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ANALYTICAL MODELING OF STRESS SHIELDING IN A PIN-LOADED HOLE WITH RADIAL CLEARANCE REINFORCED BY A FGM RING

E. Radi[1,2]

[1] *DISMI - Dipartimento di Scienze e Metodi dell'Ingegneria, Università di Modena e Reggio Emilia, Via G. Amendola 2, I-42122 Reggio Emilia, Italy. Email: enrico.radi@unimore.it*

[2] *En&Tech - Centro di Ricerca Interdipartimentale per la Ricerca Industriale ed il Trasferimento Tecnologico, 42122 Reggio Emilia, Italy*

Abstract. A novel analytical formulation is developed for frictionless contact between a loaded rigid pin and a circular hole with radial clearance, reinforced by a functionally graded ring embedded in an infinite isotropic elastic medium. The ring shear modulus follows a radial power law and matches that of the surrounding medium at the outer interface, while Poisson's ratio is uniform. Michell-type expansions and the contact and interface conditions reduce the problem to dual trigonometric series and an infinite linear algebraic system, solved by truncation. An inverse procedure prescribes the contact half-angle and determines the associated pressure distribution and resultant pin load. The formulation quantifies, for the first time within this class of pin-hole contact problems, how radial grading and ring thickness modify the nonlinear load-contact-angle relation and redistribute stresses within the reinforcement and surrounding solid. Suitable grading can reduce stress concentrations in selected regions while maintaining a continuous elastic-modulus profile across the outer interface. Comparisons with finite element results and a generalized *M*-integral provide complementary checks of the solution. The formulation supplies a benchmark for clearance-fit pinned joints with graded reinforcement and a basis for selecting stiffness profiles to control elastic stress concentrations. Assessing improvements in load-bearing capacity additionally requires information on the spatial variation of material strength.

Keywords: Functionally Graded Material, Advancing contact, Analytical solution, Clearance, Stress shielding, Optimal design.

1. Introduction

In a pinned connection with radial clearance, stress concentrations typically take place along the contact region, potentially compromising the integrity of the joint, especially for brittle materials. To enhance the wear resistance and fatigue strength of bearing components, such as cylindrical joints and rivet holes, a common engineering practice consists of surface hardening the material surrounding the hole (Mencik, 2013) by techniques such as carburizing, nitriding, or induction hardening (Osborn,

1941; Aswad et al., 2021). These treatments are intended to produce a hard surface layer while preserving the ductility of the underlying material, thereby improving wear resistance and durability under high loads, reducing friction, and hindering crack initiation and propagation under cyclic loading. However, surface-hardening processes may also generate defects and residual stresses that can adversely affect fatigue performance (Persson et al., 1997).

An attempt to model the increase in stiffness in the vicinity of a pin-loaded hole induced by a surface-hardening treatment was recently proposed by Radi (2027), within the framework of the surface elasticity theory developed by Gurtin and Murdoch (1978). It was shown that a thin hardened surface layer can significantly alter the contact-pressure distribution transmitted across the pin-hole interface and mitigate the hoop-stress concentration around the hole. An alternative strategy for increasing the load-bearing capacity and durability of cylindrical joints is to reinforce the hole with a ring made of a high-strength material and placed in direct contact with the pin (Chau and Wei, 2001; Akbarpour and Hallström, 2019). A drawback of conventional multi-material reinforcement, however, is the abrupt mismatch of elastic properties at material interfaces, which may generate high interfacial shear stresses and promote debonding.

Functionally graded materials (FGMs) provide a natural means of overcoming this limitation because their mechanical properties can vary continuously in space rather than changing abruptly across an interface. Their composition may gradually evolve from a hard to a compliant phase, from a metallic to a ceramic phase, or through more general combinations of constituent materials. In bearing applications, such grading makes it possible to tailor hardness, wear resistance, stiffness, and toughness throughout a finite region surrounding the contact surface rather than confining the modification to a very thin hardened layer. The controlled variation of constituent fractions therefore offers considerable freedom for developing novel structural concepts (Zhang et al., 2019; Kumar et al., 2023; Zhang et al., 2018). For example, incorporating ceramic-rich phases near the contact region can improve wear resistance, while a gradual transition toward the properties of the surrounding material can smooth the stress and strain fields and reduce the tendency toward crack initiation and propagation compared with conventional layered or surface-hardened designs (Giannakopoulos and Suresh, 1997a,b; Ciavarella, 2024; Çömez, 2015). In the specific context of a pin-loaded hole, an FGM reinforcing ring whose stiffness progressively approaches that of the surrounding elastic solid may therefore combine the local load-bearing capability of a stiff insert with a smooth transfer of stresses into the host material. This mechanism is here referred to as *stress shielding*. Such a graded reinforcement may improve fatigue resistance and durability under cyclic loading (Suresh et al., 1999; Suresh, 2001). These advantages must nevertheless be balanced against the greater manufacturing complexity and cost associated with producing prescribed spatial gradients of material properties.

Recent developments in additive and advanced manufacturing, including powder-bed fusion, material extrusion, and sheet lamination, have substantially increased the feasibility of fabricating FGMs with continuously varying elastic properties, for instance through the controlled blending of different constituent materials (Alkunte et al., 2024). Radial inhomogeneity may arise through directional cooling and the associated microstructural gradients (Mohammadi and Dryden, 2008), phase segregation during centrifugal casting (Ebhota et al., 2016; Pradeep and Rameshkumar, 2021), or laser-based surface modification (Bi et al., 2013). Radially graded elastic solids have been extensively investigated in axisymmetric configurations, including tubes, rings, and disks (Tutuncu, 2007; Abdalla, 2023), as well as curved beams (Mohammadi and Dryden, 2008; Arslan and Haskul, 2015; Eroglu, 2016; Pei and Li, 2021). A number of analytical investigations have also addressed non-axisymmetric loading of radially inhomogeneous solids, including the works of Li and Liu (2014), Arasan et al. (2022), Batra and Nie (2010), Nie and Batra (2010), Vasu and Bhandakkar (2018), Guleid et al. (2022), and Chawde and Bhandakkar (2021).

The mechanics of a loaded pinned joint constitutes a classical non-axisymmetric contact problem and is of broad relevance to articulated mechanical connections. In practical joints, the pin is often considerably stiffer than the material surrounding the hole and the hole radius is slightly larger than the pin radius to allow assembly and relative rotation. This radial clearance fundamentally changes the contact mechanics: contact is restricted to an unknown portion of the hole boundary, and the extent of the contact zone increases non-linearly with the applied pin load. Excessive clearance may consequently promote high local contact stresses, wear, fatigue damage, and ultimately joint failure. From a mathematical viewpoint, the problem is a mixed boundary-value problem with an unknown moving contact boundary, and its treatment generally requires either advanced numerical methods (Liu et al., 2007; Duc, 2025; Li et al., 2022) or the solution of dual series/integral equations. Closed-form or semi-analytical developments commonly assume frictionless contact (Persson, 1964; Noble and Hussain, 1969; Rao et al., 1977; Rao, 1978; Ho and Chau, 1997; Ciavarella and Decuzzi, 2001a,b; Hou and Hills, 2001), whereas the inclusion of friction generally requires a numerical treatment (Iyer, 2001; Sundaram and Farris, 2010a,b; Duc, 2025). A further simplification, appropriate when the pin stiffness is much larger than that of the surrounding material, is to model the pin as rigid (de Jong, 1977; Mizushima and Hamada, 1983; Sundaram and Farris, 2010b; Nguyen et al., 2025).

