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On groups in which Engel sinks are cyclic

Cristina Acciarri and Pavel Shumyatsky

ABSTRACT. For an element g of a group G , an Engel sink is a subset $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, g, g, \dots, g]$ belong to $\mathcal{E}(g)$. We conjecture that if G is a profinite group in which every element admits a sink that is a procyclic subgroup, then G is procyclic-by-(locally nilpotent). We prove the conjecture in two cases – when G is a finite group, or a soluble pro- p group.

1. Introduction

A group G is called an *Engel group* if for every $x, g \in G$ the equation $[x, g, g, \dots, g] = 1$ holds, where g is repeated in the commutator sufficiently many times depending on x and g . (Throughout the paper, we use the left-normed simple commutator notation $[a_1, a_2, a_3, \dots, a_r] = [\dots[[a_1, a_2], a_3], \dots, a_r]$.) Of course, any nilpotent group is an Engel group. For finite groups the converse is also known to be true: a finite Engel group is nilpotent by Zorn's theorem [15]. Given arbitrary elements x, g in a group G , here and in what follows, for any $n \geq 1$, we will denote by $[x, {}_n g]$ the commutator of the form $[x, \underbrace{g, \dots, g}_n]$.

Recently, groups that are ‘almost Engel’ in the sense of restrictions on so-called Engel sinks were given some attention. An Engel sink of

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an element $g \in G$ is a set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, g, g, \dots, g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $n(x, g)$ such that

$$[x, {}_n g] \in \mathcal{E}(g) \quad \text{for all } n \geq n(x, g).$$

Engel groups are precisely the groups for which we can choose $\mathcal{E}(g) = \{1\}$ for all $g \in G$. In [5] finite, profinite, and compact groups in which every element has a finite Engel sink were considered. It was proved that compact groups with this property are finite-by-(locally nilpotent). Similar result for linear groups was established in [9] (see also [12] for a shorter proof). Recall that a group G is locally nilpotent if every finitely generated subgroup of G is nilpotent. According to an important theorem, due to Wilson and Zelmanov [14], a profinite group is locally nilpotent if and only if it is Engel.

In [6] finite groups in which there is a bound for the ranks of the subgroups generated by Engel sinks were considered. Recall that the rank of a finite group is the minimum number r such that every subgroup can be generated by r elements. It was shown that if G is a finite group such that for every $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a subgroup of rank r , then the rank of $\gamma_\infty(G)$ is bounded in terms of r . Here $\gamma_\infty(G)$ stands for the intersection of all terms of the lower central series of G .

The goal of this article is to establish some substantial evidence in favor of the following conjecture.

Conjecture 1.1. *Let G be a profinite group in which every element admits an Engel sink that generates a procyclic subgroup. Then G is procyclic-by-(locally nilpotent).*

First, we consider finite groups in which all elements admit Engel sinks generating cyclic subgroups.

Theorem 1.2. *Let G be a finite group in which every element admits an Engel sink generating a cyclic subgroup. Then $\gamma_\infty(G)$ is cyclic.*

Recall that a profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The reader is referred to textbooks [7] and [13] for background information on profinite groups. In the context of such groups all the usual concepts of groups theory are interpreted topologically. In particular, by a subgroup of a profinite group we always mean a closed subgroup. The next result deals with soluble pro- p groups in which every element admits an Engel sink generating a procyclic subgroup.

Theorem 1.3. *Let G be a soluble pro- p group in which every element admits an Engel sink generating a procyclic subgroup. Then G has a normal procyclic subgroup K such that G/K is locally nilpotent.*

In the next section we deal with the proof of Theorem 1.2. The proof of Theorem 1.3 is given in Section 3.

2. Proof of Theorem 1.2

We start with a collection of well-known facts about coprime automorphisms that we will use throughout the article. Given a group G acted on by a group A we write $C_G(A)$ for the subgroup of fixed points of A in G and $[G, A]$ for the subgroup generated by all elements of the form $x^{-1}x^a$, where $x \in G$ and $a \in A$.

Lemma 2.1. *Let A be a group of automorphisms of a finite group G such that $(|G|, |A|) = 1$. Then*

- (i) $G = C_G(A)[G, A]$;
- (ii) $[G, A, A] = [G, A]$;
- (iii) $C_{G/N}(A) = C_G(A)N/N$ for any A -invariant normal subgroup N of G ;
- (iv) If G is cyclic of prime-power order, then A is cyclic;
- (v) If G is cyclic of 2-power order, then $A = 1$.

