Truncation dates** (m) $F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$	$F^{(5)}$	$F^{(6)}$	$F^{(7)}$	$F^{(8)}$
$t=0$	− 100.00	− 100.00	− 100.00	− 100.00	− 100.00	− 100.00	− 100.00	− 100.00
$t=1$	104.00	30.00	30.00	30.00	30.00	30.00	30.00	30.00
$t=2$	0.00	77.70	− 20.00	− 20.00	− 20.00	− 20.00	− 20.00	− 20.00
$t=3$	0.00	0.00	99.65	40.00	40.00	40.00	40.00	40.00
$t=4$	0.00	0.00	0.00	62.04	10.00	10.00	10.00	10.00
$t=5$	0.00	0.00	0.00	0.00	53.60	− 30.00	− 30.00	− 30.00
$t=6$	0.00	0.00	0.00	0.00	0.00	86.11	60.00	60.00
$t=7$	0.00	0.00	0.00	0.00	0.00	0.00	27.41	20.00
$t=8$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	7.71
VA and period effect	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$
$f^{(m)}(i,F)$	1.253	2.144	1.002	−0.315	0.822	1.718	2.540	2.466
$\Delta^T f_m(i,F)$	1.253	0.891	−1.143	−1.316	1.137	0.895	0.822	−0.074

[a] The element (t,m) of this matrix represents the cash flow at time t of the investment truncated at time m.

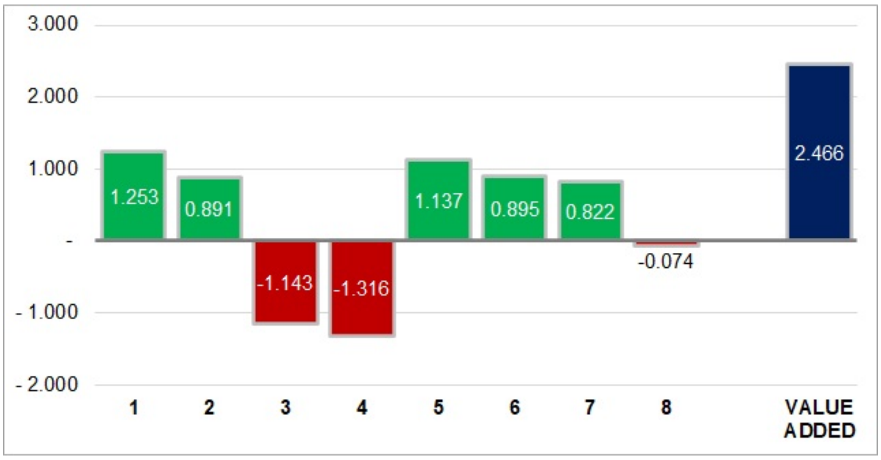

Figure 3: Attribution chart: Decomposition of value added into period effects

the beginning of a value-destroying period (i.e., $i_t < i_t^*$), then the investor's decisions will amplify (reduce) the bad manager's performance.[5]

Table 9 reports the truncated investments at time m, with $1 \leq m \leq 8$ and the resultant period effects, which are more clearly highlighted in Figure 3 with a column chart. The upper side of the table collects the cash flows

$$F^{(m)} = (F_0, F_1, \dots, F_{m-1}, E_m, 0, \dots, 0),$$

whereas the lower part shows the value added $f^{(m)}(i,F)$ and the period effects $\Delta^T f_m(i,F)$ (numbers are rounded).

In the next section, we refine the analysis by further decomposing the period effects using the clean FCSIs on the truncated investments. This will give rise to $p \cdot n$ attribution values, collected in the *Attribution Matrix*.

[5]This is because a higher (smaller) scale of the investment amplify (reduce) the (good or bad) performance.

8 The Attribution Matrix

Let $\Delta_j^T f^{(m)}$ denote the total effect of α_j on $f^{(m)}$ (i.e., the total clean FCSI of $f^{(m)}$); it measures the global impact of parameter α_j on the value added in the interval $[0, m]$. Likewise, $\Delta_j^T f^{(m-1)}$ measures the global impact of parameter α_j on the value added in the interval $[0, m-1]$. Therefore, the difference

$$\Delta_j^T f_m = \Delta_j^T f^{(m)} - \Delta_j^T f^{(m-1)} \tag{30}$$

measures that part of VA which is generated in period m, that is in the interval $[m-1, m]$, by parameter α_j, $j = 1, 2, \ldots, p$. We call $\Delta_j^T f_m$ the *attribution value* of α_j in period m; it is the effect of α_j on the economic value generated in period m (i.e., between $m-1$ and m), with $1 \leq m \leq n$.

For any given decision made by the manager, represented by α_j, $j = 1, 2, \ldots, n$, the sum of the attribution values amounts to the manager decision effect of α_j on VA; analogously, for any given decision made by the client, represented by α_j, $j = n+1, n+2, \ldots, 2n-1$, the sum of the attribution values amounts to the client decision effect of α_j on VA; formally,

$$\Delta_j^T f = \sum_{m=1}^{n} \Delta_j^T f_m. \tag{31}$$

For proving it, it is sufficient to note that $\Delta_j^T f^{(0)} = 0$ and $\Delta_j^T f^{(n)} = \Delta_j^T f$; therefore,

$$\begin{aligned}
\Delta_j^T f &= \Delta_j^T f^{(n)} - \Delta_j^T f^{(0)} \\
&= \Delta_j^T f^{(1)} - \Delta_j^T f^{(0)} \\
&+ \Delta_j^T f^{(2)} - \Delta_j^T f^{(1)} \\
&+ \Delta_j^T f^{(3)} - \Delta_j^T f^{(2)} \\
&\quad \vdots \\
&+ \Delta_j^T f^{(n-1)} - \Delta_j^T f^{(n-2)} \\
&+ \Delta_j^T f^{(n)} - \Delta_j^T f^{(n-1)} \\
&= \text{[by (30)]} \\
&= \Delta_j^T f_1 + \Delta_j^T f_2 + \ldots + \Delta_j^T f_n \\
&= \sum_{m=1}^{n} \Delta_j^T f_m.
\end{aligned} \tag{32}$$

Symmetrically, for any given period t, the sum of the attribution values is the period effect, that is, the contribution of period t to VA:

$$\sum_{j=1}^{p} \Delta_j^T f_t = \Delta^T f_t. \tag{33}$$

To prove it, we just remind that, for every project truncated at t, the sum of its clean total FCSIs ($\Delta_j^T f^{(t)}$) amounts to the value added of the truncated project, $f^{(t)}$ (see eq. (20)).

Hence,

$$
\begin{aligned}
\Delta^T f_t &= f^{(t)} - f^{(t-1)} \\
&= (\Delta_1^T f^{(t)} + \Delta_2^T f^{(t)} + \ldots + \Delta_p^T f^{(t)}) - (\Delta_1^T f^{(t-1)} + \Delta_2^T f^{(t-1)} + \ldots + \Delta_p^T f^{(t-1)}) \\
&= (\Delta_1^T f^{(t)} - \Delta_1^T f^{(t-1)}) + (\Delta_2^T f^{(t)} - \Delta_2^T f^{(t-1)}) + \ldots + (\Delta_p^T f^{(t)} - \Delta_p^T f^{(t-1)}) \\
&= \sum_{j=1}^{p} (\Delta_j^T f^{(t)} - \Delta_j^T f^{(t-1)}) \\
&= \sum_{j=1}^{p} \Delta_j^T f_t.
\end{aligned}
$$

Owing to (27), the sum of the period effects coincides with the sum of the decision effects, therefore offering a twofold decomposition of the economic created value. To better appreciate it, we gather the attribution values in a $p \times n$ *Attribution Matrix* (AM) such the element (j, t) reports the attribution value $\Delta_j^T f_t$, which expresses the value added by parameter α_j in period t, with $j = 1, 2, \ldots p$ and $t = 1, 2, \ldots, n$. Table 10 reports the AM, which is ideally partitioned into two submatrices, one regarding the manager effects (rows $1, 2, \ldots, n$), the other one regarding the client effects (rows $n+1, n+2, \ldots, 2n-1$).

For instance, referring to our example in Section 7 where $p = 15$ and $n = 8$, $\Delta_3^T f_7$ represents the value added in period 7 by the investment decisions made by the manager in period 3 ($\alpha_3 = x_3$). Likewise, the attribution value $\Delta_{12}^T f_4$ represents the value added in period 4 by the contribution or distribution decision made by the investor in period 4 ($\alpha_{12} = y_4$).

For a given column t, summing by row one gets the contribution of all the decisions made by the manager and the investor in the assessment interval $[0, n]$ to the value created in period t (i.e., in the interval $[t-1, t]$) (period effect). For a given row $j = 1, 2, \ldots, n$, summing by column one gets the contribution to VA generated by the decisions made in period j by the manager (manager decision effect); likewise, for a given row $n + j$, summing by column one gets the contribution to VA generated by the decisions made at time j by the investor (client decision effect) (see Figure 4).

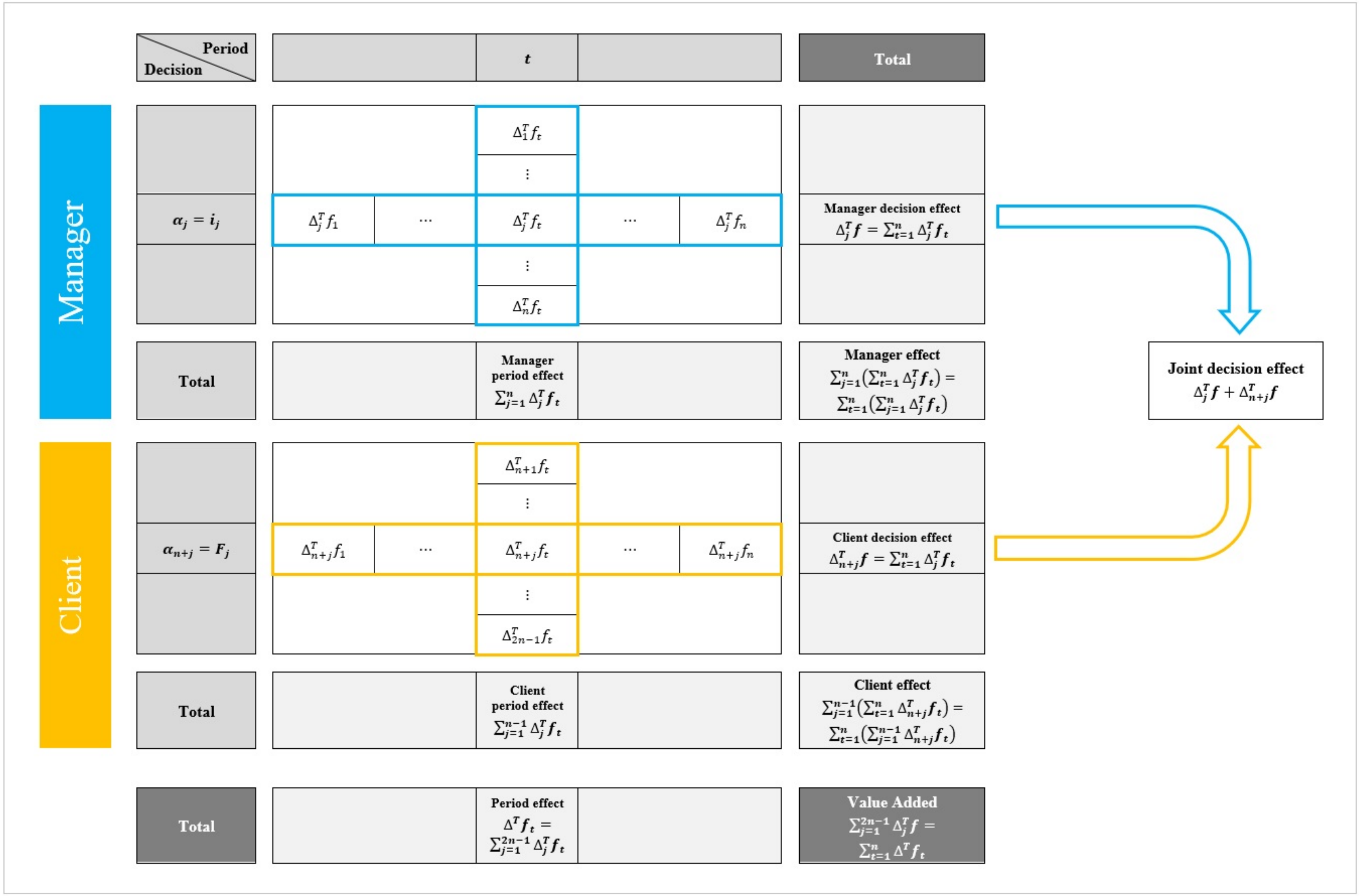

Figure 4: Attribution Matrix: Summary of decision effects, period effects, manager effect, and client effect.

Table 10: The Attribution Matrix

α	$\Delta_j^T f_1$	$\Delta_j^T f_2$	$\dots$	$\Delta_j^T f_n$	$\Delta_j^T f$
α_1	$\Delta_1^T f_1$	$\Delta_1^T f_2$	$\dots$	$\Delta_1^T f_n$	$\Delta_1^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_j	$\Delta_j^T f_1$	$\Delta_j^T f_2$	$\dots$	$\Delta_j^T f_n$	$\Delta_j^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_n	$\Delta_n^T f_1$	$\Delta_n^T f_2$	$\dots$	$\Delta_n^T f_n$	$\Delta_n^T f$
α_{n+1}	$\Delta_{n+1}^T f_1$	$\Delta_{n+1}^T f_2$	$\dots$	$\Delta_{n+1}^T f_n$	$\Delta_{n+1}^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_{n+j}	$\Delta_{n+j}^T f_1$	$\Delta_{n+j}^T f_2$	$\dots$	$\Delta_{n+j}^T f_n$	$\Delta_{n+j}^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_p	$\Delta_p^T f_1$	$\Delta_p^T f_2$	$\dots$	$\Delta_p^T f_n$	$\Delta_p^T f$
$\Delta^T f_t$	$\Delta^T f_1$	$\Delta^T f_2$	$\dots$	$\Delta^T f_p$	Δf

Summing by rows *and* by columns, one gets the VA:

$$(1 \quad 1 \quad \dots \quad 1) \cdot \begin{pmatrix} \Delta_1^T f_1 & \Delta_1^T f_2 & \dots & \Delta_1^T f_n \\ \Delta_2^T f_1 & \Delta_2^T f_2 & \dots & \Delta_j^T f_n \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_p^T f_1 & \Delta_p^T f_2 & \dots & \Delta_p^T f_n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \text{VA} \tag{34}$$

or, equivalently, $\sum_{j=1}^{p} \sum_{t=1}^{n} \Delta_j^T f_t = \text{VA}$. In words, the sum of all elements of the AM amounts to the investment's value added (see bottom-right corner of Figure 4).

We can also define the normalized attribution values, $\Phi_j^T f_t$, as

$$\Phi_j^T f_t = \frac{\Delta_j^T f_t}{\text{VA}}. \tag{35}$$

We gather the normalized attribution values in a normalized AM in Table 11, where the sum $\sum_{t=1}^{n} \Phi_j^T f_t = \Phi_j^T f$ is the normalized decision effect and, analogously, the sum $\sum_{j=1}^{p} \Phi_j^T f_t = \Phi^T f_t$ is the normalized period effect.

It is trivial to derive the perfect decomposition of the value added in normalized terms:

$$\sum_{j=1}^{p} \Phi_j^T f = 100\% = \sum_{t=1}^{n} \Phi^T f_t$$

and

$$\sum_{j=1}^{p} \sum_{t=1}^{n} \Phi_j^T f_t = 100\%.$$

Remark 4. It is worth noting that, for any $k \in \{1, 2, \dots, n\}$, if $i_k = i_k^*$, then both row k and column k of the AM are zero vectors. Formally,

- row k: $\Delta_k^T f_1 = \Delta_k^T f_2 = \dots = \Delta_k^T f_n = 0$ (α_k has no impact on any period)

Table 11: The normalized Attribution Matrix

α	$\Phi_j^T f_1$	$\Phi_j^T f_2$	$\dots$	$\Phi_j^T f_n$	$\Phi_j^T f$
α_1	$\Phi_1^T f_1$	$\Phi_1^T f_2$	$\dots$	$\Phi_1^T f_n$	$\Phi_1^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_j	$\Phi_j^T f_1$	$\Phi_j^T f_2$	$\dots$	$\Phi_j^T f_n$	$\Phi_j^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_n	$\Phi_n^T f_1$	$\Phi_n^T f_2$	$\dots$	$\Phi_n^T f_n$	$\Phi_n^T f$
α_{n+1}	$\Phi_{n+1}^T f_1$	$\Phi_{n+1}^T f_2$	$\dots$	$\Phi_{n+1}^T f_n$	$\Phi_{n+1}^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_{n+j}	$\Phi_{n+j}^T f_1$	$\Phi_{n+j}^T f_2$	$\dots$	$\Phi_{n+j}^T f_n$	$\Phi_{n+j}^T f$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
α_p	$\Phi_p^T f_1$	$\Phi_p^T f_2$	$\dots$	$\Phi_p^T f_n$	$\Phi_p^T f$
$\Phi^T f_t$	$\Phi^T f_1$	$\Phi^T f_2$	$\dots$	$\Phi^T f_n$	100.00%

- column k: $\Delta_1^T f_k = \Delta_2^T f_k = \ldots = \Delta_p^T f_k = 0$ (no decision has any impact on period k).

(see proof in the Appendix).

Remark 5. An interesting feature of the normalized AM is that it is invariant under changes in the evaluation date. Specifically, if the NPV is selected as the model output, then the AM associated with NPV is equal to the AM associated with VA, premultiplied by the discounting factor $\prod_{t=1}^{n}(1+i_t^*)^{-1}$. The normalized AM found by such a matrix is equal to the normalized AM associated with VA described in Table 11. In other words, whether one refers value creation at time $t=0$ (NPV) or at time $t=n$ (VA) or at any other date t, $0<t<n$, the normalized attribution values do not change.

Finally, we summarize the contribution of the two decision makers on the value created in a period by defining the *manager period effect* and the *client period effect*: For any fixed period m, the manager period effect is the sum of the n attribution values attributable to the manager, $\sum_{j=1}^{n}\Delta_j^T f_m$ (and, in normalized version, $\sum_{j=1}^{n}\Phi_j^T f_m$); likewise, for any fixed period m, the client period effect is the sum of the $n-1$ attribution values attributable to the client, $\sum_{j=n+1}^{2n-1}\Delta_j^T f_m$ (and, in normalized version, $\sum_{j=n+1}^{2n-1}\Phi_j^T f_m$). The period effect is equal to the sum of the manager period effect and the client period effect:

$$\Delta^T f_m = \overbrace{\sum_{j=1}^{n}\Delta_j^T f_m}^{\text{manager period effect}} + \overbrace{\sum_{j=n+1}^{2n-1}\Delta_j^T f_m}^{\text{client period effect}}. \tag{36}$$

9 Worked example (continued)

In this section we build the AM for the investment presented in Section 7.

Table 12 reports the AM. Inspecting the AM, it is clear that a decision made in a given period has no effect on previous periods, so the resulting attribution value is zero.

For example, $\Delta_5^T f_2 = 0$ means that the decisions made by the manager in period 5 has no effect on the value added in period 2. Also, focusing on the manager's attribution value, $\Delta_j^T f_t$ with $1 \le j \le 8$, it is worth noting that $\Delta_j^T f_j$ is

- positive if the manager's decisions in period j are such that fund's holding period rate i_j exceeds the benchmark return i_j^*
- negative if the manager's decisions in period j are such that fund's holding period rate i_j falls short the benchmark return i_j^*.

For example, in period 6, the manager's decisions give rise to a positive performance (since $i_6 > i_6^*$) and the impact of the manager's decisions is $\Delta_6^T f_6 = 1.078$.

Table 12: Attribution matrix for the 8-period investment

$\alpha = (x, y)$	$\Delta_j^T f_1$	$\Delta_j^T f_2$	$\Delta_j^T f_3$	$\Delta_j^T f_4$	$\Delta_j^T f_5$	$\Delta_j^T f_6$	$\Delta_j^T f_7$	$\Delta_j^T f_8$	$\Delta_j^T f$
$\alpha_1 = x_1$	1.253	0.006	−0.006	−0.012	0.012	0.006	0.019	−0.006	1.272
$\alpha_2 = x_2$	0	1.066	−0.006	−0.006	0.003	0.007	0.015	−0.005	1.074
$\alpha_3 = x_3$	0	0	−1.197	0.010	−0.009	−0.006	−0.018	0.006	−1.215
$\alpha_4 = x_4$	0	0	0	−1.878	0.003	−0.014	−0.026	0.010	−1.905
$\alpha_5 = x_5$	0	0	0	0	1.829	0.015	0.025	−0.010	1.859
$\alpha_6 = x_6$	0	0	0	0	0	1.078	0.015	−0.006	1.088
$\alpha_7 = x_7$	0	0	0	0	0	0	2.310	−0.014	2.296
$\alpha_8 = x_8$	0	0	0	0	0	0	0	−0.648	−0.648
$\alpha_9 = y_1$	0	−0.181	0.184	0.353	−0.364	−0.186	−0.556	0.182	−0.567
$\alpha_{10} = y_2$	0	0	−0.118	−0.223	0.228	0.120	0.354	−0.116	0.244
$\alpha_{11} = y_3$	0	0	0	0.439	−0.453	−0.232	−0.692	0.227	−0.710
$\alpha_{12} = y_4$	0	0	0	0	−0.112	−0.054	−0.165	0.054	−0.277
$\alpha_{13} = y_5$	0	0	0	0	0	0.162	0.484	−0.159	0.488
$\alpha_{14} = y_6$	0	0	0	0	0	0	−0.944	0.311	−0.634
$\alpha_{15} = y_7$	0	0	0	0	0	0	0	0.101	0.101
$\Delta^T f_t$	1.253	0.891	−1.143	−1.316	1.137	0.895	0.822	−0.074	2.466

Table 13: Normalized attribution matrix for the 8-period investment

$\alpha = (x, y)$	$\Phi_j^T f_1$	$\Phi_j^T f_2$	$\Phi_j^T f_3$	$\Phi_j^T f_4$	$\Phi_j^T f_5$	$\Phi_j^T f_6$	$\Phi_j^T f_7$	$\Phi_j^T f_8$	$\Phi_j^T f$
$\alpha_1 = x_1$	50.81%	0.24%	−0.25%	−0.48%	0.49%	0.25%	0.75%	−0.25%	51.57%
$\alpha_2 = x_2$	0.00%	43.23%	−0.23%	−0.25%	0.14%	0.27%	0.62%	−0.22%	43.56%
$\alpha_3 = x_3$	0.00%	0.00%	−48.56%	0.41%	−0.38%	−0.26%	−0.71%	0.24%	−49.26%
$\alpha_4 = x_4$	0.00%	0.00%	0.00%	−76.16%	0.11%	−0.55%	−1.07%	0.41%	−77.27%
$\alpha_5 = x_5$	0.00%	0.00%	0.00%	0.00%	74.17%	0.60%	1.03%	−0.41%	75.39%
$\alpha_6 = x_6$	0.00%	0.00%	0.00%	0.00%	0.00%	43.72%	0.62%	−0.22%	44.12%
$\alpha_7 = x_7$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	93.68%	−0.55%	93.13%
$\alpha_8 = x_8$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	−26.29%	−26.29%
$\alpha_9 = y_1$	0.00%	−7.33%	7.47%	14.31%	−14.75%	−7.56%	−22.53%	7.39%	−22.99%
$\alpha_{10} = y_2$	0.00%	0.00%	−4.78%	−9.03%	9.23%	4.86%	14.36%	−4.72%	9.91%
$\alpha_{11} = y_3$	0.00%	0.00%	0.00%	17.82%	−18.36%	−9.40%	−28.05%	9.20%	−28.79%
$\alpha_{12} = y_4$	0.00%	0.00%	0.00%	0.00%	−4.54%	−2.21%	−6.69%	2.18%	−11.25%
$\alpha_{13} = y_5$	0.00%	0.00%	0.00%	0.00%	0.00%	6.59%	19.64%	−6.45%	19.79%
$\alpha_{14} = y_6$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	−38.29%	12.60%	−25.70%
$\alpha_{15} = y_7$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	4.08%	4.08%
$\Phi^T f_t$	50.81%	36.15%	−46.34%	−53.38%	46.11%	36.31%	33.35%	−3.01%	100.00%

As for the investor's attribution values, $\Delta_j^T f_t$, with $1 \le t \le 8$ and $9 \le j < t + 8$, the attribution values $\Delta_j^T f_t$ are

– positive if

(i) the fund's holding period rate i_t exceeds the benchmark return i_t^* and the investor contributes cash to the fund at time $j-8$

(ii) the fund's holding period rate i_t falls short of the benchmark return i_t^* and the investor withdraws cash from the fund at time $j-8$

– negative if

(i) the fund's holding period rate i_t falls short of the benchmark return i_t^* and the investor contributes cash to the fund at time $j-8$

(ii) the fund's holding period rate i_t exceeds the benchmark return i_t^* and the investor withdraws cash from the fund at time $j-8$

(if $j \geq t+8$, then $\Delta_j^T f_t = 0$). This means that the effect of the client's decisions on a given period depends on whether the client contributes (withdraws) cash into the fund at the beginning of a value-creating period, so determining a positive (negative) effect, or the client contributes (withdraws) cash into the fund at the beginning of a value-destroying period, so determining a negative (positive) effect (see also Remark 3). For example, $\Delta_9^T f_2 = -0.181 < 0$ because, at time $t = 1$ (i.e. $9-8$), the investor withdraws 30 from the investment and period $t = 2$ is a value-creating period ($5\% = i_2 > i_2^* = 4\%$); therefore, reducing the investment scale has a negative impact. However, the same decision has a positive effect in period $t = 3$ ($\Delta_9^T f_3 = 0.184 > 0$), because period 3 is a value-destroying one ($2\% = i_3 < i_3^* = 3\%$); in other words, the reduction of the investment scale at the end of period 1 partially offsets the negative performance of period 3. And so on for the following periods. Overall, in the assessment interval $[0, 8]$, the drawdown decision made by the investor at time $t = 1$ has a net negative effect, equal to $\Delta_9^T f = -0.567$. Consider now the investor's decision of contributing 20 at time $t = 2$. The impact of such a decision in period 3 is negative ($\Delta_{10}^T f_3 = -0.118 < 0$) because that period is a value-destroying period ($2\% = i_3 < i_3^* = 3\%$); therefore, augmenting the investment scale is not a good decision. A negative impact of that decision on the following period $t = 4$ occurs as well ($\Delta_{10}^T f_4 = -0.223 < 0$) for the same reason. However, in period $t = 5$, that contribution has a positive effect ($\Delta_{10}^T f_5 = 0.228 > 0$), so the decision of increasing the scale at the end of period 2 has a positive effect after three periods. Overall, the net effect of the investor's decision made at time $t = 2$ is positive: $\Delta_{10}^T f = 0.224 > 0$.

Given a row, summing by columns, one gets the overall effect of a decision made by the manager or the client (decision effect); given a column, summing by rows one gets the overall effect onto a single period of the decisions made by the manager and the investor in all periods (period effect). For example, the overall effect of the decisions made by the manager in period $t = 4$ is $\Delta_4^T f = -1.905$ and the overall effect of all the decisions made by manager and investors in the various periods onto period $t = 7$ is $\Delta^T f_7 = 0.822$. Table 13 reports the normalized AM, obtained by dividing each cell of the AM by the investment value added VA = 2.466. As previously noted, the highest normalized decision effect is $\Phi_7^T f = 93.13\%$. Accordingly, the managerial decision in period 7 is the most relevant one for the overall investment performance. Inspecting the normalized AM, we understand that most of that value is generated in period 7

($\Phi_7^T f_7 = 93.68\%$), whereas the effect of the same decision on period 8 is negligible. The period which is most impacted by all the decisions is period 4, responsible for a normalized value destruction equal to $\Phi^T f_4 = -53.38\%$. Period 3, 4, and 8 are value-destroying periods, whereas the other periods are value-creating. The normalized AM also shows that the impact of the manager's decisions is mostly concentrated on the same period where the decisions are made; conversely, the impact of the investor's decision in one period may have a greater impact in some later period. This depends on the magnitude of the excess return $i_t - i_t^*$ which depends on the manager's decisions.[6] Finally, we remind that the normalized AM is the same if one disaggregates the NPV instead of the VA.

Finally, we compute the manager period effects and client period effects and report their absolute and normalized values in a concise AM (see Tables 14 and 15. See also the corresponding column chart in Figure 5). The manager period effect is higher (in absolute terms) than the client period effect for every $t = 1, 2, \ldots, 8$, suggesting that the manager's decisions are considerably more impactful than the client's decisions. The highest impacts are in period $t = 7$, where the (positive) manager effect is $\sum_{j=1}^{n} \Delta_j^T f_7 = 2.340$ (corresponding to about 95% of the overall value added) and the (negative) client effect is $\sum_{j=n+1}^{2n-1} \Delta_j^T f_7 = -1.518$ (corresponding to about 62% of the overall value added, with opposite sign). Table 16 summarizes the two-dimensional decomposition (decision vs. period) of manager effect and client effect.

Table 14: Concise Attribution Matrix

Effect	$\Delta_j^T f_1$	$\Delta_j^T f_2$	$\Delta_j^T f_3$	$\Delta_j^T f_4$	$\Delta_j^T f_5$	$\Delta_j^T f_6$	$\Delta_j^T f_7$	$\Delta_j^T f_8$	$\Delta_j^T f$
Manager period effect	1.253	1.072	−1.209	−1.886	1.838	1.086	2.340	−0.673	3.821
Client period effect	0	−0.181	0.066	0.570	−0.701	−0.190	−1.518	0.599	−1.355
$\Delta^T f_t$	1.253	0.891	−1.143	−1.316	1.137	0.895	0.822	−0.074	2.466

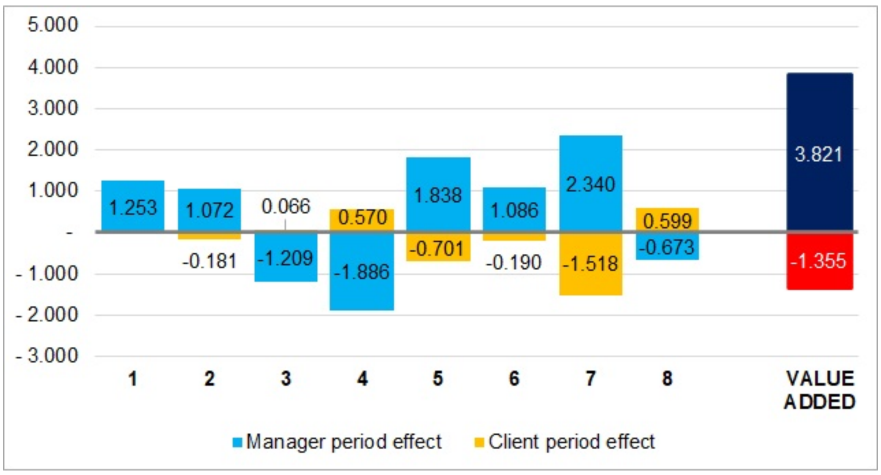

Figure 5: Attribution chart: Decomposition of value added into manager period effects and client period effects. The sum of the manager period effects is the manager effect (3.821) and the sum of the client period effects is the client effect (−1.355).

[6] We remind that the investor's effects depend on the manager's decisions; the individual contribution of an injection/withdrawal onto VA is zero.

Table 15: Normalized concise Attribution Matrix

Effect	$\Phi_j^T f_1$	$\Phi_j^T f_2$	$\Phi_j^T f_3$	$\Phi_j^T f_4$	$\Phi_j^T f_5$	$\Phi_j^T f_6$	$\Phi_j^T f_7$	$\Phi_j^T f_8$	$\Phi_j^T f$
Manager period effect	50.81%	43.48%	−49.03%	−76.48%	74.53%	44.03%	94.91%	−27.29%	154.95%
Client period effect	0%	−7.33%	2.69%	23.10%	−28.42%	−7.72%	−61.56%	24.29%	−54.95%
$\Phi^T f_t$	50.81%	36.15%	−46.34%	−53.38%	46.11%	36.31%	33.35%	−3.01%	100.00%

Table 16: Twofold decomposition of manager effect and client effect

Effect	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	Manager effect
Manager period effect	1.253	1.072	−1.209	−1.886	1.838	1.086	2.340	−0.673	3.821
Manager decision effect	1.272	1.074	−1.215	−1.905	1.859	1.088	2.296	−0.648	3.821
									Client effect
Client period effect	0	−0.181	0.066	0.570	−0.701	−0.190	−1.518	0.599	−1.355
Client decision effect	−0.567	0.244	−0.710	−0.277	0.488	−0.634	0.101	0	−1.355

10 Concluding remarks

Performance of an investment in a given span of time depends on the decisions made in each period by two decision makers, the manager and the investor/client. We employ a recent technique of sensitivity analysis, the Finite Change Sensitivity Index (Borgonovo 2010a, 2010b, Magni et al. 2020) for finding the (manager and client) decision effects, that is, the contributions to the overall investment's performance of the decision made by manager or investor in a period. Summing the manager decision effects one gets the *manager effect* and summing the client decision effects one gets the *client effect*. Then, we employ a truncation approach to investment for finding the contribution to the period investment's performance of all the decisions made by either decision maker over the investment lifespan (period effects). Such a contribution is equal to the (capitalized) residual income of the investment. Each period effect is broken down into manager period effect and client period effect. Finally, we combine the two perspectives and builds an Attribution Matrix (AM) which contains the attribution values. Each attribution value provides the contribution of a decision made by either decision maker in any period onto the investment's performance in any (same or other) period. In generating the AM we have taken into account the interactions between the manager's decisions, which affects the investment holding period rates, and the client's decisions, which determine the cash injected into or withdrawn from the investment, for each and every period. Figure 6 summarizes the steps for building the AM and Table 17 summarizes all the effects we have introduced (and quantified). Future researches may be addressed to generalize the approach and make the attribution analysis in terms of asset classes. This entails splitting up the manager decision effects considering the holdings in the various asset classes.

Investment description:

Cash Flows (determined by Client):		F_0, F, with $F = (F_1, F_2, \dots F_{n-1})$
Rates of return (determined by Manager):		Benchmark: $i^* = (i_1^*, i_2^*, \dots i_n^*)$,
		Realized $i = (i_1, i_2, \dots i_n)$
Function Parameters:	benchmark values:	$\alpha^0 = (\alpha_1^0, \alpha_2^0, \dots \alpha_p^0) = (i_1^*, i_2^*, \dots i_n^*, 0, 0, \dots 0)$
	realized values:	$\alpha^1 = (\alpha_1^1, \alpha_2^1, \dots \alpha_p^1) = (i_1, i_2, \dots i_n, F_1, F_2, \dots F_{n-1})$

Terminal value E_n of the investment:

$$E_n = -\sum_{t=0}^{n-1} (1+i_{t+1})(1+i_{t+2}) \dots (1+i_n) \cdot F_t \quad (4)$$

(see Section 2 for details)

Value Added VA of the investment:

$$\mathrm{VA} = f(i, F) = \sum_{t=0}^{n-1} \left((1+i_{t+1}^*)(1+i_{t+2}^*) \dots (1+i_n^*) - (1+i_{t+1})(1+i_{t+2}) \dots (1+i_n) \right) \cdot F_t \quad (7)$$

(see Section 3 for details)

STEP 1 – Split the Value Added into p Decision Effects with the clean FCSI, $\Delta_j^T f$, $1 \le j \le p$:

$$\Delta_j^T f = \Delta_j^1 f + \Delta_j^I f \quad (19)$$

where:

$$\Delta_j^1 f = f(\alpha_j^1, \alpha_{(-j)}^0) - f(\alpha^0), \quad \forall j = 1, 2, \dots, p \quad (13)$$

$(\alpha_j^1, \alpha_{(-j)}^0)$: parameters set at benchmark values $\alpha^0 = (i_1^*, i_2^*, \dots i_n^*, 0, 0, \dots 0)$, except $\alpha_j = \alpha_j^1$ (realized value)

$$\Delta_j^I f = \frac{\Delta_j^{\bar{1}} f - \Delta_j^1 f}{\sum_{l=1}^{p} (\Delta_l^{\bar{1}} f - \Delta_l^1 f)} \cdot \left(\Delta f - \sum_{l=1}^{p} \Delta_l^1 f \right), \quad \forall j = 1, 2, \dots, p \quad (18)$$

$$\Delta_j^{\bar{1}} f = f(\alpha^1) - f(\alpha_j^0, \alpha_{(-j)}^1), \quad \forall j = 1, 2, \dots, p \quad (16)$$

$(\alpha_j^0, \alpha_{(-j)}^1)$: parameters set at realized values $\alpha^1 = (i_1, i_2, \dots i_n, F_1, F_2, \dots F_{n-1})$, except $\alpha_j = \alpha_j^0$ (benchmark value)

j	Decision	Decision Effect $\Delta_j^T f$	
1	$\boldsymbol{\alpha_1 = i_1}$	$\Delta_1^T f$	Manager Effect = $\sum_{j=1}^{n} \Delta_j^T f$
2	$\boldsymbol{\alpha_2 = i_2}$	$\Delta_2^T f$	
⋮	⋮	⋮	
n	$\boldsymbol{\alpha_n = i_n}$	$\Delta_n^T f$	
n+1	$\boldsymbol{\alpha_{n+1} = F_1}$	$\Delta_{n+1}^T f$	Client Effect = $\sum_{j=n+1}^{p} \Delta_j^T f$
n+2	$\boldsymbol{\alpha_{n+2} = F_2}$	$\Delta_{n+2}^T f$	
⋮	⋮	⋮	
p	$\boldsymbol{\alpha_p = F_{n-1}}$	$\Delta_p^T f$	
	Total	$\sum_{j=1}^{p} \Delta_j^T f = VA$	

Joint Decision Effect in period $j = \Delta_j^T f + \Delta_{j+n}^T f$, for $j = 1, 2, \dots n$ (21)

(see Sections 4 and 5 for details, Section 7 for an example)

STEP 2 – Split the Value Added into n Period Effects $\Delta^T f_t$, $1 \le t \le n$:

$$\Delta^T f_m = f^{(m)} - f^{(m-1)} \quad (26)$$

where:

$$f^{(m)} = \sum_{t=0}^{m-1} F_t \cdot (1+i_{t+1}^*)(1+i_{t+2}^*) \dots (1+i_n^*) + E_m \cdot (1+i_{m+1}^*)(1+i_{m+2}^*) \dots (1+i_n^*) \quad (25)$$

Period	1	2	⋯	n	Total
Period Effect $\Delta^T f_t$	$\Delta^T f_1$	$\Delta^T f_2$	⋯	$\Delta^T f_n$	$\sum_{t=1}^{n} \Delta^T f_t = \mathrm{VA}$

(see Section 6 for details, Section 7 for an example)

STEP 3 – Build the Attribution Matrix:

Sub-step 3.1 - Split the Value Added in $p \times n$ Attribution Values with the clean FCSI, $\Delta_j^T f_m$:

$$\Delta_j^T f_m = \Delta_j^T f^{(m)} - \Delta_j^T f^{(m-1)} \quad (30)$$

where $\Delta_j^T f^{(m)}$ can be calculated using the formulas (13), (16), (18) and (19) in the previous step 1) applied to $f^{(m)}$.