The possibility of reducing the stress concentration factor (SCF) around a hole by means of material grading was first explored by Kubair and Bhanu-Chandar (2008), who performed a finite-element analysis of an FGM plate containing a circular hole and subjected to remote uniaxial tension. They showed that the SCF can be reduced by progressively increasing Young's modulus away from the hole according to exponential or power-law grading. Mohammadi et al. (2011) subsequently derived

an analytical solution for a circular hole in an infinite FGM plane subjected to remote biaxial tension and pure shear, assuming exponentially varying elastic properties. The beneficial effect of an FGM annular reinforcement around a hole in an otherwise homogeneous infinite plate under remote uniaxial tension was investigated analytically by Sburlati (2013) and Sburlati et al. (2014). Nie et al. (2018) and Nie and Batra (2019) considered panels in which the hole was lined with compressible and incompressible FGM rings, respectively, and showed that appropriate radial grading can markedly reduce both the contact pressure and the maximum hoop stress in the surrounding panel. Similar conclusions were reported by Andreev and Cybin (2014). Abdalla et al. (2023) investigated linear radial variations of Young's modulus, while Goyat et al. (2019) compared several grading laws using the extended finite-element method and found power-law grading particularly effective in reducing the SCF. More recently, Hart and Terokhin (2025a,b) numerically investigated a non-monotonic tri-linear variation of the elastic modulus in annular graded inclusions and reported reductions in the SCF exceeding 50%. Optimization of the radial variation has also been considered; for example, Abdalla et al. (2024) showed that a sigmoidal distribution of Young's modulus may be selected to minimize the hoop stress along prescribed radial directions.

Among the various descriptions of radial grading, the power-law is especially attractive for analytical modelling because it provides sufficient flexibility to represent non-linear variations while retaining a comparatively tractable mathematical structure (Zhou et al., 2021). The unit grading exponent recovers a linear variation as a special case (Abdalla et al., 2014), whereas other exponent values allow the stiffness gradient to be concentrated either near the hole or toward the outer boundary of the graded region. Exponential and sigmoidal laws provide additional flexibility and may better represent special manufacturing processes, although they usually lead to more involved analytical formulations. An alternative is to approximate an arbitrary grading profile by a sequence of sublayers in which the material properties vary linearly (Ke and Wang, 2006; Alinia et al., 2019). Since the present work is primarily concerned with obtaining an analytical solution and identifying the mechanical parameters controlling stress shielding, a power-law variation provides a useful compromise between physical flexibility and mathematical tractability.

The use of FGM interlayers in pin-loaded configurations has received considerably less attention. Bahri et al. (2017) examined the reduction of the SCF in a perfectly fitted pin-hole connection by introducing an FGM interlayer between the pin and the plate. Because the stress concentration is largely driven by the stiffness mismatch between the pin and the surrounding solid, they showed that a radially graded intermediate layer can smooth this mismatch and reduce the SCF. More recently, Radi et al. (2025) obtained an analytical solution for frictionless clearance contact between a rigid circular pin and a circular FGM lug with power-law grading. Their results showed that the Hertzian

contact-pressure profile is recovered only for sufficiently small loads and narrow contact regions, whereas at higher loads the pressure becomes nearly uniform over the central portion of the contact zone and develops two local maxima near its ends. The amplitude of these pressure peaks decreases with increasing lug thickness, and the pressure distribution progressively approaches a Hertzian-like form for sufficiently thick lugs. That study also demonstrated that increasing both stiffness and strength near the pin-contacting surface can enhance the load-bearing capacity of a graded lug.

Despite these developments, the combined effects of radial clearance, an embedded FGM reinforcing ring, and the surrounding homogeneous elastic medium have not yet been addressed analytically. To the best of the author's knowledge, no analytical solution is presently available for the advancing frictionless contact between a loaded rigid pin and a circular hole reinforced by an FGM ring embedded in an infinite elastic plane.

The objective of this study is to determine how radial stiffness grading in an embedded reinforcing ring modifies the contact extent and redistributes stresses in a clearance-fit pinned joint. The formulation relates the grading index and ring geometry to the contact pressure and the stress fields in the reinforcement and surrounding elastic medium. It therefore provides a basis for selecting stiffness profiles that reduce elastic stress concentrations. Assessment of the corresponding strength or fatigue benefits requires additional material data.

The stress and displacement fields in both the homogeneous plane and the FGM ring are represented through Michell-type series. For the graded annulus, the fields are constructed using the power-law solution developed by Arasan et al. (2022). The frictionless contact conditions lead to a system of dual trigonometric series equations. Following the methodology introduced by Noble and Hussain (1969) and subsequently employed by Block and Keer (2007), Radi and Strozzi (2023), Radi et al. (2025), and Radi and Güler (2026a,b), the dual-series problem is transformed into an infinite linear algebraic system for the unknown coefficients of the Michell-type expansions and solved by controlled truncation. The influence of the loading level, ring geometry, and material-grading parameters on the contact angle, contact-pressure distribution, and stress fields in both the FGM ring and the surrounding homogeneous solid is then investigated. Particular emphasis is placed on identifying combinations of ring thickness and grading exponent that reduce the maximum circumferential stress transmitted to the host material without generating excessive stress within the FGM reinforcement. A robust consistency check of the analytical solution is obtained through an extended M-integral formulation. Because the classical M-integral is path dependent in an inhomogeneous elastic medium, an additional area integral is derived using the Gauss-Green theorem so that their sum remains invariant for closed paths surrounding the hole. Independent validation against finite-element predictions is additionally provided in the Supplementary Material.

2. Problem formulation

The problem of a loaded rigid pin in frictionless quasi-conformal contact with an FGM ring embedded in an infinite isotropic elastic plane, illustrated in Fig. 1, is considered. The inner and outer radii of the FGM ring are denoted by r_i and r_o, respectively. The elastic shear modulus μ of the FGM ring displays a power-law radial variation according to the relation

$$\mu = \mu_0 \, (r/r_o)^m, \tag{2.1}$$

where μ_0 is the shear modulus of the homogeneous elastic plane and m is the grading index.

To perform a fully analytical investigation under plane stress or plane strain loading conditions, we assume that the FGM ring and the elastic plane have the same constant Poisson's ratio ν. We consider a small radial clearance $\delta = r_i - r_p$ between the rigid pin and the FGM ring, where r_p is the pin radius, thus leading to an advancing contact problem. Indeed, the contact length between the two components increases as the load P applied to the pin increases. The variation of the shear modulus along the radial direction is sketched in Fig. 2 for specific positive and negative values of the grading index m.