The assumption of coprimeness is unnecessary in the following lemma.

Lemma 2.2. *Let G be a cyclic group. The group of automorphisms of G is abelian.*

Recall that a normal subgroup N of a finite group G is a normal p -complement (for a prime p) if $N = O_{p'}(G)$ and G/N is a p -group. The well-known theorem of Frobenius states that G possesses a normal p -complement if and only if $N_G(H)/C_G(H)$ is a p -group for every nontrivial p -subgroup H of G (see [3, Theorem 7.4.5]).

Obviously, in a finite group G every element has the smallest Engel sink, so throughout this section, we use the term Engel sink for the minimal Engel sink, denoted by $\mathcal{E}(g)$, of an element $g \in G$.

Lemma 2.3. *Let G be a finite group in which for each $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup. Then G has a normal 2-complement.*

PROOF. Suppose that this is false. Then G has an element x of odd order and a 2-subgroup H such that x normalizes but not centralizes H . Let $E = H \cap \mathcal{E}(x)$. Observe that x normalizes $\langle E \rangle$. In view of Lemma 2.1(v), we deduce that x centralizes $\langle E \rangle$. Therefore for every $h \in H$ we have $[h, x, x, \dots, x] = 1$ if x is repeated in the commutator sufficiently many times. In other words, x is Engel in the group $H\langle x \rangle$ and we deduce that $[H, x] = 1$. This yields a contradiction. \square

In view of the Feit-Thompson Theorem on solubility of groups of odd order [2] the following corollary is straightforward.

Corollary 2.4. *Let G be a finite group in which the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup for each $g \in G$. Then G is soluble.*

Recall that a group G is metanilpotent if it has a normal subgroup N such that both N and G/N are nilpotent. It is easy to see that a finite group G is metanilpotent if and only if $\gamma_\infty(G)$ is nilpotent. The next result is well known (see for example [1, Lemma 2.4] for the proof).

Lemma 2.5. *Let G be a finite metanilpotent group. Assume that P is a Sylow p -subgroup of $\gamma_\infty(G)$ and H is a Hall p' -subgroup of G . Then $P = [P, H]$.*

We will now prove Theorem 1.2 under the additional assumption that G is metanilpotent.

Lemma 2.6. *Let G be a finite metanilpotent group in which for each $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup. Then $\gamma_\infty(G)$ is cyclic.*

PROOF. Since $\gamma_\infty(G)$ is nilpotent, it is sufficient to show that each Sylow subgroup of $\gamma_\infty(G)$ is cyclic. Thus, let P be a Sylow subgroup of $\gamma_\infty(G)$ for some prime p . In view of Lemma 2.5 we have $P = [P, H]$, where H is a Hall p' -subgroup of G . Without loss of generality we can assume that $G = PH$. Replacing if necessary P by $P/\Phi(P)$ and H by $H/C_H(P)$, we can assume that P is an elementary abelian p -group (a vector space over the field with p elements) on which the nilpotent group H acts faithfully by linear transformations.

Taking into account that H is nilpotent, we note that $\mathcal{E}(h) = [P, h]$ for each nontrivial $h \in H$. Therefore, if $H = \langle g \rangle$ is cyclic, then $P = \mathcal{E}(g)$ is cyclic, too. Hence, we assume that H is noncyclic.

Suppose first that H contains a noncyclic abelian subgroup A . Choose a nontrivial element $a_1 \in A$. The cyclic subgroup $[P, a_1]$ is A -invariant and, by Lemma 2.1(iv), the quotient $A/C_A([P, a_1])$ is cyclic. In particular $C_A([P, a_1]) \neq 1$ so we choose a nontrivial element $a_2 \in C_A([P, a_1])$. Since a_2 centralizes $[P, a_1]$, it follows that $[P, a_1][P, a_2]$ is not cyclic. Moreover, it is clear that a_1 centralizes $[P, a_2]$. Hence, $[P, a_1][P, a_2] \leq [P, a_1a_2]$. This shows that $\mathcal{E}(a_1a_2)$ is not cyclic, a contradiction. Therefore all abelian subgroups of H are cyclic.