Decision \ Period	1	2	⋯	n	Decision Effect $\Delta_j^T f$
$\boldsymbol{\alpha_1 = i_1}$	$\Delta_1^T f_1$	$\Delta_1^T f_2$	⋯	$\Delta_1^T f_n$	$\Delta_1^T f = \sum_{t=1}^{n} \Delta_1^T f_t$
$\boldsymbol{\alpha_2 = i_2}$	$\Delta_2^T f_1$	$\Delta_2^T f_2$	⋯	$\Delta_2^T f_n$	$\Delta_2^T f = \sum_{t=1}^{n} \Delta_2^T f_t$
⋮	⋮	⋮		⋮	⋮
$\boldsymbol{\alpha_n = i_n}$	$\Delta_n^T f_1$	$\Delta_n^T f_2$	⋯	$\Delta_n^T f_n$	$\Delta_n^T f = \sum_{t=1}^{n} \Delta_n^T f_t$
$\boldsymbol{\alpha_{n+1} = F_1}$	$\Delta_{n+1}^T f_1$	$\Delta_{n+1}^T f_2$	⋯	$\Delta_{n+1}^T f_n$	$\Delta_{n+1}^T f = \sum_{t=1}^{n} \Delta_{n+1}^T f_t$
$\boldsymbol{\alpha_{n+2} = F_2}$	$\Delta_{n+2}^T f_1$	$\Delta_{n+2}^T f_2$	⋯	$\Delta_{n+2}^T f_n$	$\Delta_{n+2}^T f = \sum_{t=1}^{n} \Delta_{n+2}^T f_t$
⋮	⋮	⋮		⋮	⋮
$\boldsymbol{\alpha_p = F_{n-1}}$	$\Delta_p^T f_1$	$\Delta_p^T f_2$	⋯	$\Delta_p^T f_n$	$\Delta_p^T f = \sum_{t=1}^{n} \Delta_p^T f_t$
Period Effect $\Delta^T f_t$	$\Delta^T f_1 = \sum_{j=1}^{p} \Delta_j^T f_1$	$\Delta^T f_2 = \sum_{j=1}^{p} \Delta_j^T f_2$	⋯	$\Delta^T f_n = \sum_{j=1}^{p} \Delta_j^T f_n$	Total: $\sum_{j=1}^{p} \Delta_j^T f = \sum_{t=1}^{n} \Delta^T f_t =$ VA

Sub-step 3.2 - Split the periods effects $\Delta^T f_m$ in manager period effects and client period effects:

$$\text{Manager period effect} = \sum_{j=1}^{n} \Delta_j^T f_m$$

$$\text{Client period effect} = \sum_{j=n+1}^{2n-1} \Delta_j^T f_m$$

such that

$$\overbrace{\Delta^T f_m}^{\text{period effect}} = \overbrace{\sum_{j=1}^{n} \Delta_j^T f_m}^{\text{manager period effect}} + \overbrace{\sum_{j=n+1}^{2n-1} \Delta_j^T f_m}^{\text{client period effect}} \quad (36)$$

where $\Delta_j^T f_m$ can be calculated using formula (30).

(see Section 8 for details, Section 9 for an example)

Figure 6: Attribution Matrix in a nutshell

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Appendix: Proof of Remark 4

Part I. We show that, if $i_k = i_k^*$, then $\Delta_k^T f_1 = \Delta_k^T f_2 = \ldots = \Delta_k^T f_n = 0$.

Since $i_k = i_k^*$, the following equalities hold:

$$\alpha_k^1 = \alpha_k^0, \quad (\alpha_k^1, \alpha_{(-k)}^0) = \alpha^0, \quad (\alpha_k^0, \alpha_{(-k)}^1) = \alpha^1.$$

Therefore, from (13), the individual effect of $\alpha_k = i_k$ on the value added of the truncated investment at each time t is null: $\Delta_k^1 f^{(t)} = 0$ for every $t \in \{1, 2, \ldots, n\}$ and, from (16), also the total effect of $\alpha_k = i_k$ on each truncated investment is zero as well: $\Delta_k^{\mathcal{T}} f^{(t)} = 0$ for every t. Since $\Delta_k^{\mathcal{I}} f^{(t)} = \Delta_k^{\mathcal{T}} f^{(t)} - \Delta_k^1 f^{(t)}$, then the interaction effect of $\alpha_k = i_k$ is null, i.e. $\Delta_k^{\mathcal{I}} f^{(t)} = 0$ for all t. Consequently, remembering (17), even the Clean Interaction effect $\Delta_k^I f^{(t)}$ is zero for each truncation date t. Hence, from (19), the Clean total effect of $\alpha_k = i_k$, is $\Delta_k^T f^{(t)} = 0, \forall t$. Finally, by (30), the attribution value of parameter α_k in each period t is null: $\Delta_k^T f_t = 0$ for every $t \in \{1, 2, \ldots, n\}$. (QED)

Part II: We show that, if $i_k = i_k^*$, then $\Delta_1^T f_k = \Delta_2^T f_k = \ldots = \Delta_p^T f_k = 0$.

Since $i_k = i_k^*$ and owing to (25), the values added of the truncated investments at $t = k-1$ and $t = k$ are equal:

$$f^{(k)}(\alpha) = f^{(k-1)}(\alpha) \tag{37}$$

for every input vector α such that $i_k = i_k^*$.

From (12) and (37), the change in value added from the passive investment α^0 to the active investment α^1 is the same in case of truncation at $t = k-1$ or at $t = k$, that is, $\Delta f^{(k)} = f^{(k)}(\alpha^1) - f^{(k)}(\alpha^0) = f^{(k-1)}(\alpha^1) - f^{(k-1)}(\alpha^0) = \Delta f^{(k-1)}$.

Furthermore, via (13) and (37), the individual effect of each factor α_j is equal on $f^{(k)}(\alpha)$ and on $f^{(k-1)}(\alpha)$, that is,

$$\Delta_j^1 f^{(k)} = f^{(k)}(\alpha_j^1, \alpha_{(-j)}^0) - f^{(k)}(\alpha^0) = f^{(k-1)}(\alpha_j^1, \alpha_{(-j)}^0) - f^{(k-1)}(\alpha^0) = \Delta_j^1 f^{(k-1)} \ \forall j \in \{1, 2, \ldots, p\}.$$

Analogously, each interaction among s-tuples (with $s \geq 2$) is the same from the truncated investments at k or $k-1$, that is, $\Delta_{j_1, j_2, \ldots, j_s} f^{(k)} = \Delta_{j_1, j_2, \ldots, j_s} f^{(k-1)}$ and, consequently, the interaction effect of each parameter is equal for the two functions, that is, $\Delta_j^{\mathcal{I}} f^{(k)} = \Delta_j^{\mathcal{I}} f^{(k-1)}$ for every $j = 1, 2, \ldots, p$. Hence, via (17), the Clean interaction effects coincide: $\Delta_j^I f^{(k)} = \Delta_j^I f^{(k-1)}$ for every j. Applying (19), the values added of the truncated investments at $t = k-1$ and $t = k$ share the same Clean Total FCSIs: $\Delta_j^T f^{(k)} = \Delta_j^T f^{(k-1)}$ for every $j = 1, 2, \ldots, p$. Therefore, from (30), the attribution value of each parameter α_j in period k is zero: $\Delta_j^T f_k = \Delta_j^T f^{(k)} - \Delta_j^T f^{(k-1)} = 0 \ \forall j \in \{1, 2, \ldots, p\}$. (QED)

Symbols and abbreviations

Symbol	Description
Section 2	
NPV	Net Present Value
n	Liquidation time
VA	Value Added of the financial investment
F_t	Cash flows into and out of the fund
E_t	End-of-period portfolio value at time t
B_t	Beginning-of-period portfolio value at time t
i_t	Rate of return of period t, i.e. the interval $[t-1, t]$
$F = (F_1, F_2, \dots, F_{n-1})$	Vector collecting the intermediate cash flows
$i = (i_1, i_2, \dots, i_n)$	Vector of the fund's holding period rates
$i^* = (i_1^*, i_2^*, \dots, i_n^*)$	Vector of benchmark's holding period rates
Section 3	
$x = (x_1, x_2, \dots, x_n)$	Vector of single-period rates
$y = (y_1, y_2, \dots, y_{n-1})$	Vector of interim contributions and distributions
(x^0, y^0)	$(i^*, 0)$
(x^1, y^1)	(i, F)
$f(x, y)$	Value added of a generic investment
$f(i, F)$	Value added of the active investment
$f(i^*, 0)$	Value added of the passive (benchmark) investment
Section 4	
FCSI	Finite Change Sensitivity Index
p	Number of inputs (for an investment, $p = 2n - 1$)
$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$	Vector of inputs
$\alpha^0 = (\alpha_1^0, \alpha_2^0, \dots, \alpha_p^0)$	Base value of inputs
$\alpha^1 = (\alpha_1^1, \alpha_2^1, \dots, \alpha_p^1)$	Realized value of inputs
$f(\alpha^1) - f(\alpha^0)$	Output change when inputs change from α^0 to α^1
$(\alpha_j^1, \alpha_{(-j)}^0)$	Input vector with all the inputs set at their base value α^0, except α_j which is given the realized value α_j^1
$(\alpha_j^0, \alpha_{(-j)}^1)$	Input vector with all the inputs set at their realized value α^1, except α_j which is given the realized value α_j^0
$(\alpha_j^1, \alpha_k^1, \alpha_{(-j,k)}^0)$	Input vector where α_j and α_k are set to the realized values, while the remaining $p-2$ parameters are set at their base value
$\Delta_j^1 f$	First order FCSI of parameter α_j
$\Phi_j^1 f$	Normalized First order FCSI of parameter α_j
$\Delta_{j,k} f$	Interaction between α_j and α_k
$(\alpha_j^1, \alpha_k^1, \alpha_h^1, \alpha_{(-j,k,h)}^0)$	Input vector with α_j, α_k and α_h are set to the realized values, while the remaining $p-3$ parameters are set at their base value
$\Delta_{j,k,h} f$	Interaction between α_j, α_k and α_h
$\Delta_j^{\mathcal{T}} f$	(Borgonovo's) total order FCSI of parameter α_j
$\Delta_j^{\mathcal{I}} f$	(Borgonovo's) interaction effect of parameter α_j
$\Phi_j^{\mathcal{T}} f$	(Borgonovo's) normalized total order FCSI of parameter α_j
$\Delta_j^T f$	Clean total order FCSI of parameter α_j
$\Delta_j^I f$	Clean interaction effect of parameter α_j
$\Phi_j^T f$	Normalized Clean total order FCSI of parameter α_j

Continued on next page

Continued from previous page

Section 5	
$\alpha \ = (x, y)$	
$\quad = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{n-1})$	Vector of inputs
$\alpha^0 = (i^*, 0)$	
$\quad = (i_1^*, i_2^*, \dots, i_n^*, 0, 0, \dots, 0)$	Passive investment policy in the benchmark with zero interim cash flows
$\alpha^1 = (i, F)$	
$\quad = (i_1, i_2, \dots, i_n, F_1, F_2, \dots, F_{n-1})$	Active investment policy in the fund with nonzero interim cash flows
$\Delta_j^T f$	Decision effect (i.e. contribution of paramater α_j to VA)
$\Phi_j^T f = \Delta_j^T f/\text{VA}$	Normalized decision effect
Section 6	
$F^{(m)} = (F_0, F_1, F_2, \dots, F_{m-1}, E_m, 0, 0, \dots, 0)$	Cash-flow stream of the investment truncated at time m
$\Delta F^{(m)} = F^{(m)} - F^{(m-1)}$	Incremental cash-flow stream if investment is continued from $m-1$ to m
$\text{NPV}^{(m)}$	NPV of the truncated project at time m
$f^{(m)}$	Value added (at time $t = n$) of the project truncated at time m
$\Delta^T f_m = f^{(m)} - f^{(m-1)}$	Period effect (i.e. contribution of period t to VA)
$\Phi^T f_m = \Delta^T f_m/\text{VA}$	Normalized period effect
Section 8	
AM	Attribution Matrix
$\Delta_j^T f^{(m)}$	Total effect of α_j on $f^{(m)}$ (clean FCSI of $f^{(m)}$)
$\Delta_j^T f_m = \Delta_j^T f^{(m)} - \Delta_j^T f^{(m-1)}$	Attribution value: Part of VA which is generated in period m, i.e. in the interval $[m-1, m]$, by parameter α_j
$\Phi_j^T f_m = \Delta_j^T f_m/\text{VA}$	Normalized attribution value

Table 17: Summary of effects

Effect	**Meaning**	**Computation**
Manager decision effect	Effect of the manager decisions made in period j on VA	Clean FCSI of α_j on VA, with $j = 1, \ldots, n$
Client decision effect	Effect of the client decisions made at time j on VA	Clean FCSI of α_j on VA, with $j = n+1, \ldots, 2n-1$
Joint decision effect	Joint effect of the (manager and client) decisions made in period j on VA	Sum of manager decision effect and client decision effect
Manager effect	Effect of all the manager decisions made in the interval $[0, n]$ on VA	Sum of all the manager decision effects
Client effect	Effect of all the client decisions made in the interval $[0, n]$ on VA	Sum of all the client decision effects
Manager period effect	Effect of all the manager decisions made in the interval $[0, n]$ on the value created in period t	Sum of all the attribution values attributable to the manager in period t
Client period effect	Effect of all the client decisions made in the interval $[0, n]$ on the value created in period t	Sum of all the attribution values attributable to the client in period t
Period effect	Effect of all the (manager and client) decisions made in the interval $[0, n]$ on the value created in period t	Sum of manager period effect and client period effect

Impact of financing and payout policy on the economic profitability of solar photovoltaic plants

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Impact of financing and payout policy on the economic profitability of solar photovoltaic plants

Carlo Alberto Magni * Davide Baschieri † Andrea Marchioni ‡

Abstract

This paper presents a comprehensive evaluation model for appraising an investment in a solar photovoltaic plant which encompasses both operational and financial management. We illustrate the intricate network of logical relations among technical (estimated) variables and financial (decision) variables and show that establishing transparent links between the former and the latter enhances the accuracy and soundness of the model. The results indicate that understanding the conceptual and formal relations of operating variables and financial decisions is necessary for correctly measuring shareholder value creation and making rational decisions, even for those projects (such as solar energy projects) where the operating, technical component is of paramount importance. We show that a firm's decision of replacing conventional energy with solar energy may be affected by managerial decisions regarding the firm's payout/retention policy and its financing policy to support the project. The model discloses insights on how to fine-tune the financing and distribution decisions in order to maximize the value creation for shareholders. We apply the model to a real-life case and quantify the effect of financial decisions on the project's net present value, showing that the financing and distribution policies may amplify or shrink the impact of changes in other inputs and may even revert an otherwise unprofitable project into a value-creating one.

Keywords. Investment analysis, photovoltaic solar energy, value creation, net present value, distribution policy, financing decision.

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1 Introduction

Sustainabile operations are becoming a major trend in the manufacturing system and the fourth industrial revolution offers innovations which potentially could accelerate a green economic development, also because of the technological advancement in the fields of decentralised energy production and storage of electrical energy (Bai and Sarkis 2017, Bai et al 2020, Wichmann, Johannes, and Spengler 2019). Photovoltaic (PV) technologies have been playing a central role in the development of a worldwide sustainable energy system, with their recent remarkable performance enhancement and cost reduction, transforming solar energy in electrical energy and combatting climate change and environmental pollution (Lupangu and Bansal 2017, Sinke 2019, Lei et al. 2019, Ezbakhe and Pérez-Foguet 2021, Kang et al. 2020), also supporting electrification opportunities in less developed and developing countries, isolated communities, and rural areas (Yu 2017, Ferrer-Martí et al. 2013, Henao et al. 2012). In spite of its environmental benefits, the adoption of solar energy by the industrial, commercial, and residential sectors is strongly affected by economic considerations (e.g., Dong et al. 2017, Cucchiella et al. 2018, Pham et al 2019). The mapping which links the key performance drivers and the investment's economic profitability requires a deep understanding of the intricate network of relations among technical aspects, accounting magnitudes, forecasting of financial data, and assumptions on financial decisions, which makes the project's evaluation particularly complex. It is then important to provide decision-aiding tools capable of measuring the project's economic profitability, taking into account uncertainty and providing insights on possible managerial actions that may affect the decision to adopt solar energy.

Several studies in the photovoltaic discipline have recently investigated technical, economical, and institutional challenges to turn potential into reality (Welling 2016, Lupangu and Bansal 2017, Lei et al. 2019, Gorjian et al. 2019). From a managerial perspective, Bhattacharya et al. (2020) propose a risk management tool for solar energy producers, investigating both natural hedges embedded in cash flows and cross hedging strategies with temperature-based weather derivatives; Ferrer-Martí et al. (2013) and Billionnet et al. (2016) study the optimal design of a hybrid wind–photovoltaic system made of photovoltaic panels, wind turbines and battery elements for serving a given demand and minimizing the cost, and, analogously, Li et al. (2017) optimize the sizing of solar and wind generating units of hybrid systems aiming to minimize the levelized energy cost. Jufri et al. (2019) recently introduced a detection system for monitoring the abnormal conditions in the photovoltaic plants and maintaining their productivity; Mauritzen (2020) study quality differences in terms of production degradation over time between photovoltaic panels produced by different manufacturers, supported by the theory of asymmetric information; Moret et al. (2020) provide a robust optimization framework for decision support under uncertainty in energy models including photovotaic systems. From a financial perspective, Abdallah et al. (2013) present an economic model for evaluating the option of installing small-scale photovoltaic plants on facility rooftops, and Büyüközkan and Güleryüz (2016) introduce a multi-criteria decision making tool for selecting the most appropriate renewable energy resources (including the solar one), from an investor-focused point of view, by considering evaluation criteria in the technical, economic, political, social, and environmental areas.

Despite the substantial amount of contributions studying the economic consequences of technical features of solar PV plants, somewhat neglected in the literature is the role of the firm's financial decisions in increasing or decreasing the firm's value and, possibly, in turning a favorable situation to an unfavorable one or vice versa. Building upon Magni (2020), we propose a framework for modeling investment decisions in solar PV systems and capturing the effect of the financial variables on the project's economic profitability, explaining why, for a given set of technical inputs, the decision on how to raise funds for covering the financial needs and the decision on the amount of cash distributed to shareholders as opposed to cash retained in the firm may affect the decision to switch to solar energy. The model acknowledges the distinction between *estimation variables* and *decision variables* on one hand and between *operating variables* and *financial variables* on the other hand. The estimation variables necessitate some estimation process to be determined (e.g., operating and maintenance costs, disposal costs, interest rate on debt financing) while the decision variables are under the managers' control (e.g., timing and size of distributions to shareholders, recourse to debt borrowing or to cash withdrawals for covering the financial needs). The operating variables express the factors which have a direct impact on the firm's costs and revenues as a result of the adoption of solar energy (e.g., solar panel efficiency, avoided electric bill, energy prices, amount of self-consumption, credit terms for energy sales to the grid). The financial variables regard the factors which affect the mix of financing sources, the cash flow raised from the capital providers, and the cash flow distributed to debtholders and shareholders as opposed to the cash flow retained in the firm (e.g., interest rate on debt, interest rate on liquid assets, risk-adjusted cost of capital, payout ratio, retention ratio). This paper precisely shows that, for a given selected set of assumptions on the operating variables, the firm's decisions on the payout policy (i.e., the cash distributed to the firm's shareholders) and the financing mix may have a significant role in adding or subtracting value and even in turning an otherwise unprofitable project into a value-creating one (or vice versa) and that the impact of such decisions may be larger or smaller depending on the value of the other input factors.

These results suggest that some time and effort should be devoted by the firm's management to model the distribution policy and the borrowing policy explicitly and measure its effects on the project's value. In such a way, the firm may calibrate a suitable financing-and-distribution policy which maximizes shareholder value creation. Since the accounting-and-finance structure of a solar PV plant is equivalent to any other engineering project, the results obtained clarify that the role of the financial decisions embedded in any capital asset project deserves more attention than it usually arouses in traditional financial modelling (see Tham and Vélez-Pareja 2004 for an exception).

The remainder of the paper is structured as follows: In Section 2 we present the model setting, breaking down the input factors into estimation variables and decision variables, and introduce the notions of operating income, operating cash flow, and free cash flow to equity (FCFE). In Section 3 we show the link between the FCFE and the associated financing and distribution decisions made by the firm. Section 4 illustrates how to carve out the project's cash flow from the estimation variables and how each year's decisions affects the accounting and financial magnitudes of the next year. In Section 5 we operationalize the logical structure illustrated in the previous sections by showing, for a solar PV plant, how to pass from inputs to cash flows. Section 6 makes use of the estimated cash flows and the net-present-value (NPV) approach to estimate the shareholder value created by the project and to make an economically rational decision. In Section 7 we apply the model and the evaluation methodology to a ground-mounted stand-alone solar PV plant. Section 8 carries out a scenario analysis for computing the project's economic profitability resulting from different financing and distribution decisions. Section 9 uses sensitivity analysis to quantify the individual impacts and the interaction effect of financing policy and distribution policy on shareholder value creation. Some remarks conclude the paper.

2 Operating Cash Flow and Free Cash Flow to Equity

The accounting-and-finance model we propose is based on a comprehensive economic evaluation of the option of switching to solar energy for a firm currently importing energy from electric grid. The framework is based on a twofold classification of the variables affecting benefits and costs. On one hand, we distinguish *estimation inputs* and *decision inputs*; on the other hand, we differentiate the *operating inputs* from the *financial inputs*:

- estimation inputs are stochastic variables whose representative values (e.g., mean values, most probable values) require an estimation process involving expert knowledge
- decision inputs deal with decisions which must be made explicitly in order to build the financial model of the project
- operating inputs have to do with the firm's operating activities and the related change in accounting and financial magnitudes under the assumption of project undertaking
- financial inputs have to do with fund raising and distribution of cash to capital providers, with the interest rates (on debt and on reinvestment of cash), and with the minimum attractive rate of return required by the investors for undertaking the project.

Owing to this taxonomy, the estimation variables may be operating (e.g., useful life of plants, solar degradation panel rate, operating and maintenance costs, annual energy consumption, energy prices, etc.) or financial (e.g., interest rate on debt, interest rate on retained cash, required return on operating assets, etc.). Likewise, decision variables may be operating or financial. The operating decisions have to do with technical aspects of the project (e.g., decisions on the amount of operating and maintenance costs) or with economic aspects such as the management of the net operating working capital and the operating cycle; the financial decisions deal with

- the *financing policies*, which are decisions on the financing mix to cover the financial deficits. The latter may be covered with debt capital, equity capital or internal financing, defined as the recourse to existing liquid assets such as cash or cash equivalents (e.g., cash withdrawals from bank accounts or sales of marketable securities)
- the *distribution* (or *payout*) *policies*, that are decisions on the amount of distribution to shareholders of cash generated by the project and decisions on the amount of cash retained in the firm and reinvested in the liquid assets.

We assume that the decision on the operating variables are given and focus on the financial decisions regarding the coverage of financial deficits and distribution of available cash to the firm's equityholders, which we call the *embedded decisions*, since the accounting-and-finance model describing the project cannot be completed without their determination. In order to understand the role of the embedded decisions on the output, we illustrate the model and then, in Section 6, clarify how to evaluate the project and make an accept-reject decision. Following, we describe the setting of the decision process and some fundamental accounting and financial magnitudes alongside the logical connections between the operating variables and financial variables on one hand, and the estimation variables and decision variables on the other hand.

The model starts from the input variables, which are used to build three pro forma statements for each one of the $n+1$ dates (0 to n): Statement of capitals (or balance sheet), statement of incomes, statement of cash flows. The first one collects the capital invested and raised by the firm for undertaking the project, the second one reports the incomes, and the third one reports the cash flows generated by the project. Letting n be the duration of the solar PV plant, a total of $3(n+1)$ statements must be built.

To draw up the statements, the analyst should first focus on the operating components. Let Rev_t be the incremental revenues derived from the sale of excess energy, and OpC_t be the incremental operational costs (O&M, insurance costs, opportunity costs such as lost rents, etc.) brought about by the plant. Let Dep_t be the depreciation charge of the solar PV plant. The pre-tax operating income, also called *earning before interest and taxes* (EBIT), is determined as

$$\mathrm{EBIT}_t = \mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t. \tag{1}$$

Subtracting the income taxes, T_t, one finds the after-tax operating income:

$$I^o_t = \mathrm{EBIT}_t - \mathrm{T}_t \qquad t = 0, 1, \dots, n \tag{2}$$

where T is obtained as the product of the marginal corporate tax, τ, on the earnings before taxes (EBT):

$$\mathrm{T}_t = \tau \overbrace{(\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t + I^l_t - I^d_t)}^{\mathrm{EBT}} \tag{3}$$

with I^l_t and I^d_t denoting, respectively, the interest income on liquid assets (cash and cash equivalents, marketable securities, other financial assets) and the interest expense on debt, obtained as

$$I_t^l = i^l \cdot C_{t-1}^l \tag{4}$$
$$I_t^d = i^d \cdot C_{t-1}^d \tag{5}$$

where

i^l = interest rate of liquid asset
i^d = interest rate of debt
C_{t-1}^l = balance of liquid assets at time $t-1$
C_{t-1}^d = debt outstanding at time $t-1$.

Once estimated the after-tax operating income,[1] the analyst must estimate the operating cash flow, that is, the cash flow generated (or absorbed, if negative) by the project's operations. To this end, it suffices to subtract the change in the operating capital invested in the project from the after-tax operating income. For doing so, the analyst has to add the depreciation charges (Dep_t) which are the opposite of fixed assets' variation, and subtract the change in net operating working capital (NOWC).[2] Letting C_t^o denote the capital invested in the operations at the beginning of period $[t, t+1]$, the operating cash flow is

$$\text{OCF}_t = I_t^o - \Delta C_t^o \tag{7}$$
$$= I_t^o + \text{Dep}_t - \Delta\text{NOWC}_t \tag{8}$$

where Δ denotes variation, so that $\Delta C_t^o = C_t^o - C_{t-1}^o$ (with $C_{-1}^o = 0$).

The OCF represents cash available for distribution to the capital providers (shareholders and debtholders). Part of it is used to service the debt and the residual amount is the so-called *Free Cash Flow to Equity* (FCFE). Mathematically,

$$\text{FCFE}_t = \text{OCF}_t - \text{CFD}_t. \tag{9}$$

When FCFE is positive, it indicates the maximum amount of cash that can be distributed to shareholders without making recourse to additional debt or to cash withdrawals from the firm's existing liquid assets; when it is negative, it indicates that the OCF provided by the operations is not sufficient to service the debt and represents the maximum amount that can be contributed by the shareholders to cover the financial shortage. In other words, FCFE is a financial surplus potentially distributable to shareholders if it is positive, whereas it expresses a financial deficit potentially contributable by shareholders if it is

[1]If one subtracts income taxes from EBT one gets the after-tax earnings, also known as *net income*, which is the profit accrued to equityholders:

$$I_t^e = \text{EBT}_t - \text{T}_t. \tag{6}$$

[2]In a solar PV plant, NOWC is represented by the accounts receivable generated by the sale of excess energy and the accounts payable generated by the purchase of energy from the grid whenever the plant does not meet the firm's electricity needs. Usually, the firm has no degree of freedom on the working capital because, in general, the payment conditions to the service operator are established by the operator. Also, the firm has little bargaining power regarding the credit terms relative to the sale of excess energy. (For the role of working capital in selecting an appropriate measure of value creation, see Magni and Marchioni 2020.)

negative.

3 FCFE and the embedded decisions

The FCFE is the hub of the matter. It is the financial variable which triggers the financing and distribution decisions embedded in the model. Specifically, the firm's analysts must determine, for each period, how a financial deficit (FCFE < 0) will be covered (financing policy) and how a financial surplus (FCFE > 0) should be employed (payout/retention policy). Modeling such decisions for every year explicitly is important because, as we now see, a decision made in one year affects next year's after-tax cash flows and, hence, determines the project's economic profitability.[3] Furthermore, the explicit account of the embedded decisions enables the firm's analysts to study the interrelations of payout policy and financing policy and their impact on value creation, which helps find an optimal financial policy which maximizes shareholder wealth.

In general, once the OCF is estimated and the cash flow to debt is subtracted, two situations may occur for each date:

- $\text{FCFE}_t > 0$: a financial surplus occurs (cash may be distributed); a decision on distribution of cash flow to equityholders is required which also determines automatically the amount of internal reinvestment (retained cash)
- $\text{FCFE}_t < 0$: a financial deficit occurs (cash must be contributed); a decision on contribution of cash flow from equityholders is required which also determines automatically the amount of internal financing (cash withdrawal).[4]

Let CFE_t denote the cash flow actually distributed to equityholders when $\text{FCFE}_t > 0$ or the cash flow actually contributed by shareholders when $\text{FCFE}_t < 0$; a positive CFE indicates that cash is distributed to equityholders and a negative CFE signals that cash is contributed by equityholders.

As noted above, a decision on the CFE is also a decision on the amount of cash withdrawn from the firm's liquid assets to cover the financial deficit (whenever $\text{FCFE}_t < 0$) or the amount of cash retained in the firm and invested in liquid assets (whenever $\text{FCFE}_t > 0$). The decisions on distribution/retention and on equity/debt financing must be explicitly modeled for every year, in order to get the estimation of the next year's incomes and cash flows. Indeed, next year's operating income and cash flow depend on the amount of next year's taxes paid and the latter is affected by the next year's interest income, which in turn depends on this year's balance of liquid assets. For example, if

[3]Traditional modeling often neglects internal financing and usually assumes (implicitly or explicitly) that 100% of a financial surplus is distributed to shareholders and 100% of a financial deficit is covered by equity or debt. In practice, firms often use cash withdrawals from existing liquid assets (internal financing) to finance the installation of solar PV projects and do not distribute all the cash available for distribution but reinvest it, wholly or partially, into liquid assets. To abide by realistic assumptions is important to avoid over- or under-estimation of the project's economic profitability.

[4]Since the FCFE may well be positive or negative, the FCFE is a Free Cash Flow *to* Equity in the former case and a Free Cash Flow *from* Equity in the latter case. In other words, the firm might be said to be *free* to distribute FCFE to shareholders in the former case and to be *free* to ask for equity contribution from shareholders in the latter case.

$\mathrm{OCF}_t = 100$ and $\mathrm{CFD}_t = 40$, the cash available for distribution is $\mathrm{FCFE}_t = 60$. Suppose the firm decides to distribute to shareholders 70% of FCFE and suppose the beginning-of-period balance of liquid assets is $C^l_{t-1} = 500$ and the interest rate on liquid assets is $i^l_t = 1\%$. Then, the cash flow distributed to the firm's shareholders is $\mathrm{CFE}_t = 70\% \cdot 60 = 42$, which means that the retained cash is $60 - 42 = 18$. Only now it is possible to determine the time-t balance of liquid assets by summing the interest income and the retained cash: $C^l_t = 500 + 1\% \cdot 500 + 18 = 523$, which determines the time-t+1 interest income and, therefore, the income taxes and, hence, the time t+1 operating income and cash flow. Viceversa, if $\mathrm{OCF}_t = 100$ and $\mathrm{CFD}_t = 120$, then $\mathrm{FCFE}_t = -20$, which represents a financial deficit. Suppose the firm decides to cover 70% of this financial shortage with equity. The cash flow contributed by the equityholders is $\mathrm{CFE}_t = 70\% \cdot (-20) = -14$, which represents an outlay for shareholders. The residual amount, $20 - 14 = 6$ is financed internally, via cash withdrawal from liquid assets. Hence, the time-t balance of liquid assets is $C^l_t = 500 + 1\% \cdot 500 - 6 = 499$, which in turn impacts the operating income and cash flow in time $t + 1$.

In general, depending on the situation, the balance of liquid asset is set, respectively, as

- $C^l_t = C^l_{t-1} + I^l_t +$ retained cash (internal reinvestment) if $\mathrm{FCFE}_t > 0$
- $C^l_t = C^l_{t-1} + I^l_t -$ cash withdrawal (internal financing) if $\mathrm{FCFE}_t < 0$

The two above equations may be compressed into a single recursive equation:

$$C^l_t = C^l_{t-1} + I^l_t - \mathrm{NOCF}_t \qquad t = 0, 1, \dots, n, \qquad C^l_{-1} = 0 \tag{10}$$

where the NOCF denotes a non-operating cash flow indicating a cash withdrawal if $\mathrm{NOCF}_t > 0$ and cash retained if $\mathrm{NOCF}_t < 0$ As noted, the NOCF is automatically determined by the decisions on equity contribution or distribution:

$$\mathrm{NOCF}_t = \mathrm{CFE}_t - \mathrm{FCFE}_t \tag{11}$$

so that the balance of liquid assets is essentially affected by CFE and FCFE as follows:

$$C^l_t = C^l_{t-1} + I^l_t = C^l_{t-1} + (\mathrm{FCFE}_t - \mathrm{CFE}_t).$$

4 The logical loop

The firm's analysts evaluating an investment opportunity should build a model which computes the streams of OCF, NOCF, and CFD. These cash-flow streams will be used for the calculation of the economic value created, as will be shown in section 6.

The project's cash-flows are dynamically interconnected via a logical loop such that the OCF of the current year, OCF_t, affects the cash available for distribution, FCFE, which affects the non-operating cash flow, NOCF_t, which in turn affects the balance of liquid assets, C^l_t, which affects the next year's interest income, I^l_{t+1} and, in turn, the amount of taxes, T_{t+1} and, hence, the operating income, $I^o_{t+1} = \mathrm{EBIT}_{t+1} - \mathrm{T}_{t+1}$, which in turn affects next year's operating cash flow OCF_{t+1}. The logical loop that needs to be accounted for in the model is then as follows:

$$
\begin{aligned}
\mathrm{OCF}_t &\Longrightarrow \mathrm{FCFE}_t \\
&\Longrightarrow \overbrace{\mathrm{CFE}_t}^{\text{decision}} \\
&\Longrightarrow \mathrm{NOCF}_t \\
&\Longrightarrow C^l_t \\
&\Longrightarrow I^l_{t+1} \\
&\Longrightarrow \mathrm{T}_{t+1} \\
&\Longrightarrow I^o_{t+1} \Longrightarrow \mathrm{OCF}_{t+1}
\end{aligned}
\tag{12}
$$

for $t = 0, 1, 2, \ldots, n-1$. Analytically, using (2)-(8), the loop linking OCF_t and OCF_{t+1}, mediated by the embedded decision about CFE_t, may be expressed as follows:

$$
\begin{aligned}
\mathrm{OCF}_{t+1} &= I^o_{t+1} - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1} - \mathrm{T}_{t+1} - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1} - \tau(\mathrm{EBIT}_{t+1} + I^l_{t+1} - I^d_{t+1}) - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1}(1-\tau) + \tau I^d_{t+1} - \tau I^l_{t+1} - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1}(1-\tau) + \tau I^d_{t+1} - \tau i^l_{t+1} C^l_t - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1}(1-\tau) + \tau I^d_{t+1} - \tau i^l_{t+1}(C^l_{t-1}(1+i^l_t) - \mathrm{NOCF}_t) - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1}(1-\tau) + \tau I^d_{t+1} - \tau i^l_{t+1}(C^l_{t-1}(1+i^l_t) + \mathrm{FCFE}_t - \overbrace{\mathrm{CFE}_t}^{\text{embedded decision}}) - \Delta C^o_{t+1} \\
&= \mathrm{EBIT}_{t+1}(1-\tau) + \tau I^d_{t+1} - \tau i^l_{t+1}(C^l_{t-1}(1+i^l_t) + \mathrm{OCF}_t - \mathrm{CFD}_t - \mathrm{CFE}_t) - \Delta C^o_{t+1}
\end{aligned}
\tag{13}
$$

for $t = 0, 1, \ldots, n-1$.

At time n (terminal date), the project is over and the entire available cash is distributed to equityholders, which is equal to the sum of the last FCFE and the terminal balance of liquid assets (i.e., net balance derived from the cash previously retained and withdrawn, with accumulated interest incomes):

$$
\mathrm{CFE}_n = \overbrace{C^l_{n-1} + I^l_n}^{\substack{\text{terminal balance of}\\ \text{liquid assets}}} + \overbrace{\mathrm{FCFE}_n}^{\mathrm{OCF}_n - \mathrm{CFD}_n}. \tag{14}
$$

It is worth noting that the CFE at time n is the result of decisions made at every date $t = 0, 1, , \ldots, n-1$. These decisions affect the balance of liquid assets at every date, as well as the magnitude of the equity book value at every date. The final liquidation CFE is nothing but the total amount of cash available to the firm, which derives from the liquid assets and from the operations of the last period, net of the debt service of the last period.

The firm's analysts should calculate the balances of all the capitals involved (operating assets, liquid assets, debt, and equity), all the incomes (operating income, interest on liquid assets and on debt, net income), all the cash flows (OCF, NOCF, CFD, and CFE).

Hence, they should collect them in three pro forma statements for each time t: The statement of capitals (or balance sheet), the statement of incomes and the statement of cash flows. The internal consistency of the model must be certified by the following three balancing equations:

$$\text{Statement of capitals} \qquad \overbrace{C_t^o + C_t^l}^{\substack{\text{Capital invested} \\ \text{in the project}}} = \overbrace{C_t^d + C_t^e}^{\substack{\text{Capital raised} \\ \text{from capital providers}}} \tag{15}$$

$$\text{Statement of incomes} \qquad \overbrace{I_t^o + I_t^l}^{\substack{\text{Income generated} \\ \text{by the project}}} = \overbrace{I_t^d + I_t^e}^{\substack{\text{Income accrued} \\ \text{to capital providers}}} \tag{16}$$

$$\text{Statement of cash flows} \qquad \overbrace{\text{OCF}_t + \text{NOCF}_t}^{\substack{\text{Cash flow contributed} \\ \text{by the project}}} = \overbrace{\text{CFD}_t + \text{CFE}_t}^{\substack{\text{Cash flow distributed} \\ \text{to capital providers}}} \tag{17}$$

(see also Magni 2020, Ch. 2).

The logical steps required to determine all the project's cash flows and build the pro forma statements may be summarized as follows:

1. Use the operating inputs to estimate the EBIT (see (1))
2. Subtract the income taxes via (3) to get the after-tax operating income via (2)
3. Add depreciation charges and subtract the change in NOWC to get the OCF (see (7)-(8))
4. Subtract the CFD to get the FCFE (see (9))
5. If the FCFE is positive, make a decision on how to split the available cash between distribution to shareholders and cash retention in the firm. If the FCFE is negative, make a decision on how to split the financial deficit between equity contribution and internal financing. This decision determines the CFE
6. Calculate the NOCF via (11) and determine the balance of liquid assets via (10)
7. Determine the next year's interest income via (4) and the interest on debt via (5)
8. Repeat the steps above for $t = 0, 1, 2, \ldots, n-1$. (For $t = n$, step 5 is replaced by the calculation of CFE_n via (14).)