A polar coordinate system (O, r, θ) is considered with the origin O coinciding with the centre of the FGM ring and the axis $\theta = 0$ aligned with the applied load P (Fig. 1).

2.1. Stress and displacement fields within the FGM ring

The cylindrical components of the stress and displacement fields within the FGM ring, containing only symmetric, single-valued fields in θ, follow from the solution to two-dimensional elastic problems involving graded materials in radial coordinates presented by Arasan et al. (2022):

$$\frac{\sigma_{rr}(r,\theta)}{2\mu_0} = \frac{\delta}{r_o}\left\{\sum_{i=1}^{2} B_{0i}\left(\frac{r}{r_o}\right)^{a_i} + \left[\sum_{i=1}^{2} B_{1i}\left(\frac{r}{r_o}\right)^{b_{1i}} + C\frac{r_o}{r}\right]\cos\theta + \sum_{n=2}^{\infty}\sum_{i=1}^{4} B_{ni}\left(\frac{r}{r_o}\right)^{b_{ni}}\cos n\theta\right\},$$

$$\frac{\sigma_{\theta\theta}(r,\theta)}{2\mu_0} = \frac{\delta}{r_o}\left\{\sum_{i=1}^{2} B_{0i}(a_i+1)\left(\frac{r}{r_o}\right)^{a_i} + \left[\sum_{i=1}^{2} B_{1i}\left(\frac{r}{r_o}\right)^{b_{1i}}(b_{1i}+2) + C\,\Delta\frac{r_o}{r}\right]\cos\theta + \sum_{n=2}^{\infty}\sum_{i=1}^{4} B_{ni}\left(\frac{r}{r_o}\right)^{b_{ni}}(b_{ni}+2)\,\Delta_{ni}\cos n\theta\right\}, \tag{2.2}$$

$$\frac{\sigma_{r\theta}(r,\theta)}{2\mu_0} = \frac{\delta}{r_o}\left\{\left[\sum_{i=1}^{2} B_{1i}\left(\frac{r}{r_o}\right)^{b_{1i}} + C\,\Delta\frac{r_o}{r}\right]\sin\theta + \sum_{n=2}^{\infty}\sum_{i=1}^{4} B_{ni}\left(\frac{r}{r_o}\right)^{b_{ni}} n\,\Delta_{ni}\sin n\theta\right\},$$

193
$$\frac{u_r(r,\theta)}{\delta}=\left\{\sum_{i=1}^{2}B_{0i}\,\kappa_i\left(\frac{r}{r_o}\right)^{a_i+1}+\left[\sum_{i=1}^{2}\frac{B_{1i}\,\Gamma_{1i}}{2(1-\rho^2)}\left(\frac{r}{r_o}\right)^{b_{1i}+1}-2C\frac{1+\rho^2+m\rho^2}{m(m+4-2\rho^2)}\right]\cos\theta\right.$$
$$\left.+\sum_{n=2}^{\infty}\sum_{i=1}^{4}B_{ni}\left(\frac{r}{r_o}\right)^{b_{ni}+1}\frac{\chi_{1ni}\cos n\theta}{2(1-\rho^2)}\right\}\left(\frac{r}{r_o}\right)^{-m},$$

194
$$\frac{u_\theta(r,\theta)}{\delta}=\left\{\left[\sum_{i=1}^{2}\frac{B_{1i}\,\Gamma_{2i}}{2(1-\rho^2)}\left(\frac{r}{r_o}\right)^{b_{1i}+1}+2C\frac{1+\rho^2-m+2m\rho^2}{m(m+4-2\rho^2)}\right]\sin\theta\right.$$
$$\left.+\sum_{n=2}^{\infty}\sum_{i=1}^{4}B_{ni}\left(\frac{r}{r_o}\right)^{b_{ni}+1}\frac{\chi_{2ni}\sin n\theta}{2n(1-\rho^2)}\right\}\left(\frac{r}{r_o}\right)^{-m},$$

195 In Eq. (2.2), B_{0i} and B_{1i} (for i = 1, 2), and B_{ni} (for i = 1, 2, 3, 4 and $n \geq 2$) are arbitrary constants to be
196 determined from the boundary conditions. Moreover:

197 $$a_{1,2}=\frac{m}{2}-1\pm\sqrt{\frac{m^2}{4}+2m\rho^2+1-m},$$

198 $$b_{11,12}=\frac{m}{2}-1\pm\sqrt{\frac{m^2}{4}+2m\rho^2+4},$$

199 $$\Delta=\frac{m-2\rho^2(m+1)}{m+4-2\rho^2},$$

$$\kappa_i=\frac{1-(1-2\rho^2)(a_i+1)}{2(1-\rho^2)(a_i-m+1)},$$

200 $$\Gamma_{1i}=\frac{1-(b_{1i}+2)(1-2\rho^2)}{b_{1i}-m+1}, \qquad (i = 1, 2)$$

$$\Gamma_{2i}=b_{1i}+1+2\rho^2-\Gamma_{1i},$$

201 $$\Delta_{ni}=\frac{b_{ni}+1}{b_{ni}-n^2+2}, \qquad (2.3)$$

202 $$b_{n1,n2}=\frac{m}{2}-1\pm\sqrt{l_{n1}+\sqrt{l_{n2}}},$$

203 $$b_{n3,n4}=\frac{m}{2}-1\pm\sqrt{l_{n1}-\sqrt{l_{n2}}},$$

204 $$l_{n1}=n^2+\frac{m^2}{4}+m\rho^2+1,$$

205 $$l_{n2}=n^2(2+m\rho^2)^2-m^2(1-\rho^2)^2(n^2-1),$$

206 $$\chi_{1ni} = \frac{1-(b_{ni}+2)(1-2\rho^2)\Delta_{ni}}{b_{ni}-m+1},$$
$$\chi_{2ni} = (b_{ni}+2)\Delta_{ni} - 1 + 2\rho^2 - \chi_{1ni}, \qquad (n = 2,3,...; i = 1,...,4)$$

207 where

208 $$\rho = \sqrt{\frac{1-2\overline{\nu}}{2(1-\overline{\nu})}}, \qquad \overline{\nu} = \begin{cases} \nu & \text{for plane strain,} \\ \dfrac{\nu}{1+\nu} & \text{for plane stress.} \end{cases} \tag{2.4}$$

209 The representation requires limiting treatment when characteristic roots coincide or a denominator
210 vanishes, as discussed in Section 2.2.2 of Arasan et al. (2022). The homogeneous case corresponds
211 to $m = 0$, while $m = -2/\rho^2$ gives $\Delta = 1$. Moreover, Δ has a pole at $m = 2\rho^2 - 4$, whereas $\Delta = 0$ at $m = 2\rho^2/(1-2\rho^2)$. Complex roots may occur for $l_{n2} < 0$. As advised by Arasan et al. (2022), "*these solutions could be obtained by approaching the corresponding values in the standard solution, and no explicit solutions in these special cases are required*". In particular, when $m = 0$, the ring and surrounding plane have identical elastic properties, so the interface at the outer ring radius is fictitious. The homogeneous solution provided by Michell (1899) is recovered from Eqs. (2.2) for any value of λ. The displacement limit is taken after fixing the rigid-translation freedom, noting that the logarithmic first harmonic follows from the limit $[1-(r/r_o)^m]/m \to \ln(r/r_o)$, as $m \to 0$. Note also that the expression of χ_{2ni} reported by Arasan et al. (2022) contains a typo.