It follows (see for example [3, Theorem 4.10(ii), p. 199]) that H is isomorphic to $Q \times C$, where Q is a generalized quaternion group and C is a cyclic group of odd order. Let a_0 be the unique involution of H . It is clear that a_0 is contained in all maximal cyclic subgroups of H . Thus we have $[P, h] = [P, a_0]$ for any $h \in H$ and so $[P, H] = [P, a_0]$. Note that $[P, a_0]$ is an H -invariant subgroup of order p . In view of Lemma 2.1(iv), note that H induces a cyclic group of automorphisms of $[P, a_0]$.

We deduce that a_0 acts trivially on $[P, a_0]$ and hence on P . This is a final contradiction. It shows that P is cyclic, as required. \square

Recall that the Fitting height of a finite soluble group G is the minimum number $h = h(G)$ such that G possesses a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_h = G$ all of whose factors are nilpotent. We say that a system of subgroups P_1, \dots, P_k of G is a tower of height k if

- Each subgroup P_i has prime-power order.
- P_j is normalized by P_i whenever $1 \leq i \leq j \leq k$.
- $P_{i+1} = \gamma_\infty(P_{i+1}P_i)$ for each $i = 1, 2, \dots, k-1$.

Every finite soluble group of Fitting height h possesses a tower of height h (see for example [11]).

We are now ready to prove the theorem on finite groups.

PROOF OF THEOREM 1.2. Recall that G is a finite group in which for each $g \in G$ the Engel sink $\mathcal{E}(g)$ generates a cyclic subgroup. We need to show that $\gamma_\infty(G)$ is cyclic. By Corollary 2.4, the group G is soluble. Suppose that the theorem is false and let G be a counterexample of minimal order. Lemma 2.6 shows that $h(G) \geq 3$.

Choose three subgroups P_1, P_2, P_3 which form a tower of height 3. Since $P_3 = \gamma_\infty(P_3P_2)$, because of Lemma 2.6 we conclude that P_3 is cyclic. By Lemma 2.2 the subgroup P_2P_1 induces an abelian group of automorphisms of P_3 . Since $P_2 = \gamma_\infty(P_2P_1)$, we conclude that P_2 acts on P_3 trivially. In other words, P_2 centralizes P_3 . In view of the equality $P_3 = \gamma_\infty(P_3P_2)$ we have a contradiction. This completes the proof. \square

3. Proof of Theorem 1.3

Our purpose in this section is to prove Theorem 1.3. Given an element g of a group G , for each $n \geq 1$, we will denote by $E_n(g)$ the subgroup of G generated by all commutators of the form $[x, {}_n g]$, with x in G .

The next two results, whose proofs can be found in [10, Lemmas 2.1 and 2.2] respectively, state general facts about nilpotent groups and Engel elements.

Lemma 3.1. *Let $G = H\langle a \rangle$, where H is a normal nilpotent subgroup of class c and a is an n -Engel element. Then G is nilpotent with class at most cn .*

Lemma 3.2. *For any positive integers c, n there exists an integer $f = f(c, n)$ with the following property. Let $G = H\langle a \rangle$, where H is a normal nilpotent subgroup of class c . Then $\gamma_f(G) \leq E_n(a)$.*

Here and throughout the article $\gamma_f(G)$ denotes the f th term of the lower central series of G .

The following lemma concerns profinite groups and Engel elements.

Lemma 3.3. *Let $G = M\langle a \rangle$ be a profinite group with an abelian normal subgroup M and an Engel element a . Then G is nilpotent.*

PROOF. For any nonnegative integer i set

$$B_i = \{x \in M \mid [x, {}_i a] = 1\}.$$

Each set B_i is closed, and $\bigcup_{i \geq 0} B_i = M$. By Baire's Category Theorem [4, p. 200] at least one of these sets has non-empty interior. Therefore there exist an integer n , an element b in M and an open normal subgroup N contained in M such that $[y, {}_n a] = 1$ for any $y \in bN$. From this we deduce that $[x, {}_n a] = 1$ for any x in N . Since N is open in M , there exists a positive integer k such that $[z, {}_k a] \in N$ for any $z \in M$. Thus $[M, {}_{n+k} a] = 1$ and the result follows. \square

Note that, for an element g of a group G , once a sink $\mathcal{E}(g)$ is chosen, the subgroup $\langle \mathcal{E}(g) \rangle$ generated by $\mathcal{E}(g)$ is also a sink for g . In the remaining part of this article it will be convenient to use the term "sink $\mathcal{E}(g)$ of g " meaning a subgroup containing all sufficiently long commutators $[x, {}_i g]$ with $x \in G$.

