A FOCAL SUBGROUP THEOREM FOR OUTER COMMUTATOR WORDS

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Abstract. Let G be a finite group of order $p^a m$, where p is a prime and m is not divisible by p, and let P be a Sylow p-subgroup of G. If w is an outer commutator word, we prove that $P \cap w(G)$ is generated by the intersection of P with the set of mth powers of all values of w in G.

Let G be a finite group and P a Sylow p-subgroup of G. The Focal Subgroup Theorem states that $P \cap G'$ is generated by the set of commutators $\{[g,z] \mid g \in G,\ z \in P,\ [g,z] \in P\}$. This was proved by Higman in 1953 [5]. Nowadays the proof of the theorem can be found in many standard books on group theory (for example, Rose's book [7] or Gorenstein's [3]).

One immediate corollary is that $P \cap G'$ can be generated by commutators lying in P. Of course, G' is the verbal subgroup of G corresponding to the group word $[x,y] = x^{-1}y^{-1}xy$. It is natural to ask the question on generation of Sylow subgroups for other words. More specifically, if w is a group word we write G_w for the set of values of w in G and $w(G)$ for the subgroup generated by G_w (which is called the *verbal subgroup* of w in G), and one is tempted to ask the following question.

Question. *Given a Sylow p-subgroup P of a finite group G, is it true that $P \cap w(G)$ can be generated by w-values lying in P, i.e., that $P \cap w(G) = \langle P \cap G_w \rangle$?*

However considering the case where G is the non-abelian group of order six, $w = x^3$ and $p = 3$ we quickly see that the answer to the above question is negative. Therefore we concentrate on the case where w is a *commutator* word. Recall that a group word is commutator if the sum of the exponents of any indeterminate involved in it is zero. Thus, we deal with the question whether $P \cap w(G)$ can be generated by w-values whenever w is a commutator word.

The main result of this paper is a contribution towards a positive answer to this question: we prove that if w is an outer commutator word, then $P \cap w(G)$ can be generated by the *powers of values of* w which lie in P. More precisely, we have the following result.

Theorem A. *Let G be a finite group of order $p^a m$, where p is a prime and m is not divisible by p, and let P be a Sylow p-subgroup of G. If w is*

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an outer commutator word, then $P \cap w(G)$ is generated by mth powers of w-values, i.e., $P \cap w(G) = \langle P \cap G_{w^m} \rangle$.

Recall that an *outer commutator word* is a word which is obtained by nesting commutators, but using always *different indeterminates*. Thus the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is an outer commutator while the Engel word $[x_1, x_2, x_2, x_2]$ is not. An important family of outer commutator words are the simple commutators γ_i, given by

$$\gamma_1 = x_1, \qquad \gamma_i = [\gamma_{i-1}, x_i] = [x_1, \dots, x_i], \quad \text{for } i \geq 2.$$

The corresponding verbal subgroups $\gamma_i(G)$ are the terms of the lower central series of G. Another distinguished sequence of outer commutator words are the *derived words* δ_i, on 2^i indeterminates, which are defined recursively by

$$\delta_0 = x_1, \qquad \delta_i = [\delta_{i-1}(x_1, \dots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \dots, x_{2^i})], \quad \text{for } i \geq 1.$$

Then $\delta_i(G) = G^{(i)}$, the ith derived subgroup of G.

Some of the ideas behind the proof of Theorem A were anticipated already in [4] where somewhat similar arguments, due to Guralnick, led to a result on generation of a Sylow p-subgroup of G' for a finite group G admitting a coprime group of automorphisms. Later the arguments were refined in [1]. In both papers [4] and [1] the results on generation of Sylow subgroups were used to reduce a problem about finite groups to the case of nilpotent groups. It is hoped that also our Theorem A will play a similar role in the subsequent projects.

Another important tool used in the proof of Theorem A is the famous result of Liebeck, O'Brien, Shalev and Tiep [6] that every element of a non-abelian simple group is a commutator. The result proved Ore's conjecture thus solving a long-standing problem. In turn, the proof in [6] uses the classification of finite simple groups as well as many other sophisticated tools.