220

221 *2.2. Stress and displacement fields in the outer elastic homogeneous region*

222 The general expressions of the stress and displacement fields in the elastic homogeneous region were
223 given by Omar and Hassan (1991) and Singh and Bhandakkar (2019). Considering only the stress
224 terms vanishing at infinity, one has:

225 $$\frac{\sigma^0_{rr}(r,\theta)}{2\mu_0} = \frac{\delta}{r_o}\left\{ D_0\left(\frac{r_o}{r}\right)^2 + \left[D_1(2-\rho^2)\frac{r_o}{r} - 2C_1\left(\frac{r_o}{r}\right)^3 \right]\cos\theta - \sum_{n=2}^{\infty}\left[C_n\, n(n+1)\left(\frac{r_o}{r}\right)^{n+2} + D_n\,(n+2)(n-1)\left(\frac{r_o}{r}\right)^n \right]\cos n\theta \right\},$$

$$\frac{\sigma^0_{\theta\theta}(r,\theta)}{2\mu_0} = \frac{\delta}{r_o}\left\{ -D_0\left(\frac{r_o}{r}\right)^2 - \left[D_1\,\rho^2\frac{r_o}{r} - 2C_1\left(\frac{r_o}{r}\right)^3 \right]\cos\theta + \sum_{n=2}^{\infty}\left[C_n\, n(n+1)\left(\frac{r_o}{r}\right)^{n+2} + D_n\,(n-2)(n-1)\left(\frac{r_o}{r}\right)^n \right]\cos n\theta \right\},$$

$$\frac{\sigma_{r\theta}^{0}(r,\theta)}{2\mu_0}=\frac{\delta}{r_o}\left\{-\left[D_1\rho^2\frac{r_o}{r}+2C_1\left(\frac{r_o}{r}\right)^3\right]\sin\theta\right.$$
$$\left.-\sum_{n=2}^{\infty}\left[C_n(n+1)\left(\frac{r_o}{r}\right)^{n+2}+D_n(n-1)\left(\frac{r_o}{r}\right)^{n}\right]n\sin n\theta\right\},$$

$$\frac{u_r^{0}(r,\theta)}{\delta}=-D_0\frac{r_o}{r}+\left\{C_1\left(\frac{r_o}{r}\right)^2+V-D_1\left[(1+\rho^2)\ln\frac{r_o}{r}+\frac{1-\rho^2}{2}\right]\right\}\cos\theta$$
$$+\sum_{n=2}^{\infty}\left[C_n n\left(\frac{r_o}{r}\right)^{n+1}+D_n\left(\frac{2\rho^2}{1-\rho^2}+n\right)\left(\frac{r_o}{r}\right)^{n-1}\right]\cos n\theta,$$

$$\frac{u_\theta^{0}(r,\theta)}{\delta}=\left\{C_1\left(\frac{r_o}{r}\right)^2-V+D_1\left[(1+\rho^2)\ln\frac{r_o}{r}-\frac{1-\rho^2}{2}\right]\right\}\sin\theta$$
$$+\sum_{n=2}^{\infty}\left[C_n n\left(\frac{r_o}{r}\right)^{n+1}-D_n\left(\frac{2}{1-\rho^2}-n\right)\left(\frac{r_o}{r}\right)^{n-1}\right]\sin n\theta.$$

(2.5)

2.3. Imposition of the boundary conditions

The continuity conditions between the elastic homogeneous plane and the outer surface of the FGM ring, at $r = r_o$, require

$$\begin{aligned}&\sigma_{rr}^{0}(r_o,\theta)=\sigma_{rr}(r_o,\theta),\\&\sigma_{r\theta}^{0}(r_o,\theta)=\sigma_{r\theta}(r_o,\theta),\qquad \text{for}\quad 0\le|\theta|\le\pi\\&u_r^{0}(r_o,\theta)=u_r(r_o,\theta),\\&u_\theta^{0}(r_o,\theta)=u_\theta(r_o,\theta),\end{aligned}\quad, \tag{2.6}$$

and the frictionless contact conditions between the rigid pin and the inner surface of the FGM ring, at $r = r_i$ require

$$\begin{aligned}&\sigma_{r\theta}(r_i,\theta)=0,\qquad \text{for}\quad 0\le|\theta|\le\pi,\\&\sigma_{rr}(r_i,\theta)=0,\qquad \text{for}\quad \alpha\le|\theta|\le\pi,\\&u_r(r_i,\theta)=v_0\cos\theta-\delta(1-\cos\theta),\qquad \text{for}\quad 0\le|\theta|\le\alpha,\end{aligned} \tag{2.7}$$

Here α is the contact half-angle and v_0 is the pin displacement along the loading direction. The third contact condition in (2.7) is the small-clearance approximation used for cylindrical pin–hole contact by Liu et al. (2005, 2007), Ciavarella and Decuzzi (2001a), and Hou and Hills (2001), and applies when $\delta << r_i$.

By imposing the boundary conditions (2.6) and $(2.7)_1$, using the stress and displacement fields (2.2) and (2.5) for the FGM ring and the elastic plane, one obtains the following relations between the unknown constants that appear in the stress and displacement fields in the elastic plane and the constants B_{ni} (i = 1, 2, 3, 4; n = 0, 1, 2,…) entering the stress and displacement fields in the FGM ring:

$$D_0 = \sum_{i=1}^{2} B_{0i},$$

$$D_1 = \frac{1}{2}\left(1-\frac{1}{\Delta}\right)\sum_{i=1}^{2}\lambda^{b_{1i}+1}B_{1i},$$

$$C = -\frac{1}{\Delta}\sum_{i=1}^{2}\lambda^{b_{1i}+1}B_{1i}, \tag{2.8}$$

$$C_1 = \frac{1}{2}\sum_{i=1}^{2}\left[\left(1-\frac{\rho^2}{2}+\frac{\rho^2}{2\Delta}\right)\lambda^{b_{1i}+1}-1\right]B_{1i},$$

$$V = \frac{1}{2}\sum_{i=1}^{2}\left[1-\left(1+\frac{1}{\Delta}\right)\frac{\lambda^{b_{1i}+1}}{2}+\frac{4\lambda^{b_{1i}+1}(1+\rho^2+m\rho^2)}{m(m-2\rho^2-2m\rho^2)}+\frac{\Gamma_{1i}}{1-\rho^2}\right]B_{1i},$$

$$D_n = \sum_{i=1}^{4}\frac{n\,\Delta_{ni}-1}{2(n-1)}B_{ni}, \qquad C_n = \sum_{i=1}^{4}\frac{1-\Delta_{ni}\,(n+2)}{2(n+1)}B_{ni}, \qquad (n=2,3,...)$$