Figure 1 provides a graphical representation of the logical loop, from start of the system ($t = 0$) to the end of the system ($t = n$).

START

STEPS	$t = -1$	$t = 0$	$t = 1$	$t = 2$	...	$t = n$
1. Use the operating inputs to estimate EBIT (see (1))		$\mathrm{EBIT}_0 = \mathrm{Rev}_0 - \mathrm{OpC}_0 - \mathrm{Dep}_0$ (1)	$\mathrm{EBIT}_1 = \mathrm{Rev}_1 - \mathrm{OpC}_1 - \mathrm{Dep}_1$ (1)	$\mathrm{EBIT}_2 = \mathrm{Rev}_2 - \mathrm{OpC}_2 - \mathrm{Dep}_2$ (1)	...	$\mathrm{EBIT}_n = \mathrm{Rev}_n - \mathrm{OpC}_n - \mathrm{Dep}_n$ (1)
2. Subtract the income taxes via (3) to get the after-tax operating income via (2)		$\mathrm{T}_0 = \tau \cdot \underbrace{\mathrm{EBT}_0}_{(\mathrm{Rev}_0 - \mathrm{OpC}_0 - \mathrm{Dep}_0 + 0 - 0)}$ (3) $I_0^o = \mathrm{EBIT}_0 - \mathrm{T}_0$ (2)	$\mathrm{T}_1 = \tau \cdot \underbrace{\mathrm{EBT}_1}_{(\mathrm{Rev}_1 - \mathrm{OpC}_1 - \mathrm{Dep}_1 + I_1^l - I_1^d)}$ (3) $I_1^o = \mathrm{EBIT}_1 - \mathrm{T}_1$ (2)	$\mathrm{T}_2 = \tau \cdot \underbrace{\mathrm{EBT}_2}_{(\mathrm{Rev}_2 - \mathrm{OpC}_2 - \mathrm{Dep}_2 + I_2^l - I_2^d)}$ (3) $I_2^o = \mathrm{EBIT}_2 - \mathrm{T}_2$ (2)	...	$\mathrm{T}_n = \tau \cdot \underbrace{\mathrm{EBT}_n}_{(\mathrm{Rev}_n - \mathrm{OpC}_n - \mathrm{Dep}_n + I_n^l - I_n^d)}$ (3) $I_n^o = \mathrm{EBIT}_n - \mathrm{T}_n$ (2)
3. Add depreciation charges and subtract the change in NOWC to get the OCF (see (7)-(8))	$C_{-1}^o = 0$	$\mathrm{OCF}_0 = I_0^o - \Delta C_0^o$ (7) $= I_0^o + \mathrm{Dep}_0 - \Delta\mathrm{NOWC}_0$ (8)	$\mathrm{OCF}_1 = I_1^o - \Delta C_1^o$ (7) $= I_1^o + \mathrm{Dep}_1 - \Delta\mathrm{NOWC}_1$ (8)	$\mathrm{OCF}_2 = I_2^o - \Delta C_2^o$ (7) $= I_2^o + \mathrm{Dep}_2 - \Delta\mathrm{NOWC}_2$ (8)	...	$\mathrm{OCF}_n = I_n^o - \Delta C_n^o$ (7) $= I_n^o + \mathrm{Dep}_n - \Delta\mathrm{NOWC}_n$ (8)
4. Subtract the CFD to get the FCFE (see (9))		$\mathrm{FCFE}_0 = \mathrm{OCF}_0 - \mathrm{CFD}_0$ (9)	$\mathrm{FCFE}_1 = \mathrm{OCF}_1 - \mathrm{CFD}_1$ (9)	$\mathrm{FCFE}_2 = \mathrm{OCF}_2 - \mathrm{CFD}_2$ (9)	...	$\mathrm{FCFE}_n = \mathrm{OCF}_n - \mathrm{CFD}_n$ (9)
5. If the FCFE is positive, make a decision on how to split the available cash between distribution to shareholders and cash retention in the firm. If the FCFE is negative, make a decision on how to split the financial deficit between equity contribution and internal financing. This decision determines the CFE		$\Rightarrow \overbrace{\mathrm{CFE}_0}^{\text{decision}}$	$\Rightarrow \overbrace{\mathrm{CFE}_1}^{\text{decision}}$	$\Rightarrow \overbrace{\mathrm{CFE}_2}^{\text{decision}}$	...	$\mathrm{CFE}_n = C_{n-1}^l + I_n^l + \mathrm{FCFE}_n$ (12)
6. Calculate the NOCF via (11) and determine the balance of liquid assets via (10)	$C_{-1}^l = 0$	$\mathrm{NOCF}_0 = \mathrm{CFE}_0 - \mathrm{FCFE}_0$ (11) $C_0^l = 0 + 0 - \mathrm{NOCF}_0$ (10)	$\mathrm{NOCF}_1 = \mathrm{CFE}_1 - \mathrm{FCFE}_1$ (11) $C_1^l = C_0^l + I_1^l - \mathrm{NOCF}_1$ (10)	$\mathrm{NOCF}_2 = \mathrm{CFE}_2 - \mathrm{FCFE}_2$ (11) $C_2^l = C_1^l + I_2^l - \mathrm{NOCF}_2$ (10)	...	$\mathrm{NOCF}_n = \mathrm{CFE}_n - \mathrm{FCFE}_n$ (11) $C_n^l = C_{n-1}^l + I_n^l - \mathrm{NOCF}_n = 0$ (10)
7. Determine the next year's interest income via (4) and the interest on debt via (5)	$I_0^l = i^l \cdot C_{-1}^l = 0$ (4) $I_0^d = i^d \cdot C_{-1}^d = 0$ (5)	$I_1^l = i^l \cdot C_0^l$ (4) $I_1^d = i^d \cdot C_0^d$ (5)	$I_2^l = i^l \cdot C_1^l$ (4) $I_2^d = i^d \cdot C_1^d$ (5)	$I_3^l = i^l \cdot C_2^l$ (4) $I_3^d = i^d \cdot C_2^d$ (5)	...	

8. Repeat the steps above for $t = 0, 1, 2, \ldots, n-1$.

END

Figure 1: The logical loop for calculating the project's cash flows

5 Feeding the model: from inputs to cash flows

In this section, we show how to plug the input factors in the financial model described above, making some assumptions on (the estimation variables and) the embedded decisions.

Consider a firm currently importing energy from electric grid, which is offered the opportunity of switching to solar energy. The solar PV plant will be installed on a land property owned by the company and currently rented. With retail energy, the firm periodically pays a utility bill and receives a rental income from the rent of the land. If the solar PV plant is installed, the firm will stipulate a leasing contract whereby lease payments will be made periodically.[5] The plant will also require operating and maintenance costs (O&M) as well as insurance costs. After several years, at the expiration date, the lessee will pay a lump sum to acquire the plant, which may be financed with debt, equity, or internal financing (i.e., cash withdrawal from the firm's existing liquid assets). Once acquired the property of the plant, the solar PV system will continue to generate electric power for some years. At the end of its useful life, the plant will be removed, and the firm will incur disposal costs.

In terms of benefits and costs, if the retail system is replaced by the PV plant, the cash flows will increase as a result of the cost savings (avoided utility bill), but they will also decrease as a result of the operating and maintenance costs and the lost rental income (the solar panels will be ground-mounted). The two conflicting effects will determine a (positive or negative) change in the firm's EBIT and, hence, in the project's operating cash flow. At the expiration date, the firm will sustain a further expenditure for acquiring the property of the plant, which will bring about further benefits consisting of the ceased lease payments.

In Table 1, the input variables are reported (accompanied by their symbols and units of measure) with the specification of their nature (estimation, decision, operational, financial) and, in addition, with the managerial area with which they are associated.

The quantity of energy consumed for the firm's operations is estimated to be constant through time and equal to q; the current purchase price of energy is p_p, growing at a constant rate g_p per year. The utility bill is paid to the Energy Service Provider in the same year in which energy is consumed. The firm stipulates a lease contract with the following economic conditions: The lease payment, equal to P, is made periodically until the expiration date m; at time m, the firm acquires the property of the plant by paying a lump sum equal to CapEx (capital expenditure), and the solar PV system will keep on producing electric power for some years, until time n. From an accounting perspective, CapEx is a fixed asset, which is assumed to be depreciated evenly from $t = m + 1$ until $t = n$, so that the depreciation charge is Dep $=$ CapEx$/(n - m)$. We assume that the PV plant is installed at $t = 0$ in a field owned by the firm, which is currently rented at a rent equal to R growing at the constant annual rate g_c. The latter represents an opportunity cost for the firm (a foregone income).

[5] A lease contract is an operating variable if lease payments are treated as operating expenses and the asset is not reported in the balance sheet during the lease term; it is a financial variable if it is treated like a loan, in which case accounting effects are shown on the balance sheets.

Table 1: Inputs for a solar PV plant

Input	**Symbol**	**Unit of measure**	**Type**	**Nature**	**Managerial area**
Useful life of PV plant	n	years	Estimation	Operating	Project
Annual unit production (first year)	$Q_{\max}$	kWh/kWp/year	Estimation	Operating	Project
Solar panel degradation rate	g_Q	%	Estimation	Operating	Project
Disposal costs	H	€	Estimation	Operating	Project
Opportunity costs (e.g., foregone rents)	R	€/year	Estimation	Operating	Project
Growth rate for costs	g_c	%	Estimation	Operating	Project
Productivity loss in case of O&M = 0%	ProdLoss	%	Estimation	Operating	Project
Technical suggested O&M and insurance (% of plant's total cost)	SuggO&M	%	Estimation	Operating	Project
Lease expiration date	m	years	Estimation	Operating	Project
Lease payment	P	€/year	Estimation	Operating	Project
Purchase price of plant (at the expiration date)	CapEx	€	Estimation	Operating	Project
Annual energy consumption	q	kWh/year	Estimation	Operating	Company
Tax rate	τ	%	Estimation	Operating	Company
Energy purchase price	p_p	€/kWh	Estimation	Operating	Energy Market
Energy selling price	p_s	€/kWh	Estimation	Operating	Energy Market
Growth rate of energy price	g_p	%	Estimation	Operating	Energy Market
Required return on operating assets	r^o	%	Estimation	Financial	Capital market
Required return on liquid assets	r^l	%	Estimation	Financial	Capital market
Required return on debt	r^d	%	Estimation	Financial	Capital market
Interest rate on liquid assets	i^l	%	Estimation	Financial	Distribution
Interest rate on debt	i^d	%	Estimation	Financial	Financing
Loan tenure	D_m	years	Estimation	Financial	Financing
O&M and insurance (% of plant's total cost)	O&M	%	Decision	Operating	Project
First of year of CFE distribution	d_m	years	Decision	Financial	Distribution
Payout Ratio	α	%	Decision	Financial	Distribution
Equity financing	E	%	Decision	Financial	Financing
Internal financing (cash withdrawal)	L	%	Decision	Financial	Financing
Debt financing	D	%	Decision	Financial	Financing

Starting from the first period, the PV plant requires operating, maintenance and insurance costs, expressed as a percentage of the total cost of the plant, which is the product between its nameplate capacity (in kWp) and its unit cost (per kWp). Technical experts determine a suggested level of these (percentage) costs for the first year in order to maximize the energy production, which we denote as SuggO&M. We denote as O&M the actual percentage) expenses established by the management, which may be equal to or smaller than the suggested ones (i.e., O&M $\leq$ SuggO&M); both are assumed to grow at the constant annual rate g_c.

The solar panel degradation rate is g_Q. If O&M = SuggO&M, the PV system will produce $Q_{\max}$ units of energy in the first year, which decrease every year at the rate g_Q; if O&M = 0 (i.e., the company is not willing to spend for operating and maintenance costs), the energy production suffers from a percentage loss due to lack of maintenance, denoted as ProdLoss. Furthermore, technical experts expect that the effective energy production in each period t, denoted as Q_t, will be proportional to the established level of O&M costs as compared to the suggested level. Specifically,

$$Q_t = Q_{\max}(1 - g_Q)^{t-1} \cdot \Big(1 - \max(\text{ProdLoss} \cdot \frac{\text{SuggO\&M} - \text{O\&M}}{\text{SuggO\&M}}, 0)\Big).$$

If the energy produced by the plant, Q_t, is higher than the energy consumed by the firm, the firm sells the differential quantity to the Energy Service Operator at the energy selling price p_s, growing at a constant rate g_p per year; the grid operator will pay the firm in the following year (this gives rise to accounts receivable). We assume that, at time $t = n$, the energy sold is paid immediately.

As a result, if the annual produced quantity is lower than the consumed energy in year t, that is, $Q_t < q$, energy costs savings arise equal to $Q_t \cdot p_p(1 + g_p)^{t-1}$. If, instead, the produced quantity is higher than the consumed one, that is, $Q_t > q$, two benefits arise:

- energy costs savings arise equal to $q \cdot p_p(1 + g_p)^{t-1}$
- energy sales revenues equal to $(Q_t - q) \cdot p_s(1 + g_p)^{t-1}$ are generated, which determine the presence of NOWC (i.e., accounts receivable).

Overall, the effect of the energy sales revenues and energy costs savings on the operating income can be summarized with the expression $\min(q, Q_t) \cdot p_p(1 + g_p)^{t-1} + \max(0, Q_t - q) \cdot p_s(1 + g_p)^{t-1}$ and the operating working capital can be represented with the formula $\text{NOWC}_t = \max(0, Q_t - q) \cdot p_s(1 + g_p)^{t-1}$ and $\text{NOWC}_n = 0$. (See also Magni and Marchioni 2019).

At time n, disposal costs for removing the plant should be supported by the firm, whose current estimation for $t = 1$ is equal to H, expected to grow at the annual rate g_c in the time interval from 1 to n. Therefore, the expected disposal costs sustained at time n are equal to $H(1 + g_c)^{n-1}$.

To sum up, conceptually

- the firm-without-the-project pays the utility bills and receives the rent for the land (for the whole period);
- the firm-with-the-project sustains the lease payments (until $t = m$), the operating

and maintenance costs (until $t = n$), the lump sum (in $t = m$), and the disposal costs (in $t = n$), and receives cash payments for the energy sold to the Energy Service Operator.

The project is, by definition, the difference between the firm-with-the-project and the firm-without-the project. Therefore, the pre-tax operating income is formally represented by

$$\text{EBIT}_t = \begin{cases} Z - P & \text{for } 1 \leq t \leq m \\ Z - \text{Dep} & \text{for } m+1 \leq t \leq n-1 \\ Z - \text{Dep} - H(1+g_c)^{t-1} & \text{for } t = n. \end{cases} \tag{18}$$

where

$$Z = \min(q, Q_t) \cdot p_p(1+g_p)^{t-1} + \max(0, Q_t - q) \cdot p_s(1+g_p)^{t-1} - R \cdot (1+g_c)^{t-1} - \text{O\&M} \cdot (1+g_c)^{t-1}$$

The project's operating assets, C_t^o, are represented by net operating working capital, NOWC_t, and, from time m on, by fixed assets, net of depreciation, NFA_t:

$$C_t^o = \begin{cases} \overbrace{\max(0, Q_t - q) \cdot p_s(1+g_p)^{t-1}}^{\text{NOWC}_t} & \text{for } 1 \leq t \leq m-1 \\ \overbrace{\max(0, Q_t - q) \cdot p_s(1+g_p)^{t-1}}^{\text{NOWC}_t} + \overbrace{\text{CapEx} - \text{Dep} \cdot (t-m)}^{\text{NFA}_t} & \text{for } m \leq t \leq n-1 \\ 0 & \text{for } t = n. \end{cases} \tag{19}$$

Using (2) and (7)-(8), one gets the OCF in each period. The OCF may be positive or negative (or zero). We assume that, whenever OCF is negative, all the financial needs will be covered by internal financing (cash withdrawal) except at time m, where CapEx is financed

- by equity capital with a proportion equal of $E \leq 1$
- by cash withdrawals from liquid assets (internal financing) with a proportion of $L \leq 1$
- by a loan contract of tenure $D_m = n - m$ with a proportion of $D = 1 - (E + L) \leq 1$.

Whenever OCF is positive, FCFE is calculated subtracting the CFD associated to the loan stipulated at time m (see eq.(9)). If FCFE is negative, the financial needs will be covered with internal financing. If FCFE is positive a decision on payout/retention is required, as seen in the previous sections. Let d_m be the first date at which some CFE is distributed; we assume that the firm will distribute a proportion α of the smaller between the net income and the FCFE, provided that they are both positive, that is

$$\mathrm{CFE}_t = \begin{cases} 0 & \text{for } t = 1, 2, \ldots, d_m - 1 \\ \alpha \cdot \max[0, \min(I_t^e, \mathrm{FCFE}_t)] & \text{for } t = d_m, d_m + 1, \ldots, m-1, m+1, \ldots, n-1 \\ -E \cdot \mathrm{CapEx} & \text{for } t = m. \end{cases} \tag{20}$$

The decision on CFE is also a decision on the amount of cash retained by the firm. The latter is equal to the FCFE minus the CFE (see eq.(11)) so that

$$\mathrm{NOCF}_t = \begin{cases} -\mathrm{FCFE}_t & \text{for } t = 1, 2, \ldots, d_m - 1 \\ \alpha \cdot \max[0, \min(I_t^e, \mathrm{FCFE}_t)] - \mathrm{FCFE}_t & \text{for } t = d_m, d_m + 1, \ldots, m-1, m+1, \ldots, n-1 \\ -E \cdot \mathrm{CapEx} - \mathrm{FCFE}_t & \text{for } t = m. \end{cases} \tag{21}$$

Finally, the project closes at time $t = n$ and we recall that CFE at $t = n$ is not a decision variable, since the available cash resulting from the retention decisions of the previous periods is entirely distributed to shareholders according to (14).

These nontrivial conceptual and formal relationships among estimation and decision variables and the impact on incomes and cash flows testify to the complexity of the financial modeling and suggest that the analyst should build a transparent model, where the embedded decisions are explicitly considered. Failing to do so would invalidate the determination of the financial magnitudes and even the internal consistency of the model.

Once this accounting-and-finance model of the project is built, all the cash flow streams associated with the project will be available. Using these cash-flow streams, the project is evaluated and the decision on whether undertaking the project or not will be made. In the next section, we illustrate the appraising process.

6 Shareholder value creation

In the previous sections, we have shown the first part of the financial model, consisting in drawing up three pro forma statements for the capitals, the incomes, and the cash flows. The second part of the financial modeling has to do with the evaluation of the project on the basis of those statements, taking the point of view of the firm's shareholders.

Since the manager's mandate is to increase the wealth of the firm's shareholders, once the three pro forma statements have been built, the analyst must evaluate the shareholder value created by the project. As known, evaluation depends on the (opportunity) cost of capital: The economic (or market) value of any cash-flow stream is obtained by discounting its cash flows at the (assumed constant) expected rate of return on an equivalent-risk asset traded in the (assumed efficient) capital market: $V_0 = \sum_{t=1}^{n} F_t (1+r)^{-t}$. It represents the price that the cash-flow stream would have it were traded in the market.

Let r^o, r^l, r^d be, respectively, the required return on OCFs, the required return on NOCFs, and the required return on CFD, as estimated by the analyst.[6] Then, the eco-

[6]Required returns are usually estimated by summing the risk-free rate to a risk premium compensating for

nomic values of the OCF stream, NOCFs stream and CFD stream are, respectively,

$$
\begin{aligned}
V_0^o &= \sum_{t=1}^{n} \frac{\mathrm{OCF}_t}{(1+r^o)^t} \\
V_0^l &= \sum_{t=1}^{n} \frac{\mathrm{NOCF}_t}{(1+r^l)^t} \\
V_0^d &= \sum_{t=1}^{n} \frac{\mathrm{CFD}_t}{(1+r^d)^t}.
\end{aligned}
\tag{22}
$$

Subtracting the respective initial capital, one gets the net present value (NPV) of the three areas:

$$
\begin{aligned}
\mathrm{NPV}^o &= V_0^o - C_0^o && \text{operating NPV} \\
\mathrm{NPV}^l &= V_0^l - C_0^l && \text{non-operating NPV} \\
\mathrm{NPV}^d &= V_0^d - C_0^d && \text{debt NPV.}
\end{aligned}
\tag{23}
$$

The first one is the economic value generated by the operations (specifically, the production and consumption of energy, the maintenance of the plant, and the sale of excess energy to the grid operator); the second one is the economic value jointly generated by the internal financing and the reinvestment in liquid assets of the retained cash. The sum of NPV^o and NPV^l is the project's NPV, that is, the economic value created by the project as a result of the operations and the management of the non-operating cash flows. The third one is the part of the project's NPV which is grasped by debtholders. (All of these NPVs may be either positive or negative or zero.) The residual amount obtained by subtracting the debt NPV from the project's NPV is the equity NPV, that is, the economic value created by the project and accrued to shareholders, after honoring the cash flows to debtholders:

$$
\mathrm{NPV}^e = \overbrace{\mathrm{NPV}^o + \mathrm{NPV}^l}^{\text{project's NPV}} - \mathrm{NPV}^d \tag{24}
$$

or, which is the same,

$$
\mathrm{NPV}^e = \overbrace{(V_0^o + V_0^l - V_0^d)}^{\text{market value of equity}} - \overbrace{(C_0^o + C_0^l - C_0^d)}^{\text{initial equity investment}} = V_0^e - C_0^e. \tag{25}
$$

NPV decision criterion. A project is worth undertaking if and only if it creates value for its equityholders, that is, $\mathrm{NPV}^e > 0$.

Owing to (24), shareholder value may be broken down to three components: The operating assets, the liquid assets, and the debt. Equity-holders may then benefit not just from a value-creating operating activity (i.e., $\mathrm{NPV}^o > 0$), but also from an efficient management of liquid assets (i.e., $\mathrm{NPV}^l > 0$) and from the ability of borrowing at a rate i^d which is lower than the cost of debt, r^d, that is, the equilibrium rate prevailing in the capital

risk. This is established by the market, possibly integrating it with subjective consideration (see Damodaran 1999, 2006, Titman and Martin 2016, Berk and DeMarzo 2014, Magni 2020, Sect. 5. See also Boudreaux et al. 2011, Bora and Vanek 2017 for the use of buil-up models).

markets (i.e., $\text{NPV}^d < 0$). At the same time, the equity NPV may be positive even if the operating NPV is negative, as long as the management of the financial variables is efficient (resulting in $\text{NPV}^l > 0$ and $\text{NPV}^d < 0$).
If one assumes $i^d = r^d$, then the market value of debt coincides with the nominal value of debt, that is, $V_0^d = C_0^d$, which means $\text{NPV}^d = 0$. In this case, the equity NPV is the sum of operating NPV and non-operating NPV:

$$\text{NPV}^e = \text{NPV}^o + \text{NPV}^l. \tag{26}$$

If, in addition, $i^l = r^l$, then $V_0^l = C_0^l$ and $\text{NPV}^l = 0$ so that the equity NPV is equal to the operating NPV:

$$\text{NPV}^e = \text{NPV}^o. \tag{27}$$

In the next section, we apply the model to a firm facing the opportunity of switching from retail energy to solar energy.

7 Example

In this section we analyse value creation for a solar PV plant which has been recently offered to a small-sized firm by a solar PV installer company (in which one of the authors of this paper works). The specific assumptions about estimation inputs and decision inputs are reported in Table 2. From the input data, the statements of capitals (balance sheets), the income statements, and cash flow statements are drawn in the way described in the previous sections. We report them in Figure 2.[7] The three pro forma statements are logically interconnected in the non-trivial logical loop described in (12) and formalized in (13), owing to the embedded decisions: The decisions on financing and cash flow distribution affect the amount of liquid assets; this in turn affects next-period interest on liquid assets, which in turn affects next-period operating income and, therefore, the next-period OCF.

As described in the first part of this paper, the model logically chains estimated data and decisions regarding the proportions of distribution and retention and the proportions of equity financing and internal financing. For example, to build the balance of liquid assets at $t = 24$, C_{24}^l, we consider the balance of liquid assets at time $t = 23$, which is $C_{23}^l = 2,390.66$. This amount increases by the interest income $I_{24}^l = i^l \cdot C_{23}^l = 0.5\% \cdot 2,390.66 = 11.95$, and by the retained cash (i.e., the amount not distributed to the equityholders) at time $t = 24$. The latter is obtained via eq. (11) as

$$\begin{aligned}
\text{Retained cash} &= -\text{NOCF}_{24} \\
&= \text{FCFE}_{24} - \text{CFE}_{24} \\
&= \text{FCFE}_{24} - \alpha \cdot \max[0, \min(I_{24}^e,\ \text{FCFE}_{24})] \\
&= 3,279.58 - 50\% \cdot \max[0, \min(869.72,\ 3,279.58)] \\
&= 3,279.58 - 50\% \cdot 869.72 = 2,844.72.
\end{aligned}$$

[7] We assume that the debt is reimbursed with level payments and that the plant's total cost is 96,600 euro, obtained as the product of the plant's nameplate capacity (92 kWp) and its unit cost (1,050 euro per kWp).

Table 2: Assumptions

Input	Assumption
ESTIMATED VARIABLES	
Useful life of PV plant	$n = 25$ years
Annual unit production (first year)	$Q_{\max} = 1,080.00$ kWh/kWp/year
Solar panel degradation rate	$g_Q = 0.90\%$
Disposal costs	H =5,000 €
Lost rent from land property	$R = 3,000$ €/year
Growth rate for costs	$g_c = 1.25\%$
Productivity loss in case of O&M = 0%	ProdLoss= 15%
Technical suggested O&M and insurance	SuggO&M= 4%
Lease expiration date	$m = 20$ years
Lease payment	$P = 6,268.45$ €/year
Purchase price of plant (at the expiration date)	CapEx $= 25,000$€
Required return on operating assets	$r^o = 6\%$
Required return on liquid assets	$r^l = 3\%$
Required return on debt	$r^d = 3\%$
Annual energy consumption	$q = 30,000$kWh/year
Tax rate	$\tau = 27.9\%$
Energy purchase price	$p_p = 0.160$€/kWh
Energy selling price	$p_s = 0.130$€/kWh
Growth rate of energy price	$g_p = 1.25\%$
Interest rate on liquid assets	$i^l = 0.50\%$
Interest rate on debt	$i^d = 4\%$
Loan tenure (dependent variable)	$D_m = n - m = 5$ years
DECISION VARIABLES	
O&M and insurance	O&M =3.50%
First year of CFE distribution	$d_m = 15^{\text{th}}$ year
Payout Ratio	$\alpha = 50.0\%$
Equity financing	$E = 25\%$
Internal financing (cash withdrawal)	$L = 25\%$
Debt financing (loan)	$D = 1 - (E + L) = 50\%$

As a result, using (10), the balance of liquid assets at time $t = 24$ as $C^l_{24} = C^l_{23} + I^l_{24} - \text{NOCF}_{24} = 2,390.66 + 11.95 + 2,844.72 = 5,247.33$.[8] This amount enables calculating the terminal CFE at time $t = 25$ via eq. (14) as $\text{FCFE}_{25} + C^l_{24} + I^l_{25} = 6,849.34 + 5,247.33 + 26.24 = 12,122.91$, which entirely liquidates the investment project.[9]

The pro forma financial statements in Figure 2 represent the changes in the pro forma financial statements of the firm as a result of switching to solar energy. For example, the revenues express the increase in the firm's revenues, the operating costs express the increase in the firm's operating costs (note that, from year 21 to year 24, a decrease in the firm's operating costs occurs, since the cost savings due to avoided bills outweigh the plant's operating and maintenance costs). The last line of the cash-flow statement highlights the project's non-operating cash flows. In year 1, a decrease of cash occurs in order to cover the financial deficit ($\text{NOCF}_1 > 0$). From year 2 to year 19 part of FCFE generated by the project is retained in the firm ($\text{NOCF}_t < 0$). In year 20, cash is withdrawn again from liquid assets ($\text{NOCF}_{20} > 0$) to partially finance the purchase of the solar PV plant. From year 21 to year 24, part of the FCFE is retained($\text{NOCF}_t > 0$) and,

[8]Only after this computation is done, the balance sheet at time $t = 24$ may be completed by calculating the equity capital:

$$C^e_{24} = C^o_{24} + C^l_{24} - C^d_{24} = 13,510.01 + 5,247.33 - 2,699.85 = 16,057.50.$$

[9]The terminal CFE may equivalently be obtained as $C^e_{24} + I^e_{25} = 16,057.50 - 3,934.59 = 12,122.91$, confirming the logical consistency of (this part of) the model.

in year 25, cash is distributed to capital providers ($\text{NOCF}_{25} > 0$).

The expected (equity) NPV is the model output. With the assumptions made, it is slightly positive: $\text{NPV}^e = 32.84 > 0$. A better understanding of this result is presented in the following table, obtained from (24):

Equity NPV decomposition	
+ NPV of operating assets	$+\text{NPV}^o = -1,188.91$
+ NPV of liquid assets	$+\text{NPV}^l = 1,420.57$
− NPV of debt	$-\text{NPV}^d = -198.81$
= NPV of equity	$= \text{NPV}^e = 32.84$

The NPV of the OCFs is negative and tends to destroy value. However, it would be unwise to recommend rejection on the basis of this operating NPV alone. The NPV of the NOCFs (i.e., the cash withdrawals and the reinvestment in financial assets) creates more value than the operating assets destroy: The way the firm will manage the financial policy is able to compensate and turn an otherwise unprofitable project into a profitable one. The NPV of the project is then $\text{NPV}^o + \text{NPV}^l = -1,188.91 + 1,420.57 = 231.66$. Part of the project's value created is captured by debtholders; specifically, equityholders lose this part of the value created at the expense of the debtholders, but this loss is tiny, due to the limited scale of the debt. As a result, the assumptions made are such that the financial decisions more than compensate, albeit slighlty, the negative performance of the operations. This case testifies to the importance of the financial variables and, in particular, of the embedded decisions, in creating value.

The next section shows that the value created by this project may be changed (increased or decreased) by changing choices of financing and distribution.

BALANCE SHEET	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
BS INVESTMENTS																										
Operating Assets	-	8,775	8,769	8,763	8,756	8,749	8,741	8,733	8,724	8,715	8,705	8,695	8,684	8,673	8,661	8,648	8,635	8,622	8,608	8,593	33,578	28,562	23,545	18,528	13,510	-
Accounts receivable from grid operator	-	*8,775*	*8,769*	*8,763*	*8,756*	*8,749*	*8,741*	*8,733*	*8,724*	*8,715*	*8,705*	*8,695*	*8,684*	*8,673*	*8,661*	*8,648*	*8,635*	*8,622*	*8,608*	*8,593*	*8,578*	*8,562*	*8,545*	*8,528*	*8,510*	-
Net fixed assets	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	*25,000*	*20,000*	*15,000*	*10,000*	*5,000*	-
Liquid assets	-	-8,108	-7,482	-6,873	-6,281	-5,707	-5,150	-4,612	-4,093	-3,593	-3,115	-2,657	-2,221	-1,807	-1,417	-1,227	-1,050	-885	-733	-595	-6,610	-3,541	-540	2,391	5,247	-
INVESTMENTS	**-**	**667**	**1,286**	**1,889**	**2,474**	**3,042**	**3,591**	**4,121**	**4,631**	**5,121**	**5,590**	**6,038**	**6,463**	**6,865**	**7,244**	**7,421**	**7,585**	**7,737**	**7,874**	**7,998**	**26,967**	**25,021**	**23,005**	**20,918**	**18,757**	**-**
BS FINANCINGS																										
Debt	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	12.500	10.192	7.792	5.296	2.700	-
Equity	-	667	1,286	1,889	2,474	3,042	3,591	4,121	4,631	5,121	5,590	6,038	6,463	6,865	7,244	7,421	7,585	7,737	7,874	7,998	14,467	14,828	15,213	15,623	16,057	-
FNIANCINGS	**-**	**667**	**1,286**	**1,889**	**2,474**	**3,042**	**3,591**	**4,121**	**4,631**	**5,121**	**5,590**	**6,038**	**6,463**	**6,865**	**7,244**	**7,421**	**7,585**	**7,737**	**7,874**	**7,998**	**26,967**	**25,021**	**23,005**	**20,918**	**18,757**	**-**

INCOME STATEMENT	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Revenues	**-**	**5,775**	**5,731**	**5,687**	**5,642**	**5,596**	**5,549**	**5,500**	**5,451**	**5,401**	**5,350**	**5,298**	**5,245**	**5,190**	**5,135**	**5,078**	**5,021**	**4,962**	**4,902**	**4,841**	**4,779**	**4,715**	**4,651**	**4,585**	**4,518**	**4,449**
New revenue: sale of energy	-	*8,775*	*8,769*	*8,763*	*8,756*	*8,749*	*8,741*	*8,733*	*8,724*	*8,715*	*8,705*	*8,695*	*8,684*	*8,673*	*8,661*	*8,648*	*8,635*	*8,622*	*8,608*	*8,593*	*8,578*	*8,562*	*8,545*	*8,528*	*8,510*	*8,492*
Lost rent from land property	-	*-3,000*	*-3,038*	*-3,075*	*-3,114*	*-3,153*	*-3,192*	*-3,232*	*-3,273*	*-3,313*	*-3,355*	*-3,397*	*-3,439*	*-3,482*	*-3,526*	*-3,570*	*-3,614*	*-3,660*	*-3,705*	*-3,752*	*-3,799*	*-3,846*	*-3,894*	*-3,943*	*-3,992*	*-4,042*
(-) Operating costs	**-**	**-4,849**	**-4,832**	**-4,814**	**-4,796**	**-4,777**	**-4,759**	**-4,740**	**-4,721**	**-4,701**	**-4,682**	**-4,662**	**-4,642**	**-4,621**	**-4,601**	**-4,580**	**-4,559**	**-4,537**	**-4,516**	**-4,494**	**-4,472**	**1,819**	**1,842**	**1,865**	**1,888**	**-4,825**
(-) Lease annual payment	-	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	*-6,268*	-	-	-	-	-
(-) O&M cost	-	*-3,381*	*-3,423*	*-3,466*	*-3,509*	*-3,553*	*-3,598*	*-3,643*	*-3,688*	*-3,734*	*-3,781*	*-3,828*	*-3,876*	*-3,925*	*-3,974*	*-4,023*	*-4,074*	*-4,124*	*-4,176*	*-4,228*	*-4,281*	*-4,335*	*-4,389*	*-4,444*	*-4,499*	*-4,555*
(-) Disposal costs	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	*-6,737*
(-) Cost saving: self consumption of energy	-	*4,800*	*4,860*	*4,921*	*4,982*	*5,045*	*5,108*	*5,171*	*5,236*	*5,302*	*5,368*	*5,435*	*5,503*	*5,572*	*5,641*	*5,712*	*5,783*	*5,855*	*5,929*	*6,003*	*6,078*	*6,154*	*6,231*	*6,309*	*6,387*	*6,467*
EBITDA	**-**	**925**	**900**	**873**	**846**	**819**	**790**	**761**	**731**	**700**	**668**	**636**	**603**	**569**	**534**	**499**	**462**	**425**	**386**	**347**	**307**	**6,535**	**6,493**	**6,450**	**6,406**	**-375**
(-) Depreciation	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-5,000	-5,000	-5,000	-5,000	-5,000
EBIT	**-**	**925**	**900**	**873**	**846**	**819**	**790**	**761**	**731**	**700**	**668**	**636**	**603**	**569**	**534**	**499**	**462**	**425**	**386**	**347**	**307**	**1,535**	**1,493**	**1,450**	**1,406**	**-5,375**
Interest income	-	-	-41	-37	-34	-31	-29	-26	-23	-20	-18	-16	-13	-11	-9	-7	-6	-5	-4	-4	-3	-33	-18	-3	12	26
(-) interest expenses	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-500	-408	-312	-212	-108
EBT	**-**	**925**	**859**	**836**	**812**	**787**	**762**	**735**	**708**	**680**	**651**	**621**	**590**	**558**	**525**	**491**	**456**	**419**	**382**	**344**	**304**	**1,002**	**1,067**	**1,136**	**1,206**	**-5,457**
(-) Taxes	-	-258	-240	-233	-227	-220	-212	-205	-197	-190	-181	-173	-165	-156	-147	-137	-127	-117	-107	-96	-85	-279	-298	-317	-337	1,523
NET INCOME	**-**	**667.0**	**619.4**	**602.7**	**585.4**	**567.5**	**549.1**	**530.0**	**510.3**	**490.0**	**469.0**	**447.4**	**425.2**	**402.2**	**378.6**	**354.3**	**328.7**	**302.4**	**275.4**	**247.7**	**219.3**	**722.2**	**769.6**	**818.7**	**869.7**	**-3,934.6**

EBIT	**-**	**925**	**900**	**873**	**846**	**819**	**790**	**761**	**731**	**700**	**668**	**636**	**603**	**569**	**534**	**499**	**462**	**425**	**386**	**347**	**307**	**1,535**	**1,493**	**1,450**	**1,406**	**-5,375**
(-) Taxes	-	-258	-240	-233	-227	-220	-212	-205	-197	-190	-181	-173	-165	-156	-147	-137	-127	-117	-107	-96	-85	-279	-298	-317	-337	1,523
OPERATING INCOME	**-**	**667**	**660**	**640**	**620**	**599**	**578**	**556**	**533**	**510**	**487**	**463**	**438**	**413**	**388**	**361**	**335**	**308**	**280**	**251**	**222**	**1,255**	**1,195**	**1,133**	**1,070**	**-3,853**

CASH FLOW	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
(+) OCF	-	-8,108	666	646	627	606	585	564	542	520	497	473	449	425	400	374	348	321	294	266	-24,762	6,271	6,212	6,150	6,087	9,657
(-) CFD	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	12,500	-2,808	-2,808	-2,808	-2,808	-2,808
FCFE (Free Cash Flow for Equity)	**-**	**-8,108**	**666**	**646**	**627**	**606**	**585**	**564**	**542**	**520**	**497**	**473**	**449**	**425**	**400**	**374**	**348**	**321**	**294**	**266**	**-12,262**	**3,463**	**3,404**	**3,342**	**3,280**	**6,849**
(-) CFE	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-177	-164	-151	-138	-124	6,250	-361	-385	-409	-435	-12,123
(-) NOCF	**-**	**-8,108**	**666**	**646**	**627**	**606**	**585**	**564**	**542**	**520**	**497**	**473**	**449**	**425**	**400**	**197**	**183**	**170**	**156**	**142**	**-6,012**	**3,102**	**3,019**	**2,933**	**2,845**	**-5,274**
NOCF		**8,108**	**-666**	**-646**	**-627**	**-606**	**-585**	**-564**	**-542**	**-520**	**-497**	**-473**	**-449**	**-425**	**-400**	**-197**	**-183**	**-170**	**-156**	**-142**	**6,012**	**-3,102**	**-3,019**	**-2,933**	**-2,845**	**5,274**
interpretation		*cash withdrawal*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*cash withdrawal*	*retained cash*	*retained cash*	*retained cash*	*retained cash*	*cash withdrawal*

Figure 2: Balance Sheets, Income Statements, Cash-flow Statements of the solar PV project.