1. Preliminaries

If X and Y are two subsets of a group G, and N is a normal subgroup of G, it is not always the case that $XN \cap YN = (X \cap Y)N$, i.e., that $\overline{X} \cap \overline{Y} = \overline{X \cap Y}$ in the quotient group $\overline{G} = G/N$. In our first lemma we have a situation in which this property holds, and which will be of importance in the sequel.

Lemma 1.1. *Let G be a finite group, and let N be a normal subgroup of G. If P is a Sylow p-subgroup of G and X is a normal subset of G consisting of p-elements, then $XN \cap PN = (X \cap P)N$. In other words, if we use the bar notation in G/N, we have $\overline{X} \cap \overline{P} = \overline{X \cap P}$.*

Proof. We only need to worry about the inclusion $\overline{X} \cap \overline{P} \subseteq \overline{X \cap P}$. So, given an element $g \in XN \cap PN$, we prove that $g \in xN$ for some $x \in X \cap P$. Since we have $g \in XN$, we may assume without loss of generality that $g \in X$, and in particular g is a p-element. Since also $g \in PN$, there exists $z \in P$ such that $gN = zN$.

Put $H = \langle g \rangle N = \langle z \rangle N$, and observe that $H' \leq N$. Since $P \cap N$ is a Sylow p-subgroup of N and $z \in P$, it follows that $P \cap H = \langle z \rangle (P \cap N)$ is a Sylow p-subgroup of H. Now, g is a p-element of H, and so we get $g^h \in P \cap H$ for some $h \in H$. If we put $x = g^h$ then $x \in X \cap P$, since X is a normal subset of G, and $g = x^{h^{-1}} = x[x, h^{-1}] \in xH' \subseteq xN$, as desired. □

The next lemma will be fundamental in the proof of Theorem A, since it will allow us to go up a series from 1 to $w(G)$ in which all quotients of two consecutive terms are verbal subgroups of a word all of whose values are also w-values.

Lemma 1.2. *Let G be a finite group, and let P be a Sylow p-subgroup of G. Assume that $N \le L$ are two normal subgroups of G, and use the bar notation in the quotient group G/N. Let X be a normal subset of G consisting of p-elements such that $\overline{P} \cap \overline{L} = \langle \overline{P} \cap \overline{X} \rangle$. Then $P \cap L = \langle P \cap X, P \cap N \rangle$.*

Proof. By Lemma 1.1, we have $\overline{P} \cap \overline{L} = \langle \overline{P \cap X} \rangle$, and this implies that $PN \cap L = \langle P \cap X \rangle N$. By intersecting with P, we get

$$P \cap L = P \cap (PN \cap L) = P \cap \langle P \cap X \rangle N = \langle P \cap X \rangle (P \cap N),$$

where the last equality follows from Dedekind's law. This proves the result. □

We will also need the following lemma, of a different nature.

Lemma 1.3. *Let G be a finite group, and let N be a minimal normal subgroup of G. If N does not contain any non-trivial elements of G_{δ_i}, where $i \ge 1$, then $[N, G^{(i-1)}] = 1$.*

Proof. We argue by induction on i. If $i = 1$ then, since N is normal in G and does not contain any non-trivial commutators of elements of G, it follows that $[n, g] = 1$ for every $n \in N$ and $g \in G$. Thus $[N, G] = 1$, as desired.

Assume now that $i \ge 2$. The fact that N is a minimal normal subgroup of G implies that the subgroup $\langle N \cap G_{\delta_{i-1}} \rangle$ must be either equal to N or the trivial subgroup. In the former case, we have $N = \langle N \cap G_{\delta_{i-1}} \rangle$ and so $[N, G^{(i-1)}]$ is generated by elements of the form $[a, b]$ where $a \in N \cap G_{\delta_{i-1}}$ and $b \in G_{\delta_{i-1}}$. In particular, each commutator $[a, b]$ belongs to $N \cap G_{\delta_i}$ and must be 1 by the hypothesis. Hence $[N, G^{(i-1)}] = 1$. If $N \cap G_{\delta_{i-1}} = 1$, then it follows from the induction hypothesis that $[N, G^{(i-2)}] = 1$, and the result holds. □

We conclude this preliminary section by showing that Theorem A holds *for every word* under the assumption that the verbal subgroup $w(G)$ is nilpotent.