where $\lambda = r_i/r_o < 1$ is the ring aspect ratio. Moreover, the following relations are also obtained between constants B_{01} and B_{02}:

$$\sum_{i=1}^{2}(1+\kappa_i)B_{0i} = 0, \tag{2.9}$$

between constants B_{11} and B_{12}:

$$\sum_{i=1}^{2}\left[\frac{2m(1-\rho^2)\lambda^{b_{1i}+1}}{m-2\rho^2-2m\rho^2}-\frac{b_{1i}+3}{1-\rho^2}\right]B_{1i} = 0, \tag{2.10}$$

and between constants B_{ni} for i = 1, 2, 3, 4, and n = 0, 1, 2, …:

$$\sum_{i=1}^{4}\lambda^{b_{ni}}\Delta_{ni}\,B_{ni} = 0,$$

$$\sum_{i=1}^{4}\left[\chi_{1ni}-2\frac{n(1+n\rho^2)\Delta_{ni}-n-\rho^2}{n^2-1}\right]B_{ni} = 0, \tag{2.11}$$

$$\sum_{i=1}^{4}\left[\chi_{2ni}+2n\frac{n(n+\rho^2)\Delta_{ni}-1-n\rho^2}{n^2-1}\right]B_{ni} = 0.$$

From Eqs. (2.9), (2.10), and (2.11) one can find

$$B_{02} = -B_{01}\frac{1+\kappa_1}{1+\kappa_2},$$

$$B_{12} = -\beta\,B_{11}, \tag{2.12}$$

$$B_{ni} = c_{ni}\,B_{n1}, \qquad (\text{for } n=2,3,... \text{ and } i=2,3,4)$$

respectively, where

$$\beta = \frac{2m\lambda^{b_{11}+1}(1-\rho^2)^2 - (m-2\rho^2-2m\rho^2)(b_{11}+3)}{2m\lambda^{b_{12}+1}(1-\rho^2)^2 - (m-2\rho^2-2m\rho^2)(b_{12}+3)}, \tag{2.13}$$

and c_{n2}, c_{n3}, and c_{n4}, for $n \geq 2$, can be calculated from the system (2.11).

The contact conditions $(2.7)_{2,3}$ then yield the following dual series equations

$$\sum_{i=1}^{2} B_{0i}\lambda^{a_i} + \left(1-\frac{1}{\Delta}\right)\sum_{i=1}^{2} B_{1i}\lambda^{b_{1i}}\cos\theta + \sum_{n=2}^{\infty}\sum_{i=1}^{4} B_{ni}\lambda^{b_{ni}}\cos n\theta = 0, \qquad \text{for} \quad \alpha \leq |\theta| \leq \pi,$$

$$\sum_{i=1}^{2} B_{0i}\,\kappa_i\,\lambda^{a_i} + \sum_{i=1}^{2} B_{1i}\,\lambda^{b_{1i}}\left[\frac{\Gamma_{1i}}{2(1-\rho^2)} + \frac{2(1+\rho^2+m\rho^2)}{m(m-2\rho^2-2m\rho^2)}\right]\cos\theta$$

$$+ \sum_{n=2}^{\infty}\sum_{i=1}^{4} B_{ni}\frac{\lambda^{b_{ni}}\chi_{1ni}}{2(1-\rho^2)}\cos n\theta = \left[\left(1+\frac{\nu_0}{\delta}\right)\cos\theta - 1\right]\lambda^{m-1}, \qquad \text{for} \quad 0 \leq |\theta| \leq \alpha. \tag{2.14}$$

Using relations (2.12), from Eq. (2.14) one obtains

$$B_{01}\left(\lambda^{a_1} - \lambda^{a_2}\frac{1+\kappa_1}{1+\kappa_2}\right) + B_{11}\left(1-\frac{1}{\Delta}\right)\left(\lambda^{b_{11}} - \beta\lambda^{b_{12}}\right)\cos\theta + \sum_{n=2}^{\infty} B_{n1}\sum_{i=1}^{4} c_{ni}\,\lambda^{b_{ni}}\cos n\theta = 0, \quad \text{for } \alpha \leq |\theta| \leq \pi,$$

$$B_{01}\left(\kappa_1\lambda^{a_1} - \frac{1+\kappa_1}{1+\kappa_2}\kappa_2\lambda^{a_2}\right) + B_{11}\left[\frac{\lambda^{b_{11}}\Gamma_{11} - \beta\lambda^{b_{12}}\Gamma_{12}}{2(1-\rho^2)} + \frac{2(1+\rho^2+m\rho^2)(\lambda^{b_{11}} - \beta\lambda^{b_{12}})}{m(m-2\rho^2-2m\rho^2)}\right]\cos\theta \tag{2.15}$$

$$+ \sum_{n=2}^{\infty} B_{n1}\sum_{i=1}^{4} c_{ni}\,\frac{\lambda^{b_{ni}}\chi_{1ni}}{2(1-\rho^2)}\cos n\theta = \left[\left(1+\frac{\nu_0}{\delta}\right)\cos\theta - 1\right]\lambda^{m-1}, \qquad \text{for} \quad 0 \leq |\theta| \leq \alpha,$$

namely:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos n\theta = 0, \quad \text{for} \quad \alpha \leq |\theta| \leq \pi,$$

$$\frac{A_0}{2}\,\frac{\lambda^{a_1}\kappa_1(1+\kappa_2) - \lambda^{a_2}\kappa_2(1+\kappa_1)}{\lambda^{a_1}(1+\kappa_2) - \lambda^{a_2}(1+\kappa_1)} + \sum_{n=1}^{\infty}\frac{A_n\,f_n}{1-\rho^2}\cos n\theta = \left[\left(1+\frac{\nu_0}{\delta}\right)\cos\theta - 1\right]\lambda^{m-1}, \text{ for } 0 \leq |\theta| \leq \alpha, \tag{2.16}$$

where

$$A_0 = 2B_{01}\left(\lambda^{a_1} - \lambda^{a_2}\frac{1+\kappa_1}{1+\kappa_2}\right),$$

$$A_1 = B_{11}\left(1-\frac{1}{\Delta}\right)\left(\lambda^{b_{11}} - \beta\lambda^{b_{12}}\right), \tag{2.17}$$

$$f_1 = \frac{\Delta}{2(\Delta-1)(\lambda^{b_{11}} - \beta\lambda^{b_{12}})}\left[\lambda^{b_{11}}\Gamma_{11} - \beta\lambda^{b_{12}}\Gamma_{12} + \frac{4(1-\rho^2)(1+\rho^2+m\rho^2)(\lambda^{b_{11}} - \beta\lambda^{b_{12}})}{m(m-2\rho^2-2m\rho^2)}\right],$$

and

$$A_n = B_{n1}\sum_{i=1}^{4} c_{ni}\,\lambda^{b_{ni}},$$

$$f_n = \frac{1}{2}\sum_{i=1}^{4} c_{ni}\,\lambda^{b_{ni}}\,\chi_{1ni} \Big/ \sum_{i=1}^{4} c_{ni}\,\lambda^{b_{ni}}\,, \tag{2.18}$$