8 Scenario analysis

The impact of the financing-and-distribution decisions on the value of a solar PV project may be best appreciated by considering the project presented in Section 7 and showing the effect of changes in the assumptions on the equity NPV. Table 3 and Figures 3-4 present 8 scenarios with different assumptions on how financial deficits are covered and how financial surplus are employed. In particular, we have considered different proportions of financing sources for the purchase of the solar PV plant, different payout ratios and different years as first year of cash distribution. Scenario 4 is the base scenario described in Section 7. In this scenario, the financial policy offsets the negative performance of the operations. In scenario 1 to 3, where equity financing for the purchase of the PV plant is predominant, the payout ratio is low, and the firm distributes the available cash late, the financial decisions do not compensate the bad performance of the operations. However, by (i) increasing the payout ratio, (ii) anticipating the cash distributions, (iii) reducing the equity financing, and (iv) increasing the internal financing, the equity NPV is greatly increased. The variance between scenario 1 and scenario 8 is significant, as illustrated in Figure 4: The equity NPV turns from a negative -772.69€ (scenario 1) to a positive $3,041.44$€ (scenario 8) representing an increase of $3,814.13$€.

Table 3: Scenario analysis for the decision inputs

SCENARIO	1^{st} year of CFE distribution (d_m)	Payout Ratio (α)	Equity financing (E)	Internal financing (L)	Debt financing (D)	Equity NPV (NPV^e)
1	25	0%	100%	0%	0%	−772.69
2	25	0%	75%	0%	25%	−642.60
3	20	25%	50%	25%	25%	−202.75
4	15	50%	25%	25%	50%	32.84
5	10	50%	0%	50%	50%	651.21
6	5	50%	0%	75%	25%	1,331.60
7	1	75%	0%	75%	25%	2,215.90
8	1	100%	0%	100%	0%	3,041.44

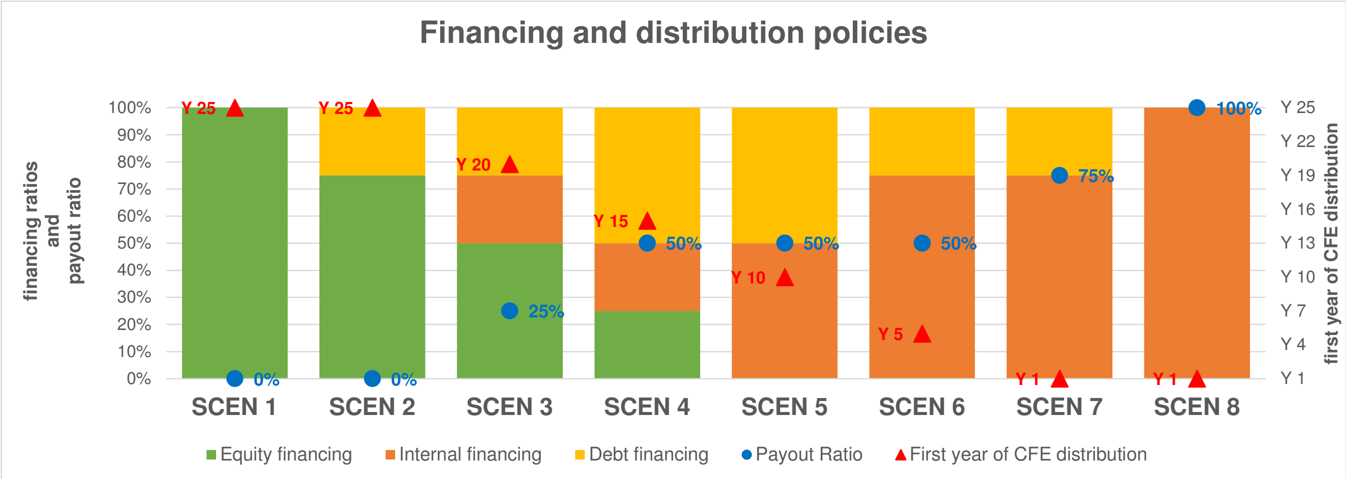

Figure 3: Scenario analysis: Financing and distribution policies (base scenario = 4)

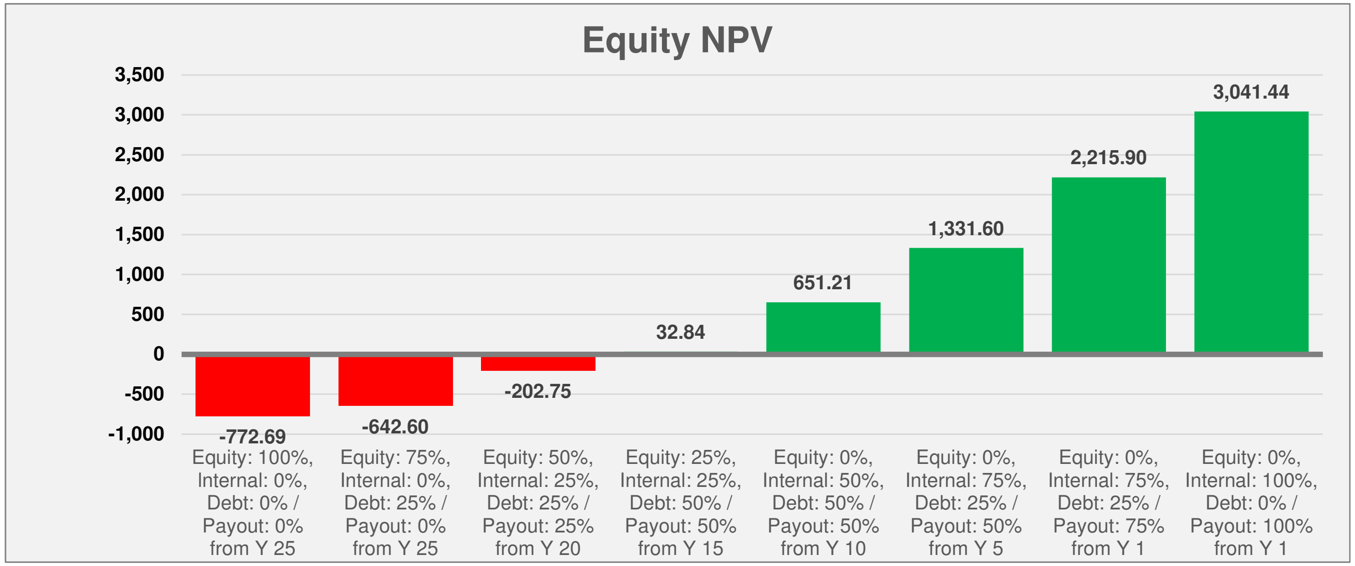

Figure 4: Scenario analysis: Equity NPV for 8 different financing and distribution policies

It is easy to see that the choice of internal financing for purchasing the solar PV plant at time m (expiration date of the lease contract) contributes positively to create value. To see it, consider scenario 8, where 100% internal financing is assumed and the resulting equity NPV is 3,041.44. With distribution policy unvaried (i.e., $d_m = 1$ and $\alpha = 100\%$) if one assumes 100% equity financing or 100% debt financing or a mix of equity and debt, then the equity NPV will be much smaller than 3,041.44. Following are some results for different mixes of financing sources in scenario 8:

100% internal financing (scenario 8)	$\text{NPV}^e = 3{,}041.44$
100% equity, 0% debt	$\text{NPV}^e = 1{,}410.84$
80% equity, 20% debt	$\text{NPV}^e = 1{,}499.57$
60% equity, 40% debt	$\text{NPV}^e = 1{,}588.29$
50% equity, 50% debt	$\text{NPV}^e = 1{,}632.65$
40% equity, 60% debt	$\text{NPV}^e = 1{,}677.02$
20% equity, 80% debt	$\text{NPV}^e = 1{,}765.74$
0% equity, 100% debt	$\text{NPV}^e = 1{,}865.36$

We mention that the equity NPV increases as the equity financing decreases and debt financing increases until rlower maximum value of $1,865.36$ is achived with 100% debt, which is still much lower than the equity NPV esulting in the original scenario 8 with 100% of internal financing.

It is worth noting that the impact of the financing-and-distribution decisions on shareholder value creation is sensitive to other input factors, which may amplify or shrink their effect. Next, we analyze a change in the interest rate on liquid asset (shrinking effect) and a change in the annual energy consumption (magnifying effect).

Table 4 shows that by increasing i^l, other things unvaried, the effect of financial decisions on equity NPV is diminished, as testified by the max-min deviation (last line). Note that, given a scenario, the higher the interest rate, the lower the equity NPV. This is because, in most periods, the balance of liquid assets is negative (indicating that the project entails a reduction in the firm's liquid assets), so i^l represents a foregone rate of return on those liquid assets.

Table 4: NPV deviations under different assumptions of interest rate on liquid assets

Interest rate (i^l)	0.5%	1.5%	2%	2.5%	3%	3.5%
SCENARIO 1	−772.69	−815.11	−867.17	−942.69	−1,044.07	−1,173.90
SCENARIO 2	−642.60	−741.43	−821.99	−926.22	−1,056.50	−1,215.46
SCENARIO 3	−202.75	−444.29	−597.63	−775.54	−980.38	−1,214.72
SCENARIO 4	32.84	−301.90	−501.94	−726.19	−976.64	−1,255.42
SCENARIO 5	651.21	109.55	−193.55	−520.05	−871.47	−1,249.41
SCENARIO 6	1,331.60	572.24	163.38	−266.20	−717.48	−1,191.51
SCENARIO 7	2,215.90	1,131.89	575.30	8.72	−568.05	−1,155.23
SCENARIO 8	3,041.44	1,664.88	975.66	285.83	−404.62	−1,095.70
Max-Min deviation	3,814.13	2,479.99	1,842.84	1,228.52	651.88	159.72

Consider now a change in the annual energy consumption. Table 5 shows that, by increasing the annual energy consumption (q), other things unvaried, the effect of financial decisions on NPV is augmented, as measured via the max-min deviation. Note that, for any given scenario, a higher energy consumption generates a higher NPV. The reason is that higher q implies higher cost savings, higher net incomes and, therefore, higher CFE, which results in higher NPVs. This effect is amplified by the financing and distribution decisions.

Table 5: NPV deviations under different assumptions of annual energy consumption

Energy consumption (q)	20,000	25,000	30,000	40,000	50,000	60,000
SCENARIO 1	−3,110.53	−1,941.61	−772.69	1,565.16	3,903.00	6,240.84
SCENARIO 2	−2,980.45	−1,811.53	−642.60	1,695.24	4,033.08	6,370.93
SCENARIO 3	−2,550.31	−1,376.53	−202.75	2,144.80	4,492.36	6,839.92
SCENARIO 4	−2,400.37	−1,184.89	32.84	2,468.31	4,903.78	7,338.79
SCENARIO 5	−1,909.79	−630.48	651.21	3,214.60	5,777.99	8,341.39
SCENARIO 6	−1,407.68	−39.35	1,331.60	4,073.49	6,815.39	9,557.28
SCENARIO 7	−916.45	647.42	2,215.90	5,352.85	8,489.80	11,626.75
SCENARIO 8	−349.03	1,342.88	3,041.44	6,438.58	9,835.71	13,232.84
Max-Min deviation	2,761.50	3,284.48	3,814.13	4,873.42	5,932.71	6,991.99

As a result, for a given set of estimation variables, different financial decisions lead to different NPVs and different sets of estimation variables make those decisions more or less impactful. Figure 5 illustrates the effects of different scenarios and different assumptions on input values (interest rates in the top charts and energy consumption in the bottom charts).

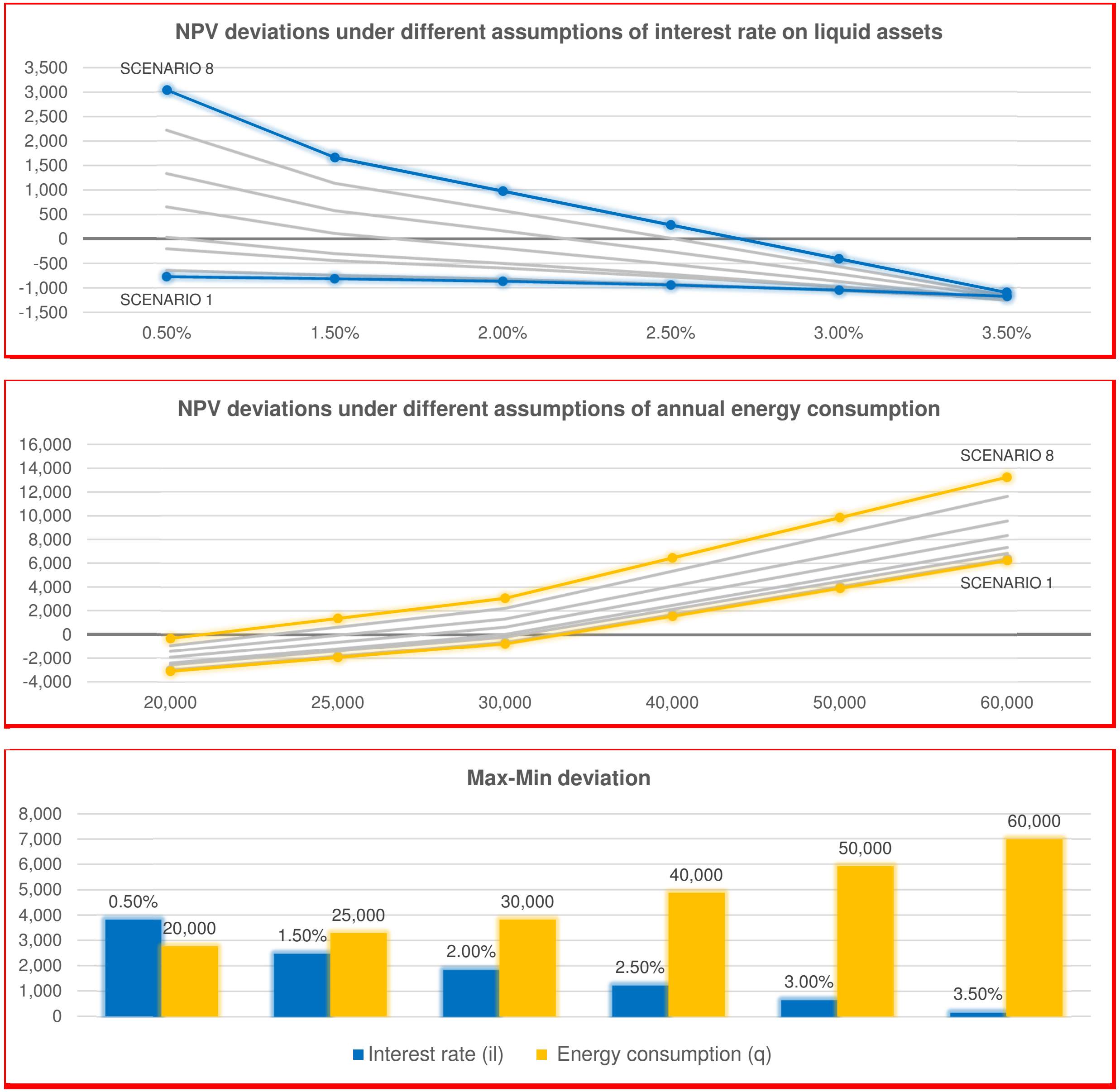

Figure 5: NPV deviations under different assumptions of interest rate on liquid asset and different assumptions of annual energy consumption, associated with different financing/payout policies (scenarios 1 to 8)

Furthermore, simultaneous changes in more than one inputs have different effects under different financial policies. For example, consider the effect of a simultaneous change in energy consumption q and interest rate i^l . Tables 6 and 7 describe the effects of different values of the pair (q, i^l) on the equity NPV for scenario 1 and scenario 8, other things unvaried. The effects are different for different scenarios and the effect of the financing and payout policy changes in the NPV in the two scenarios may be greater or smaller depending on the value of the pair (q, i^l). For example, when $(q, i^l) = (20{,}000, 3.5\%)$ the effect of the financing/payout policy is minimum and the financing/payout policy in scenario 1 is only slightly better than scenario 8 $(-5,178.95-(-5,203.53) = 24.59)$, with scenario 8 being incapable of making the project economically profitable. In contrast, when $(q, i^l) = (60{,}000, 0.5\%)$ the effect of the financing/payout policy is maximum, with a sharp increase in value creation changing the financing/payout policy from scenario 1 to scenario 8: $13{,}232.84 - 6{,}240.84 = 9{,}835.71$. In some cases, the financing/payout policy even reverts the sign of the project's economic profitability; for example, when $(q, i^l) = (30{,}000, 0.5\%)$, by changing the financing/payout policy from scenario 1 to scenario 8, the financial efficiency turns from negative ($\text{NPV}^e = -772.69$) to positive ($\text{NPV}^e = 3,041.44$) with a sharp increase of 3,814.13 (see Table 8).

Table 6: NPV for different values of q and i^l - Scenario 1

(q, i^l)	0.50%	1.50%	2.00%	2.50%	3.00%	3.50%
20,000	−3,110.53	−3,646.82	−3,970.33	−4,335.15	−4,744.91	−5,203.53
25,000	−1,941.61	−2,230.97	−2,418.75	−2,638.92	−2,894.49	−3,188.72
30,000	−772.69	−815.11	−867.17	−942.69	−1,044.07	−1,173.90
40,000	1,565.16	2,016.61	2,235.98	2,449.76	2,656.77	2,855.73
50,000	3,903.00	4,848.32	5,339.14	5,842.22	6,357.61	6,885.36
60,000	6,240.84	7,680.03	8,442.30	9,234.67	10,058.45	10,914.99

Table 7: NPV for different values of q and i^l - Scenario 8

(q, i^l)	0.50%	1.50%	2.00%	2.50%	3.00%	3.50%
20,000	−349.03	−1,940.62	−2,738.68	−3,541.94	−4,353.89	−5,178.95
25,000	1,342.88	−148.22	−894.29	−1,640.39	−2,387.85	−3,138.49
30,000	3,041.44	1,664.88	975.66	285.83	−404.62	−1,095.70
40,000	6,438.58	5,291.06	4,716.59	4,141.64	3,566.22	2,990.32
50,000	9,835.71	8,917.24	8,457.52	7,997.45	7,537.06	7,076.34
60,000	13,232.84	12,543.43	12,198.44	11,853.27	11,507.90	11,162.35

9 Contribution of financing and distribution to value creation

In the light of what we have seen in the previous sections, it should be clear that the decision variables may play a significant role in increasing or decreasing the attractiveness

Table 8: NPV deviations for different values of q and i^l - Scenario 1 vs Scenario 8

(q, i^l)	0.50%	1.50%	2.00%	2.50%	3.00%	3.50%
20,000	2,761.50	1,706.20	1,231.65	793.21	391.01	24.59
25,000	3,284.48	2,082.75	1,524.46	998.53	506.64	50.23
30,000	3,814.13	2,479.99	1,842.84	1,228.52	639.44	78.21
40,000	4,873.42	3,274.46	2,480.60	1,691.88	909.45	134.59
50,000	5,932.71	4,068.93	3,118.37	2,155.24	1,179.45	190.98
60,000	6,991.99	4,863.40	3,756.14	2,618.59	1,449.45	247.37

of the solar PV project. One may wonder whether the contribution to value creation of the financing policy is smaller or greater than the contribution of the payout policy. Given the complexity of the relations between the estimation variables and the decision variables, and even among variables within the same group, there is no general answer. However, for each situation, one can find the contribution to the change in the equity NPV of the distribution policy as opposed to the financing policy with the following simple technique. Let f be the group of financing variables (equity, liquid assets, debt) and d be the group of distribution variables (payout ratio, first year of distribution):

$$f = (E, L, D), \qquad d = (d_m, \alpha).$$

We focus on the two extreme scenarios presented in Table 3: Scenario 1 is the worst case with value destruction equal to -772.69 euro and scenario 8 is the best one with value creation equal to $3,041.44$ euro. The pair (f, d) represents a pair of macro-inputs: the vector $(f^1, d^1) = (100\%, 0\%, 0\%, 25, 0\%)$ describes the assumptions in scenario 1 and $(f^8, d^8) = (0\%, 100\%, 0\%, 1, 100\%)$ describes the assumptions in scenario 8. We denote as $h(f, d)$ the equity NPV computed by setting the financing and payout macro-inputs as (f, d). The increase in NPV from the worst case (scenario 1) to the best case (scenario 8) is

$$\Delta\text{NPV}^e = \text{NPV}^{e,8} - \text{NPV}^{e,1} = h(f^8, d^8) - h(f^1, d^1). \tag{28}$$

The individual contribution of the financing variables, denoted as ΔNPV^e_f, may be obtained by calculating the change that the NPV would have if f changed from f^1 to f^8 while leaving the values of the distribution group unvaried at the worst case d^1:

$$\Delta\text{NPV}^e_f = h(f^8, d^1) - h(f^1, d^1). \tag{29}$$

Analogously, the individual contribution of the distribution variables, denoted as ΔNPV^e_d, may be obtained by calculating the change that the NPV would have if d changed from d^1 to d^8 while leaving the values of the financing group unvaried at the worst case f^1:

$$\Delta\text{NPV}^e_d = h(f^1, d^8) - h(f^1, d^1). \tag{30}$$

The difference between the overall equity-NPV increase, ΔNPV^e, and the individual contributions of the two groups represents the interaction effect between financing and payout policy:

$$\Delta\text{NPV}^e_{f,d} = \Delta\text{NPV}^e - (\Delta\text{NPV}^e_f + \Delta\text{NPV}^e_d) \tag{31}$$

(see Saltelli et al. 2004, Borgonovo 2010, 2017 and Borgonovo, Gatti, and Peccati 2010 on measures of individual contributions and interaction effects).

In such a way, the change in the equity NPV from the worst scenario to the best scenario may be apportioned to the financing variables, to the distribution variables, and to a possible interaction effect between the two groups. Figure 6 depicts the results for the project at hand. As can be gleaned from inspection of the figure, the effect of the financing group from eq. (29) amounts to $\text{NPV}^e_f = 869.36 - (-772.69) = 1,642.04$€, which represents the 43.1% of the NPV increase; the effect of the distribution group from eq. (30) amounts to $\text{NPV}^e_d = 1,410.84 - (-772.69) = 2,183.53$€, meaning that the 57.2% of the NPV increase from the worst to the best scenario is explained by the payout policy. The interaction effect from eq. (31) is negligible since $\text{NPV}^e_{f,d} = 3,814.13 - (1,642.04 + 2,183.53) = -11.44$€.

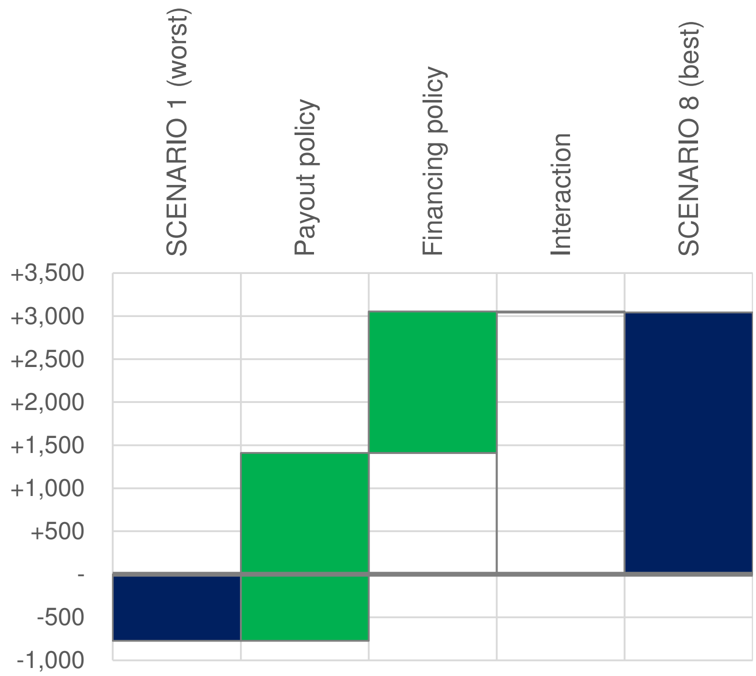

Figure 6: Contribution of payout policy and financing policy to value creation

10 Concluding remarks

Since solar energy undeniably contributes to a sustainable economy, the decision of adopting a solar energy system by firms is important to achieve a substantial cumulative effect in the environment. However, firms' decisions are mostly motivated by the economic profitability of a project and by the value created for the firm's shareholders. Building upon

Magni (2020), we present an analytical tool increasing the analysts' and managers' awareness of the importance of modeling the financial variables associated with an industrial project.

We show that, while operating variables (energy prices, O&M costs, solar panel degradation rate, etc.) are important, financial variables may have a substantial impact on the value created as well. In particular, the embedded decisions are of special importance: They deal with the amount of cash distribution to shareholders, the retained cash, the proportion of equity financing and debt financing as opposed to cash withdrawals from liquid assets. Our model takes these variables into explicit consideration and measures their impact on the firm's pro forma financial statements and, hence, on the shareholder value created. The model may be helpful in real-life applications, especially considering that, in practice, a substantial amount of solar PV plants is financed by firms with internal financing, with no recourse to equity issuance (and sometimes not even to debt financing).

We apportion the overall value created according to the various sources of value, namely, the operating activities (operating NPV), the liquid assets (non-operating NPV), and the debt borrowing (debt NPV). We show that the non-operating NPV may play a role in creating value, and may even turns an otherwise unprofitable project into a profitable one.

As a result, this paper's findings suggest that, while the technical inputs describing the functioning of the plant, the cost savings, and the sales revenues from excess energy are of paramount importance, the financial variables and the embedded decisions should not be nonetheless neglected, for their impact on the project's attractiveness may be non-negligible. Armed with an appropriate model including both estimation variables and decision variables, the firm's analysts may fine-tune the financial decisions for a given set of expected value of estimated inputs and optimize the NPV for the firm's shareholders.

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Papers presented at conferences

Project appraisal and the Intrinsic Rate of Return

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Project appraisal and the Intrinsic Rate of Return

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ABSTRACT

Building upon Magni (2011)'s approach, we propose a new rate of return measuring a project's economic profitability. It is called the intrinsic rate of return (IROR). It is defined as the ratio of project return to project's intrinsic value. The IROR approach decomposes the NPV into project scale and economic efficiency. In particular, NPV is found as the product of the project's total invested capital and the excess rate of return, obtained as the difference between the IROR and the minimum attractive rate of return (MARR). This approach provides correct project ranking and is capable of managing time-varying costs of capital. In case of levered projects, shareholder value creation is captured by the equity IROR, which we call *Intrinsic Return On Equity* (IROE) (net income divided by total equity capital invested). If the project is unlevered, the IROE and the IROR lead to the same decision; if the project is levered, and the nominal value of debt is not equal to the market value of debt, the IROE should be preferred to project IROR.

INTRODUCTION

As often reported in empirical studies, practitioners are interested in assessing economic profitability with a relative measure of worth no less than with an absolute measure of worth such as the Net Present Value (NPV). The use of a rate of return in place of or in conjunction with NPV is rather common (Remer and Nieto 1995a,b, Graham and Harvey 2001, Sandahl and Sjögren 2003). Furthermore, recent findings in the literature have revived the debate on relative measures of worth and their relations with NPV (Hazen 2003, Hartman and Schafrick 2004, Magni 2010, 2011, 2013, 2016, Lima e Silva et al. 2017, Ben-Horin and Kroll 2017). In particular, the ability of Chisini means of making sense of seemingly disparate measures of worth have been demonstrated (Magni et al. 2018) and a stronger definition of NPV-consistency has been recently advanced (Marchioni and Magni 2018). We present a new relative measure of worth for project evaluation, called the Intrinsic Rate of Return (IROR). Contrary to IRR, it does not require solving equations, it exists and is unique and is, literally, a *return on investment*, namely, the total profit generated by the project divided by the total invested capital, where the capital is expressed in terms of intrinsic or economic values. The IROR is a rational measure of worth, simple to use and intuitive, which may be used for project ranking as well as accept-reject decisions, for both levered and unlevered projects. It improves on the traditional NPV analysis for it decomposes NPV into two value drivers: The project's scale (total capital invested) and the project's economic efficiency (excess rate of return). A companion of IROR is the Intrinsic Return On Equity (IROE), which measures the equity rate of return. IROE is NPV-consistent as well, and it is preferable to IROR whenever the nominal value of debt differs from the market value of debt. Both IROR and IROE easily cope with time-varying costs of capital.

1. NPV and intrinsic value

Consider an n-period project and let Rev_t and OpC_t be the estimated incremental revenues and incremental operational costs associated with the project, respectively. The project's after-tax operating profit is

$$P_t = (\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t)(1-\tau)$$

where Dep_t is the capital's depreciation charge and τ is the marginal corporate tax rate. The estimated free cash flow (FCF) stream is $F=(F_0,F_1,\dots,F_n)$ and $C_0=-F_0$ is the project cost, such that

$$F_t=P_t+\mathrm{Dep}_t \tag{1}$$

for $t>0$, assuming that working capital is equal to 0. Let r be the cost of capital, that is, the interest rate at which funds may be invested or borrowed in a normal, competitive financial market. If the project is levered, the cost of capital is often called weighted average cost of capital (WACC). The cost of capital expresses the minimum attractive rate of return (MARR). We assume, for the time being, that it is constant. The project's net present value is defined as

$$\mathrm{NPV}=\frac{F_1}{1+r}+\cdots+\frac{F_n}{(1+r)^n}-C_0.$$

It measures the economic value created, that is, the investors' wealth increase. The project is worth undertaking if and only if $\mathrm{NPV}>0$. Consider now the following definition of IROR.

Definition (Intrinsic Rate of Return) *The IROR is equal to the ratio of total profit to total capital invested:*

$$i=\frac{\mathrm{TP}}{\mathrm{TC}}=\frac{\sum_{t=0}^{n}P_t}{\sum_{t=0}^{n}V_t} \tag{2}$$

where $V_t=\sum_{k=t+1}^{n}F_k(1+r)^{t-k}$ is the discounted sum of the prospective FCFs (with $P_0=V_n=0$). V_t expresses the *intrinsic* value of the project, that is, the value at which an equal-risk asset is traded in the market (or, equivalently, it is the price that the project would have if it were traded in the market). It is then an economic measure of the capital invested in the project at time t. Note that, recursively,

$$V_t=V_{t-1}(1+r)-F_t$$

or, proceeding backward,

$$V_t=\frac{V_{t+1}+F_{t+1}}{1+r}.$$

Once profits are estimated, FCFs are derived from (1). Then, the intrinsic value is obtained from FCFs recursively as described above. In other words, V_t is the capital intrinsically invested at the beginning of period $[t,t+1]$, $t=0,1,\dots,n-1$. Summing the invested amounts, one gets the total capital, TC, invested in the span of n years.

The IROR in (2) is economically significant for it fulfills the literal definition of a rate of return: An amount of return per unit of invested capital.

The IROR may also be framed in a different-but-equivalent way, using cash flows instead of profits. Specifically, we first prove that the total profit coincides with the project's net cash flow:

$$\sum_{t=1}^{n}P_t=\sum_{t=0}^{n}F_t. \tag{3}$$

To this end, consider that, owing to (1),

$$F_0+\sum_{t=1}^{n}F_t=-C_0+\sum_{t=1}^{n}(P_t+\mathrm{Dep}_t).$$

As $C_0=\sum_{t=1}^{n}\mathrm{Dep}_t$, then (3) is straightforward. As a result, the IROR may be alternatively viewed as a profit measure or as a cash-flow measure:

$$\overbrace{\frac{P_1+P_2+\cdots+P_n}{V_0+V_1+\cdots+V_{n-1}}}^{\text{profit to capital}}=\mathrm{IROR}=\overbrace{\frac{F_0+F_1+\cdots+F_n}{V_0+V_1+\cdots+V_{n-1}}}^{\text{cash flow to capital}}.$$

It is a ratio of total profit to invested capital or a ratio of net cash flow to invested capital.

The following decision criterion is naturally derived from the IROR.

IROR decision criterion. An investment project is worth undertaking (i.e., it creates value) if and only if $i>r$. A financing project is worth undertaking (i.e., it creates value) if and only if $i<r$.

Whether the IROR criterion is economically rational or not depends on whether it is consistent with the NPV criterion. The NPV criterion recommends acceptance if and only if $\mathrm{NPV}>0$. We now show that such a consistency indeed holds.

2. NPV-consistency of IROR

Consider the following definition.

Investment project and financing project. If $\mathrm{TC}>0$, the project is an investment project and i is an investment rate; if $\mathrm{TC}<0$, the project is a financing (or borrowing) project and i is a financing rate.

(See also Magni 2010, 2013, 2016 on the difference between investment and financing). From section 1, we know that $V_t=V_{t-1}(1+r)-F_t$, whence

$$r=\frac{V_t+F_t-V_{t-1}}{V_{t-1}}$$

for every $t\geq 1$. The WACC, r, is the market return that would be earned by investors if they invested V_{t-1}

in the market instead of investing it in the project. More precisely, the project's cash-flow stream is $(-C_0, F_1, F_2, \dots, F_n)$ while the cash-flow stream of a portfolio replicating the project's prospective FCFs is $(-V_0, F_1, F_2, \dots, F_n)$. The return stream of the project is $(P_1, P_2, \dots, P_n)$ while the return stream of the replicating portfolio is $(rV_0, rV_1, \dots, rV_{n-1})$. Using (3), the difference between the total project return and the total market return is

$$\sum_{t=1}^{n} P_t - \sum_{t=1}^{n} rV_{t-1} = \sum_{t=0}^{n} F_t - \sum_{t=1}^{n} (F_t + V_t - V_{t-1}).$$

As $V_n = 0$, this means

$$\sum_{t=1}^{n} P_t - \sum_{t=1}^{n} rV_{t-1} = V_0 - C_0.$$

However, $V_0 = \sum_{t=1}^{n} F_t (1+r)^{-t}$ and

$$V_0 - C_0 = \sum_{t=0}^{n} F_t (1+r)^{-t} = \text{NPV}.$$

Therefore,

$$\text{NPV} = \sum_{t=1}^{n} P_t - \sum_{t=1}^{n} rV_{t-1}.$$

Dividing by $\text{TC} = \sum_{t=0}^{n} V_t$,

$$\text{NPV} = \overbrace{\text{TC}}^{\text{Project Scale}} \cdot \overbrace{(i-r)}^{\text{Economic Efficiency}} \tag{4}$$

Equation (4) represents an economically significant decomposition of NPV. It says that the economic value created by the project is the result of two effects: The amount of capital that will be invested in the project (project scale) and the extent by which the project rate of return will exceed the MARR (economic efficiency). Note that this kind of information cannot be derived from a traditional NPV analysis. Equation (4) proves that the IROR is NPV-consistent.

Proposition 1. (NPV-consistency of IROR) *In an investment project,* $\text{NPV} > 0$ *if and only if* $i > r$. *In a borrowing project,* $\text{NPV} > 0$ *if and only if* $i < r$.

Note that, if the project is a financing project, then the IROR represents a financing rate, as well as r. Therefore, the project is worth undertaking if its financing cost is smaller than the borrowing cost prevailing in the market. (Financing projects may occur only if total assets are negative, which may occur whenever fixed assets are sufficiently small and the net working capital is negative and sufficiently high in absolute value. In these situations, cash is received from customers earlier than cash is paid out to suppliers.)

3. Time-varying WACCs

We now show how the MARR should be computed if the WACC is time-varying. Let $r = (r_1, r_2, \dots, r_n)$ be the stream of WACCs holding in the various years, such that $r_t = (V_t + F_t - V_{t-1})/V_{t-1}$.

In this case, the equality $\text{NPV} = \sum_{t=1}^{n} P_t - \sum_{t=1}^{n} rV_{t-1}$ shown in the previous section generalizes to

$$\text{NPV} = \sum_{t=1}^{n} P_t - \sum_{t=1}^{n} r_t V_{t-1}.$$

Equation (4) still holds, with the understanding that r is redefined as a weighted mean of the WACCs:

$$r = \frac{\sum_{t=1}^{n} r_t V_{t-1}}{\sum_{t=1}^{n} V_{t-1}}. \tag{5}$$

In other words, the MARR is the weighted average of the time-varying WACCs. An investment project is worth undertaking if and only if the IROR is greater than this MARR.

4. Equity perspective

Suppose that the project is levered and let Int_t be the interest expense associated with the debt. Let r_t^e be the required return to equity (equity cost of capital) in period t and let V_t^e be the intrinsic equity value:

$$V_t^e = \sum_{k=t+1}^{n} \frac{F_k^e}{(1 + r_{t+1}) \cdot (1 + r_{t+2}) \cdot \dots \cdot (1 + r_k)}$$

where F_k^e expresses the cash flow to equity (CFE) at time k. The latter is in turn obtained from the net income as follows. The net income is

$$\text{NI}_k = (\text{Rev}_k - \text{OpC}_k - \text{Dep}_k - \text{Int}_k)(1 - \tau)$$

or, equivalently, $\text{NI}_k = P_t - \text{Int}_k \cdot (1 - \tau)$, and the CFE is $F_k^e = \text{NI}_k + \text{Dep}_k + (D_k - D_{k-1})$, where $D_k - D_{k-1}$ is the change in the outstanding debt. We define the *equity IROR* (i^e) as the ratio of the project's overall net income to total equity (intrinsic) value:

$$\begin{aligned} i^e &= \frac{\text{TNI}}{\text{TC}^{\text{e}}} \\ &= \frac{\sum_{t=1}^{n} (\text{Rev}_t - \text{OpC}_t - \text{Dep}_t - \text{Int}_t)(1 - \tau)}{\sum_{t=0}^{n} V_t^e} \end{aligned} \tag{6}$$

with $V_n^e = 0$. We will also call this ratio *Intrinsic Return On Equity* (IROE). The equity NPV is $\text{NPV}^{\text{e}} = V_0^e + F_0^e$ or, equivalently

$$\mathrm{NPV^e} = F_0^e + \sum_{k=1}^{n} \frac{F_k^e}{(1+r_1^e)\cdot(1+r_2^e)\cdot\ldots\cdot(1+r_k^e)}.$$

Applying (4) to the equity capitals and the net incomes, one may write

$$\mathrm{NPV^e} = \mathrm{TC^e}\cdot(i^e - r^e) \tag{7}$$

where TC^e expresses the value of the equity invested in the project and

$$r^e = \frac{\sum_{t=1}^n r_t^e V_{t-1}^e}{\sum_{t=1}^n V_{t-1}^e}$$

is the weighted average of the costs of equity. This is the equity MARR.