Lemma 3.4. *Let G be a metabelian profinite group and let a be an element of G . Then, for any choice of a sink $\mathcal{E}(a)$, there exists an integer n such that $E_n(a) \leq \mathcal{E}(a)$.*

PROOF. If $\mathcal{E}(a)$ is finite, then a is Engel in G . Set $K = G'\langle a \rangle$. By Lemma 3.3 the subgroup K is nilpotent and $[G',_{n-1} a] = 1$, for some integer n . Therefore $[G,_{n-1} a] = 1$ and so $E_n(a) \leq \mathcal{E}(a)$.

Assume that $\mathcal{E}(a)$ is infinite. Let E_1 be the subgroup generated by all commutators $[x, a, \dots, a] \in \mathcal{E}(a)$ such that $x \in G'$. Note that $E_1 \leq \mathcal{E}(a)$ and E_1 is a normal subgroup of G . Moreover a is Engel in G/E_1 . In view of Lemma 3.3 the subgroup $G'\langle a \rangle$ is nilpotent modulo E_1 and the result follows. \square

Lemma 3.5. *Let G be a metabelian profinite group and $a \in G$. For each $n \geq 1$ the subgroup $E_n(a)$ is normal in G .*

PROOF. For any $i \geq 1$, any $g \in G'$ and $y \in G$ we have

$$[g,_{i-1} a]^y = [g^y,_{i-1} a] \text{ and } [g^{-1},_{i-1} a] = [g,_{i-1} a]^{-1}.$$

Moreover, for any $x, y \in G$, the equality $[x, a]^y = [xy, a][y, a]^{-1}$ holds.

We only need to prove the lemma with $n \geq 2$ since for $n = 1$ the result is well known even without the assumption that G is metabelian. For arbitrary elements $x, y \in G$, by using the standard commutator laws, write

$$[x,_{n-1} a]^y = [[x, a]^y,_{n-1} a] = [[xy, a][y, a]^{-1},_{n-1} a] = [xy,_{n-1} a][y,_{n-1} a]^{-1}.$$

The formula above shows that $[x,_{n-1} a]^y \in E_n(a)$ and the lemma follows. \square

We write C_n to denote the cyclic group of order n and \mathbb{Z}_p the additive group of p -adic integers. Recall that the group of automorphisms of \mathbb{Z}_p is isomorphic to $\mathbb{Z}_p \oplus C_{p-1}$ if $p \geq 3$ and $\mathbb{Z}_2 \oplus C_2$ if $p = 2$ (see for example [7, Theorem 4.4.7]). Note that all nontrivial subgroups of \mathbb{Z}_p have finite index in \mathbb{Z}_p (see for example [8, Proposition and Corollary 1 at p. 23]).

Lemma 3.6. *Let G be a pro- p group and K a normal infinite procyclic subgroup of G . If $a \notin C_G(K)$, then for any $i \geq 1$, the subgroup $[K, {}_i a]$ has finite index in K . In particular, if G is locally nilpotent, then K is central in G .*

PROOF. Let α be the automorphism of K induced by the conjugation by the element a . Write R for the ring of the p -adic integers and regard K as the additive group of R . There exists $b \in R$ such that $x^a = x \cdot b$, for each $x \in K$. Note that the subgroup $[K, {}_n a]$ consists of elements of the form $x \cdot (b - 1)^n$, where x ranges over K . Moreover the set $\{x \cdot (b - 1)^n \mid x \in K\}$ is infinite for each $n \geq 1$, since R has no zero divisors. The lemma follows. \square

We now can prove Theorem 1.3 in the particular case where G is metabelian. The general case will require considerably more efforts.

Proposition 3.7. *Let G be a metabelian pro- p group such that $\mathcal{E}(g)$ can be chosen procyclic for each g in G . Then G has a normal procyclic subgroup K such that G/K is locally nilpotent.*

PROOF. If G is Engel, then it is locally nilpotent and there is nothing to prove. Assume that G is not Engel and let X be the set of all non-Engel elements in G . In view of Lemmas 3.4 and 3.5 we can assume that all $\mathcal{E}(g)$ are chosen procyclic and normal in G . Indeed, by Lemmas 3.4 and 3.5, for each g in G there exists an integer n such that $E_n(g)$ is a normal subgroup in $\mathcal{E}(g)$, so we can take such $E_n(g)$ as the sink $\mathcal{E}(g)$ of g . In view of Lemma 3.6 each subgroup $[\mathcal{E}(x), {}_i x]$ has finite index in $\mathcal{E}(x)$ whenever $x \in X$.