for n = 2, 3, …. To remove the rigid body displacement v_0, the operator

$$L[\phi] = \frac{\mathrm{d}\phi}{\mathrm{d}\theta} + \int_0^{\theta} \phi(s)\,\mathrm{d}s, \tag{2.19}$$

is applied on equation $(2.16)_2$, as suggested by Omar and Hassan (1991) and Hassan (1997), thus obtaining

$$\left[\lambda^{m-1} + \frac{A_0}{2}\,\frac{\lambda^{a_1}\,\kappa_1(1+\kappa_2) - \lambda^{a_2}\,\kappa_2(1+\kappa_1)}{\lambda^{a_1}(1+\kappa_2) - \lambda^{a_2}(1+\kappa_1)}\right](1-\rho^2)\,\theta + \sum_{n=1}^{\infty} A_n f_n\left(\frac{1}{n} - n\right)\sin n\theta = 0, \quad \text{for } 0 \le |\,\theta\,| \le \alpha. \tag{2.20}$$

Then, we apply the procedure used by Block and Keer (2007), Radi and Strozzi (2023), and Radi et al. (2025) for solving a set of dual series equations similar to Eqs. $(2.16)_1$ and (2.20). To this aim, we rearrange Eq. (2.20) in the form

$$\sum_{n=1}^{\infty} A_n \sin n\theta = (c\,A_0 + d)\,\theta + \sum_{n=1}^{\infty} h_n\,A_n \sin n\theta, \qquad \text{for} \quad 0 \le |\,\theta\,| \le \alpha, \tag{2.21}$$

where

$$\begin{aligned} c &= -\frac{\lambda^{a_1}\,\kappa_1(1+\kappa_2) - \lambda^{a_2}\,\kappa_2(1+\kappa_1)}{2\,[\lambda^{a_1}(1+\kappa_2) - \lambda^{a_2}(1+\kappa_1)]}(1-\rho^2), \\ d &= -(1-\rho^2)\,\lambda^{m-1}, \\ h_n &= 1 - f_n\left(\frac{1}{n} - n\right). \end{aligned} \tag{2.22}$$

It is possible to check that

$$f_n = -\frac{1}{n} + \frac{3m - 2\rho^2(m+1)}{2n^2} + o(n^{-2}), \qquad \text{as} \quad n \to \infty\,, \tag{2.23}$$

and thus $h_n = \mathrm{O}(n^{-1})$ as $n \to \infty$. The latter asymptotic behavior is required for the convergence of the solution to Eq. (2.21). The set of dual equations $(2.16)_1$ and (2.21) can be solved by using the procedure presented by Sneddon (1966), developed by Noble and Hussain (1969), used by Block and Keer (2007), and recently adopted by Radi and Strozzi (2023), Radi (2024), and Radi et al. (2025), which yields the following relations for the coefficients A_n, for n = 0, 1, 2, …:

$$A_0 = -4\,(c\,A_0 + d)\ln\left(\cos\frac{\alpha}{2}\right) + \sum_{n=1}^{\infty} n\,h_n\,A_n\,J_n, \tag{2.24}$$

$$A_n = (c\,A_0 + d)\,J_n + \frac{1}{2}\sum_{j=1}^{\infty} j\,K_{nj}\,h_j\,A_j, \qquad \text{for} \quad n = 1, 2, \ldots \tag{2.25}$$

where (Noble and Hussain, 1969):

293 $$J_n = \frac{1}{n}[P_{n-1}(\cos\alpha) - P_n(\cos\alpha)]\,, \tag{2.26}$$

294 $$K_{nj} = \frac{(1+\cos\alpha)\sin^2\alpha}{2(n^2 - j^2)}[(n+1)P_{j-1}^{(0,1)}(\cos\alpha)P_{n-2}^{(1,2)}(\cos\alpha) - (j+1)P_{n-1}^{(0,1)}(\cos\alpha)P_{j-2}^{(1,2)}(\cos\alpha)], \tag{2.27}$$

295 and $P_n^{(a,b)}(t)$ denotes the Jacobi polynomial of order *n*. Note also that $J_n \to 0$ as $n \to \infty$.

296 The results provided in Section 3 of Noble and Whiteman (1970) have been used in calculating the
297 integrals (2.26) and (2.27) together with Eq. (3.3) reported by Derevyagin and Juricic (2020). A
298 detailed derivation of the result (2.27) is illustrated in Appendix B of Radi and Strozzi (2023). Note
299 that for $j = n$, the coefficient K_{nn} in (2.27) is undetermined and thus it must be calculated as a limit
300 (Radi and Strozzi, 2023), namely

301 $$K_{nn} = \frac{(1+\cos\alpha)\sin^2\alpha}{4n}\Big\{P_{n-1}^{(0,1)}(\cos\alpha)P_{n-2}^{(1,2)}(\cos\alpha) + (n+1)\times \left[P_{n-1}^{(0,1)}(\cos\alpha)\frac{d}{dn}P_{n-2}^{(1,2)}(\cos\alpha) - P_{n-2}^{(1,2)}(\cos\alpha)\frac{d}{dn}P_{n-1}^{(0,1)}(\cos\alpha)\right]\Big\}. \tag{2.28}$$

302 The resultant of the contact pressure distribution must be equal to the total load *P* applied to the pin,
303 namely

304 $$P = -2\,r_i\int_0^{\pi}\sigma_{rr}(r_i,\theta)\cos\theta\,\mathrm{d}\theta\,. \tag{2.29}$$

305 By using Eq. (2.2), (2.12), (2.17), and (2.18), then Eq. (2.29) yields

306 $$P = -4\,\mu_0\,\delta\lambda\int_0^{\pi}\left(\frac{A_0}{2} + \sum_{n=1}^{\infty}A_n\,\cos n\theta\right)\cos\theta\,\mathrm{d}\theta = -2\pi\,\mu_0\,\delta\lambda\,A_1\,. \tag{2.30}$$

307 The rigid body motion of the pin follows from Eq. $(2.16)_2$ for $\theta = 0$ as

308 $$v_0 = \delta\lambda^{1-m}\left[\frac{A_0}{2}\,\frac{\lambda^{a_1}\,\kappa_1(1+\kappa_2) - \lambda^{a_2}\,\kappa_2\,(1+\kappa_1)}{\lambda^{a_1}(1+\kappa_2) - \lambda^{a_2}(1+\kappa_1)} + \sum_{n=1}^{\infty}\frac{A_n\,f_n}{1-\rho^2}\right]. \tag{2.31}$$

309

310 *2.4. Approximate solution*

311 An approximate solution of the system of infinite algebraic equations (2.24) and (2.25) for the
312 unknown coefficients A_n (n = 0,1, …, N) can be obtained by truncation if a sufficiently large number
313 of terms N is considered, namely:

314 $$A_0 = -4\,(c\,A_0 + d)\ln\left(\cos\frac{\alpha}{2}\right) + \sum_{n=1}^{N} n\,h_n\,A_n\,J_n, \tag{3.1}$$

$$A_n = (c\,A_0 + d)\,J_n + \frac{1}{2}\sum_{j=1}^{N} j\,K_{nj}\,h_j\,A_j, \qquad \text{for} \quad n = 1, 2, ..., N. \tag{3.2}$$

An inverse procedure is used to determine the relation between the applied pin load P and the contact half-angle α. For each prescribed value of α, the N+1 coefficients A_0, A_1, …, A_N are obtained by solving Eqs. (3.1) and (3.2). The resultant load P corresponding to the full contact angle 2α is then calculated from Eq. (2.30).