Assuming the interest rate on debt is equal to the required return to debt (debt's cost of capital),[1] then the market value of debt coincides with the book value of debt, which implies that the equity NPV is equal to the project NPV. From (4) and (7),

$$\mathrm{TC}(i-r) = \mathrm{TC^e}(i^e - r^e)$$

This implies $i^e > r^e$ if and only if $i > r$ (assuming, as usual, that TC and TC^e have the same sign). The IROR and the IROE are reciprocally consistent.

If, instead, the interest rate on debt differs from the required rate of return to debt, then $\mathrm{NPV} \neq \mathrm{NPV^e}$. In this case, part of the value created by the project is captured (if $\mathrm{NPV} > \mathrm{NPV}^e$) or given up ($\mathrm{NPV} < \mathrm{NPV}^e$) by the debtholders and the project IROR will not be reliable as a measure of shareholder value creation any more; as shareholders' value creation is the goal of the firm, the IROE will be an appropriate intrinsic rate of return.

5. Project ranking

Choice between mutually exclusive projects and ranking of $m > 2$ projects may be accomplished by incremental analysis: If the incremental IROR of A−B is greater than the incremental MARR, then A is preferable to B. Specifically, let i^A and i^B the IRORs of project A and B and let r^A and r^B be the respective MARRs. Let also TC^A be the total intrinsic value of A and TC^B the total intrinsic value of B. Assuming, with no loss of generality, that $\mathrm{TC}^A > \mathrm{TC}^B$, then $\mathrm{NPV}^A > \mathrm{NPV}^B$ if and only if

$$\mathrm{TC}^A(i^A - r^A) > \mathrm{TC}^B(i^B - r^B)$$

which in turn holds if and only if $i^{A-B} > r^{A-B}$ where

$$i^{A-B} = \frac{\sum_{t=1}^n (P_t^A - P_t^B)}{\sum_{t=0}^n (V_t^A - V_t^B)} = \frac{\sum_{t=0}^n (F_t^A - F_t^B)}{\sum_{t=0}^n (V_t^A - V_t^B)}$$

is the incremental IROR and

$$r^{A-B} = \frac{\sum_{t=1}^n (r_t^A V_{t-1}^A - r_t^B V_{t-1}^B)}{\sum_{t=0}^n (V_t^A - V_t^B)}$$

is the incremental MARR. In other words, if investors undertake A instead of B, they earn money at an incremental rate of return equal to the incremental IROR, i^{A-B}, but, at the same time, they incur an incremental opportunity cost which is equal to the incremental MARR, r^{A-B}. If the incremental IROR exceeds the incremental MARR, then project A is preferable to project B.

6. Numerical example

Consider a 5-year project with input data as follows:

- Incremental revenues in first year: $350
- Growth rate for revenues: 6% annual
- Incremental operating costs: 30% of revenues
- Cost of the project: $800
- Dep_t: $160 (constant)
- Amount of debt: $300
- Type of debt: Bullet bond (4 years)
- Debt rate: 3%
- Required return to debt: 3%
- Required return to equity: 10% (constant)
- Tax rate: 30%

We use these data to compute the after-tax operating profit and the net income, as well as the equity capital invested, the outstanding debt, the CFE and the cash flow to debt (CFD) (see Table 1). Note that the relation among CFE, CFD and FCF is as follows: $F_t = F_t^e + F_t^d - \tau\cdot\mathrm{Int}_t$, where F_t^d denotes the CFD (see any corporate finance textbook for details) which shows the relation between tax shield and FCF.

The IROE is 19.1% and is greater than the equity MARR by $19.1\% - 10\% = 9.1\%$. The latter figure expresses the economic efficiency of the equity investment. Applied to a total equity value of $1,996, the equity NPV is found to be $\mathrm{NPV^e} = 182$. As we assume that interest rate on debt and cost of debt are equal, the nominal value of debt equates the intrinsic value of debt and the project NPV equates the equity

[1] The required return on debt is the interest rate required by the investors of a competitive, normal market who receive the same prospective cash flows as the debtholders. The interest rate on debt is the contractual rate at which the debt is actually granted by the debtholders. While the two rates are often assumed to be equal, there may be cases where they are not.

NPV, that is, $\text{NPV}^{\text{e}} = \text{NPV} = 182$. However, in the project perspective, a total \$3,196 is invested, obtained as

$$3{,}196 = 982 + 837 + 667.2 + 469.5 + 240.5$$

or, equivalently, as the sum of total equity value, \$1,996, and total debt value, \$1,200 ($= \$300 \cdot 4$). As the total afer-tax operating profit is \$406.8$= 59.5 + 69.8 + 80.7 + 92.3 + 104.5$, dividing the latter by \$3,196 one gets the project IROR, which is equal to 12.73%. The WACC is computed as a weighted average of the cost of equity and the (after-tax) cost of debt, where the weights are the intrinsic value of equity and debt:

$$r_t = \frac{0.1 \cdot V_{t-1}^e + 0.03 \cdot V_{t-1}^d (1 - 0.3)}{V_{t-1}}$$

with $V_{t-1} = V_{t-1}^e + V_{t-1}^d$. It is time-varying because, while cost of equity and cost of debt are time-invariant, the intrinsic value of equity and debt changes over time.

In turn, the mean of the r_t's, weighted by the respective intrinsic values V_{t-1} (see eq. (5)) is equal to the project MARR, which is equal to $r = 7.03\%$, smaller than the IROR by 5.7%. This is the economic efficiency of the project. Applying this figure to the total intrinsic value, the NPV is found back.

7. CONCLUSIONS

The intrinsic rate of return (IROR) is a simple metric, since it is a mere ratio of total profit to total invested capital or, equivalently, the ratio of net cash flow to total invested capital. Therefore, it is, at the same time, an income-based as well as a cash-flow-based measure. It is ready-to-use and understandable by any practitioner. It may be applied to any engineering project as well as a financial investment, for both ex ante decision-making and ex post performance measurement. Multiplied by the total capital invested, it provides the shareholders' wealth increase. Contrary to IRR, it exists, is unique, no equation is required, and it is based on the economically meaningful measure of profit and intrinsic value. It is capable of coping with time-varying WACCs and of correctly ranking competing projects via incremental analysis.

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Table 1

Year		0	1	2	3	4	5
Revenues	Rev_t		350.0	371.0	393.3	416.9	441.9
Operating Costs	OpC_t		105.0	111.3	118.0	125.1	132.6
Depreciation	Dep_t		160.0	160.0	160.0	160.0	160.0
Pre-tax operating profit	$\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t$		85.0	99.7	115.3	131.8	149.3
Taxes on operating profit	$\tau \cdot (\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t)$		25.5	29.9	34.6	39.5	44.8
After-tax operating profit	P_t		59.5	69.8	80.7	92.3	104.5
Pre-tax operating profit	$\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t$		85.0	99.7	115.3	131.8	149.3
Interest	Int_t		9.0	9.0	9.0	9.0	0.0
Earnings before taxes (EBT)	$\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t - \mathrm{Int}_t$		76.0	90.7	106.3	122.8	149.3
Taxes on EBT	$\tau \cdot (\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t - \mathrm{Int}_t)$		22.8	27.2	31.9	36.8	44.8
Net income	NI_t		53.2	63.5	74.4	86.0	104.5
Equity capital		500	340	180	20	160	0
Debt capital		300	300	300	300	0	0
FCF	F_t	−800	219.5	229.8	240.7	252.3	264.5
CFE	F_t^e	−500	213.2	223.5	234.4	−54.0	264.5
CFD	F_t^d	−300	9.0	9.0	9.0	309.0	0.0
EQUITY perspective							
Intrinsic value	V_t^e	682.0	537.0	367.2	169.5	240.5	0.0
Total intrinsic value	TC^e	1,996.0					
Total net income	TNI	381.6					
IROE	i^e	19.1%					
MARR	r^e	10.0%					
equity NPV	NPV^e	182.0					
PROJECT perspective							
Intrinsic value	V_t	982.0	837.0	667.2	469.5	240.5	0.0
Total intrinsic value	TC	3,196.0					
Total operating profit	TP	406.8					
WACC	r_t		7.6%	7.2%	6.4%	5.0%	10.0%
IROR	i	12.73%					
MARR	r	7.03%					
Project NPV	NPV	182.0					

The accounting-and-finance of a solar photovoltaic plant: Economic efficiency of a replacement project

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The accounting-and-finance of a solar photovoltaic plant: Economic efficiency of a replacement project

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Energy project analysis, investment evaluation, value creation.

ABSTRACT

In this work we illustrate a simple logical framework serving the purpose of assessing the economic profitability and measuring value creation in a solar photovoltaic (PhV) project and, in general, in a replacement project where the cash-flow stream is nonnegative, with some strictly positive cash flows. We use the projected accounting data to compute the average ROI, building upon Magni (2011, 2019) (see also Magni and Marchioni 2018), which enables retrieving information on the role of the project's economic efficiency and the role of the project scale on increasing shareholders' wealth. The average ROI is a genuinely internal measure and does not suffer from the pitfalls of the internal rate of return (IRR), which may be particularly critical in replacement projects such as the purchase of a PhV plant aimed at replacing conventional retail supplies of electricity.

INTRODUCTION

Investment decisions may be evaluated adopting absolute measures of worth such as net present value (NPV), residual income, value added, or relative measures of worth, such as rates of return or benefit-cost indices. The NPV is regarded as a rational measure of value creation, since it correctly quantifies the net effect of the project on shareholders' current wealth (Brealey, Myers and Allen 2011). However, rates of return are more intuitive. For instance, to say "the project has generated a 10% return, better than 8% market return" is more intuitive than saying "at a discount rate of 8%, the project NPV is $150" (Remer, Stokdyk and VanDriel 1993, Remer and Nieto 1995a,b, Graham and Harvey 2001, Ryan and Ryan 2002, Ross, Westerfield and Jordan 2011). Also, a rate of return informs about economic efficiency, that is, how good or bad money is invested, whereas NPV blends economic efficiency and project scale into a unique number.

Among the various rates of return, the internal rate of return (IRR) is a common metric. Unfortunately, it may not exist (or be multiple), especially in replacement projects, where the cash-flow stream is often nonconventional (i.e., cash-flow stream changes sign more than once or never changes sign). We build upon Magni (2011, 2019) and the internal-average-rate-of-return (IARR) approach with pro forma accounting profits and book values to accomplish a comprehensive analysis of a PhV project whose cash-flow stream results in a nonconventional cash-flow stream with no IRR.

This paper aims at introducing theoretical and applicative tools for the analysis, in a firm perspective, of a replacement project in the field of renewable energy, considering the case where conventional retail electricity system (based on supplies from utility) may be replaced by a standalone solar PhV system purchased from a producer, installed on a land property owned by the firm. We describe the project as an incremental economic system, that is, as a deviation of the firm-with-the project from the firm-without-the-project (status quo) in terms of accounting magnitudes. We assume that, in the status quo, a utility bill is paid periodically and a rent from the land is received. The solar PhV plant implies a leasing contract whereby lease payments and operating and maintenance costs are made periodically. After several years, at the expiration date, the lessee may pay a lump sum to acquire the plant, and the system will continue to generate electric power for some years. At the end of its useful life, the plant is removed and the firm incurs disposal costs. If the retail system is replaced by the PhV plant, the incomes, book values and resultant cash flows increase as a result of the lease payment and the terminal outlay for acquiring the plant but decrease by effect of the cost savings (the utility bill). This paper uses the accounting estimations and a benchmark portfolio to assess the PhV's economic profitability via the internal-average-rate-of-return (IARR) approach, and, in particular, the average ROI. This latter exists and is unique and enables understanding how much of the value created is due to the economic efficiency of the project and how much of it is due to the scale of the project.

1. INVESTMENT AND FINANCING SIDE OF A PROJECT

Let P be a n-period investment project. Economically, a project consists of two sides: The investment side and the financing side. The invested side refers to the invested capital, which is divided into two main classes:

- Operating capital, C_t^o, consisting of of net fixed assets and net operating working capital: $C_t^o = \mathrm{NFA}_t^o + \mathrm{WC}_t^o$
- Non-operating or liquid assets, C_t^l (excess cash, marketable securities, and other financial activities).

The financing side of the project refers to the financing raised by the firm for undertaking the project. It can be conveniently divided into two components:

- Debt capital, C_t^d (loans, bonds, notes payable, etc.)
- Equity capital, C_t^e (capital raised by the firm from the firm's owners).

Investment side and financing side balance out, that is,

$$\overbrace{\mathrm{NFA}_t^o + \mathrm{WC}_t^o}^{C_t^o} + C_t^l = C_t = C_t^d + C_t^e$$

The project's income I_t and the project's cash flow F_t are the source of variation of the capital. Both can be split up into operating components and non-operating component (asset side) and into equity component and debt component (financing side). The project's income is the sum of the operating income and the interest income and, at the same time is equal to the sum of the net income and the interest expenses:

$$I_t^o + I_t^l = I_t = I_t^d + I_t^e.$$

Likewise, the project's cash flow is equal to the sum of the operating cash flow and the non-operating (i.e., liquid) cash flow and, at the same time, equal to to the sum of the cash flow to debt and cash flow to equity:

$$F_t^o + F_t^l = F_t = F_t^d + F_t^e.$$

The evolution of capital through time is described by a dynamical equation according to which capital increases with the income produced and decreases with the cash flow extracted from the economic system,

$$C_t = C_{t-1} + I_t - F_t$$

and $C_t^j = C_{t-1}^j + I_t^j - F_t^j$, with $j = o, l, e, d$. At the end of the project, $C_n = 0$ since the transactions are over.

Let Rev_t be the incremental revenues, OpC_t the incremental operational costs, Dep_t the depreciation charge of fixed assets, and τ the corporate tax rate. The after-tax project income is

$$I_t = (\mathrm{Rev}_t - \mathrm{OpC}_t - \mathrm{Dep}_t + I_t^l)(1-\tau) + \tau I_t^d. \tag{1}$$

The project's cash flows, F_t, is the difference between the operating income and the change in operating capital, such that

$$F_t = I_t - \Delta C_t = I_t + \mathrm{Dep}_t - \Delta \mathrm{WC}_t - \Delta C_t^l. \tag{2}$$

2. NPV AND RELATIVE PERFORMANCE METRICS

Let r_t be the project's cost of capital (required rate of return), that is, the interest rate at which funds may be invested or borrowed in a normal, competitive financial market, often called weighted average cost of capital (WACC). The economic (or intrinsic) value of the project at time *t*, V_t, is the value that the project would have if it were traded in the market and is equal to the discounted sum of future cash flows:

$$\mathrm{V}_t = \frac{F_{t+1}}{(1+r_{t+1})} + \frac{F_{t+2}}{(1+r_{t+1})(1+r_{t+2})} + \cdots + \frac{F_n}{(1+r_{t+1})(1+r_{t+2})\dots(1+r_n)}.$$

The value created by a project is measured via its net present value (NPV), which is the difference between the initial economic value, V_0, and the initial cost, $-F_0$. Therefore,

$$\mathrm{NPV} = V_0 - (-F_0) = F_0 + \frac{F_1}{1+r_1} + \frac{F_2}{(1+r_1)(1+r_2)} + \cdots + \frac{F_n}{(1+r_1)(1+r_2)\dots(1+r_n)}.$$

Let r_t^j, $j = o, l, d, e$ be the cost of capital for operating assets, non-operating assets, debt, and equity, respectively. The economic value of each constituent class in *t*, V_t^j, is

$$\mathrm{V}_t^{\mathrm{j}} = \frac{F_{t+1}^j}{(1+r_{t+1}^j)} + \frac{F_{t+2}^j}{(1+r_{t+1}^j)(1+r_{t+2}^j)} + \cdots + \frac{F_n^j}{(1+r_{t+1}^j)(1+r_{t+2}^j)\dots(1+r_n^j)}$$

and its NPV is

$$\mathrm{NPV}^j = F_0^j + \frac{F_1^j}{1+r_1^j} + \frac{F_2^j}{(1+r_1^j)(1+r_2^j)} + \cdots + \frac{F_n^j}{(1+r_1^j)(1+r_2^j)\dots(1+r_n^j)}.$$

The NPV of the project equals the sum of the NPVs of the financings:

$$\mathrm{NPV} = \mathrm{NPV}^{\mathrm{e}} + \mathrm{NPV}^{\mathrm{d}}.$$

Furthermore, it is worth noting that the project's cost of capital, r_t, is equal to the weigthed average of the cost of equity r_t^e and the cost of debt r_t^d, with weights represented by the market values:

$$r_t = \frac{r_t^e V_{t-1}^e + r_t^d V_{t-1}^d}{V_{t-1}^e + V_{t-1}^d}.$$

If the cost of capital is constant through time, $r_t = r$, the project's NPV results

$$\mathrm{NPV} = F_0 + \frac{F_1}{1+r^j} + \frac{F_2}{(1+r^j)^2} + \cdots + \frac{F_n}{(1+r^j)^n}.$$

A project is worth undertaking if and only if it creates value for its equityholders, that is $\mathrm{NPV} > 0$.

Among the various relative performance metrics, the internal rate of return (IRR) of the project is defined as the discount rate x which solves the equation NPV=0:

$$\mathrm{NPV}(x) = \frac{F_1}{1+x} + \cdots + \frac{F_n}{(1+x)^n} - C_0 = 0.$$

Although the IRR is commonly adopted by practioners, its pitfalls are significant, including multiplicity, non-existence, and ambiguous financial nature when the project is not a conventional project.

Consider now the following definition of the Internal Average Rate of Return (IARR).

Definition (Internal Average Rate of Return) *The IARR is equal to the ratio of total profit to total capital invested:*

$$i = \frac{I}{C} = \frac{\sum_{t=0}^{n} I_t}{\sum_{t=0}^{n} C_t}. \tag{3}$$

The IARR in (3) is economically significant for it fulfills the literal definition of a rate of return: An amount of return per unit of invested capital. Since profits and capitals are pro forma accounting values, this rate is an average accounting rate of return. More precisely, it is an average Return On Investment (ROI).

The average ROI exists and is unique, so it may be used in place of IRR by those practioners who are willing to calculate a reliable, internal relative measure of worth without incurring the difficulties of IRR.

Since $\sum_{t=0}^{n} I_t = \sum_{t=0}^{n} F_t$, the average ROI may be alternatively viewed as a cash-flow measure:

$$\frac{\overbrace{I_0 + I_1 + \cdots + I_n}^{\text{profit to capital}}}{C_0 + C_1 + \cdots + C_n} = i = \frac{\overbrace{F_0 + F_1 + \ldots + F_n}^{\text{cash flow to capital}}}{C_0 + C_1 + \ldots + C_n}.$$

It is a ratio of total profit to invested capital or a ratio of net cash flow to invested capital.

For decision-making purposes, the average ROI should be compared to a suitable minimum attractive rate of return (MARR), which is the rate of return that investors would earn if they invested the same amount of the project in a market (value-neutral) portfolio replicating the project's cash flows (from time 1 to time n). The replicating portfolio's return is $r_t V_{t-1}$. In total, the firm would earn a total income $I^V = \sum_{t=1}^{n} r_t V_{t-1}$. This means that the firm invests $C = \sum_{t=0}^{n} C_t$ at the average ROI, i, while foregoing the opportunity of investing the same total amount at the rate ρ such that

$$\rho = \frac{I^V}{C}.$$

Such a foregone rate is the MARR. It is easy to verify that $I - I^V = \text{NPV}$; therefore,

$$\text{NPV} = \overbrace{C}^{\text{Project Scale}} \cdot \overbrace{(i - \rho)}^{\text{Economic Efficiency}} \tag{4}$$

(see Magni 2019), which represents an economically significant decomposition of NPV: The economic value created depends on the product between the amount of invested capital (project scale) and the extent by which the project rate of return will exceed the MARR (economic efficiency). The following decision criterion is naturally derived.

Decision criterion. A project is worth undertaking if and only if the average ROI is greater than the MARR, $i > \rho$.

3. SOLAR PhV PLANT

In this section, we describe the economic system of a replacement project whereby the conventional retail energy supply is replaced with a renewable energy plant. In particular, we consider the case of a firm currently purchasing electric power from a utility. It faces the opportunity of entering into an m-year leasing contract for operating a standalone solar photovoltaic (PhV) system.

Suppose the quantity of energy consumed for the firm's operations is constant through time and equal to q; the current purchase price of energy is p_1, growing at a constant rate g_{p_1} per year. The utility bill is payed periodically, in the same year in which energy is consumed.

The leasing contract contains the following economic conditions: The lease payment, equal to L, is made periodically; at time m (expiration date) the firm may acquire the plant paying a lump sum equal to CapEx, and the system will keep producing electric power for some years, until time n. CapEx represents a capital expenditure, with an assumed straight-line depreciation from $t = m + 1$ until $t = n$ equal to $\text{Dep} = \text{CapEx}/(n - m)$.

The PhV plant is installed at $t = 0$ in a field owned by the firm, which could otherwise be rented on the property market at a costant rent equal to R per year. The latter represents an opportunity cost for the firm (a foregone income).

The PhV system produces Q units of energy in the first year, which decreases every year at the rate g_Q. If the energy produced by the plant is higher than the energy consumed by the firm, the firm sells the differential quantity to the Energy Service Operator at the energy selling price p_2, growing at a constant rate g_{p_2} per year, with payment in the following year. We assume that, at time $t = n$, the energy sold is paid immediately. Therefore, if the produced quantity is lower than the consumed energy in year t, that is, $Q(1 - g_Q)^{t-1} < q$, energy costs savings arise equal to $Q(1 - g_Q)^{t-1} \cdot p_1(1 + g_{p_1})^t$; if the produced quantity is higher than the consumed one, that is, $Q(1 - g_Q)^{t-1} > q$, energy costs savings arise equal to $q \cdot p_1(1 + g_{p_1})^t$ as well as energy sales revenues equal to $(Q(1 - g_Q)^{t-1} - q) \cdot p_2(1 + g_{p_2})^t$, determining the presence of operating working capital. Hence, the income effect of the energy sales revenues and costs savings in the two different scenarios can be summarized with the expression

$$\min\left(q\,, Q(1 - g_Q)^{t-1}\right) \cdot p_1(1 + g_{p_1})^t + \max\left(0\,, Q(1 - g_Q)^{t-1} - q\right) \cdot p_2(1 + g_{p_2})^t$$

and the operating working capital can be represented with the formula $\text{WC}_t = \max\left(0, Q(1 - g_Q)^{t-1} - q\right) \cdot p_2(1 + g_{p_2})^t$ and $WC_n = 0$.

Starting from year $M < m$, the PhV plant requires operating and maintenance costs which are expected to be constant and equal to O&M. At time n, the plant is removed with disposal costs equal to H and salvage value equal to zero.

In summary, the firm-without-the-project pays the utility bills and receives the rent for the land (for the whole period); in contrast, the firm-with-the-project sustains the lease payments (until $t = m$), the operating and maintenance costs (from $t = M$ to $t = n$), the lump sum (in $t = m$), and the disposal costs (in $t = n$), and receives payments for the energy sold to the Energy Service Operator. Considering that a project represents, by definition, the difference between the firm-with-the-project and the firm-without-the-project, the project's incomes may be calculated as in (1):

- $I_t = \left(\min\left(q\,,\ Q(1-\ g_Q)^{t-1}\right)\cdot p_1(1+g_{p_1})^t + \max\left(0\,,\ Q(1-\ g_Q)^{t-1}-q\right)\cdot p_2(1+g_{p_2})^t - L - R + I_t^l\right)(1-\tau) + \tau I_t^d$
  for $1 \le t \le M-1$
- $I_t = \left(\min\left(q\,,\ Q(1-\ g_Q)^{t-1}\right)\cdot p_1(1+g_{p_1})^t + \max\left(0\,,\ Q(1-\ g_Q)^{t-1}-q\right)\cdot p_2(1+g_{p_2})^t - L - R - \text{O\&M} + I_t^l\right)(1-\tau) + \tau I_t^d$
  for $M \le t \le m$
- $I_t = \left(\min\left(q\,,\ Q(1-\ g_Q)^{t-1}\right)\cdot p_1(1+g_{p_1})^t + \max\left(0\,,\ Q(1-\ g_Q)^{t-1}-q\right)\cdot p_2(1+g_{p_2})^t -\ R\ - \text{Dep} + I_t^l\right)(1-\tau) + \tau I_t^d$
  for $m+1 \le t \le n-1$
- $I_t = \left(\min\left(q\,,\ Q(1-\ g_Q)^{t-1}\right)\cdot p_1(1+g_{p_1})^t + \max\left(0\,,\ Q(1-\ g_Q)^{t-1}-q\right)\cdot p_2(1+g_{p_2})^t -\ R - \text{Dep}\ - H + I_t^l\right)(1-\tau) + \tau I_t^d$
  for $t = n$.

The project's assets are represented by working capital, liquid assets and, from time m, fixed assets:

- $C_t = \max\left(0, Q(1-\ g_Q)^{t-1}-q\right)\cdot p_2(1+g_{p_2})^t + C_t^l$
  for $1 \le t \le m-1$
- $C_t = \max\left(0, Q(1-\ g_Q)^{t-1}-q\right)\cdot p_2(1+g_{p_2})^t + \text{CapEx} - \text{Dep}\cdot(t-\ m) + C_t^l$
  for $m \le t \le n-1$
- $C_t = 0$
  for $t = n$.

Finally, the cash flows are obtained as $F_t = I_t - \Delta C_t, \forall t = 0,1,\dots,n.$

We assume that the project is financed with internal financing, that is, with retained cash. This implies, $C_t^d = I_t^d = F_t^d = 0\ \forall\ t = 0,1,\dots,n$ and $C_t = C_t^e,\ I_t = I_t^e,\ F_t = F_t^e$ for all t. The rate of return on liquid assets is constant and equal to i^l, hence $I_t^l = i^l\cdot C_{t-1}^l$. The assumption of zero debt means that, whenever the operating cash flows is negative, $F_t^o < 0$, the operating disbursement is covered by absorbing resources from the liquid assets, that is, $F_t^l = -F_t^o > 0$, therefore, implying that the project's cash flow is zero, $F_t = 0$. In contrast, when the operating cash flow is positive, $F_t^o > 0$, the distribution policy is the following:

- Until $t = m$, cash is retained and invested in liquid assets
- From $t = m+1$, the positive operating cash flow is fully distributed to equityholders within the period, $F_t^e = F_t^o = F_t > 0$.

A time n, the project is terminated, such that every asset and liabity goes back to zero.

We assess the economic profitability and value creation of the replacement project via the average ROI and the NPV, and show that the IRR notion fatally collapses under the assumption of internal financing. We assume that the costs of capital of operating assets and liquid assets are constant through time, equal to r^o and r^l respectively.

Assuming that the project's terminal cash flow is nonnegative, $F_n \ge 0$, it is easy to see that the assumption of internal financing implies that the project's cash-flow stream is nonnegative, that is, $F_t \ge 0$ for all t. Therefore, NPV $\ge$ 0, which implies that the analyzed replacement project is worth undertaking. However, the equation NPV $= 0$ has no solution and the IRR does not exist. In contrast, the average ROI exists and is unique, with unambiguous financial nature determined by the sign of the total capital. The NPV may be decomposed into project scale (i.e., total capital), and economic efficiency (difference between average ROI and MARR).

4. NUMERICAL EXAMPLE

Consider a replacement project with the following input data:

- Energy purchase price = Energy selling price: $p_1 = p_2 = 0.1$ (€/kWh)
- Growth rate of energy-purchase-price = Growth rate of energy-selling-price: $g_{p_1} = g_{p_2} = 1.5\%$
- Yearly consumed energy: $q = 800{,}000$ kWh
- Rent of the land property: $R =$ €10,000
- Length of the leasing contract: $m = 20$ years
- Lease payment: $L =$ €40,000
- Capital expenditure CapEx = €150,000
- Energy produced by PhV plant in the first year: $Q = 1{,}000{,}000$ kWh
- Efficiency loss per year: $g_Q = 3\%$
- Useful life of PhV plant: $n =$30 years
- Maintenance costs: O&M = €1,000
- Starting of O&M operations: $M =$11-th year
- Disposal costs: $H =$ €30,000
- Tax rate: $\tau = 25\%$
- Interest rate on liquid assets = Cost of capital for liquid assets: $i^l = r^l = 4\%$
- Cost of capital for operating assets: $r^o = 12\%$

We use these data to determine the income statements, balance sheets, and financial prospects of the project through time (see Table 1, 2, and 3 respectively).

The substitution of the retail energy system with the PhV plant creates value for equityholders, since NPV $= 215{,}027.22 > 0$. It is worth noting that the cash-flow stream of the project and equity coincides and is non-negative, with some positive cash flows. In particular, the project's cash flow is zero in the first 20 years, then it is positive up to (and including) the last year (see Table 3). As anticipated, this implies that the project IRR does not exist. The use of the average ROI overcomes the problem of non-existence of IRR. The total capital invested in the project is $C = C^e = 15{,}132{,}751.7 > 0$, and, therefore, the project is an investment, and the average ROI (equal to the average ROE) is $i = i^e = 8.35\%$, and the MARR is $\rho = \rho^e = 6.93\%$. The economic efficiency is $8.35\% - 6.93\% = 1.42\%$, which, multiplied by the total capital $C = 15{,}132{,}751.7$, gives back the NPV= $215{,}027.22$. The investors invest an overall capital of €15,132,751.70 at a 8.35% rate of return, which is higher than the MARR by 1.42 percentage points.

It is worth noting that it is not even possible to calculate the IRR of the operating cash-flow stream, notwithstanding the fact that it does change sign. The reason is that it changes sign twice, from positive to negative and then positive again (see Table 3), and the magnitudes of the changes are such that no real-valued IRR exists. In contrast, the IARR approach enables computing the *operating* average ROI ratio by dividing the net operating cash flow (i.e., the algebraic sum of the operating cash flows), $\sum_{t=0}^{30} F_t^o = 694{,}377.03$, by the total operating assets, $C^o = 908{,}424.11$. The result is $i^o = \frac{694{,}377.03}{908{,}424.11} = 76.44\%$. The project's average ROI is the weighted mean of the operating average ROI and the interest rate on liquid assets, where the weights are the total operating assets and the total liquid assets:

$$i = \frac{\overbrace{76.44\%}^{i^o} \cdot \overbrace{908{,}424.11}^{C^o} + \overbrace{4\%}^{i^l} \cdot \overbrace{14{,}224{,}327.59}^{C^l}}{908{,}424.11 + 14{,}224{,}327.59} = 8.35\%.$$

5. CONCLUSIONS

We have provided an accounting-and-finance model capable of correctly describing the economic transactions underlying the replacement project and we have offered a logically-consistent system for supporting the investment appraisal and the decision-making process. Specifically, we have shown that the Internal Rate of Return (IRR) is not reliable, in general, since the lease+purchase of the PhV may generate nonconventional patterns of cash-flow streams such that no change in sign occurs, which implies that the IRR does not exist. We have shown that the IARR approach and, in particular, the average ROI, provides a simple rate of return, naturally linked with NPV. The NPV is broken down into project scale (overall capital invested) and economic efficiency (difference between average ROI and MARR).

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Table 1: Income statements

t	+ Energy sales revenues	+ Energy cost savings (avoided bills)	− Lease payments	− Lost rental income	− O&M	− Disposal costs	= EBITDA	− Depreciation	= EBIT	+ Interest income	= EBT	− Taxes	= Net Income
0													
1	20,000.00	80,000.00	− 40,000.00	− 10,000.00			50,000.00		50,000.00		50,000.00	12,500.00	37,500.00
2	17,255.00	81,200.00	− 40,000.00	− 10,000.00			48,455.00		48,455.00	700.00	49,155.00	12,288.75	36,866.25
3	14,515.87	82,418.00	− 40,000.00	− 10,000.00			46,933.87		46,933.87	2,284.45	49,218.32	12,304.58	36,913.74
4	11,781.97	83,654.27	− 40,000.00	− 10,000.00			45,436.24		45,436.24	3,870.56	49,306.81	12,326.70	36,980.11
5	9,052.67	84,909.08	− 40,000.00	− 10,000.00			43,961.75		43,961.75	5,459.12	49,420.88	12,355.22	37,065.66
6	6,327.32	86,182.72	− 40,000.00	− 10,000.00			42,510.04		42,510.04	7,050.92	49,560.97	12,390.24	37,170.72
7	3,605.30	87,475.46	− 40,000.00	− 10,000.00			41,080.76		41,080.76	8,646.77	49,727.53	12,431.88	37,295.65
8	885.97	88,787.59	− 40,000.00	− 10,000.00			39,673.56		39,673.56	10,247.47	49,921.04	12,480.26	37,440.78
9		88,288.11	− 40,000.00	− 10,000.00			38,288.11		38,288.11	11,853.88	50,141.99	12,535.50	37,606.49
10		86,924.06	− 40,000.00	− 10,000.00			36,924.06		36,924.06	13,393.58	50,317.63	12,579.41	37,738.22
11		85,581.08	− 40,000.00	− 10,000.00	− 1,000.00		34,581.08		34,581.08	14,903.10	49,484.19	12,371.05	37,113.14
12		84,258.85	− 40,000.00	− 10,000.00	− 1,000.00		33,258.85		33,258.85	16,387.63	49,646.48	12,411.62	37,234.86
13		82,957.05	− 40,000.00	− 10,000.00	− 1,000.00		31,957.05		31,957.05	17,877.02	49,834.08	12,458.52	37,375.56
14		81,675.37	− 40,000.00	− 10,000.00	− 1,000.00		30,675.37		30,675.37	19,372.05	50,047.41	12,511.85	37,535.56
15		80,413.48	− 40,000.00	− 10,000.00	− 1,000.00		29,413.48		29,413.48	20,873.47	50,286.95	12,571.74	37,715.21
16		79,171.09	− 40,000.00	− 10,000.00	− 1,000.00		28,171.09		28,171.09	22,382.08	50,553.17	12,638.29	37,914.88
17		77,947.90	− 40,000.00	− 10,000.00	− 1,000.00		26,947.90		26,947.90	23,898.67	50,846.57	12,711.64	38,134.93
18		76,743.61	− 40,000.00	− 10,000.00	− 1,000.00		25,743.61		25,743.61	25,424.07	51,167.68	12,791.92	38,375.76
19		75,557.92	− 40,000.00	− 10,000.00	− 1,000.00		24,557.92		24,557.92	26,959.10	51,517.02	12,879.25	38,637.76
20		74,390.55	− 40,000.00	− 10,000.00	− 1,000.00		23,390.55		23,390.55	28,504.61	51,895.16	12,973.79	38,921.37
21		73,241.21		− 10,000.00	− 1,000.00		62,241.21	− 15,000.00	47,241.21	24,061.47	71,302.68	17,825.67	53,477.01
22		72,109.64		− 10,000.00	− 1,000.00		61,109.64	− 15,000.00	46,109.64	25,023.92	71,133.56	17,783.39	53,350.17
23		70,995.54		− 10,000.00	− 1,000.00		59,995.54	− 15,000.00	44,995.54	26,024.88	71,020.42	17,755.11	53,265.32
24		69,898.66		− 10,000.00	− 1,000.00		58,898.66	− 15,000.00	43,898.66	27,065.88	70,964.54	17,741.13	53,223.40
25		68,818.73		− 10,000.00	− 1,000.00		57,818.73	− 15,000.00	42,818.73	28,148.51	70,967.24	17,741.81	53,225.43
26		67,755.48		− 10,000.00	− 1,000.00		56,755.48	− 15,000.00	41,755.48	29,274.45	71,029.93	17,757.48	53,272.45
27		66,708.66		− 10,000.00	− 1,000.00		55,708.66	− 15,000.00	40,708.66	30,445.43	71,154.09	17,788.52	53,365.56
28		65,678.01		− 10,000.00	− 1,000.00		54,678.01	− 15,000.00	39,678.01	31,663.25	71,341.25	17,835.31	53,505.94
29		64,663.28		− 10,000.00	− 1,000.00		53,663.28	− 15,000.00	38,663.28	32,929.78	71,593.06	17,898.26	53,694.79
30		63,664.23		− 10,000.00	− 1,000.00	− 30,000.00	22,664.23	− 15,000.00	7,664.23	34,246.97	41,911.20	10,477.80	31,433.40

Table 2: Balance Sheets

t	Operating assets	Net fixed assets	Working capital	Liquid assets	Total assets	Equity
0						
1	20,000.00		20,000.00	17,500.00	37,500.00	37,500.00
2	17,255.00		17,255.00	57,111.25	74,366.25	74,366.25
3	14,515.87		14,515.87	96,764.12	111,279.99	111,279.99
4	11,781.97		11,781.97	136,478.12	148,260.10	148,260.10
5	9,052.67		9,052.67	176,273.08	185,325.75	185,325.75
6	6,327.32		6,327.32	216,169.16	222,496.48	222,496.48
7	3,605.30		3,605.30	256,186.82	259,792.12	259,792.12
8	885.97		885.97	296,346.93	297,232.90	297,232.90
9				334,839.39	334,839.39	334,839.39
10				372,577.62	372,577.62	372,577.62
11				409,690.76	409,690.76	409,690.76
12				446,925.62	446,925.62	446,925.62
13				484,301.18	484,301.18	484,301.18
14				521,836.74	521,836.74	521,836.74
15				559,551.95	559,551.95	559,551.95
16				597,466.83	597,466.83	597,466.83
17				635,601.76	635,601.76	635,601.76
18				673,977.52	673,977.52	673,977.52
19				712,615.28	712,615.28	712,615.28
20	150,000.00	150,000.00		601,536.65	751,536.65	751,536.65
21	135,000.00	135,000.00		625,598.12	760,598.12	760,598.12
22	120,000.00	120,000.00		650,622.04	770,622.04	770,622.04
23	105,000.00	105,000.00		676,646.92	781,646.92	781,646.92
24	90,000.00	90,000.00		703,712.80	793,712.80	793,712.80
25	75,000.00	75,000.00		731,861.31	806,861.31	806,861.31
26	60,000.00	60,000.00		761,135.76	821,135.76	821,135.76
27	45,000.00	45,000.00		791,581.19	836,581.19	836,581.19
28	30,000.00	30,000.00		823,244.44	853,244.44	853,244.44
29	15,000.00	15,000.00		856,174.22	871,174.22	871,174.22
30						

Table 3: Financial prospects

t	Operating cash flow	Non operating cash flow	Project's cash flow	Cash flow to capital providers	Cash flow to equityholders
0					
1	17,500.00	−17,500.00			
2	38,911.25	−38,911.25			
3	37,368.42	− 37,368.42			
4	35,843.44	− 35,843.44			
5	34,335.84	− 34,335.84			
6	32,845.15	− 32,845.15			
7	31,370.90	− 31,370.90			
8	29,912.64	−29,912.64			
9	26,638.58	−26,638.58			
10	24,344.65	−24,344.65			
11	22,210.03	−22,210.03			
12	20,847.23	−20,847.23			
13	19,498.53	− 19,498.53			
14	18,163.51	−18,163.51			
15	16,841.74	−16,841.74			
16	15,532.80	−15,532.80			
17	14,236.26	−14,236.26			
18	12,951.69	−12,951.69			
19	11,678.66	−11,678.66			
20	− 139,583.24	139,583.24			
21	44,415.54		44,415.54	44,415.54	44,415.54
22	43,326.25		43,326.25	43,326.25	43,326.25
23	42,240.44		42,240.44	42,240.44	42,240.44
24	41,157.53		41,157.53	41,157.53	41,157.53
25	40,076.92		40,076.92	40,076.92	40,076.92
26	38,998.00		38,998.00	38,998.00	38,998.00
27	37,920.13		37,920.13	37,920.13	37,920.13
28	36,842.69		36,842.69	36,842.69	36,842.69
29	35,765.02		35,765.02	35,765.02	35,765.02
30	12,186.43	890,421.19	902,607.62	902,607.62	902,607.62

Table 4: Valuation metrics

Valuation metric	Project
NPV	215,027.22
Average ROI	8.35%
MARR	6.93%
Economic efficiency	1.42%
Total capital	15,132,751.70
IRR	Does not exist

Performance measurement and decomposition of value added

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Performance measurement and decomposition of value added

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Abstract

In this paper, we benchmark an investment actively managed (e.g., fund, portfolio) against a reference portfolio passively managed replicating the investment's cash flows in order to measure the value added by the active investment and decompose it according to the influence of the investment choices (i.e., selection and allocation of assets) made in the various periods. The active investment choices are reflected in the investment's returns as opposed to the benchmark returns earned by the passive strategy. We precisely quantify the impact of the holding period rates on the value added and rank them accordingly, in order to identify the most (and the least) influential ones. The analysis is performed by applying the Finite Change Sensitivity Index (FCSI) method (Borgonovo 2010a, 2010b), a recently-conceived technique of sensitivity analysis, which we refine by means of a duplication-clearing procedure which allows a perfect (i.e., with no residue) decomposition of the value added.