Given $a, b \in X$, suppose first that $\mathcal{E}(a) \cap \mathcal{E}(b) = 1$. On the one hand, a acts on $\mathcal{E}(a)$ in such a way that, for any $i \geq 1$, the subgroup $[\mathcal{E}(a), {}_i a]$ has finite index in $\mathcal{E}(a)$. On the other hand, a centralizes $\mathcal{E}(b)$, since the intersection of the two sinks is trivial. A similar remarks applies to b . Note that ab acts on $\mathcal{E}(a) \oplus \mathcal{E}(b)$ in the following way: it acts as the element a on $\mathcal{E}(a)$ and as b on $\mathcal{E}(b)$. This implies that, for any n , the subgroup $E_n(ab)$ contains a subgroup, which is the direct sum

of a finite index subgroup in $\mathcal{E}(a)$ and a finite index subgroup in $\mathcal{E}(b)$, isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Thus $\mathcal{E}(ab)$ is not procyclic, a contradiction.

Hence, $\mathcal{E}(a) \cap \mathcal{E}(b) \neq 1$, for any $a, b \in X$. Let $K = \mathcal{E}(a)$ for some $a \in X$. We see that, for any $g \in G$, the image in G/K of the sink $\mathcal{E}(g)$ is finite. Indeed, if g is Engel in G , then the claim is obvious. Otherwise $g \in X$ and the image of $\mathcal{E}(g)$ in G/K is isomorphic to $\mathcal{E}(g)/(\mathcal{E}(a) \cap \mathcal{E}(g))$ which is finite. It follows that G/K is Engel, hence locally nilpotent by the Wilson-Zelmanov theorem. The proof is complete. \square

Next, we consider another particular case of Theorem 1.3.

Lemma 3.8. *Let G be a soluble pro- p group such that $\mathcal{E}(g)$ can be chosen procyclic for each g in G . Assume that G has a normal nilpotent subgroup M and $a \in G$ such that $G = M\langle a \rangle$. Then there exists n such that $E_n(a)$ is procyclic and normal in G . Moreover there exists i such that $\gamma_i(G)$ has finite index in $E_n(a)$.*

PROOF. We argue by induction on the nilpotency class of M . If M is abelian, then in view of Lemma 3.5 there exists n such that $E_n(a)$ is procyclic and normal in G . Since $G/E_n(a)$ is nilpotent, the result holds. Suppose that M is nonabelian and set $Z = Z(M)$. By induction assume that there is n such that $L = ZE_n(a)$ is normal in G and L/Z is procyclic. Since L/Z is procyclic, the subgroup L is abelian. Now looking at the action of $\langle a \rangle$ on L and using the fact that L is abelian, Lemma 3.4 shows that if j is big enough, then the subgroup $E_{n+j}(a)$ is procyclic. By Lemma 3.2 there exists f such that $\gamma_f(G) \leq E_{n+j}(a)$. Thus, $\gamma_f(G)$ is a normal procyclic subgroup and so all subgroups of $\gamma_f(G)$ are normal in G . In particular, $E_f(a)$ is normal and procyclic. Since by Lemma 3.1 the factor-group $G/E_f(a)$ is nilpotent, there exists i such that $\gamma_i(G) \leq E_f(a)$. This completes the proof. \square

In the sequel we will use, without mentioning explicitly, the following fact: let H be a subgroup of a profinite group G and let x be an element of G such that $H^x \leq H$. Then $H^x = H$. This is because if

$H^x < H$, then the inequality would also hold in some finite image of G , which yields a contradiction.

The next result is a key observation that will be applied many times throughout the proof of the main result.

Lemma 3.9. *Let G be a profinite group and K a procyclic pro- p subgroup of G such that $K \cap K^x \neq 1$ for each $x \in G$. Then K contains a nontrivial subgroup L (of finite index) which is normal in G .*

PROOF. If K is finite, the result is obvious, so we assume that K is infinite. Recall that $K \cap K^x$ has finite index in K for each $x \in G$. For each i set

$$S_i = \{x \in G \mid K \cap K^x \text{ has index at most } p^i \text{ in } K\}.$$