3. Loaded pin in a hole in a homogeneous isotropic elastic plane

Let r_o, r_p, μ_0, and ν denote the hole radius, the pin radius, the shear modulus, and the Poisson's ratio of the isotropic elastic plane, respectively. Let $\delta = r_o - r_p$ be the small radial clearance between the rigid pin and the hole.

The stress and displacement fields within the isotropic elastic plane are assumed in the form (2.5), where no rigid motion of the plane is assumed, namely $V = 0$. Then, the frictionless contact conditions between the rigid pin and the hole surface at $r = r_0$ require

$$\begin{aligned} &\sigma^0_{r\theta}(r_o, \theta) = 0, \qquad \text{for} \quad 0 \le |\theta| \le \pi, \\ &\sigma^0_{rr}(r_o, \theta) = 0, \qquad \text{for} \quad \alpha \le |\theta| \le \pi, \\ &u^0_r(r_o, \theta) = v_0 \cos\theta - \delta(1 - \cos\theta), \qquad \text{for} \quad 0 \le |\theta| \le \alpha, \end{aligned} \tag{4.1}$$

where α is the unknown contact angular extent and v_0 denotes the rigid body motion of the pin along the loading direction.

By imposing the boundary conditions $(4.1)_1$, one obtains the following relations for the unknown constants that appear in the fields (2.5):

$$C_1 = -\frac{\rho^2}{2} D_1, \qquad C_n = -\frac{n-1}{n+1} D_n. \tag{4.2}$$

The contact conditions $(4.1)_{2,3}$ then yield the following dual series equations

$$\frac{D_0}{2} + D_1 \cos\theta - \sum_{n=2}^{\infty} D_n (n-1) \cos n\theta = 0, \qquad \text{for} \quad \alpha \le |\theta| \le \pi, \tag{4.3}$$

$$1 - D_0 - \frac{D_1}{2}\cos\theta + 2\sum_{n=2}^{\infty} D_n \left(\frac{1}{1-\rho^2} - \frac{1}{n+1} \right) \cos n\theta = \left(1 + \frac{v_0}{\delta} \right) \cos\theta, \quad \text{for } 0 \le |\theta| \le \alpha. \tag{4.4}$$

The operator L defined in Eq. (2.19) is then applied to Eq. (4.4) to eliminate the rigid body displacement v_0. Combining the resulting equation with Eq. (4.3) yields

$$\frac{A_0}{2} + A_1 \cos\theta + \sum_{n=2}^{\infty} A_n \cos n\theta = 0, \qquad \text{for} \quad \alpha \le |\theta| \le \pi, \tag{4.5}$$

$$\frac{1-\rho^2}{2}(1-A_0)\theta+\sum_{n=2}^{\infty}A_n\left(1+\frac{\rho^2}{n}\right)\sin n\theta=0,\qquad \text{for}\quad 0\le|\theta|\le\alpha, \tag{4.6}$$

where

$$A_0=D_0,\qquad A_1=D_1,\qquad A_n=-(n-1)D_n,\qquad \text{for } n=2,3,\ldots. \tag{4.7}$$

Finally, Eq. (4.6) can be rewritten in the form

$$\sum_{n=1}^{\infty}A_n\sin n\theta=-\frac{1-\rho^2}{2}(1-A_0)\,\theta-\rho^2\sum_{n=1}^{\infty}\frac{A_n}{n}\sin n\theta=0,\qquad \text{for}\quad 0\le|\theta|\le\alpha, \tag{4.8}$$

The set of dual equations (4.5) and (4.8) can be solved by using the procedure employed in Sections 2 and 3, which yields the following relations for the coefficients A_n, for n = 0, 1, 2, …:

$$A_0=2(1-\rho^2)\,(1-A_0)\ln\left(\cos\frac{\alpha}{2}\right)-\rho^2\sum_{n=1}^{\infty}A_n\,J_n, \tag{4.9}$$

$$A_n=-\frac{1-\rho^2}{2}(1-A_0)\,J_n-\frac{\rho^2}{2}\sum_{j=1}^{\infty}K_{nj}\,A_j,\qquad \text{for}\quad n=1,2,\ldots \tag{4.10}$$

where J_n and K_{nj} are defined in (2.26) and (2.27). An approximate solution of the system of infinite algebraic equations (4.9) and (4.10) for the unknown coefficients A_n, for n = 0, 1, 2, …, N, can be obtained by truncation if a sufficiently large number of terms N is considered. Then, the pin load P is given by Eq. (2.30).

According to Eq. (4.4) for θ = 0 and using Eq. (4.7), the rigid body displacement of the pin is

$$v_0=-\delta\left[A_0+\frac{A_1}{2}+\sum_{n=2}^{N}\frac{2A_n}{n-1}\left(\frac{1}{1-\rho^2}-\frac{1}{n+1}\right)\right]. \tag{4.11}$$

4. Results

The main objective of this study is to investigate which type of FGM reinforcement can be used to reduce the stress concentration factor in a pinned connection with radial clearance. For illustrative purposes, a uniform Poisson's ratio ν = 0.3 is assumed both for the elastic plane and the FGM ring. Initially, the plane strain condition is taken into consideration, and then the plane stress condition is briefly addressed in Sect. 5.1. The stress and displacement fields in the FGM ring and in the infinite isotropic elastic plane are obtained by considering a set of values of the contact half-angle α and then calculating the corresponding resultant load applied to the pin. To achieve an adequate degree of accuracy in the results, the truncation order N = 60 is assumed for the system (3.1) and (3.2), corresponding to 61 coefficients.

The convergence of the series expansion is illustrated in Fig. 3, which shows the variation of the ratios A_n/A_0 (for $0 \leq n \leq N$) as the number of terms increases (N = 10, 20, 40, 60), with m = 1, λ = 0.6, and various contact region extents. As the number of terms N increases, the higher-order terms A_n become negligible compared to the leading term A_0, particularly for larger contact regions. It can be observed that a large number of terms is required to obtain a sufficiently accurate solution only if the contact region is very narrow, since narrower angular features require more harmonics to resolve. The convergence improves with increasing contact angle, likely because of the assumption of conformal contact embedded in Eq. $(2.7)_3$. Similar convergence trends are also observed for various values of the material and geometric parameters m and λ, indicating that convergence requirements are primarily dictated by the extent of the contact region. The results plotted in Fig. 3 thus justify truncating the system (3.1) and (3.2) to the first 60 terms for a contact half-angle $\alpha \geq 0.1\ \pi$. Lanczos sigma factors (Lanczos, 1956) are used for graphical reconstruction, whereas the resultant load is evaluated from the unfiltered coefficient A_1 using Eq. (2.30).