We conduct the analysis for a given contribution-and-distribution policy, characterized by a fixed sequence of deposits and withdrawals. We show that, if the contribution-and-distribution policy changes, the effect of the investment choices made in the various periods on the value added changes as well.

Keywords. Value added, performance measurement, investment policy, sensitivity analysis.

1 Introduction

A number of metrics are used in practice for measuring the performance of an investment (portfolio of assets, fund, etc.) and a substantial amount of contributions have recently dealt with pros and cons of various metrics from several points of view, all of which taking into account the role of a benchmark return in assessing the investment's value added (see Long and Nickels 1996, Gredil et al 2014, Magni 2014, Altshuler and Magni 2015, Jiang 2017, Cuthbert and Magni 2018). However, despite the considerable attention drawn on the appropriateness of a performance criterion, the problem of measuring the impact of the investment choices (i.e., selection and allocation of assets) made in a period on the investment's value added have been neglected. Since decisions about selection and allocation of assets in a given period generate a well-determined holding period rate, this problem boils down to measuring the effect of each investment's holding period rate on the investment's value added.

This paper is a first attempt to fill the need of measuring the impact of the investment policy on the value added. As anticipated, we use the effect of a holding period rate on the value added to measure the effect of the decisions made in that period on the overall investment's value added. We measure the impact of each rate and rank the rates according to their impact on value added, thereby identifying the ones that have been most influential. In this way, the analysis enables measuring the effect of the investment decisions made in every period on the investment's performance and understand in which periods the most important (and less important) decisions have been made. To accomplish this objective, we assume that the contribution-and-distribution policy is given (i.e., we assume the sequence of deposits and withdrawals is given) and describe an investment's value added as the change in the capital terminal value obtained by switching from a passive investment in a benchmark portfolio to an active investment generating returns which are different from the benchmark returns. Then, we make use of a recently-conceived technique of sensitivity analysis, which apportions a discrete change in a model output to the discrete changes in the model inputs: The so-called Finite Change Sensitivity Index (FCSI), introduced in Borgonovo (2010a, 2010b). We suitably supplement this technique with a fine-tuning of the FCSI procedure which enables achieving a perfect (i.e., with no residue) decomposition of the investment's value added.

The remainder of the paper is structured as follows. Section 2 introduces the setting and, in particular, presents the benchmark portfolio and its role in the definition of an investment's value added. Section 3 introduces the Finite Change Sensitivity Index and the way it triggers a decomposition of the finite change of an objective function. Since the FCSI duplicates the interaction effects, we fine-tune it with a simple duplication-clearing procedure and provide the Clean FCSI. Section 4 uses the Clean FCSI technique for apportioning the effect of the investment decisions made in the various periods to the investment's value added. Section 5 illustrates the procedure with a numerical example. Some remarks conclude the paper.

2 Benchmark portfolio and value added

Following is a simple description of a model for the (discrete) evaluation of the investment, consisting of a portfolio of assets. An investor invests a capital B_0 at time $t = 0$. By selecting the assets and allocating them in every period, the portfolio's value is increased or decreased. Furthermore, the investor makes decisions about capital contributions or distributions in the various periods, which increase or decrease the amount of capital invested in the portfolio.

We assume that the investment starts at time $t = 0$ and analyze its performance in the time interval $[0, n]$ where, for convenience, we assume that n is the current date.

Let E_t be the end-of-period portfolios's value and B_t its beginning-of-period value. Let F_t be the investor's contribution/distribution into/from the portfolio at time $t = 0, 1, \dots, n-1$. From the point of view of the investor, a contribution is an outflow ($F_t < 0$), a distribution is an inflow ($F_t > 0$). In particular, at time 0, the contributed amount is an outflow, so $F_0 = -B_0 < 0$. Then, the following relations hold:

$$\begin{aligned} B_t &= E_t - F_t \\ i_t &= \frac{E_t - B_{t-1}}{B_{t-1}} \\ E_t &= B_{t-1} \cdot (1 + i_t) \end{aligned} \tag{1}$$

where i_t denotes the rate of return in the period. The first equation says that the beginning-of-period value is obtained by deducting the capital call or adding the contribution made by the investor; the second relation says that the investment's holding period rate expresses the relative increase in the capital value; the third relation says that the ending value is obtained from the beginning value by marking it up by the return rate i_t. The selection and allocation policy affects i_t, which in turn affects E_t and, hence, B_t. The investor's choices about withdrawals and deposits affects B_t and, hence, E_t. Therefore, both types of policies affect the capital values, but only the investment policy affects i_t. The latter is then an appropriate measure of the effect on the value added of the investment policy in a given period.

Let us focus on the terminal date, $t = n$, and on its closing value, $E_n = B_{n-1}(1+i_n)$.[1] Using (1) and solving for $t = n$, one can express E_n as a function of the return rates and the cash flows prior to n:

$$E_n = -\sum_{t=0}^{n-1} F_t(1 + i_{t+1})(1 + i_{t+2}) \dots (1 + i_n). \tag{2}$$

The above relation tells us that the terminal investment's value is the compounded amount of the contributions (net of distributions) made by the investor.

Consider now a benchmark index whose holding period rate is denoted as i_t^*, and a reference (benchmark) portfolio which acts as the opportunity cost of capital for the investment. More precisely, let us consider what would have occurred if the investor had made the same contributions/distributions in the benchmark portfolio. Under this

[1]If the investment is liquidated at time n, then $E_n = F_n$.

assumption, the investor follows a passive strategy and replicates the investment's cash flows: Every contribution to the investment is matched by an equal contribution in the benchmark portfolio and every distribution from the investment is matched by an equal distribution from the benchmark. In general, the benchmark portfolio's value is different from the investment's value at every date t, which means that the holding period rates i_t and i_t^* are different. The difference between the two returns is determined by the active choices of asset selection and stock allocation in period t. In such a way, the benchmark portfolio is a replica of the investment's cash flows up to (and including) time $n-1$. At time n, the investment's residual value will differ from the benchmark's residual value.

Formally, let $F_t^* = F_t$ be the cash flows in the reference portfolio, $t = 0, \dots, n-1$. We denote as B_t^* and E_t^* the beginning-of-period and end-of-period market value of this benchmark portfolio. Then, the following relations mimic the ones presented in (1):

$$\begin{aligned} B_t^* &= E_t^* - F_t^* \\ i_t^* &= \frac{E_t^* - B_{t-1}^*}{B_{t-1}^*} \\ E_t^* &= B_{t-1}^* \cdot (1 + i_t^*). \end{aligned} \tag{3}$$

In $t = n$, the net value of the benchmark portfolio is $E_n^* = B_{n-1}^*(1 + i_n^*)$. Analogously to eq. (2), the benchmark terminal net asset value E_n^* depends on the previous cash flows and the benchmark index return rates:

$$E_t^* = -\sum_{t=0}^{n-1} F_t(1 + i_{t+1}^*)(1 + i_{t+2}^*) \dots (1 + i_n^*). \tag{4}$$

As the investment and the benchmark portfolio release the same sequence of inflows and outflows up to time $n-1$, the investment outperforms the benchmark if and only if the terminal value of the fund is greater than the terminal value of the replicating portfolio: $E_n > E_n^*$. The difference $E_n - E_n^*$ is the value added, denoted as VA:

$$\mathrm{VA} = E_n - E_n^* = \sum_{t=0}^{n-1} F_t \cdot \Big((1+i_{t+1}^*)(1+i_{t+2}^*) \dots (1+i_n^*) - (1+i_{t+1})(1+i_{t+2}) \dots (1+i_n)\Big). \tag{5}$$

Therefore, the investment outperforms the benchmark if and only if the value added is positive, $\mathrm{VA} > 0$.

For a given sequence of injections and withdrawals $(F_0, F_1, \dots, F_{n-1})$ and a given sequence of benchmark returns $(i_1^*, i_2^*, \dots, i_n^*)$, the value added by such an investment depends on the active investment decisions, which is reflected in the return vector $(i_1, i_2, \dots, i_n)$.

3 Finite Change Sensitivity Indices

Sensitivity analysis (SA) is the study of how the variance of the output of a model (numerical or otherwise) can be apportioned to different input key parameters (Saltelli et al. 2004). As such, it aims at quantifying how much of an output change is attributed to a given parameter or a set of parameters. It is widely employed in finance and management (Huefner 1972), for instance in analysing the value creation of industrial projects (Borgonovo and Peccati 2004, 2006; Borgonovo, Gatti, and Peccati 2010; Percoco and Borgonovo 2012; Marchioni and Magni 2018), the composition of optimal financial portfolios (Luo, Seco and Wu 2015), and the effects of corporate debt (Donders, Jara and Wagner 2018; Délèze and Korkeamäki 2018).

There exist several SA techniques defined in the literature (see Borgonovo and Plischke 2016, Pianosi et al. 2016, Saltelli et al. 2008, 2004 for reviews of SA methods). Among others, the Finite Change Sensitivity Indices (FCFIs) have been recently conceived for analyzing the effect of the finite changes in the model inputs onto the finite changes of a model output. Formally, let f be the objective function, which maps the vector of inputs (parameters, key drivers) $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ onto the model output $y(x)$:

$$f : \mathbb{R}^n \to \mathbb{R}, \quad y = f(x), \quad x = (x_1, x_2, \dots, x_n)\,. \tag{6}$$

Let the inputs vary from $x^0 = (x_1^0, \dots, x_n^0)$, the so-called base value, to $x^1 = \left(x_1^1, x_2^1, \dots, x_n^1\right)$, the realized value. The corresponding model outputs are $f(x^0)$ and $f(x^1)$, so that the output variation is $\Delta f = f(x^1) - f(x^0)$. Let $(x_j^1, x_{(-j)}^0) = (x_1^0, x_2^0, \dots, x_{j-1}^0, x_j^1, x_{j+1}^0, \dots, x_n^0)$ be the vector consisting of all the inputs set at their base value x^0, except parameter x_j which is given the realized value x_j^1. Analogously, let

$$(x_j^1, x_k^1, x_{(-j,k)}^0) = (x_1^0, x_2^0, \dots, x_{j-1}^0, x_j^1, x_{j+1}^0, \dots, x_{k-1}^0, x_k^1, x_{k+1}^0, \dots, x_n^0)$$

be the input vector where x_j and x_k are set to the realized values, while the remaining $n-2$ are set at their base value, and so forth for all s-tuples of inputs, $s = 1, 2, \dots, n$.

Borgonovo (2010a, 2010b) defines two versions of FCSIs: First Order FCSI and Total Order FCSI. The First Order FCSI of parameter x_j measures the individual effect of x_j (Borgonovo 2010a), $\Delta_j^1 f = f(x_j^1, x_{(-j)}^0) - f(x^0)$, and, in normalized version, $\Phi_j^{1,f} = \frac{\Delta_j f}{\Delta f}$. On the other side, the Total Order FCSI quantifies the total effect of the parameter, including both its individual contribution and its interactions with other parameters. Let $\Delta_{j,k} f$ be the interaction between x_j and x_k, that is the portion of $f(x_j^1, x_k^1, x_{(-j,-k)}^0) - f(x^0)$ not explained by the individual effects $\Delta_j^1 f$ and $\Delta_k^1 f$: $\Delta_{j,k} f = f(x_j^1, x_k^1, x_{(-j,-k)}^0) - f(x^0) - \Delta_j^1 f - \Delta_k^1 f$. Similarly, let $\Delta_{j,k,h} f$ be the interaction among the inputs x_j, x_k and x_h, which is the portion of $f(x_j^1, x_k^1, x_h^1, x_{(-j,-k,-h)}^0) - f(x^0)$ not explained by the individual effects and by the interactions between any pair: $\Delta_{j,k,h} f = f(x_j^1, x_k^1, x_h^1, x_{(-j,-k,-h)}^0) - f(x^0) - \Delta_j^1 f - \Delta_k^1 f - \Delta_h^1 f - \Delta_{j,k} f - \Delta_{j,h} f - \Delta_{k,h} f$ (analogously for a s-tuple, with $s > 3$). The variation of f from x^0 to x^1 is equal to the sum of individual effects and interactions, counted only once, between parameters and groups of parameters:

$$\Delta f = \overbrace{\sum_{i=j}^{n} \Delta_j^1 f}^{\text{individual effects}} \quad +$$

$$\underbrace{\overbrace{\sum_{j_1<j_2} \Delta_{j_1,j_2} f}^{\text{pairs}} + \overbrace{\sum_{j_1<j_2<j_3} \Delta_{j_1,j_2,j_3} f}^{\text{triplets}} + \cdots + \overbrace{\sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f}^{s\text{-tuples}} + \ldots + \overbrace{\Delta_{j_1,j_2,\ldots,j_n} f}^{n\text{-tuple}}}_{\text{overall interaction effects}},$$

where $\sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f$ is the sum of the interactions between s-tuples.

Borgonovo (2010a) defines the Total Order FCSI of x_j, $\Delta_j^{\mathcal{T}} f$, as the sum of First Order FCSI of x_j, $\Delta_j^1 f$, and the interaction effect of x_j, identified as $\Delta_j^{\mathcal{I}} f$ and called Interaction FCSI. The latter is the sum of every interaction involving x_j:

$$\Delta_j^{\mathcal{I}} f = \sum_{\substack{j_1<j_2 \\ j\in\{j_1,j_2\}}} \Delta_{j_1,j_2} f + \ldots + \sum_{\substack{j_1<j_2\ldots<j_s \\ j\in\{j_1,j_2,\ldots,j_s\}}} \Delta_{j_1,j_2,\ldots,j_s} f + \ldots + \Delta_{j_1,j_2,\ldots,j_n} f.$$

Therefore,

$$\Delta_j^{\mathcal{T}} f = \Delta_j^1 f + \Delta_j^{\mathcal{I}} f = \Delta_j^1 f + \sum_{\substack{j_1<j_2 \\ j\in\{j_1,j_2\}}} \Delta_{j_1,j_2} f + \ldots + \sum_{\substack{j_1<j_2\cdots<j_s \\ j\in\{j_1,j_2,\ldots,j_s\}}} \Delta_{j_1,j_2,\ldots,j_s} f + \ldots + \Delta_{j_1,j_2,\ldots,j_n} f \tag{7}$$

and, in normalized version, $\Phi_j^{\mathcal{T}} = \frac{\Delta_j^{\mathcal{T}} f}{\Delta f}$.

Computationally, the calculation of the Interaction FCSIs may be extremely burdensome if the model does not contain a very small number of inputs.[2] Borgonovo (2010a, Proposition 1) shows that the following result holds:

$$\Delta_j^{\mathcal{T}} f = f(x^1) - f(x_j^0, x_{(-j)}^1), \ \forall j = 1, 2, \ldots, n, \tag{8}$$

where $(x_j^0, x_{(-j)}^1)$ denotes the vector with each input equal to the realized value x^1, except for x_j which is set equal to x_j^0. This enables computing the total FCSI of x_j with no need of summing the First Order FCSI of x_j and the Interaction FCSI of x_j.

Unfortunately, the Total Order FCSI does not provide a complete decomposition of the output change:

$$\sum_{l=1}^{n} \Delta_l^{\mathcal{T}} f \neq \Delta f = f(x^1) - f(x^0) \quad \text{or, equivalently,} \ \sum_{l=1}^{n} \Phi_l^{\mathcal{T}} \neq 1.$$

In other words, the sum of Total FCSIs explains less (or more) than 100% of the output change. To understand why, consider that, in the sum of the Interaction FCSIs, $\sum_{l=1}^{n} \Delta_l^{\mathcal{I}} f$, the pairwise interactions of x_j and x_k appear twice (in $\Delta_j^{\mathcal{T}} f$ and in $\Delta_k^{\mathcal{T}} f$); the three-wise interactions of x_j, x_k, and x_h appear three times (in $\Delta_j^{\mathcal{T}} f$, in $\Delta_k^{\mathcal{T}} f$, and in $\Delta_h^{\mathcal{T}} f$); and so on for all the s-wise interactions, $s = 2, 3, \ldots, n$. This implies that the sum of Interaction

[2]The number of interactions between parameters and groups of parameters is equal to $2^n - n$.

FCSIs does not equate the overall interaction effects:

$$\sum_{l=1}^{n} \Delta_l^{\mathcal{I}} f \neq \underbrace{\overbrace{\sum_{j_1<j_2} \Delta_{j_1,j_2} f}^{\text{pairs}} + \overbrace{\sum_{j_1<j_2<j_3} \Delta_{j_1,j_2,j_3} f}^{\text{triplets}} + \cdots + \overbrace{\sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f}^{s\text{-tuples}} + \ldots + \overbrace{\Delta_{j_1,j_2,\ldots,j_n} f}^{n\text{-tuple}}}_{\text{overall interaction effects}}$$

and, therefore, $\sum_{l=1}^{n} \Delta_l^{\mathcal{T}} f \neq \Delta f$.

However, it is possible to introduce a duplication-clearing factor which eliminates the redundant, multiple interactions and allows a complete and exact decomposition of the output change. We define the *Clean* Interaction FCSI of x_j, $\Delta_j^I f$, as the product of the Interaction FCSI $\Delta_j^{\mathcal{I}} f$ and a suitable corrective factor:

$$\Delta_j^I f = \Delta_j^{\mathcal{I}} f \cdot \frac{\sum_{j_1<j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_n} f}{\sum_{j=1}^{n} \Delta_j^{\mathcal{I}} f}. \tag{9}$$

Considering that $\Delta_j^{\mathcal{I}} f = \Delta_j^{\mathcal{T}} f - \Delta_j^1 f$ and

$$\sum_{j_1<j_2} \Delta_{j_1,j_2} f + \cdots + \sum_{j_1<j_2\cdots<j_s} \Delta_{j_1,j_2,\ldots,j_s} f + \cdots + \Delta_{j_1,j_2,\ldots,j_n} f = \Delta f - \sum_{j=1}^{n} \Delta_j^1 f,$$

one may reframe (9) as

$$\Delta_j^I f = \frac{\Delta_j^{\mathcal{T}} f - \Delta_j^1 f}{\sum_{l=1}^{n} (\Delta_l^{\mathcal{T}} f - \Delta_l^1 f)} \cdot \big(\Delta f - \sum_{l=1}^{n} \Delta_l^1 f\big). \tag{10}$$

In other words, the Clean Interaction FCSI is computed by imputing a share of the overall true interaction effect $(\Delta f - \sum_{l=1}^{n} \Delta_l^1 f)$ to parameter x_j. This share is obtained as the ratio of the Interaction FCSI of x_j and the sum of all Interaction FCSIs.

We define the Clean Total Order FCSI of parameter x_j, $\Delta_j^T f$, as the sum of individual contribution and Clean Interaction FCSI of x_j:

$$\Delta_j^T f = \Delta_j^1 f + \Delta_j^I f \tag{11}$$

and, in normalized version, $\Phi_j^T = \frac{\Delta_j^T f}{\Delta f}$. It is easy to see that the Clean Total FCSIs completely explain the output variation:

$$\sum_{l=1}^{n} \Delta_l^T f = \Delta f,$$

and, in normalized version, $\sum_{l=1}^{n} \Phi_l^T = 1$.

The sign of a Clean Total FCSI, $\Delta_j^T f$, signals the directional effect of an input change onto the output change: A positive (negative) index signals that the change in the input has the effect of increasing (decreasing) the output. The absolute value of the Clean Total FCSI quantifies the magnitude of the effect; one may then rank the input factors according to their influence on the change in the objective function: Input x_j has higher rank than x_j if and only if $|\Delta_j^T f| > |\Delta_j^T f|$. We denote the rank of parameter x_j as R_j.

The rank vector is $R = (R_1, R_2, \ldots, R_n)$.

4 Attribution of value added

Let $x = (x_1, x_2, \ldots, x_n)$ be the vector of time-varying return rates of an investment with cash flows F_t from $t = 0$ to $n-1$. Generalizing equations (2) and (4), the terminal net asset value implied by the return rates vector x, denoted as $f(x)$, is, for a given sequence of cash flows $(F_0, F_1, \ldots, F_{n-1})$, equal to

$$f(x) = -\sum_{t=0}^{n-1} F_t(1 + x_{t+1})(1 + x_{t+2}) \ldots (1 + x_n). \tag{12}$$

Let $x^0 = i^*$ be the stream of benchmark returns (base value). The active investment policy followed in the various periods has the effect of moving the rates from $x^0 = i^*$ to $x^1 = i$ (realized case). This in turn has the effect of changing the terminal value from $f(x^0) = f(i^*)$ to $f(x^1) = f(i)$. However,

$$f(i^*) = E_n^* \tag{13}$$

$$f(i) = E_n. \tag{14}$$

Therefore, the value added by the investment may be written as

$$\text{VA} = E_n - E_n^* = f(i) - f(i^*). \tag{15}$$

As a result, the value added is equal to a finite change of f. Therefore, one may apply the FCSI technique integrated by the duplication-clearing procedure for decomposing VA in terms of period rates. It is then possible to identify the periods whose investment choices have most affected the investment's performance. In particular, for any given sequence of contributions and distributions, the value added may be considered as the sum of all the effects of the active selection and allocation choices made in the various periods, as opposed to a passive strategy consisting in investing in a benchmark portfolio.

For accomplishing a complete, exact decomposition of the value added, we use the Clean FCSIs. Note that the piece of information provided by Φ_j^T is not whether and how much the investment outperforms or underperforms the benchmark in period t, but whether the investment decisions made in period t have contributed, overall, to outperform or underperform the benchmark in the time interval $[0, n]$ and how much of the value added is attributable to them. This piece of information necessarily takes account of the interactions with the decisions made in the other periods. The decisions made in period t determine i_t, which measures the relative growth in the investment's value at time t and, therefore, affect (along with the other rates) the magnitude of the value added not only in period t, but also in the following periods $t+1$, $t+2$, …, n. The Clean Total FCSI, $\Delta_j^T f$, precisely provides the amount of value added that is determined by the investment policy in period t.

The analysis above assumes that the policy of contributions and distributions is fixed and equal to $(F_0, F_1, \ldots, F_{n-1})$. Consider now a different sequence of contributions and distributions:

$$(G_0, G_1, \ldots, G_{n-1}) \neq (F_0, F_1, \ldots, F_{n-1})$$

and let

$$g(x) = -\sum_{t=0}^{n-1} G_t(1+x_{t+1})(1+x_{t+2})\ldots(1+x_n) \tag{16}$$

be the investment's terminal value. In general, the functions $f(x)$ and $g(x)$ are different, which implies that the value added will be different as well: $f(i)-f(i^*) \neq g(i)-g(i^*)$. In addition, the Clean Total FCSIs of the parameters under f and g will generally be different, implying that the same choices about investments in a given period have a different impact on the value added depending on the choices about injections/withdrawals made by the investor. Therefore, it may occur the case where a given parameter x_j triggered by a given investment policy in period j has a substantial impact on value added for a contribution-and-distribution policy and a negligible impact on value added for a different contribution-and-distribution policy.

In the following section we present a worked example where we measure the impact of the period investment decisions under two different assumptions about contributions and distributions.

5 Worked example

We consider an investment management agreement whereby an investor endows a fund manager the capital amount $B_0 = -F_0 = 100$. The investment lasts $n = 8$ periods and is described in Table 1. The contribution and distribution policy is under full control of the investor, who determines the timing and amount of withdrawals and deposits from $t = 1$ to $t = 7$. The investment policy of the fund manager in period t brings about a return rate equal to i_t in period t, $t = 1, 2, \ldots, 8$. In the same period, the benchmark index's return is i^*_t. From (2) and (4), the terminal values of the fund and of the replicating portfolio are $E_8 = 7.71$ and $E^*_8 = 5.25$, respectively, implying that, given the sequence of contributions and distributions, the value added is VA $= 2.47 = 7.71 - 5.25 > 0$.

Table 1: Input data

Time t	**Fund's cash flows** F_t	**Fund's returns** i_t	**Benchmark's returns** i^*_t
0	−100		
1	30	4%	3%
2	−20	5%	4%
3	40	2%	3%
4	10	4%	6%
5	−30	3%	1%
6	60	3%	2%
7	20	5%	2%
8		4%	5%

We now decompose the value added in terms of the influences of the active investment choices made in the various periods with respect to a passive investment earning the benchmark return with the same array of contributions and distributions. This is done by evaluating the effect of the change of the terminal value when the return vector is changed from the benchmark return vector, i^*, to the fund's return vector, i. To this end, we consider the objective function

$$f(x) = -\sum_{t=0}^{7} F_t(1 + x_{t+1}) \dots (1 + x_8)$$

with

$$x^0 = i^* = (3\%, 4\%, 3\%, 6\%, 1\%, 2\%, 2\%, 5\%)$$

and

$$x^1 = i = (4\%, 5\%, 2\%, 4\%, 3\%, 3\%, 5\%, 4\%).$$

Table 2 collects the results of the analysis. The first column collects the vector of input parameters, $(x_1, x_2, \dots, x_8)$, which are determined by the investment choices made in the various periods. The second column describes the First Order FCSIs, the third column is the Total Order FCSI determined via eq. (8), the fourth one collects the Interaction FCSIs calculated as difference between third column and fourth column; the fifth column clears the duplications and supplies the Clean Interaction FCSI, which is computed as in (10); the sixth column represents the Clean Total Order FCSI as defined in (11); the seventh column reports the normalized Clean Total Order FCSI, and, finally the eight column shows the inputs' ranking.

Table 2: Decomposition of the value added
and inputs' ranking

x_j	$\Delta_j^1 f$	$\Delta_j^{\mathcal{T}} f$	$\Delta_j^{\mathcal{I}} f$	$\Delta_j^I f$	$\Delta_j^T f$	Φ_j^T	R_j
x_1	1.25	1.29	0.04	−0.02	1.23	49.96%	2
x_2	0.88	0.91	0.03	−0.02	0.86	34.98%	6
x_3	−1.12	−1.18	−0.06	0.03	−1.09	−44.24%	4
x_4	−1.30	−1.38	−0.08	0.05	−1.25	−50.70%	1
x_5	1.14	1.17	0.03	−0.02	1.13	45.74%	3
x_6	0.89	0.91	0.03	−0.01	0.87	35.40%	5
x_7	0.77	0.81	0.04	−0.02	0.75	30.34%	7
x_8	−0.05	−0.07	−0.02	0.01	−0.04	−1.48%	8

The most influential input on the value added is the return rate in period 4, x_4, with $\Delta_4^T f = -1.25$ and $\Phi_4^T = -50.70\%$, implying that it has had a negative effect on the VA and that its magnitude is about half of the value added. In other words, the investment decisions made in the fourth period have overall contributed negatively to the fund's performance and have had the greatest impact on the value added.

It is worth noting that the individual contribution of x_4 to the value added is obtained with the following argument: Suppose the investor invests passively in the benchmark index from time $t = 0$ to time $t = 3$, then switches to the fund manager's active investment at time $t = 3$ and then switches back to the benchmark index at time $t = 4$. This strategy results in the following terminal value:

$$
\begin{aligned}
E_8 &= f(0.03, 0.04, 0.03, \mathbf{0.04}, 0.01, 0.02, 0.02, 0.05) \\
&= 100(1.03)(1.04)(1.03)(1.04)(1.01)(1.02)(1.02)(1.05) \\
&- 30(1.04)(1.03)(1.04)(1.01)(1.02)(1.02)(1.05) \\
&+ 20(1.03)(1.04)(1.01)(1.02)(1.02)(1.05) \\
&- 40(1.04)(1.01)(1.02)(1.02)(1.05) \\
&- 10(1.01)(1.02)(1.02)(1.05) \\
&+ 30(1.02)(1.02)(1.05) \\
&- 60(1.02)(1.05) \\
&- 20(1.05) = 3.95.
\end{aligned}
$$

If no switching occurs, the terminal capital value is

$$
\begin{aligned}
E_8^* &= f(0.03, 0.04, 0.03, \mathbf{0.06}, 0.01, 0.02, 0.02, 0.05) \\
&= 100(1.03)(1.04)(1.03)(1.06)(1.01)(1.02)(1.02)(1.05) \\
&- 30(1.04)(1.03)(1.06)(1.01)(1.02)(1.02)(1.05) \\
&+ 20(1.03)(1.06)(1.01)(1.02)(1.02)(1.05) \\
&- 40(1.06)(1.01)(1.02)(1.02)(1.05) \\
&- 10(1.01)(1.02)(1.02)(1.05) \\
&+ 30(1.02)(1.02)(1.05) \\
&- 60(1.02)(1.05) \\
&- 20(1.05) = 5.25.
\end{aligned}
$$

The difference, $\Delta_4^1 f = 3.95 - 5.25 = -1.3$, represents the individual contribution of x_4, that is, the impact of the decisions made in period 4 on the value added, taken in isolation from the other inputs. The clean interaction effect is calculated as in eq. (10) and supplies a partial compensating effect, $\Delta_4^I f = 0.05$. Overall, the contribution to value of the active investment policy of the fourth period on the investment's value added is $\Delta_4^T = -1.25$. In relative terms, x_4's weight is $\Phi_4^T = -50.7\%$.

The second and third most influential inputs are the return rates in periods 1 and 5, x_1 and x_5, which have had a positive effect on value added. In particular, their total contributions are, respectively, $\Delta_1^T = 1.23$ and $\Delta_5^T = 1.13$. In relative terms, their weights are $\Phi_1^T = 49.96\%$ and $\Phi_5^T = 45.74\%$. Next come x_3 (negative impact), x_6, x_2, x_7 (positive impact) and x_8 (negative effect). The latter explains just -1.48% of VA. The Clean Total Order FCSIs exactly decompose the value added:

$$
\overbrace{1.23 + 0.86 - 1.09 - 1.25 + 1.13 + 0.87 + 0.75 - 0.04}^{\text{sum of Clean Total FCSIs}} = 2.47
$$

$$
\overbrace{49.96\% + 34.98\% - 44.24\% - 50.70\% + 45.74\% + 35.40\% + 30.34\% - 1.48\%}^{\text{sum of normalized Clean Total FCSI (percentage)}} = 100\%.
$$

Consider now a different contribution and distribution policy, determined by the sequence $(G_0, G_1, \ldots, G_{n-1})$ such that $G_0 = F_0 = -100$ and $G_t = 0$ for $t = 1, 2, \ldots 7$, and assume that the selection and allocation choices do not vary. The investment's value added varies; in particular, using (16), the fund's and the benchmark portfolio's values at time 8 are, respectively,

$$E_8 = g(i) = 100(1.04)^3(1.05)^2(1.02)(1.03)^2 = 134.20$$

and

$$E_8^* = g(i^*) = 100(1.03)^2(1.04)(1.06)(1.01)(1.02)^2(1.05) = 129.04,$$

implying that the value added is

$$\mathrm{VA} = g(i) - g(i^*) = 134.2 - 129.04 = 5.16.$$

The value added has increased with respect to the previous case. The FCSI analysis with duplication-clearing procedure is reported in Table 3, showing that, in the case of no interim contributions and distributions, the same investment choices have a very different impact on the value added. The most influential return rate is x_7 ($R_7 = 1$), which has a positive effect on VA. As previously seen, its rank in the case where $(F_0, F_1, \ldots, F_{n-1})$ represented the choices about deposits and withdrawals was only $R_7 = 7$. This means that investment decisions made by the manager in period 7 have the greatest impact if the investor does not make any interim contribution/distribution, whereas they have negligible effect in case of the timing and amounts of cash flows are $(F_0, F_1, \ldots, F_{n-1})$. Conversely, the first-period rate, x_1, which reflects the investment decisions made in period 1, has rank 6 ($R_1 = 6$), whereas it represented the second most influential parameter in the previous case.

Table 3: Decomposition of the value added and inputs' ranking (no interim cash flows)

x_j	$\Delta_j^1 f$	$\Delta_j^{\mathcal{T}} f$	$\Delta_j^{\mathcal{I}} f$	$\Delta_j^I f$	$\Delta_j^T f$	Φ_j^T	R_j
x_1	1.25	1.29	0.04	0.02	1.27	24.63%	6
x_2	1.24	1.28	0.04	0.02	1.26	24.39%	7
x_3	−1.25	−1.32	−0.06	−0.03	−1.28	−24.86%	5
x_4	−2.43	−2.58	−0.15	−0.07	−2.50	−48.54%	3
x_5	2.56	2.61	0.05	0.02	2.58	49.99%	2
x_6	1.27	1.30	0.04	0.02	1.28	24.87%	4
x_7	3.80	3.83	0.04	0.02	3.81	73.91%	1
x_8	−1.23	−1.29	−0.06	−0.03	−1.26	−24.39%	8

6 Concluding remarks

This paper proposes a method for evaluating the effect of the investment policy on an investment's performance, as measured by the value added. Specifically, we show how to quantify the part of the value added generated by the investment decisions made in the various periods, given a fixed sequence of cash flows (contributions and distributions). We compare an active investment strategy with a passive investment strategy in a benchmark portfolio and formalize it in terms of difference between terminal values in case of active investment and passive investment, respectively. This difference, which equals the investment's value added, depends on the relations between the sequence of benchmark returns and the sequence of investment's returns. To accomplish the task, we make use of the Finite Change Sensitivity Index (FCSI) technique (Borgonovo 2010a, 2010b) suitably fine-tuned for clearing the double-counting of the interaction effects implied therein. This brings about the Clean Total FCSI which quantifies and ranks the efficacy of the investment policy via the ranking of the effect of the investment returns on the investment's value added. We also find that, for a given investment policy, not only different contribution-and-distribution policies give rise to different performances but also the effect of the investment decisions have a different impact on the value added. This means that decisions about contributions and distributions and decisions about selection and allocation of assets are strictly intertwined. Further researches may be conducted to assess the degree and the direction of the interaction between investment policy and contribution/distrribution policy.

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Sortino(γ): A Modified Sortino Ratio with Adjusted Threshold

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Sortino(γ): A Modified Sortino Ratio with Adjusted Threshold

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Abstract

A portfolio's Sortino ratio is strongly affected by the risk-free vs. risky assets mix, except for the case where the threshold, T is equal to the risk-free rate. Therefore, if T differs from the risk-free rate, the portfolio's Sortino ratio could potentially be increased by merely changing the mix of the risk-free and the risky components. The widely used Sharpe ratio, on the other hand, does not share this caveat.

We introduce a modified Sortino ratio, Sortino(γ), which is invariant with respect to the portfolio's risk-free vs. risky assets mix, and hence eliminates the above deficiency. The selected threshold $T(\gamma)$, mimics the portfolio composition in the sense that it equals to the risk-free rate plus γ times the portfolio's equity risk premium. Higher selected γ reflects higher risk/loss aversion. We propose a procedure for optimizing the composition of the risky portion of the portfolio to maximize the Sortino(γ) ratio. In addition, we show that Sortino(γ) is consistent with first and second order stochastic dominance with riskless asset rules.

HIGHLIGHTS

- We introduce a modified Sortino ratio, Sortino(γ), whose threshold $T(\gamma)$ is tied to the portfolio mix of risk-free vs. risky assets.
- Sortino(γ) is invariant with respect to the portfolio's of risk-free vs. risky assets mix. Therefore it can be maximized only by improving the composition of the portfolio's risky component, and a maximization process is presented.
- Sortino(γ) is consistent with first and second stochastic dominance with riskless asset rules.

Keywords: Performance ratios, Sortino ratio, Risk aversion, Loss aversion, FSDR rule, SSDR rule.

JEL CLASSIFICATION: G11, D81

I. Introduction

The standard deviation (StDev) of returns is a proper measure of risk only in the limited case of normal return distributions. For all other distributions, preference by the mean variance criterion (MVC) that uses the StDev as its risk measure, is neither necessary nor sufficient condition for preference by all expected utility investors[1]. Indeed, the StDev as a measure of risk has been heavily criticized by many, including Markowitz (1959, pp. 286-288), the originator of the application of the MVC to portfolio optimization. Thus, many researchers suggested the replacement of the StDev with downside risk measures[2]. However, despite its deficiencies and the heavy criticism, the StDev is the risk measure employed by the Sharpe ratio which is probably the most popular performance ratio, and it is also the risk factor in the well-known Capital Asset Pricing Model (CAPM)[3]. Its popularity is probably due, at least in part, to the simple mathematical algorithm needed to construct the optimal portfolios that minimize StDev for any given vector of expected returns given the variance covariance matrix, as well as due to the resulting independence between of the portfolio's optimal risky assets composition and the degree to which the portfolio uses the riskless asset for lending and/or borrowing (i.e., the monetary Separation property).

One of the commonly used downside performance ratios, is the Sortino ratio. The numerator of the Sortino ratio is the expected return of the risky portfolio minus a defined threshold, T, and the denominator is the root of the expected squared return deviations below T [4]. Unfortunately, where T differs from the risk-free rate, the Sortino ratio of a portfolio is affected by the risk-free vs. risky assets mix and this effect increases with the deviation of T from the riskless rate[5]. Thus, in the case where T differs from the risk-free rate, a portfolio's Sortino ratio is sensitive to its equity level and the optimal composition of the equity components of the portfolio cannot be separated from its optimal mix between the risky and the risk-free component. Our paper presents a modified Sortino ratio, Sortino(γ), which is invariant to the portfolio's equity level, for all relevant threshold values.

Our modification is based on replacing the constant T threshold, which is not responsive to the portfolio's equity level, with $T(\gamma)$ which equals the weighted average of the portfolio's expected rate of return and the risk-free rate, using weights of γ and $(1-\gamma)$, respectively. Under the trivial assumption that the portfolio's expected rate of return exceeds the risk-free rate, the higher the γ the higher is $T(\gamma)$.