Theorem 1.4. *Let G be a finite group of order $p^a m$, where p is a prime and m is not divisible by p, and let P be a Sylow p-subgroup of G. If w is any word such that $w(G)$ is nilpotent, then*

$$P \cap w(G) = \langle P \cap G_{w^m} \rangle.$$

Proof. By Bezout's identity, there exist two integers λ and μ such that $1 = \lambda p^a + \mu m$. If we put $u = w^{p^a}$ and $v = w^m$, then for every $g \in G_w$ we have

$$g = (g^{p^a})^{\lambda} \cdot (g^m)^{\mu} \in \langle G_u \rangle \cdot \langle G_v \rangle.$$

Hence

$$w(G) = \langle G_u, G_v \rangle. \tag{1}$$

Note that all elements of G_u have p'-order, and all elements of G_v have p-power order. Since $w(G)$ is nilpotent, it follows that $\langle G_u \rangle$ is a p'-subgroup of $w(G)$, $\langle G_v \rangle$ is a p-subgroup, and G_u and G_v commute elementwise. As a consequence of this and (1), we get

$$w(G) = \langle G_u \rangle \times \langle G_v \rangle, \tag{2}$$

and $\langle G_u \rangle$ and $\langle G_v \rangle$ are a Hall p'-subgroup and a Sylow p-subgroup of $w(G)$, respectively. We conclude that $P \cap w(G) = \langle G_v \rangle$, which proves the theorem. □

2. The proof of Theorem A

The first step in the proof of Theorem A is to verify it for δ_i, which is done in Theorem 2.3 below. For this, we will rely on the result by Liebeck, O'Brien, Shalev and Tiep [6] that proved Ore's conjecture, according to which every element of a non-abelian simple group is a commutator, and *a fortiori*, also a value of δ_i for every i. We will also need the following result of Gaschütz (see page 191 of [8]).

Theorem 2.1. *Let G be a finite group, and let P be a Sylow p-subgroup of G. If N is a normal abelian p-subgroup of G, then N is complemented in G if and only if N is complemented in P.*

In the proof of Theorem A for both δ_i and an arbitrary outer commutator word, we will argue by induction. Then it will be important to take into account the following remark.

Remark 2.2. Let G be a group of order $p^a m$ for which we want to prove Theorem A in the case of a given word w. Assume that K is a group whose order $p^{a^*} m^*$ is a divisor of $p^a m$ (for example, a subgroup or a quotient of G), and let P^* be a Sylow p-subgroup of K. If Theorem A is known to hold for K and w, then we have $P^* \cap w(K) = \langle P^* \cap K_{w^{m^*}} \rangle$. Since m/m^* is a positive integer which is coprime to p, it follows that $P^* \cap w(K) = \langle (P^* \cap K_{w^{m^*}})^{m/m^*} \rangle$, and so also that $P^* \cap w(K) = \langle P^* \cap K_{w^m} \rangle$. In other words, in the statement of Theorem A for K, we can replace the power word w^{m^*} corresponding to the order of K with the word w^m, which corresponds to the order of G.

We can now proceed to the proof of Theorem A for δ_i.

Theorem 2.3. *Let G be a finite group of order $p^a m$, where p is a prime and m is not divisible by p, and let P be a Sylow p-subgroup of G. Then, for every $i \geq 0$, we have*

$$P \cap G^{(i)} = \langle P \cap G_{\delta_i^m} \rangle.$$

Proof. We argue by induction on the order of G. The result is obvious if either $i = 0$ or $G^{(i)} = 1$, so we assume that $i \geq 1$ and $G^{(i)} \neq 1$.