The sets S_i are closed. By Baire Category Theorem at least one of these sets has non-empty interior. Therefore there is an open normal subgroup N , an element $d \in G$, and a fixed p -power p^i such that $K \cap K^x$ has index at most p^i in K for every $x \in dN$. Let $K_0 = K^{p^i}$ be the subgroup of index p^i in K . We see that N normalizes K_0 . Since N is open, it follows that K_0 has only finitely many conjugates in G . Let L be their intersection. Obviously, L is normal in G . Since $K_0 \cap K_0^x$ has finite index in K_0 for each $x \in G$, the subgroup L is nontrivial. \square

Now we are ready to deal with the proof of Theorem 1.3. We want to establish that if G is a soluble pro- p group such that $\mathcal{E}(g)$ can be chosen procyclic for each g in G , then G has a normal procyclic subgroup K such that G/K is locally nilpotent.

PROOF OF THEOREM 1.3. The argument will be by induction on the derived length of G . Set $H = G'$. By induction, H has a normal procyclic subgroup K such that H/K is locally nilpotent.

CLAIM 1. H is locally nilpotent.

If K is finite, the claim holds. So we assume that K is infinite. It is sufficient to show that H has a procyclic subgroup K_0 , which is normal

in G , such that H/K_0 is locally nilpotent. Indeed, once the existence of such subgroup K_0 is established, observe that $K_0 \leq Z(H)$ because $G/C_G(K_0)$ embeds into $\text{Aut}(\mathbb{Z}_p)$ which is abelian. Hence H is locally nilpotent. Thus, assume that K is not normal in G .

For any $x \in G$ the quotient H/K^x is locally nilpotent. If there exists x such that $K^x \cap K = 1$, then H , being isomorphic to a subgroup $H/K \times H/K^x$, must be locally nilpotent, as desired. Therefore we will assume that $K^x \cap K \neq 1$ for any $x \in G$.

In view of Lemma 3.9 K contains a nontrivial subgroup L which is normal in G . Since H/K is locally nilpotent and L has finite index in K , it follows that H/L is locally nilpotent too. Moreover, since L is normal in G , it follows that L is in the center of H and so H is locally nilpotent. This establishes Claim 1.

CLAIM 2. Assume that G has a normal nilpotent subgroup M such that G/M is nilpotent and finitely generated. Then G has a normal procyclic subgroup M_0 such that G/M_0 is nilpotent.

Indeed, choose a_1, \dots, a_s in G such that $G = \langle M, a_1, \dots, a_s \rangle$. We argue by induction on the nilpotency class of G/M and also use induction on s .

Assume first that G/M is abelian. The case $s = 1$ follows from Lemma 3.8 so suppose that $s \geq 2$. Let $V_j = M\langle a_j \rangle$, for $1 \leq j \leq s$. Observe that for each j the subgroup V_j is normal in G and, in view of Lemma 3.8, there exists $i(j)$ such that $\gamma_{i(j)}(V_j)$ is procyclic. If for any j the subgroup $\gamma_{i(j)}(V_j)$ is finite (or trivial), then each V_j is nilpotent and so G is nilpotent too. Thus we can assume that some $\gamma_{i(j)}(V_j)$ are procyclic infinite. Moreover, if for some j and k the subgroups $\gamma_{i(j)}(V_j)$ and $\gamma_{i(k)}(V_k)$ are infinite and satisfy $\gamma_{i(j)}(V_j) \cap \gamma_{i(k)}(V_k) = 1$, then we get a contradiction. Indeed, set $N = \gamma_{i(j)}(V_j) \oplus \gamma_{i(k)}(V_k)$ and consider the action of $a_j a_k$ on N . Arguing as in the proof of Proposition 3.7, we see that $\mathcal{E}(a_j a_k)$ is not procyclic, since for any n the subgroup $E_n(a_j a_k)$ contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. We therefore assume that all infinite subgroups $\gamma_{i(j)}(V_j)$ intersect pairwise nontrivially. In particular

their intersection V is a nontrivial normal procyclic subgroup such that G/V is nilpotent. This concludes the argument in the case where G/M is abelian.