The paper is organized as follows. In Section 2 we show that if the conventional threshold T is below the portfolio's expected return but differs from the risk-free rate, the Sortino ratio increases or

[1] The stochastic dominance rules for all rational investors (First degree Stochastic Dominance rule - FSD) and for all rational risk averse investors (Second degree Stochastic Dominance rule - SSD) provide necessary and sufficient (optimal) efficiency rules for preference. However, the practical application of these rules for constructing optimal portfolios and obtaining market equilibrium conditions is quite limited.

[2] For a review of many downside risk measures, see Sortino and Price 1994, Sortino and Forsey 1996, Nawrocki 1999, Pedersen and Satchell 2002, Pedersen and Rudholm-Alfvin 2003, Sortino 2009.

[3] The basic CAPM was developed by Treynor 1961, 1965, Sharpe 1964, Lintner 1965, and Mossin 1966).

[4] The ratio belongs to a wider set of performance ratios, Kappa, that employ the lower partial moment as a measure of risk (Kaplan and Knowles 2004).

[5] In what follows, and for the purpose of abbreviation, we often refer to the proportion of the portfolio's risky (equity) component as the "equity level" or the "risk level" of the portfolio.

decreases monotonically and respectively with the portfolio's proportion of the risky vs. the riskless component. This undesired feature of a performance measure potentially allows portfolio managers to increase the ex-ante ratio by merely changing its equity level, namely, by altering the mix of the riskless vs. risky assets rather than by improving the composition of the portfolio's risky component. In Section 3 we present the modified performance measure, Sortino(γ), which employs the threshold $T(\gamma)$. In this section we show that the resulting ratio is invariant with respect to the portfolio's split between the risky and riskless components[6]. Section 4 presents the procedure for obtaining the optimal risky portfolio which maximizes Sortino(γ) for a given γ. Section 5 shows that dominance by stochastic dominance with riskless asset rules (FSDR and SSDR) implies dominance by $S(\gamma)$. Dominance by SDR rules compare preferences for all expected utility investors with none-decreasing utility (FSDR) and for all investors with none-decreasing utility as well as none-increasing marginal utility (SSDR) provided they can borrow and lend against the risky portfolio using the same given riskless rate. Section 6 presents a summary and offers some conclusions.

II. Sortino ratio and the level of the equity component

The ex-ante Sortino ratio of a portfolio with a threshold T is given by[7]:

$$(1) \quad S_P(T) = \frac{E(\tilde{R}_P) - T}{\left[\underset{\tilde{R}_P \le T}{E}(T - \tilde{R}_P)^2\right]^{0.5}}.$$

$S_P(T)$ is the portfolio's Sortino ratio, $\tilde{R}_P$ is the (random) rate of return on the portfolio and E is the expected value operator. While the riskless rate of return is perhaps the most likely choice for a threshold, thresholds which are higher or lower than the riskless rate are used in the literature[8]. Denote the portfolio's proportion of the risky asset and the proportion of the risk-free asset by α and (1-α), respectively, and let $\tilde{R}_e$ and R_f represent the (random) rate of return on the equity component and the rate of return on the risk-free asset, respectively. Since $\tilde{R}_P = \alpha\tilde{R}_e + (1-\alpha)R_f$ we can rewrite Eq. (1) as follows:

$$(2) \quad S_P(T) = \frac{\alpha E(\tilde{R}_e) + (1-\alpha)R_f - T}{\left\{\underset{\alpha\tilde{R}_e + (1-\alpha)R_f \le T}{E}\left[T - \left(\alpha\tilde{R}_e + (1-\alpha)R_f\right)\right]^2\right\}^{0.5}}.$$

Proposition 1 below, shows that the traditional Sortino ratio is invariant with respect to α when the threshold rate is equal to the risk-free rate.

Proposition 1. If $T = R_f$, then for all α, $S_P(T) = S_e(T)$ which is the Sortino ratio of the all-equity portfolio (i.e, $\alpha = 1$):

---

[6] In our theoretical model we assume that there is a riskless rate and it is the same for borrowing and lending and the same for the investment fund and the individual investors. Under these assumptions, a reasonable performance measure should not be affected by the selected proportion of the riskless asset vs. the risky assets in the portfolio, regardless of whether the choice is made by the fund manager or by the ultimate ("individual") investor.

[7] The presentation below is an ex-ante version while in practice, the ratio is estimated using sample observations.

[8] For example: Frugier (2016) and Hu et al. (2015) and others use 0% as a threshold. Booth and Broussard (2017) consider thresholds from -0.01 to -0.10, and Perelló (2007) examines thresholds from -30% to +30%.

(3) $S_P(T = R_f) = S_e(T = R_f) = \frac{E(\tilde{R}_e) - R_f}{\left[\underset{\tilde{R}_e - R_f \le 0}{E}(R_f - \tilde{R}_e)^2\right]^{0.5}}$

The proof of the proposition is immediate as Eq. (2) is reduced to Eq. (3) when $T = R_f$.

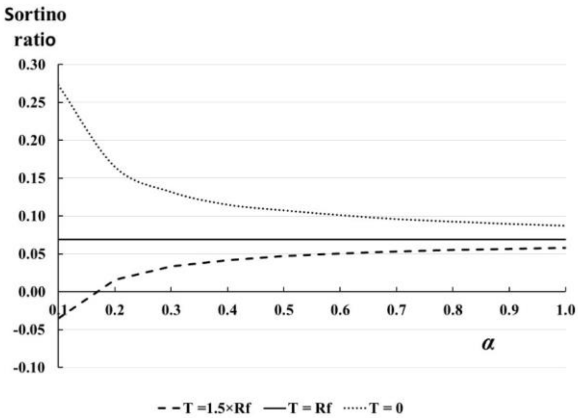
However, when $T \neq R_f$ and also $T < E(\tilde{R}_P)$ and $\alpha > 0$ then, Proposition 2 holds:

Proposition 2. Given that $T < E(\tilde{R}_P)$ and $\alpha > 0$ then, $S_P(T) < S_P(R_f)$ and increases with α, if and only if, $T > R_f$**.** The opposite holds for $T < R_f$. The proof is presented in an Appendix.

Note that the condition $T < E(\tilde{R}_P)$ guarantees a threshold below the expected return of the portfolio and the condition $\alpha > 0$ eliminates an overall short position of the portfolio. These are two very reasonable requirements.

Figure 1 presents estimated Sortino ratios using bootstrapping simulations on the S&P-500 index rates as a function of α. The data and simulations details are in the Figure's caption.

Figure 1
Sortino ratio as a function of α with three alternative threshold values
Based on 2000 random draws from120 monthly returns on the S&P-500 index, February 2008 to January 2018.



It is clear from Proposition 2 and Figure 1 that selection of *T* below (above) the risk-free rate, may lead fund managers who seek to increase their fund's Sortino ratio, to adopt too low (high) equity investment strategy. The Sortino ratio is particularly sensitive to changes of α at low α levels.

In the next section we present our modified Sortino(γ) ratio which employs a threshold $T(\gamma)$ that equals γ times the expected return of the portfolio and (1-γ) times the risk-free rate. It is shown that the modified Sortino ratio is invariable with respect to the proportion of risk-free asset in the portfolio.

The economic logic for choosing $T(\gamma)$ as a threshold rate, is that the threshold for measuring the downside risk of a portfolio should be adjusted to the portfolio's risk premium. This is because it is likely that as the selected overall expected volatility of the portfolio increases, the investor's propensity to absorb losses increases as well. We thus define the threshold rate in terms of the risk-free rate plus γ times the portfolio's expected premium above the risk-free rate. If $\gamma = 1$, any return lower than the expected portfolio return is considered to be in the "loss" region when calculating the downside risk. If, for example, γ = 0.5, the downside risk measure considers all the returns which are lower than the risk-free rate plus 50% of the portfolio's expected risk premium, and when γ = 0, all returns below the risk-free rate are regarded as a loss and count as part of the downside risk measure. Negative γ values maybe unlikely for rational investors because they place the threshold below the riskless rate while the riskless rate is always an open alternative and ignoring the loss between the risk-free rate and the threshold even when the latter is greater than 0, may be supported, at most, on psychological grounds. However, if a negative γ is selected, such as γ = - 0.2, the downside risk measure considers only the returns which are lower than the risk-free rate minus 20% of the portfolio's risk premium. Non-positive thresholds exist for the following γ values:

(4) $$\gamma \le -\frac{R_f}{E(\tilde{R}_P)-R_f}$$

Proposition 3 prove that Sortino(γ) is invariant with respect to α.

Proposition 3. S(γ) ratio is invariant with respect to the portfolio's equity level, α.

Proof:

(5) $$S(T(\gamma)) \equiv S(\gamma) = \frac{E(\tilde{R}_P)-T(\gamma)}{\left\{\underset{\tilde{R}_P<T(\gamma)}{E}[T(\gamma)-\tilde{R}_P]^2\right\}^{0.5}}$$

Eq. (7) can be specified as:

(6) $$S(\gamma) = \frac{\alpha\left[\left(E(\tilde{R}_e)-R_f\right)-\gamma\left(E(\tilde{R}_e)-R_f\right)\right]}{\left\{\underset{\alpha(\tilde{R}_e)+(1-\alpha)R_f<\gamma\alpha(E(\tilde{R}_e)-R_f)+R_f}{E}\left[\left(\gamma\alpha(E(\tilde{R}_e)-R_f)+R_f\right)-\left(\alpha(\tilde{R}_e)+(1-\alpha)R_f\right)\right]^2\right\}^{0.5}}$$

which is the same as:

(7) $$S(\gamma) = \frac{(1-\gamma)\left[E(\tilde{R}_e)-R_f\right]}{\left\{\underset{\tilde{R}_e-R_f<\gamma(E(\tilde{R}_e)-R_f)}{E}\left[\gamma\left(E(\tilde{R}_e)-R_f\right)-(\tilde{R}_e-R_f)\right]^2\right\}^{0.5}}$$

The last formulation of $S(\gamma)$ is invariant with respect to α as claimed by the proposition.
If γ is positive (zero) the threshold is set higher than (equal to) the risk-free rate. Negative γ implies threshold below the risk-free rate.

IV. The portfolio's optimal risky component for a given γ

Since S(γ) is invariant with respect to α, its ex-ante maximization can be attained only by changing the composition of the portfolio's risky component. Define the equity "risk premium ratio" as the ratio of the (random) equity component's risk premium to its expected value, and denote it $\widetilde{rpr_e}$:

(8) $$\widetilde{rpr_e} = \frac{\tilde{R}_e-R_f}{E(\tilde{R}_e)-R_f}$$

Proposition 4. For a given γ the ratio S(γ) is maximized by minimizing the expected downside square deviations of the "risk premium ratio" from γ, namely:

(9) $$\underset{\underset{\sim}{q}}{MIN}\left[\underset{\tilde{R}_e<\gamma E(\tilde{R}_e)+(1-\gamma)R_f}{E}(\widetilde{rpr_e}-\gamma)^2\right]$$

Where $\underline{q}$ is the vector of the proportions invested in the individual risky securities.

The proof is based on Eq. (9) that can be re-written as:

$$(10) \quad S(\gamma) = \frac{1-\gamma}{\left[\underset{\tilde{R}_e<\gamma E(\tilde{R}_e)+(1-\gamma)R_f}{E}(\gamma-\widetilde{rpr_e})^2\right]^{0.5}}$$

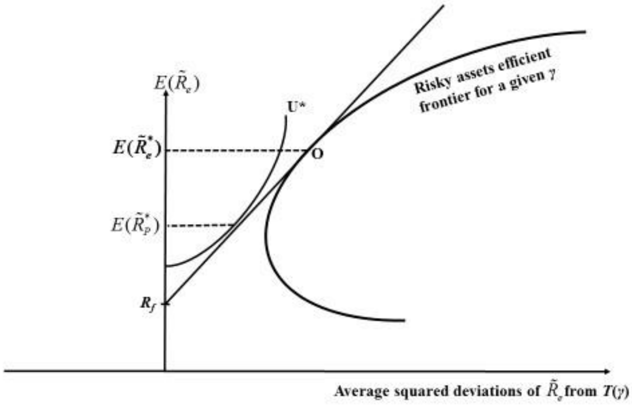
As argued, the conventional Sortino ratio is not invariant with respect to α (except for $T = R_f$) and therefore, its reward vs. downside risk frontier changes with α as well. In contrast, $S(\gamma)$ is invariant to the choice of α and therefore one may apply Eq. (11) subject to any given expected return and obtain the minimum downside risk for each expected return, thereby creating the efficient mean-downside risky frontier of the risky portion of the portfolio for the chosen γ. Consequently, for any $T(\gamma)$, one can use the minimization process in Eq. (11) to find the optimal composition of the risky component of the portfolio. The portfolio's optimal split between the risk-free asset and the optimal risky component is determined subjectively by the investor. Figure 2 depicts the result of this optimization process: it represents tradeoffs for a given positive γ. The portfolio's optimal risky component, composed only with the equities, has an expected return of $E(\tilde{R}_e^*)$. This optimal portfolio is determined objectively and is applicable only for investors who select a specific γ. The overall optimal subjective combination of the risky assets and the risk-free asset for the investor who selected the said γ, has an expected rate of return $E(\tilde{R}_P^*)$. The optimal overall portfolio is found at the tangency point between the investor's relevant indifference curve and the tangent line that run from R_f toward (and beyond) the tangency point with the efficient risky frontier at point O.

V. Consistency with stochastic dominance with riskless asset rules (SDR)

Stochastic dominance (SD) rules provide necessary and sufficient conditions for preference between any two-alternative return (or income) distributions, $\tilde{X}$ and$\tilde{Y}$, for a wide range of assumptions regarding the investor's utility function. The First degree Stochastic Dominance (FSD) rule assumes only non-decreasing utility function while the Second degree Stochastic Dominance (SSD) rule assumes also a non-increasing marginal utility, i.e., risk aversion. The SD rules are partial ordering rules since, in general, it is not guaranteed that <u>all</u> investors with the assumed utilities prefer the same one alternative over another.

Let the preference of $\tilde{X}$ over $\tilde{Y}$ by the conventional Sortino ratio $S(T)$, be denoted as $\tilde{X} \underset{S(T)}{\geq} \tilde{Y}$ and let the same preference by $S(\gamma)$, be denoted $\tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$. These preferences present complete ordering which, a-priori, may be inconsistent with SD rules. Namely, in general, Sortino ordering may not be sufficient for dominance by SD rules. With respect to $S(T)$, Balder and Schweizer (2017) (BS) showed that if $\tilde{X} \underset{SSD}{D} \tilde{Y}$ and $E(\tilde{X}) \geq T \geq E(\tilde{Y})$ then $\tilde{Y} \underset{S(T)}{\geq} \tilde{X}$.
Levy and Kroll (1976) extended the SD rules to portfolios of risky assets that could be diversified with the riskless asset and denoted these rules SDR rules (i.e., Stochastic Dominance with Riskless asset rules). The First and Second degree SDR rules, are denoted FSDR and SSDR rules, respectively. They proved that if there is a combination of a proportion α invested in $\tilde{X}$ and (1-α) invested in the riskless asset such that this combination dominates $\tilde{Y}$ by FSD or SSD, then for any other combination of $\tilde{Y}$ with the riskless asset, there is at least one other combination of $\tilde{X}$ with the riskless asset that dominates it by FSD or SSD, respectively.

Figure 2
The efficient risky frontier, the optimal expected rate of return of the portfolio's risky component, $E(\tilde{R}_e^*)$, and the optimal expected rate of return of the overall portfolio $E(\tilde{R}_p^*)$, for a chosen γ



It should be noted that the partial ordering by SDR rules is potentially much more effective than the SD rules. For example, assume that $\tilde{X}$ and $\tilde{Y}$ are uniformly distributed returns: $\tilde{X} \sim U(0{,}20)$ and $\tilde{Y} \sim U(5{,}10)$. In this example, there is no FSD or SSD dominance relationships between $\tilde{X}$ and $\tilde{Y}$. The expected return of $\tilde{X}$ is greater than that of $\tilde{Y}$ (10 > 7.5) hence $\tilde{Y}$ clearly does not dominate $\tilde{X}$, but also the lowest outcome of $\tilde{X}$ is smaller than that of $\tilde{Y}$ (0 < 5) and thus $\tilde{X}$ does not dominate $\tilde{Y}$. However, if each of the risky assets could be diversified with a risk-free asset whose return is 7.2%, then, for example, a portfolio of 30% $\tilde{X}$ and 70% R_f is also distributed uniformly, $\tilde{X}_{\alpha=30\%} \sim U(5.04{,}11.04)$, and it dominates $\tilde{Y}$ by FSD. Likewise, by SDR rules, for any combination of $\tilde{Y}$ and R_f, one can find at least one combination of $\tilde{X}$ with R_f that dominates it.

This example shows that considering diversification between risky and risk-free alternatives, a lack of dominance by the FSD or SSD rules between two distributions may nevertheless exhibit dominance relationship by the FSDR or SSDR rules, respectively.

Proposition 5. If $\tilde{X} \underset{FSD}{D} \tilde{Y}$ and there are no short sales of either $\tilde{X}$ or $\tilde{Y}$, then $\tilde{X} \underset{S(T)}{\geq} \tilde{Y}$ for every T and by $\tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$ for every γ.

Proof: The proof is almost immediate. Such dominance implies that for each cumulative distribution of order P ($0 \leq P \leq 1$) the $\tilde{X}(P) \geq \tilde{Y}(P)$. Thus, for each constant T or $T(\gamma)$ as calculated by Eq. (4) we have $T - \tilde{X}(p) \leq T - \tilde{Y}(p)$. Denote by $P_{\tilde{X}}(T)$ and $P_{\tilde{Y}}(T)$) the P order probabilities that lead to the T value of $\tilde{X}$ and $\tilde{Y}$, respectively. Due to the FSD assumption, also $P_{\tilde{X}}(T) \leq P_{\tilde{Y}}(T)$ and thus the average square deviations between T and $\tilde{X}$, is also smaller than the respective average square deviations between T and $\tilde{Y}$. Namely:

$$(11)\quad \int_0^{P_{\tilde{X}}(T)} p\big(T - \tilde{X}(p)\big)^2 dp \leq \int_0^{P_{\tilde{Y}}(T)} p\big(T - \tilde{Y}(p)\big)^2 dp$$

Proposition 6. $\tilde{X} \underset{FSDR}{D} \tilde{Y} \Rightarrow \tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$ for all $\gamma < 1$.

Proof. If there is FSDR of $\tilde{X}$ over $\tilde{Y}$ then there is a combination of $\tilde{X}$ and the risk-free asset, that dominates a given combination of $\tilde{Y}$ with the risk-free asset, and thus we are back in a situation which is presented in Proposition 5. It is guaranteed that for any other combination of $\tilde{Y}$ with the risk-free asset there is at least one other combination of $\tilde{X}$ with the risk-free asset that dominates it, and the conditions of Proposition 5 hold again.

Proposition 7. $\tilde{X} \underset{SSDR}{D} \tilde{Y} => \tilde{X} \underset{S(\gamma)}{\geq} \tilde{Y}$ for all $\gamma < 1$.

Proof. If there is SSDR of $\tilde{X}$ over $\tilde{Y}$ then there is a combination of $\tilde{X}$ and the risk-free asset, that dominates a given combination of $\tilde{Y}$ with the risk-free asset, and thus we are back in a situation which is presented in Proposition 5. It is guaranteed that for any other combination of $\tilde{Y}$ with the risk-free asset there is at least one other combination of $\tilde{X}$ with the risk-free asset that dominates it, and the conditions of Proposition 5 hold again.

VI. Concluding remarks

Sortino ratio is defined as the excess expected return over a given threshold T divided by the square root of the expected squared return deviations below T and it is one of the most popular downside performance measures among practitioners. Since investors vary with respect to their attitude toward loss, they use different thresholds to define the rate that separates the loss from the reward and indeed the Sortino literature allows a wide range of T values. Our paper shows that if T is above (below) the riskless rate, the Sortino ratio increases (decreases) with a portfolio's equity level. This undesirable shortcoming allows one to increase the portfolio's degree of leverage.

Our modified Sortino ratio, uses the target $T(\gamma)$ which equals γ times the expected return of the portfolio plus (1- γ) times the risk-free rate. Since the expected ex-ante return of a risky portfolio is higher than the risk-free return, the threshold $T(\gamma)$ which reflects the investor's sensitivity to loss, increases with γ. In contrast to the conventional Sortino ratio, our modified ratio is invariant with respect to the portfolio's equity level, α, and depends only on the selected "loss benchmark" γ. Hence, an ex-ante change of Sortino(γ) ratio, for a given γ, is possible only through better composition of the risky portion of the portfolio.

The paper presents a simple criterion for minimizing the downside risk for any chosen expected return and γ, allowing the investor to separate the optimal mix of the risky and riskless components of the portfolio from the optimal composition of the portfolio's risky component.

We also show that ranking portfolios' performance by first and second degree stochastic dominance with riskless asset rules (FSDR and SSDR respectively), implies ranking by $S(\gamma)$. Stochastic dominance with riskless asset rules (SDR) examine dominance between risky portfolios where it is assumed that each of the distributions being compared, may be diversified with the risk-free asset. These SDR rules potentially show dominance where stochastic dominance without riskless asset rules signal no dominance. Dominance by SDR rules implies dominance by our $S(\gamma)$ criterion.

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Appendix: Proof of Proposition 2

We begin with the Sortino ratio of a two-asset portfolio consisting of a proportion α invested in a risky asset (equity) and a proportion (1-α) invested in a riskless asset:

$$S_P(T) \equiv S = \frac{E(\tilde{R}_P) - T}{\left\{\underset{\tilde{R}_P \le T}{E}(T - \tilde{R}_P)^2\right\}^{0.5}} =$$
$$= \frac{E\left[\alpha\tilde{R}_e + (1-\alpha)R_f\right] - T}{\left\{\underset{\alpha\tilde{R}_e + (1-\alpha)R_f \le T}{E}\left[T - \left(\alpha\tilde{R}_e + (1-\alpha)R_f\right)\right]^2\right\}^{0.5}}$$

Define $T \equiv R_F + \Delta$ and for $\alpha > 0$ we may write:

$$S = \frac{E\left(\tilde{R}_e - R_f - \frac{\Delta}{\alpha}\right)}{\left[\underset{\tilde{R}_e - R_f - \frac{\Delta}{\alpha} \le 0}{E}\left(\frac{\Delta}{\alpha} - \tilde{R}_e + R_f\right)^2\right]^{0.5}}$$

Denoting $u_\alpha = \tilde{R}_e - R_f - \frac{\Delta}{\alpha}$, we rewrite the ratio as:

$$S = \frac{E(\tilde{u}_\alpha)}{\left[\underset{\tilde{u}_\alpha \le 0}{E}(-\tilde{u}_\alpha)^2\right]^{0.5}}$$

$$E(\tilde{R}_P) > T \quad \Rightarrow \alpha E(\tilde{R}_e) + (1-\alpha)R_f > R_f + \Delta \quad \Rightarrow \alpha\left(E(\tilde{R}_e) - R_f\right) - \Delta > 0$$
$$\Rightarrow E(\tilde{R}_e) - R_f - \frac{\Delta}{\alpha} > 0 \Rightarrow E(\tilde{u}_\alpha) > 0$$

In addition, we note that $\frac{\partial E(u_\alpha)}{\partial \alpha} = \frac{\partial u_\alpha}{\partial \alpha} = \frac{\Delta}{\alpha^2}$.

For $\Delta \neq 0$ we obtain:

$$\frac{\partial S}{\partial \alpha} = \frac{\frac{\Delta}{\alpha^2}\left\{\left[\underset{\tilde{u}_\alpha \le 0}{E}(-\tilde{u}_\alpha)^2\right]^{0.5} + E(\tilde{u}_\alpha) \times 0.5 \times \left[\underset{\tilde{u}_\alpha \le 0}{E}(-\tilde{u}_\alpha)^2\right]^{-0.5} \times 2 \times \underset{\tilde{u}_\alpha \le 0}{E}(-\tilde{u}_\alpha)\right\}}{\underset{\tilde{u}_\alpha \le 0}{E}(-\tilde{u}_\alpha)^2}$$

The denominator of the derivative is clearly positive. The first term in the numerator's curly brackets is positive as well. The expected value of $\tilde{u}_\alpha$ is likewise positive as noted above. And the last term in the numerator of the curly brackets is positive by definition, which ensures that the entire expression inside the numerator's curly brackets is positive. It follows that the sign of the derivative, $\frac{\partial S}{\partial \alpha}$, is determined by the sign of Δ. Positive Δ indicates a threshold higher than the risk-free rate in which case the derivative is positive while negative Δ indicates a threshold lower than the risk-free rate in which case the derivative is negative.

Comprehensive financial modeling of solar PV systems

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COMPREHENSIVE FINANCIAL MODELING OF SOLAR PV SYSTEMS

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ABSTRACT. The adoption of a photovoltaic system has positive environmental effects, but the main driver of the choice in the industrial and commercial sector is economic profitability. Switching from acquisition of energy to production of energy is an investment with costs (e.g. leasing annual payment, O&M costs, capital expenditure) and benefits (e.g. savings in the electric bill, sale of the energy exceeding consumptions). In this work, we use an accounting-and-finance model to calculate the Equity Net Present Value in different scenarios and a sensitivity-analysis method (Finite Change Sensitivity Index) to explain the reasons for differences in results. This technique enables identifying the contribution of any input factor in the output value variation. In this way, the investor can draw attention on the most significant critical variables in the initial estimations to ensure success in forecasting.

Keywords: photovoltaic, economic analysis, financial modelling, financing, estimation, decision.

1 AIM AND APPROACH USED

Solar energy undeniably brings about environmental benefits, but the adoption of solar energy by the industrial, commercial, and residential sectors is strongly affected by economic considerations (e.g., Cucchiella et al 2018 [3], Dong et al 2017 [4]). The mapping which links the key performance drivers and the investment’s economic profitability entails understanding of the intricate network of relations among technical aspects, accounting magnitudes, forecasting of financial data, and assumptions on financing decisions, which makes the determination of economic profitability particularly complex. It is then important to provide decision-aiding tools capable of measuring the investment return, taking into account uncertainty and providing insights on possible managerial actions that may affect the decision to adopt solar energy.

Building upon Magni and Marchioni (2019) [8], we propose a comprehensive framework for modeling investment decisions in solar photovoltaic (PV) systems, aimed at helping analysts, advisors, firms’ managers to assess the economic impact of solar energy, manage uncertainty, distinguish the high-impact drivers from the low-impact drivers, calibrate the structure of the model (increasing the depth of analysis for those drivers which have major effects on the investment financial efficiency), and choose various alternative proposals (e.g., alternative capturing technologies).

Specifically, the proposed model makes use of Magni’s (2020) [6] accounting-and-finance system to engineering economic decisions. It accomplishes a detailed analysis of the sources of value creation in both absolute and relative terms, always supplying the net present value (NPV), the rate of return, and the financial efficiency, thereby overcoming the limitations of the internal rate of return (IRR), usually recommended in benefit-cost analysis (Sartori et al 2014 [10], Mangiante et al 2020 [9]), but most likely to be undetermined in this kind of projects.

The model acknowledges the distinction between *estimation variables* and *decision variables* on one hand and between *operating variables* and *financial variables* on the other hand: The estimation variables necessitate some estimation process to be determined (e.g., operating and maintenance costs, disposal costs, interest rate on debt financing) while the decision variables are under the managers’ control (e.g., timing and size of distributions to shareholders, recourse to debt borrowing or to cash withdrawals for covering the financial needs). The operating variables express the factors which have a direct impact on the firm’s costs and revenues as a result of the adoption of solar energy (e.g., solar panel efficiency, the avoided electric bill, energy price, amount of self-consumption, credit terms for energy sales to the grid). The financial variables regard the factors which affect the mix of financing sources and the amount of incremental liquid assets in the firm’s balance sheets (e.g., interest rate on liquid assets, risk-adjusted cost of capital, distribution to equityholders).

We also aim at validating the model by means of sensitivity analysis (SA), which confirms that the presence or absence of relevant drivers may affect the increase in investors’ wealth and may affect the decision. In particular, we assess the contribution of financial variables and decision variables to the output variability. With the aid of the recently developed Clean FCSI (Magni et al 2020 [7]), based on Borgonovo’s (2010) [2] FCSI, we aim to detect the most critical drivers and understand which driver is more likely to cause a change in the decision. SA will also be of help to analysts for calibrating the model: if the contribution to value of some parameters is small, then there is no need of modeling those inputs in more detail;

in contrast, if some parameters contribute significantly to value creation, then the analyst may consider a further development of the model for gaining deeper insights. Clean FCSI will also be of help to show that interactions among all the variables substantially affect the investment's economic profitability. This testifies to the importance of modeling the project to take account of all relevant value drivers and to make analysts aware of the effect of estimation process on the accept/reject decision.

2 SCIENTIFIC INNOVATION AND RELEVANCE

This work presents a comprehensive approach to financial modeling of investments in solar energy which differentiates itself from the traditional financial modeling derived from finance. The innovation of the approach may be summarized as follows:

1. as opposed to traditional models, the proposed model acknowledges that the investment value (and related decision) depends on both operating variables and financial variables. Also, it depends on decision variables such as the distribution of cash to shareholders and the reinvestment of cash, which may affect the return on solar investment. The proposed model is transparent, for it takes distribution policy in explicit consideration as well as borrowing policy, and appraises the interaction with the operating variables, reflecting their impact on the firm's pro forma financial statements and, hence, on the investment value and return
2. in real life, a substantial amount of solar PV plants is financed by firms with internal funds (i.e., cash withdrawals from bank accounts) and/or by debt, with no recourse to equity issuance. In traditional financial modeling, this form of financing is not taken into explicit account. The proposed model takes account of any mix of financing sources, either internal (cash withdrawals) or external (debt and/or equity)
3. contrary to traditional financial modeling, the proposed model apportions the overall investment value according to the various sources of value, namely, the operating activities, the financial activities (reinvestment of excess cash and cash withdrawals), and the debt borrowing
4. in this kind of investments, it is likely that financial efficiency may not be determined with traditional tools such as the internal rate of return (IRR) (see Magni and Marchioni 2019 [8]). Equipped with Magni's (2010) [5] Average Internal Rate of Return, the proposed model always provides an appropriate measure of financial efficiency, in terms of Return On Investment (entity perspective) or Return On Equity (equity perspective)
5. we validate the model with the aid of SA, which also supplies helpful information to calibrate the model for a more careful treatment of the highest-impact value drivers and confirm the relevance of the interaction effects and the importance of fine-tuning the estimation process.

3 RESULTS

The accounting-and-finance model we propose is able to make a thorough evaluation of the various aspects of the option of switching to solar energy for an agent (e.g., a firm) currently importing energy from electric grid. Switching to a solar PV system entails cost savings equal to the electric bill and incremental costs due to the purchase of the solar PV system. This may be purchased with an upfront payment or, as frequently occurs, with lease contracts (or power purchase agreements); at the end of the contract, the lessee may pay a lump to acquire the plant. The lump sum will be financed either with debt, equity, or internal financing (withdrawal from liquid assets, i.e., cash and cash equivalents). The amount of power which will be produced in excess of self-consumption will be sold to the grid operator, generating cash inflows after some period (depending on the credit terms); in contrast, if energy consumption is smaller than energy production, the firm will buy the residual energy from the grid. For example, consider the case of a ground-mounted solar panel system to be installed in a currently rented land, associated with a lease contract and with no equity financing. We use data for a solar PV plant proposed by GRAF Spa, a solar PV installer company, to an Italian firm located in Northern Italy.

Table I: Equity NPV in two different scenarios

Variables	**Scenario 1**	**Scenario 2**
Operating variables (estimation)		
Nameplate capacity [kWp]	92	92
Unit cost [€/kWp]	1,050	1,050
Useful life of PV plant [years]	22	28
Annual unit prod. (Y 1) [kWh/kWp/y]	1,000	1,130
Solar panel degradation rate [%/y]	1.15%	0.65%
Lease term length [years]	20	20
Lease interest rate [%]	4%	4%
Purchase price of plant (year 20) [€]	25,000	25,000
O&M, insurance, etc. [%]	4.00%	2.75%
Disposal costs [€]	3,000	2,500
Lost rent from land property [€/y]	1,500	1,250
Growth rate for costs [%]	1.50%	0.50%
Annual energy consumption [kWh/y]	62,500	87,500
Tax rate [%]	30%	20%
Energy purchase price [€/kWh]	0.140	0.180
Energy selling price [€/kWh]	0.105	0.155
Growth rate of energy price [%]	0.50%	2.00%
Credit terms for energy purchases [dd]	0	0
Credit terms for energy sales [dd]	365	365
Financial variables (estimation)		
Interest rate on liquid assets [%]	4.00%	-0.50%
Interest rate on debt [%]	6.00%	2.00%
Required return on oper. assets [%]	6.00%	6.00%
Required return on liquid assets [%]	2.00%	2.00%
Required return on debt [%]	3.00%	3.00%
Financial variables (decision)		
Internal financing (cash) [%]	60%	60%
Debt borrowing [%]	40%	40%
Equity financing [%]	0%	0%
First CFE distribution [y]	1	1
Payout ratio [%]	50%	50%
Equity NPV [€]	**−15,494.88**	**84,570.02**

In Table I, column 2 (scenario 1) reports the estimated input data, for a given set of financing and distribution policy. These input data are used for drawing up three pro forma financial statements (balance sheets, income statements, cash flow statements) which are logically interconnected in a non-trivial way, since decisions on financing and cash flow distribution will affect the amount of liquid assets and debt outstanding in the firm. This in turn affects next-period interest on debt and on liquid assets, which in turn affects next-period income and, therefore, the equity. With these data, shareholders' wealth increase, as measured by the shareholder net present value (NPV), is negative and equal to −15,494.88, so the project is not worth undertaking. (It is worth noting that neither the project IRR nor the operating IRR nor the equity IRR exist).[i]

Consider now a different set of estimated parameters, as described in column 3 (scenario 2). Shareholder value created increases by almost 100,000 to 84,570, so making the project highly profitable.

Table II breaks down the equity NPV into operating NPV (i.e., NPV of the operating assets), non-operating NPV (i.e., NPV of the liquid assets), and debt NPV (i.e. NPV of the debtholders).

Table II: Equity NPV

	Scenario 1	**Scenario 2**
+ Operating NPV	−12,110.92	+108,603.47
+ Non-operating NPV	−3,142.14	−24,264.57
− Debt NPV	−(+241.83)	−(−231.12)
= Equity NPV	**−15,494.88**	**84,570.02**

The FCSI helps explain why this dramatic change occurs, providing the change in NPV due to the change in estimate of the drivers (columns 2 and 3 in table III. See Magni et al 2020 [7] for details on FCSI). It is worth noting that the most important driver of change is a financial driver, the interest rate on liquid assets (rank 1). This means that attention should be drawn on the estimation of such a variable and it is worth modeling such an aspect in greater detail and/or refining the estimation process. Energy prices and O&M (operating drivers) are next in importance (ranks 2, 3, and 4). Somewhat unexpected is the negligible effect of the efficiency loss (rank 12). Disposal costs are also negligible (rank 13). Even the sharp deviation of estimate in the interest rate on debt is irrelevant (rank 14), suggesting that, in this case, the conditions of the loan contract are non-significant.

Once calibrated the model and obtained a reliable set of estimated data, the analyst should fine- tune the borrowing policy and the distribution policy in order to increase the project's value and get the best output for the investors. Preliminary results show that a change in such policies may have a remarkable effect on the output and, in some cases, may even cause a change in the decision to adopt solar energy (and distribution policy may have an even greater effect than borrowing policy).

Table III: Changes in NPV (%) and Rank of input factors

Variable	**Change in NPV (%)**	**Rank**
Operating variables (estimation)		
Useful life of PV plant	−6.09%	9
Annual unit prod. (Y 1)	7.27%	8
Solar panel degradation rate	0.70%	12
O&M, insurance, etc.	13.10%	4
Disposal costs	0.16%	13
Lost rent from land property	3.28%	11
Growth rate for costs	5.61%	10
Annual energy consumption	10.17%	5
Tax rate	−9.04%	6
Energy purchase price	19.91%	2
Energy selling price	14.18%	3
Growth rate of energy price	8.79%	7
Financial variables (estimation)		
Interest rate on liquid assets	31.99%	1
Interest rate on debt	−0.03%	14

4 CONCLUSIONS

Since solar energy undeniably contributes to a sustainable economy, the decision of adopting a solar energy system by firms is important to achieve a substantial cumulative effect in the environment. However, firms' decisions are mostly motivated by financial efficiency and shareholder value creation. We present an operational tool increasing analysts' and managers' awareness on the financial impact of solar energy on these economic measures. This model blends accounting and finance and takes account of the subtle network of relations between operating variables and financial variables on one hand, and estimation variables

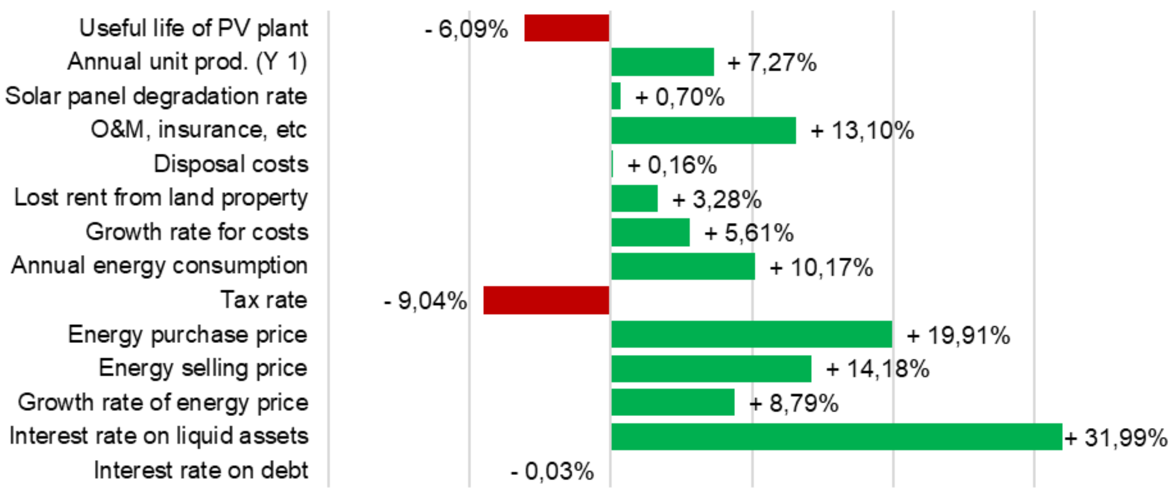

Figure I: Changes in NPV (%)

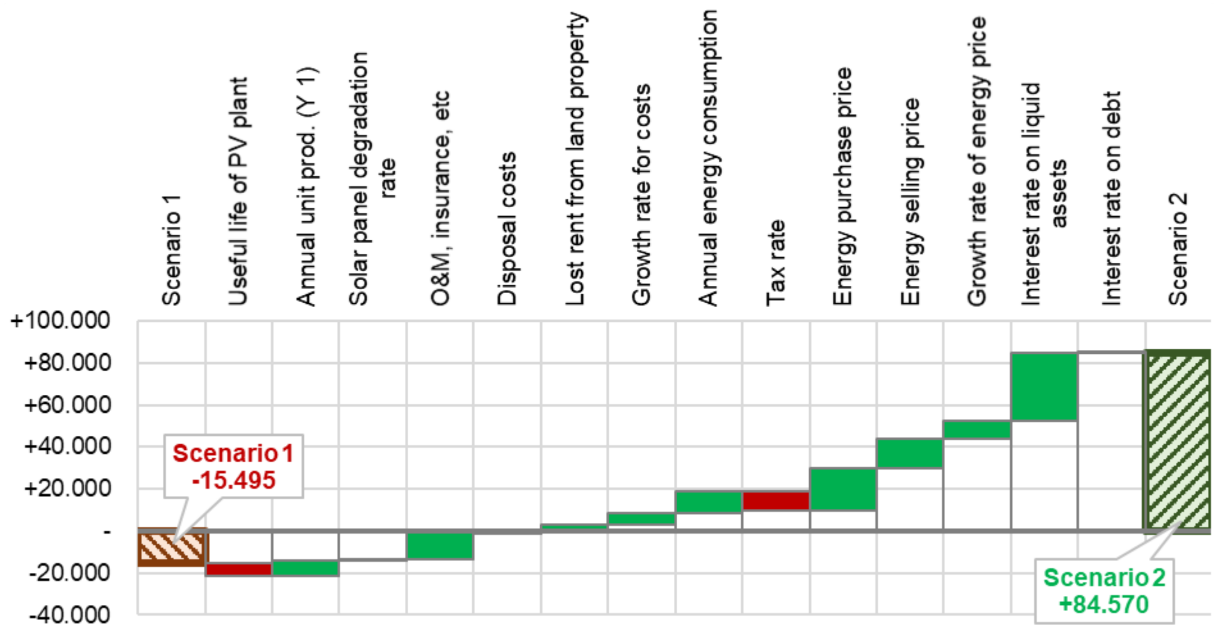

Figure II: Changes in NPV

and decision variables on the other hand. In particular, it explicitly takes account of the impact of internal financing as opposed to equity financing as well as of the reinvestment of retained cash as opposed to a full payout policy. The model is associated with a sensitivity-analysis technique which validates the model and provides managerial insights on the most critical drivers, which helps calibration of the model to the firm's needs. It also helps analysts to fine-tune the firm's borrowing and distribution, for any given set of estimated input data, in order to increase the financial benefits of solar energy.