Let $N \neq 1$ be a normal subgroup of G which is contained in $G^{(i)}$. Then the result holds in $\overline{G} = G/N$, and we have $\overline{P} \cap \overline{G}^{(i)} = \langle \overline{P} \cap \overline{G}_{\delta_i^m} \rangle$. By applying Lemma 1.2, we get

$$P \cap G^{(i)} = \langle P \cap G_{\delta_i^m}, P \cap N \rangle. \tag{3}$$

Now we assume that N is a minimal normal subgroup of G, and we consider three different cases, depending on the structure of N.

(i) N is a direct product of isomorphic non-abelian simple groups.

By the positive solution to Ore's conjecture, we have $N = N_{\delta_i}$. Hence we get $P \cap N \subseteq N_{\delta_i}$, and since the map $z \mapsto z^m$ is a bijection in $P \cap N$, it follows that $P \cap N \subseteq P \cap N_{\delta_i^m}$. Now the result is immediate from (3).

(ii) $N \cong C_q \times \cdots \times C_q$, where q is a prime different from p.

In this case, $P \cap N = 1$ and the result obviously holds.

(iii) $N \cong C_p \times \cdots \times C_p$.

In this case, we have $N \leq P$ and so $P \cap N = N$. Since $\langle N \cap G_{\delta_i} \rangle$ is a normal subgroup of G and N is a minimal normal subgroup, it follows that either $\langle N \cap G_{\delta_i} \rangle = N$ or $N \cap G_{\delta_i} = 1$. In the former case, we have $N = \langle (N \cap G_{\delta_i})^m \rangle$, since N is a finite p-group, and so $N = \langle N \cap G_{\delta_i^m} \rangle$ and the theorem follows again from (3). So we are necessarily in the latter case, and then by Lemma 1.3, we have $[N, G^{(i-1)}] = 1$.

If G is not perfect, then the theorem holds by induction in G', and so $P \cap G^{(i+1)} = P \cap (G')^{(i)}$ can be generated by values of δ_i^m lying in P. If $G^{(i+1)} \neq 1$ then we can use (3) with $G^{(i+1)}$ in the place of N, and we are done. On the other hand, if $G^{(i+1)} = 1$ then $G^{(i)}$ is abelian, and the result is a consequence of Theorem 1.4.

Thus we may assume that G is perfect. Then $P \cap G^{(i)} = P$. Also $[N, G] = [N, G^{(i-1)}] = 1$, and N is central in G. Being a minimal normal subgroup of G, this implies that $|N| = p$. If $N \leq \Phi(P)$ then it follows from (3) that $P = \langle P \cap G_{\delta_i^m} \rangle$, as desired. Hence we may assume that N is not contained in a maximal subgroup M of P. Since $|N| = p$, it follows that M is a complement of N in P. By Theorem 2.1, it follows that N has also a complement in G, say H. Since $N \leq Z(G)$, we conclude that $G = H \times N$, a contradiction with the fact that G is perfect. This completes the proof. $\square$

We will deal with arbitrary outer commutator words using some concepts from the paper [2], where outer commutator words are represented by binary rooted trees in the following way: indeterminates are represented by an isolated vertex, and if $w = [u, v]$ is the commutator of two outer commutator words u and v, then the tree T_w of w is obtained by drawing the trees T_u and T_v, and a new vertex (which will be the root of the new tree) which is then connected to the roots of T_u and T_v. For example, the following are the trees for the words γ_4 and δ_3 (we also label every vertex with the outer commutator word which is represented by the tree appearing on top of that vertex):

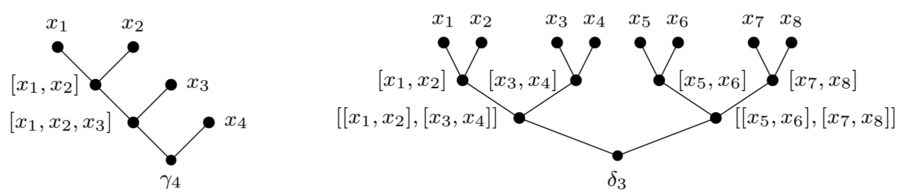

FIGURE 1. The trees of the words γ_4 and δ_3.