Next, suppose that G/M has nilpotency class at least two, so in particular s is bigger than one. Let $W_j = \langle a_j \rangle HM$, for $1 \leq j \leq s$. Note that any subgroup W_j modulo M is a finitely generated subgroup, since it is a subgroup of a finitely generated nilpotent group. Furthermore W_j modulo M has nilpotency class smaller than the nilpotency class of G/M , since it is generated by the image of H and a_j . Thus, by induction, any W_j has a normal procyclic subgroup B_j such that W_j/B_j is nilpotent. So, there exists $l(j)$ such that $\gamma_{l(j)}(W_j) \leq B_j$. As in the previous paragraph, if all B_j are finite (or trivial), then G is nilpotent. If the infinite B_j intersect nontrivially, then the claim follows since their intersection B is a nontrivial normal procyclic subgroup such that G/B is nilpotent. Suppose that for some i, j the subgroups B_i and B_j are infinite and $B_i \cap B_j = 1$. Note that Claim 1 implies that both B_i and B_j are centralized by H . Set $N = B_i \oplus B_j$ and look at the action of $a_i a_j$ on N . We see that for any n the subgroup $E_n(a_i a_j)$ contains a subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Thus $\mathcal{E}(a_i a_j)$ is not procyclic, a contradiction. This concludes the proof of Claim 2.

Let R be the last nontrivial term of the derived series of G . By induction on the derived length of G assume that for G/R the theorem holds. Thus G has a normal subgroup S , containing R , such that S/R is procyclic and G/S is locally nilpotent. Obviously, we can choose S in such way that $S \leq H$. Let $a \in S$ such that $S = R\langle a \rangle$. In view of Claim 1, a is an Engel element. Thus applying Lemma 3.3 we deduce that S is nilpotent.

CLAIM 3. Let $a_1, \dots, a_s \in G$ and set $J = R\langle a_1, \dots, a_s \rangle$. Then J has a normal procyclic subgroup J_0 such that J/J_0 is nilpotent.

If J/R is nilpotent, then the claim follows from Claim 2. Assume that J/R is not nilpotent. Set $J_1 = \langle J, a \rangle$, where a is as above. Note

that $S \leq J_1$ and J_1/S is nilpotent since G/S is locally nilpotent. Hence, again by Claim 2 there exists a normal procyclic subgroup N_0 in J_1 such that J_1/N_0 is nilpotent. In particular JN_0/N_0 is nilpotent too, so we can take $J_0 = J \cap N_0$. This concludes the proof of Claim 3.

We now embark on the final part of the proof of the theorem. Assume that the group G is not locally nilpotent. Choose elements $a_1, \dots, a_s \in G$ such that $T = \langle a_1, \dots, a_s \rangle$ is not nilpotent. Recall that S is a nilpotent normal subgroup of G such that G/S is locally nilpotent. By Claim 2 the group ST has a normal procyclic subgroup K_0 such that ST/K_0 is nilpotent. Without loss of generality we assume that there is a positive integer i_0 such that $K_0 = \gamma_{i_0}(ST)$. Note that K_0 here must be infinite since T is not nilpotent. Moreover we can replace K_0 by $S \cap K_0$ and simply assume that $K_0 \leq S$. Indeed, since ST/K_0 and ST/S are both nilpotent, we have $\gamma_i(ST) \leq S \cap K_0$, for some positive integer i .

Given any finite subset Y of G , we write T_Y for the subgroup $\langle Y, T \rangle$. By Claim 2 the group ST_Y has a normal procyclic subgroup K_Y such that ST_Y/K_Y is nilpotent. Again there is a positive integer i_Y such that $K_Y = \gamma_{i_Y}(ST_Y)$. Note that all subgroups K_Y are infinite and have infinite intersection with K_0 . Indeed, any subgroup ST_Y contains ST , the subgroup ST is nilpotent modulo the intersection of K_Y with K_0 , so if this intersection were trivial, then ST_Y would be nilpotent, a contradiction. As before, since G/S is locally nilpotent, we choose all K_Y inside S .

Now choose an arbitrary element $x \in G$ and set

$$Y(x) = \{a_1^x, \dots, a_s^x, a_1, \dots, a_s\}.$$

We see that $K_{Y(x)}$ has infinite intersection with each of the subgroups K_0 and K_0^x . Hence $K_0 \cap K_0^x$ is nontrivial and this holds for any choice of $x \in G$. Thus, by Lemma 3.9, K_0 contains a nontrivial subgroup L_0 which is normal in G .

Note that for any choice of a finite subset Y of G , the subgroup L_0 intersects K_Y by a finite index subgroup, since K_0 intersects K_Y

nontrivially and L_0 has finite index in K_0 . Therefore every subgroup T_Y is nilpotent modulo L_0 , since K_Y becomes finite modulo L_0 . Hence G is locally nilpotent modulo L_0 and this concludes the proof. \square

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