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[i] It is interesting to note that the initial cash flow to equityholders is equal to 0 (they do not provide any capital injection) while it is positive in the following years. This is the reason why the equity IRR does not exist.

Investment and financing perspectives for a solar photovoltaic project

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Investment and financing perspectives for a solar photovoltaic project

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Abstract. In this work we illustrate a simple logical framework serving the purpose of measuring value creation in a real-life solar photovoltaic project, funded with a lease contract, a loan contract and internal financing (i.e., withdrawal from liquid assets). We use the projected accounting data to compute the value created. We assess the project from both an investment perspective (operating assets and liquid assets) and a financing perspective (debt and equity). Furthermore, focusing on value creation for equityholders, we calculate the expected contribution on shareholders' wealth increase of operating and financing activity. In particular, we highlight the role of the distribution policy in financial modeling by describing the strict logical connections between estimated data and financial decisions.

Keywords: photovoltaic solar energy, project evaluation, net present value, distribution policy

1 Economic setting

Switching from traditional energy sources to renewable energy has a beneficial impact in terms of ecological sustainability (Ezbakhe and Pérez-Foguet 2021, Kang et al. 2020, Lei et al. 2019, Sinke 2019, Lupangu and Bansal 2017). However, firms willing to switch from retail energy to renewable energy are also concerned with the impact on economic profitability (Pham et al. 2019, Cucchiella et al. 2018, Dong et al. 2017). Therefore, an appropriate financial modeling and profitability metrics are required which correctly assess the effect on shareholders' wealth (Magni and Marchioni 2019, Baschieri, Magni and Marchioni 2020). In this study, we consider the appraisal of a solar photovoltaic (PhV) project proposed by an Italian installer company to a small firm, located in Northern Italy, which aims to switching from retail energy to solar energy and draw up a financial model which connects operating variables and financing variables.

Let Rev_t be the incremental revenues derived from the sale of excess energy, OpC_t be the incremental operational costs brought about by the plant, Dep_t be the depreciation charge of the solar PhV plant, I_t^l the interest income derived from reinvestment of liquid assets, I_t^d the interest expenses associated with debt, and τ the corporate tax rate. Formally, the project income is $I_t = \left(Rev_t - OpC_t - Dep_t + I_t^l\right)(1 - \tau) + \tau I_t^d$. As is standard in finance, the project's cash flows, F_t, can be computed by subtracting the change in capital from the income, so that $F_t = I_t - \Delta C_t$. Let r_t be the project's cost of capital (minimum required rate of return).

The net present value (NPV) quantifies the net effect of the project on the investors' current wealth (Brealey, Myers and Allen 2011):

$$NPV = F_0 + \frac{F_1}{1+r_1} + \frac{F_2}{(1+r_1)(1+r_2)} + \cdots + \frac{F_n}{(1+r_1)(1+r_2)\dots(1+r_n)}. \tag{1}$$

Capital amounts, incomes and cash flows of the project are intertwined in a non-trivial way via the pro forma financial statements, namely the balance sheets, the income statements and the cash-flow statements. These depend on estimated data regarding the operating activity but also on the firm's financing policy, that is, borrowing policy and distribution policy. Three sources of financing are possible:

- debt financing
- equity financing
- internal financing (i.e., withdrawal from liquid assets).

As for the distribution policy, the operating cash flows generated by the project may well be (wholly or partially) retained by the firm. and, if they are invested in financial assets, they produce interest incomes. Let $j = o, l, d, e$ be the operating assets, liquid assets, debt, and equity of the project, respectively. The first two components, o and l, represent the investment side of the project whereas the last two categories, d and e, describe its financing side. Each area is associated with its own net present value (NPV), as represented in Figure 1.

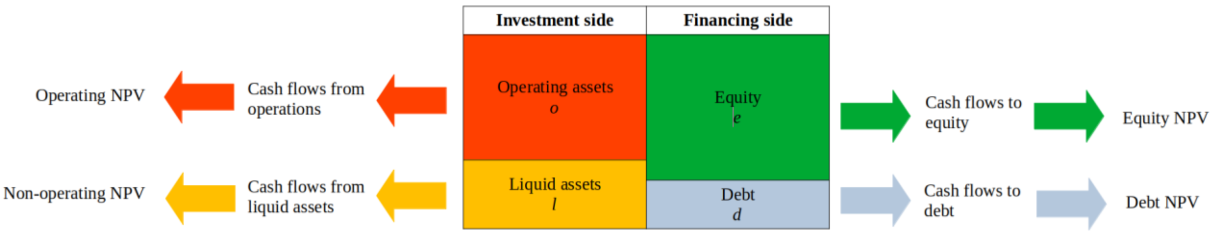

Figure 1: NPV of investments and financing sources

The NPV of each asset class j can be computed as

$$NPV^j = F_0^j + \frac{F_1^j}{1+r_1^j} + \frac{F_2^j}{(1+r_1^j)(1+r_2^j)} + \cdots + \frac{F_n^j}{(1+r_1^j)(1+r_2^j)\dots(1+r_n^j)}$$

where F_t^j and r_t^j are the cash flows and costs of capital corresponding to each asset class. As shown in Magni (2020), the NPV of the project may be viewed under an investment perspective and a financing perspective:

$$\overbrace{NPV^o + NPV^l}^{\text{investment perspective}} = \overbrace{NPV}^{\text{project NPV}} = \overbrace{NPV^e + NPV^d}^{\text{financing perspective}}, \tag{2}$$

where

NPV^o = NPV of operating assets
NPV^l = NPV of liquid assets
NPV^e = NPV of equityholders
NPV^d = NPV of debtholders.

Since the managers' primary mandate is wealth increase of equityholders, the measure we focus on is the equity NPV, NPV^e. From (2),

$$NPV^e = NPV^o + NPV^l - NPV^d, \quad (3)$$

meaning that equityholders may benefit not just from a value-creating operating activity ($NPV^o > 0$), but also from an efficient management of liquid assets such that they are invested at a rate of return greater than the cost of capital of liquid assets ($NPV^l > 0$), and from the ability of borrowing at lower rate than the cost of debt, that is, the equilibrium rate prevailing in the capital markets ($NPV^d < 0$).[1]

In this work, we model the technical and financial description of a real-life case of solar PhV system. We measure the contribution of operating and financial areas on the overall value creation of the investment project and on the wealth increase for equityholders.

2 Solar PhV plant

We describe a real-life industrial case where an Italian company located in Northern Italy faces the opportunity of replacing a conventional retail electricity system (based on supplies from a grid operator) with a standalone solar PhV system purchased from an Italian producer and installer. The plant will be installed on a land property owned by the company and currently rented. With retail energy, the firm periodically pays a utility bill and receives a rental income from the rent of the land. The solar PhV plant implies a leasing contract whereby lease payments and operating and maintenance costs are made periodically. After several years, at the expiration date, the lessee will pay a lump sum to acquire the plant, and the system will continue to generate electric power for some years. The lump sum is paid through the issuance of new debt capital and withdrawal from liquid assets. At the end of its useful life, the plant will be removed, and the firm will incur disposal costs. If the retail system is replaced by the PhV plant, the incomes and cash flows will increase as a result of the ceased lease payment and the cost savings (the utility bill), but will increase as a result of operating and maintenance costs, the terminal outlay for acquiring the plant, and the lost rental income.

The model is described as follows: the quantity of energy consumed for the firm's operations is estimated to be constant through time and equal to q; the current purchase price of energy is p_p, growing at a constant rate g_p per year. The utility bill is payed periodically, in the same year in which energy is consumed. The leasing contract contains the following economic conditions: the lease payment, equal to L, is made periodically; at time m (expiration date) the firm may acquire the plant paying a lump sum equal to $CapEx$, and the system will keep producing electric power for some years, until time n. $CapEx$ represents the capital expenditure for buying the plant and is depreciated evenly from $t = m + 1$ until $t = n$, so that the depreciation charge is $Dep = CapEx/(n - m)$. As anticipated, the PhV plant is installed at $t = 0$ in a field owned by the firm, which could otherwise be rented on the property market at a rent equal to R growing at the constant annual rate g_c. The latter represents an opportunity cost for the firm (a foregone income).

1 The debt NPV is the part of the value generated by the project captured by debtholders: if it is negative, then equityholders grasp that value. Usually, such an NPV is zero or positive, so part of the value generated by the project is shared with the debtholders.

Starting from the first period, the PhV plant requires operating, maintenance and insurance costs. Technical experts determine a suggested level of these costs for the first year in order to maximize the energy production, which we denote as $SuggO\&M$. We denote as $O\&M$ the actual expenses, which may be equal to or smaller than the suggested ones (i.e., $O\&M \leq SuggO\&M$), both assumed to grow at the constant annual rate g_c

If $O\&M = SuggO\&M$, the PhV system will produce Q_{max} units of energy in the first year, which decrease every year at the rate g_Q. In contrast, if $O\&M = 0$ (i.e., the company is not willing to spend for operating and maintenance costs), the energy production suffers from a percentage loss due to lack of maintenance, denoted as $ProdLoss$. Furthermore, technical experts expect that the effective energy production in each period t, denoted as Q_t, is proportional to the level of actual $O\&M$ costs as compared to the suggested level. Specifically

$$Q_t = Q_{max}\left(1 - g_Q\right)^{t-1} \cdot \left(1 - \max\left(ProdLoss \cdot \frac{SuggO\&M - O\&M}{SuggO\&M}, 0\right)\right).$$

If the energy produced by the plant, Q_t, is higher than the energy consumed by the firm, the firm sells the differential quantity to the Energy Service Operator at the energy selling price p_s, growing at a constant rate g_p per year, with payment in the following year. We assume that, at time $t = n$, the energy sold is paid immediately. Therefore, if the produced quantity is lower than the consumed energy in year t, that is, $Q_t < q$, energy costs savings arise equal to $Q_t \cdot p_p\left(1 + g_p\right)^{t-1}$; if the produced quantity is higher than the consumed one, that is, $Q_t > q$, energy costs savings arise equal to $q \cdot p_p\left(1 + g_p\right)^{t-1}$ as well as energy sales revenues equal to $(Q_t - q) \cdot p_s\left(1 + g_p\right)^{t-1}$, determining the presence of operating working capital. Hence, the income effect of the energy sales revenues and costs savings in the two different scenarios can be summarized with the expression

$$\min(q, Q_t) \cdot p_p\left(1 + g_p\right)^{t-1} + \max(0, Q_t - q) \cdot p_s\left(1 + g_p\right)^{t-1}$$

and the operating working capital can be represented with the formula $WC_t = max(0, Q_t - q) \cdot p_s\left(1 + g_p\right)^{t-1}$ and $WC_n = 0$. At time n, the plant is removed with disposal costs equal to H growing at the constant annual rate g_c.

To sum up, the firm-without-the-project pays the utility bills and receives the rent for the land (for the whole period); in contrast, the firm-with-the-project sustains the lease payments (until $t = m$), the operating and maintenance costs (until $t = n$), the lump sum (in $t = m$), and the disposal costs (in $t = n$), and receives payments for the energy sold to the Energy Service Operator. Considering that a project represents, by definition, the difference between the firm-with-the-project and the firm-without-the-project, the project's incomes are:

$$\begin{aligned} I_t = \Big[&\min(q, Q_t) \cdot p_p\left(1 + g_p\right)^{t-1} + \max(0, Q_t - q) \cdot p_s\left(1 + g_p\right)^{t-1} - L - R \cdot (1 + g_c)^{t-1} \\ &- O\&M \cdot (1 + g_c)^{t-1} + I_t^l\Big](1 - \tau) + \tau I_t^d \end{aligned}$$

$$\text{for } 1 \leq t \leq m$$

$$I_t = \Big[\min(q, Q_t)\cdot p_p\big(1+g_p\big)^{t-1} + \max(0, Q_t - q)\cdot p_s\big(1+g_p\big)^{t-1} - R\cdot(1+g_c)^{t-1} - O\&M \cdot (1+g_c)^{t-1} - Dep + I_t^l\Big](1-\tau) + \tau I_t^d \qquad \text{for } m+1 \le t \le n-1$$

$$I_t = \Big[\min(q, Q_t)\cdot p_p\big(1+g_p\big)^{t-1} + \max(0, Q_t - q)\cdot p_s\big(1+g_p\big)^{t-1} - R\cdot(1+g_c)^{t-1} - O\&M \cdot (1+g_c)^{t-1} - Dep - H\cdot(1+g_c)^{t-1} + I_t^l\Big](1-\tau) + \tau I_t^d \qquad \text{for } t = n.$$

The project's assets are represented by working capital, liquid assets (C_t^l) and, from time m, fixed assets:

$$C_t = \overbrace{\max(0, Q_t - q)\cdot p_s\big(1+g_p\big)^{t-1}}^{\text{working capital}} + \overbrace{C_t^l}^{\text{liquid assets}} \qquad \text{for } 1 \le t \le m-1$$

$$C_t = \overbrace{\max(0, Q_t - q)\cdot p_s\big(1+g_p\big)^{t-1}}^{\text{working capital}} + \overbrace{CapEx - Dep\cdot(t-m)}^{\text{fixed assets}} + \overbrace{C_t^l}^{\text{liquid assets}} \qquad \text{for } m \le t \le n\text{-}1$$

$$C_t = 0 \qquad \text{for } t = n$$

where the balance of liquid assets at the end of period t, C_t^l, is obtained from the liquid balance at the beginning of period, C_{t-1}^l, increased by the interest income I_t^l and by the cash contribution into the liquid assets account at time t, equal to $-F_t^l$, that is, $C_t^l = C_{t-1}^l + I_t^l - F_t^l$ (for the derivation of liquid assets see also the numerical application below). Finally, as already mentioned, the forecasted cash flows are obtained as $F_t = I_t - \Delta C_t,\ \forall t = 0,1,\dots,n$.

Considering the financing policy, until the expiration date of the leasing contract$m$, the project is fully financed with internal financing, that is, with retained cash. The rate of return on liquid assets is constant and equal to i^l, hence the interest income is $I_t^l = i^l \cdot C_{t-1}^l$. At time m, the operating disbursement is covered by absorbing resources from the liquid assets (internal financing), according to a proportion W, and by a loan contract for the complementary proportion $1 - W$. After time m, further disbursements are fully satisfied via internal financing.

The dividend distribution to equityholders, F_t^e, starts at a time d_m, according to the payout ratio α, to be applied to the smallest between the net income and the potential dividend (i.e., the difference between the operating cash flow and the cash flow to debt, $F_t^o - F_t^d$), provided that they are both positive, that is $F_t^e = \alpha\cdot\max\big[0, \min\big(I_t^e, F_t^o - F_t^d\big)\big]$. The cash contribution into the liquid assets account at time t, $-F_t^l$, is the retained cash, that is, the amount not distributed to the equityholders, therefore $-F_t^l = (F_t^o - F_t^d) - \alpha\cdot\max\big[0, \min\big(I_t^e, F_t^o - F_t^d\big)\big]$.At time n, the project is terminated, such that every asset and liability go back to zero.

The income statements, balance sheets, and cash-flow statements of the solar PhV plant are derived from the technical and financial model described above The overall value creation is calculated via eq. (1) by discounting the cash flows F_t and, analogously, the NPVs of the asset classes $j = o, l, d, e$ are determined by considering the corresponding cash flows F_t^j. The decomposition of the project NPV and the explanation of the equityholders' value creation are computed via (2) and (3).

In the next section, we present the technical and financial data of the photovoltaic project and illustrate the practical applications of the financial measures for making a decision.

3 Value creation and decomposition of the solar PhV plant

The industrial case of the solar PhV project is described with the following operating and financial input data.

Operating inputs:

- Useful life of PV plant: n =28 years
- Total cost of the plant = € 96,600.00
- Annual unit production in the first year at the technically suggested O&M (including insurance costs): Q_{max} =103,960 kWh
- Efficiency loss (per year): $g_Q = 0.65\%$
- Actual O&M and insurance: $O\&M$ = 2.75% of total cost of the plant
- Technically suggested O&M and insurance: $SuggO\&M = 4\,\%$ of total cost of the plant
- Productivity loss due to lack of maintenance (with O&M=0): $ProdLoss = 15\,\%$
- Disposal costs: H = €2,500.00
- Lost rent from land property: R = €1,250.00
- Growth rate for costs: g_c = 0.50%
- Lease term length: m = 20 years
- Purchase price of PV plant: $CapEx$ = €25,000.00
- Leasing annual payment: L = €6,268.45
- Annual energy consumption: q = 87,500 kWh
- Tax rate: τ = 20.00%
- Energy purchase price: $p_p = 0.180$(€/kWh)
- Energy selling price: $p_s = 0.155$ (€/kWh)
- Growth rate of energy price: g_p = 2.00%

Financial inputs:

- First of year of CFE distribution: d_m = 1st year
- Payout Ratio: α = 50.0% of the minimum between the net income and the potential dividends
- Internal financing: W = 60% of the purchase price of PhV plant
- Debt borrowing: $1 - W$ = 40% of the purchase price of PhV plant
- Interest rate on liquid assets i^l = 0%
- Interest rate on debt: i^d = 2.00%
- Required return on operating assets (constant): r^o = 6.00%
- Required return on liquid assets (constant): r^l = 2.00%
- Required return on debt (constant): r^d = 3.00%

The corresponding pro forma balance sheets, income statements and cash-flow statements are presented in Tables 1-3. Discounting the overall cash flows F_t, it results that the project NPV is $NPV = 84{,}338 > 0$, signaling that the PhV solar plant creates value. The decomposition of the value created under the investing and financing perspectives is described in the table below, via eq. (2).

Investment perspective			Financing perspective		
NPV^o	$=$	$+108{,}125$	NPV^e	$=$	$+88{,}635$
NPV^l	$=$	$-19{,}721$	NPV^d	$=$	-231
NPV	$=$	$88{,}404$	NPV	$=$	$88{,}404$

According to the investment perspective (left side of the table), the operations create value by $NPV^o = 108{,}125 > 0$, which is partly offset by the significant value destruction due to the liquidity management with $NPV^l = -19{,}721 < 0$ (due to an inefficient allocation of capital with $i^l = 0\% < r^l = 2.00\%$).

Considering the financing perspective (right side of the table), equityholders increase their wealth by $NPV^e = 88{,}635 > 0$, higher than the project NPV, $NPV = 88{,}404$, due to a value-creating borrowing policy, such that $NPV^d = -231 < 0$ (because the loan interest rate i^d is lower than the cost of debt capital r^d). This means that equityholders gain value at the expense of the debt-holders, but this transfer of value is tiny, due to the very small difference between the interest rate on debt (2%) and the maximum acceptable financing rate (3%), as well as the limited scale of the debt.

Finally, we decompose the wealth increase of equityholders into the contributions of operations, liquidity and debt, according to (3), obtaining the following partition.

$+NPV^o$	$=$	$108{,}125$
$+NPV^l$	$=$	$-19{,}721$
$-NPV^d$	$=$	$-(-231)$
$=NPV^e$	$=$	$88{,}635$

The equity NPV is lower than the operating NPV because investments in liquid assets significantly destroy value whereas value transfer from debtholders to equityholders is almost irrelevant (as also depicted in Figure 2.)

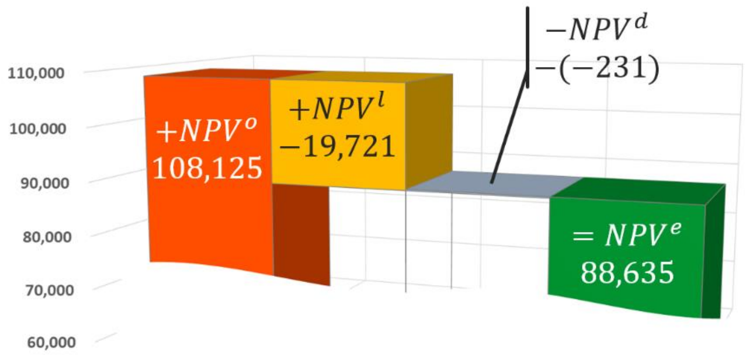

Figure 2: Decomposition of equity NPV

4 Financial efficiency of the solar PhV plant

As opposed to the NPV which does not suffer from any shortcoming, we note that the Internal Rate of Return (IRR), which is the most employed relative performance ratio in capital budgeting, does not exist for the overall project nor for the equity investment, as a consequence of the non-conventional cash flows streams $(F_0, F_1, \dots, F_n)$ and $(F_0^e, F_1^e, \dots, F_n^e)$, the first one having more than one change in sign and the second one having no change in sign.

Since the IRR fails, a viable solution for measuring the rate of return (and, therefore, the financial efficiency) of the project and of the equity investment is offered by the so-called average internal rate of return (AIRR) approach, introduced in Magni (2010, 2013), based on the estimated incomes and capital amounts, coherently defined as the ratio of the overall (discounted) income over the overall (discounted) capital. The AIRR of the project quantifies the project's rate of return over the total invested capital:

$$AIRR = \frac{\sum_{t=1}^{n} \frac{I_t}{(1+r_1)(1+r_2)\dots(1+r_t)}}{C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+r_1)(1+r_2)\dots(1+r_t)}} = \frac{113{,}956}{589{,}145} = 19.34\% \tag{4}$$

and, analogously, the equity AIRR measures the relative performance for equityholders, expressed as the ratio of net income to total equity invested:

$$AIRR^e = \frac{\sum_{t=1}^{n} \frac{I_t^e}{(1+r_1)(1+r_2)\dots(1+r_t)}}{C_0^e + \sum_{t=1}^{n} \frac{C_t^e}{(1+r_1^e)(1+r_2^e)\dots(1+r_t^e)}} = \frac{113{,}717}{575{,}270} = 19.77\% \tag{5}$$

where r_t and r_t^e are explicitly derived from the costs of capital of operating assets, non-operating assets, and debt (see Magni 2020, Ch. 8 for details on the calculation of the project costs of capital).
Furthermore, Magni (2010, 2013) proves that the AIRR approach is NPV-consistent[2] and is possible to decompose the value creation of the project into a financial efficiency component (defined as the difference between the AIRR of the project and the average cost of capital r) and an investment scale component, therefore enriching the informational content of the valuation. More precisely,

$$\begin{aligned} NPV = & \overbrace{(AIRR - r)}^{\text{financial efficiency}} \cdot \overbrace{\left(C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+r_1)(1+r_2)\dots(1+r_t)}\right)}^{\text{scale}} \\ & = (19.34\% - 4.34\%) \cdot 589{,}145 = 15.01\% \cdot 589{,}145 = € \, 84{,}404. \end{aligned} \tag{6}$$

where r is the project's average cost of capital. Symmetrically, the equity NPV is decomposed via the AIRR approach as the product of financial efficiency for equityholders and the scale of the equity investment:

[2]See also Marchioni and Magni (2018) for a definition of strong NPV-consistency of rates of return.

$$NPV^e = \overbrace{(AIRR^e - r^e)}^{\substack{\text{equity} \\ \text{financial efficiency}}} \cdot \overbrace{\left(C_0^e + \sum_{t=1}^{n} \frac{C^e{}_t}{(1+r^e{}_1)(1+r^e{}_2)\dots(1+r^e{}_t)}\right)}^{\substack{\text{equity} \\ \text{scale}}} \tag{7}$$
$$= (19.77\% - 4.66\%) \cdot 575{,}270 = 15.41\% \cdot 575{,}270 = € \ 88{,}635$$

where r^e is the average cost of equity capital.

Considering the equityholders' perspective, each euro invested in the project produces an equity return equal to 19.77%, remarkably higher than the alternative return equal to 4.66% that could be obtained on the financial market for investments of comparable risk. The financial efficiency of equity is positive, equal to 15.41%, representing the relative advantage for equityholders in investing in the PhV plant instead of alternative available investments. Overall, the equityholders invest € 575,270 at an above-normal return of 15.41%, so realizing a wealth increase equal to $€575{,}270 \cdot 15.41\% = €88{,}635$.

5 The role of distribution policy

It is worth noting that, in such a model, the estimated data are logically chained to decisions regarding distribution policy and retained cash. For example, to build the balance of liquid assets at the end of period $t = 14$, C_{14}^l, one needs start from the balance at the beginning of that period, $C_{13}^l = €45{,}997$. Assuming that the cash retained in the firm will not generate any interest income, the balance will increase by the retained cash (i.e., the amount not distributed to the equityholders) at time $t = 14$, which is equal to

$$\overbrace{-F_{14}^l}^{\text{retained cash}} = \overbrace{(F_{14}^o - F_{14}^d)}^{\text{potential dividends}} - \overbrace{\alpha \cdot \max\left[0, \min\left(I_{14}^e, F_{14}^o - F_{14}^d\right)\right]}^{\text{cash flow to equity}} = €4{,}362$$

Therefore, we obtain the balance of liquid assets at the end of period as

$$C_{14}^l = C_{13}^l - F_{14}^l = €45{,}997 + €4{,}362 = €50{,}358.$$

In this application, the distribution policy remarkably affects the economic results, with $NPV^l = -19{,}721$, because of high differences between the interest rate on liquid assets and minimum acceptable rate of return on liquid assets and high balances of liquid assets in several different periods of the investment. Only after computing the balance of liquid assets, the equity book value may be calculated as $C_{14}^e = C_{14}^o + C_{14}^l - C_{14}^d$.

Logically, the disregard of the distribution policy would have invalidated the logical consistency of the model. It is necessary to first calculate the potential dividends, then subtract the part of it which is not distributed and add it to the cash balance, as we have shown above. This brings about a network of complex relationships among the accounting magnitudes, which makes it necessary to draw up the cash-flow statement. The latter enables the analyst to calculate the cash flow associated with the liquid assets, F_t^l, which depends on the cash flow distributed to equityholders, F_t^e, which in turn depends on the operating cash flow. However, the latter can be computed only on the basis of elements of the income statement (the operating income) and elements of the balance sheets (operating assets). In turn, the balance sheet cannot be completed without the cash-flow statement, because, as we remind, the equity capital is equal to $C_t^e = C_t^o + C_t^l - C_t^d$ and C^l cannot be computed without computing F_t^l (i.e., without using the cash-flow statement). This nontrivial relationships among these three financial statements also testifies to the connections between estimated data (operating variables) and decision variables (distribution policy and reinvestment of retained cash). As a result, pro forma balance sheet and income statement are not sufficient; the cash flow statement is required for a sound and logically consistent model (and, therefore, a correct valuation of the project).[3]

6 Conclusions

In the current work we have provided a logically consistent model for the investment appraisal of a real-life photovoltaic energy project. Contrary to traditional modeling, we take account of the subtle relations interconnecting operating variables and financing variables, which depend on decisions (borrowing decision and distribution policy). We have considered the firm's decisions on distribution in the cash-flow statement, which is necessary to draw up the balance sheet (and, therefore, the income statement of the next period). We have decomposed the value created under two different perspectives, namely, the investment view which considers operating and liquid assets, and the financing view, which analyzes the equity and debt components, highlighting that the equity NPV may be significanty different from the operating NPV due to the remarkable role of financial decisions about liquid assets and debt.

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[3] Some authors discount the potential dividends, $(F_t^o - F_t^d)$ at the cost of equity capital r_t^e thereby avoiding the calculation of the balance of liquid assets. However, this does not produce the correct equity NPV, unless the retained cash is invested at the cost of equity, an often implausible assumption, or the firm distributes 100% of the potential dividends to equityholders, which is not always the case (and is not the case of the Italian company we consider) (see also Magni 2020, p. 344).

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Table 1: Balance sheets (thousands of Euro)

BALANCE SHEET	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
BS ASSETS																													
Operating Assets	-	1.8	1.7	1.7	1.6	1.5	1.4	1.4	1.3	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.1	25.0	21.9	18.8	15.6	12.5	9.4	6.3	3.1	-
Accounts receivable from grid operator	-	*1.8*	*1.7*	*1.7*	*1.6*	*1.5*	*1.4*	*1.4*	*1.3*	*1.2*	*1.1*	*1.0*	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.1*	*0.0*	-	-	-	-	-	-	-	-
Net fixed assets	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	*25.0*	*21.9*	*18.8*	*15.6*	*12.5*	*9.4*	*6.3*	*3.1*	-
Liquid assets	-	2.1	5.2	8.4	11.7	15.1	18.6	22.2	25.9	29.7	33.6	37.6	41.7	46.0	50.4	54.8	59.4	64.1	69.0	73.9	69.0	77.2	85.5	93.9	102.4	111.1	119.8	128.6	-
ASSETS	**-**	**3.8**	**6.9**	**10.0**	**13.2**	**16.6**	**20.0**	**23.5**	**27.1**	**30.9**	**34.7**	**38.6**	**42.7**	**46.8**	**51.1**	**55.4**	**59.9**	**64.5**	**69.2**	**74.1**	**94.0**	**99.1**	**104.3**	**109.5**	**114.9**	**120.4**	**126.0**	**131.8**	**-**
BS LIABILITIES																													
Loan current debt	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	10.0	8.8	7.6	6.4	5.2	3.9	2.7	1.3	-0.0
Equity	-	3.8	6.9	10.0	13.2	16.6	20.0	23.5	27.1	30.9	34.7	38.6	42.7	46.8	51.1	55.4	59.9	64.5	69.2	74.1	84.0	90.2	96.6	103.1	109.7	116.5	123.4	130.4	-
LIABILITIES	**-**	**3.8**	**6.9**	**10.0**	**13.2**	**16.6**	**20.0**	**23.5**	**27.1**	**30.9**	**34.7**	**38.6**	**42.7**	**46.8**	**51.1**	**55.4**	**59.9**	**64.5**	**69.2**	**74.1**	**94.0**	**99.1**	**104.3**	**109.5**	**114.9**	**120.4**	**126.0**	**131.8**	**-0.0**

Table 2: Income statements (thousands of Euro)

INCOME STATEMENT	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Revenues	**-**	**0.5**	**0.5**	**0.4**	**0.3**	**0.2**	**0.2**	**0.1**	**-0.0**	**-0.1**	**-0.2**	**-0.3**	**-0.4**	**-0.5**	**-0.6**	**-0.7**	**-0.9**	**-1.0**	**-1.1**	**-1.2**	**-1.4**	**-1.4**	**-1.4**	**-1.4**	**-1.4**	**-1.4**	**-1.4**	**-1.4**	**-1.4**
New revenue: sale of energy	-	*1.8*	*1.7*	*1.7*	*1.6*	*1.5*	*1.4*	*1.4*	*1.3*	*1.2*	*1.1*	*1.0*	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.1*	*0.0*	-	-	-	-	-	-	-	-
Lost rent from land property	-	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.3*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*	*-1.4*
(-) Operating costs	**-**	**6.8**	**7.1**	**7.4**	**7.7**	**8.1**	**8.4**	**8.7**	**9.1**	**9.4**	**9.8**	**10.1**	**10.5**	**10.9**	**11.3**	**11.7**	**12.1**	**12.5**	**12.9**	**13.3**	**13.8**	**20.3**	**20.6**	**20.9**	**21.2**	**21.5**	**21.9**	**22.2**	**19.6**
(-) Lease annual payment	-	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	*-6.3*	-	-	-	-	-	-	-	-
(-) O&M cost	-	*-2.7*	*-2.7*	*-2.7*	*-2.7*	*-2.7*	*-2.7*	*-2.7*	*-2.8*	*-2.8*	*-2.8*	*-2.8*	*-2.8*	*-2.8*	*-2.8*	*-2.8*	*-2.9*	*-2.9*	*-2.9*	*-2.9*	*-2.9*	*-2.9*	*-2.9*	*-3.0*	*-3.0*	*-3.0*	*-3.0*	*-3.0*	*-3.0*
(-) Disposal costs	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	*-2.9*
(-) Cost saving: self consumption of energy	-	*15.8*	*16.1*	*16.4*	*16.7*	*17.0*	*17.4*	*17.7*	*18.1*	*18.5*	*18.8*	*19.2*	*19.6*	*20.0*	*20.4*	*20.8*	*21.2*	*21.6*	*22.1*	*22.5*	*22.9*	*23.3*	*23.6*	*23.9*	*24.2*	*24.5*	*24.9*	*25.2*	*25.5*
EBITDA	**-**	**7.4**	**7.6**	**7.8**	**8.1**	**8.3**	**8.6**	**8.8**	**9.1**	**9.3**	**9.6**	**9.8**	**10.1**	**10.4**	**10.6**	**10.9**	**11.2**	**11.5**	**11.8**	**12.1**	**12.4**	**18.9**	**19.2**	**19.5**	**19.8**	**20.1**	**20.4**	**20.7**	**18.2**
(-) Depreciation	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-3.1	-3.1	-3.1	-3.1	-3.1	-3.1	-3.1	-3.1
EBIT	**-**	**7.4**	**7.6**	**7.8**	**8.1**	**8.3**	**8.6**	**8.8**	**9.1**	**9.3**	**9.6**	**9.8**	**10.1**	**10.4**	**10.6**	**10.9**	**11.2**	**11.5**	**11.8**	**12.1**	**12.4**	**15.8**	**16.1**	**16.4**	**16.7**	**17.0**	**17.3**	**17.6**	**15.1**
Interest income	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
(-) interest expenses	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-0.2	-0.2	-0.2	-0.1	-0.1	-0.1	-0.1	-0.0
EBT	**-**	**7.4**	**7.6**	**7.8**	**8.1**	**8.3**	**8.6**	**8.8**	**9.1**	**9.3**	**9.6**	**9.8**	**10.1**	**10.4**	**10.6**	**10.9**	**11.2**	**11.5**	**11.8**	**12.1**	**12.4**	**15.6**	**15.9**	**16.3**	**16.6**	**16.9**	**17.2**	**17.6**	**15.0**
(-) Taxes	-	-1.5	-1.5	-1.6	-1.6	-1.7	-1.7	-1.8	-1.8	-1.9	-1.9	-2.0	-2.0	-2.1	-2.1	-2.2	-2.2	-2.3	-2.4	-2.4	-2.5	-3.1	-3.2	-3.3	-3.3	-3.4	-3.4	-3.5	-3.0
NET INCOME	**-**	**5.9**	**6.1**	**6.3**	**6.5**	**6.6**	**6.8**	**7.0**	**7.2**	**7.4**	**7.7**	**7.9**	**8.1**	**8.3**	**8.5**	**8.7**	**9.0**	**9.2**	**9.4**	**9.7**	**9.9**	**12.5**	**12.7**	**13.0**	**13.3**	**13.5**	**13.8**	**14.1**	**12.0**

Table 3: Cash flow statements (thousands of Euro)

CASH FLOW	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
(+) CFO	-	4.1	6.1	6.3	6.5	6.7	6.9	7.1	7.3	7.5	7.7	8.0	8.2	8.4	8.6	8.8	9.1	9.3	9.6	9.8	-15.0	15.8	16.0	16.3	16.5	16.7	17.0	17.2	15.2
(-) CFD	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	10.0	-1.4	-1.4	-1.4	-1.4	-1.4	-1.4	-1.4	-1.4
FCFE (Free Cash Flow for Equity)	**-**	**4.1**	**6.1**	**6.3**	**6.5**	**6.7**	**6.9**	**7.1**	**7.3**	**7.5**	**7.7**	**8.0**	**8.2**	**8.4**	**8.6**	**8.8**	**9.1**	**9.3**	**9.6**	**9.8**	**-5.0**	**14.5**	**14.7**	**14.9**	**15.1**	**15.4**	**15.6**	**15.9**	**13.8**
(-) CFE	-	-2.1	-3.0	-3.1	-3.2	-3.3	-3.4	-3.5	-3.6	-3.7	-3.8	-3.9	-4.0	-4.1	-4.3	-4.4	-4.5	-4.6	-4.7	-4.8	-	-6.2	-6.4	-6.5	-6.6	-6.8	-6.9	-7.0	-142.5
(-) CFL	**-**	**2.1**	**3.1**	**3.2**	**3.3**	**3.4**	**3.5**	**3.6**	**3.7**	**3.8**	**3.9**	**4.0**	**4.1**	**4.2**	**4.4**	**4.5**	**4.6**	**4.7**	**4.8**	**5.0**	**-5.0**	**8.2**	**8.3**	**8.4**	**8.5**	**8.6**	**8.7**	**8.8**	**-128.6**

Note: Notwithstanding the existence and uniqueness of NPV and NPV^e, neither the IRR of the project cash-flow stream nor the IRR of the equity cash-flow stream exists.

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