Each of these trees has a visual height, which coincides with the largest distance from the root to another vertex of the tree. Observe that the full binary tree of height i corresponds to the derived word δ_i. The following two concepts, also introduced in [2], will be essential in our proof of Theorem A.

Definition 2.4. Let u and w be two outer commutator words. We say that u is an *extension* of w if the tree of u is an upward extension of the tree of w. If $u \neq w$, we say that u is a *proper extension* of w.

An important remark is that, if u is an extension of w, then $G_u \subseteq G_w$.

Definition 2.5. If w is an outer commutator word whose tree has height i, the *defect* of w is the number of vertices that need to be added to the tree of w in order to get the tree of δ_i. In other words, if the tree of w has V vertices, the defect of w is $2^{i+1} - 1 - V$.

Thus the only words of defect 0 are the derived words. Our proof of Theorem A also depends on the following result, which is implicit in the proof of Theorem B of [2].

Theorem 2.6. *Let $w = [u, v]$ be an outer commutator word of height i, different from δ_i. Then at least one of the subgroups $[w(G), u(G)]$ and $[w(G), v(G)]$ is contained in a product of verbal subgroups corresponding to words which are proper extensions of w of height i.*

Let us now give the proof of Theorem A.

Proof of Theorem A. We argue by induction on the defect of the word w. If the defect is 0, then w is a derived word, and the result is true by Theorem 2.3. Hence we may assume that the defect is positive. If the height of w is i, let $\Phi = \{\varphi_1, \ldots, \varphi_r\}$ be the set of all outer commutator words of height i which are a proper extension of w. Since every word in the set Φ has smaller defect than w, the theorem holds for all φ_i.

Put $N_0 = 1$, $N_i = \varphi_1(G) \ldots \varphi_i(G)$ for $1 \leq i \leq r$, and $N = N_r$. Let us write $w = [u, v]$, where u and v are outer commutator words. Since $[w(G), w(G)]$ is contained in both $[w(G), u(G)]$ and $[w(G), v(G)]$, it follows from Theorem 2.6 that $[w(G), w(G)] \leq N$. Thus if $\overline{G} = G/N$, the verbal subgroup $w(\overline{G})$ is abelian, and so Theorem A holds in $\overline{G}$, according to Theorem 1.4. Hence $\overline{P} \cap w(\overline{G}) = \langle \overline{P} \cap \overline{G}_{w^m} \rangle$, and by applying Lemma 1.2, we get $P \cap w(G) = \langle P \cap G_{w^m}, P \cap N \rangle$.

Consequently, it suffices to show that $P \cap N$ can be generated by values of w^m. We see this by proving that $P \cap N_i = \langle P \cap N_i \cap G_{w^m} \rangle$ for every $i = 0, \ldots, r$, by induction on i. There is nothing to prove if $i = 0$, so we assume that $i \geq 1$. If $\overline{G} = G/N_{i-1}$, we have $\overline{N}_i = \varphi_i(\overline{G})$. Since the theorem is true for φ_i, it follows that $\overline{P} \cap \overline{N}_i = \langle \overline{P} \cap \overline{G}_{\varphi_i^m} \rangle$. By Lemma 1.2, we get

$$P \cap N_i = \langle P \cap G_{\varphi_i^m}, P \cap N_{i-1} \rangle.$$

Observe that, since φ_i is an extension of w, we have $G_{\varphi_i^m} \subseteq G_{w^m}$. Since also $G_{\varphi_i^m} \subseteq \varphi_i(G) \leq N_i$, we can further say that $G_{\varphi_i^m} \subseteq N_i \cap G_{w^m}$. Hence

$$P \cap N_i = \langle P \cap N_i \cap G_{w^m}, P \cap N_{i-1} \rangle,$$

and the result follows from the induction hypothesis. □

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