The effects of the clearance amount, grading of material properties, and geometry on the stress fields are assessed by varying the grading index m and the aspect ratio $\lambda = r_i/r_o$ of the FGM ring. As m tends to 0, the special case of uniform shear modulus presented in Section 4 is recovered. In this case, the predictions of the present approach are consistent with those for a rigid pin in a hole in a homogeneous elastic plane provided by Noble and Hussain (1969), Ho and Chau (1997), and Omar and Hassan (1991). Moreover, Fig. 4 compares the present homogeneous rigid-pin solution with the results of Hou and Hills (2001) for a pin-to-plane elastic modulus ratio of 3. Because the pin compliances differ, this comparison illustrates agreement in trends for the special case of m = 0 rather than constituting an exact comparison of identical boundary-value problems.

The normalized variations of the contact half-angle α with the pin load factor $P/(\mu_0\ \delta)$ under plane strain conditions are plotted in Figs. 5a,b, for specific values of the grading index m, both for a thick FGM ring with r_i/r_o = 0.6 (Fig. 5a) and a thinner one with r_i/r_o = 0.8 (Fig. 5b). These plots are obtained from Eq. (2.30) calculated for a discrete set of values of the contact half-angle α. As the pin load P increases or the clearance δ decreases, the contact half-angle increases non-linearly and approaches a limit value lower than $\pi/2$. If the pin and the plane are made of the same isotropic elastic material, Hou and Hills (2001) showed that the theoretical maximum contact half-angle is approximately 84°, namely 0.467π. The contact angle also increases as the grading index m increases, namely as the elastic modulus of the FGM ring decreases moving from the outer to the inner rim (see Fig. 2), so that the inner part of the ring becomes softer. Indeed, for a positive grading index, the elastic shear modulus increases within the FGM ring moving from the surface in contact with the pin towards the interface with the elastic plane. There, it reaches the same value μ_0 as the infinite elastic plane (see

Fig. 2). Under the same pin load, an increase in the contact angle then implies a more uniform distribution of the contact pressure. In general, a lower elastic stiffness of the FGM ring close to the contact surface also implies a larger reduction of the material strength just where the highest stress concentration occurs. Therefore, the load-bearing capacity of the FGM ring with a positive grading index depends on these two effects: a reduction in the stress concentration factor and an associated plausible reduction in the material strength.

The normalized relations between the pin load and the pin displacement are plotted in Figs. 5c,d for the same set of values of the grading index m and aspect ratio r_i/r_o considered in Fig. 5a,b. These plots are obtained from Eq. (2.31) calculated for a discrete set of values of the contact half-angle α. For sufficiently large pin load, these variations are almost linear, even though the advancing contact problem is non-linear. This result agrees with the findings of the FEM investigations by Duc (2025). Furthermore, the pin displacement caused by the same load is greater for positive grading indices than for negative ones. However, the slope of these plots does not vary monotonically with the grading index m.

The normalized distributions of the contact pressure p and hoop stress $\sigma_{\theta\theta}$ along the contact surface between a rigid pin and an FGM ring with $r_i/r_o = 0.6$ and 0.8 are shown in Figs. 6. Both quantities are normalized by the constant shear modulus μ_0 of the surrounding elastic plane. Results are reported for different grading indices m, under the same pin load $P = 20\ \mu_0\delta$. Under the same pin load, the maximum contact pressure is slightly lower for large negative grading indices than for positive ones, especially in the case of a thick FGM ring.

For the cases considered here, the calculated contact-pressure profiles have a single maximum at the centre of the contact region and vanish at its edges. A centrally peaked profile has been observed also for a rigid pin in a hole in an elastic isotropic plane (Ho and Chau, 1997; Mizushima and Hamada, 1983), as well as for the contact problem between two deformable cylindrical solids with FGM coatings whose elastic moduli vary exponentially (Guler and Erdogan, 2006). This shape differs from the two off-centre maxima reported for some lug configurations by Radi and Strozzi (2023) and Radi et al. (2025).

For relatively small contact angles, the calculated pressure distribution remains close to the classical Hertzian profile

$$p_H(\theta) = \frac{P}{\pi r_i\, J_1(\alpha)} \sqrt{1 - \frac{\theta^2}{\alpha^2}}, \qquad \text{for } |\theta| \le \alpha, \tag{5.1}$$

where the pin load P is provided by Eq. (2.30). A comparison between the normalized contact pressure distributions obtained from the dual-series solution and the Hertzian distribution (5.1), evaluated for the same contact half-angle α and pin load P is presented in Fig. 7 for several values of

the grading index m, for $\alpha = 0.2$ and 0.4, and for the two aspect ratios $r_i/r_o = 0.6$ and 0.8. A fixed contact angle has been considered in each plot of Fig. 7 and thus the curves obtained for different grading indices correspond to different pin loads, according to the results in Fig. 3a,b. Fig. 7 shows that the Hertzian limit is approached as the contact angle becomes small, but only for a homogeneous elastic material ($m = 0$). As the contact half-angle or the magnitude of m increase, progressively larger departures from the Hertz approximation are observed, reflecting the increasing influence of contact conformity and radial material grading. In particular, the sign of m affects the shape of the pressure distribution differently: positive and negative grading produce opposite deviations from the homogeneous response. The finite thickness of the FGM ring also plays a significant role: the deviations are generally more pronounced for $r_i/r_o = 0.6$ than for $r_i/r_o = 0.8$, indicating that non-Hertzian effects become stronger as the ring thickness increases. These observations are consistent with the results of Hou and Hills (2001), who showed that the Hertzian and conformal-contact solutions converge as the contact angle tends to zero, whereas increasing contact conformity leads to progressively non-Hertzian pressure distributions, evolving away from the classical semi-elliptical profile. They are also consistent with the conclusions of Liu et al. (2005), according to which the applicability of the Hertz approximation to cylindrical joints with clearance is essentially restricted to conditions of relatively large clearance and small applied pin load.

The grading index also has a large influence on the hoop stress along the inner rim of the FGM ring. Indeed, for large positive grading indices, a marked reduction in the peak of the hoop stress is observed in Figs. 6c,d near the end of the contact region, namely at $\theta \approx \alpha$. This reduction becomes even more significant for thicker FGM rings. Moreover, for large positive grading indices, the hoop stress may become negative, and hence compressive, in a limited region near $\theta = 0$, where it reaches its minimum value.

Since tensile hoop stress may promote the formation and propagation of radial cracks beneath the contact region, especially where it attains its maximum tensile value, these findings indicate that the use of an FGM ring with a large positive grading index can substantially reduce the stress concentration along the inner rim and thus shield the hole from crack initiation and propagation.

As noted above, however, the strength of the FGM ring depends on the local shear modulus and is expected to increase with it (Ciavarella, 2024). Thus, when the reference shear modulus μ_0 is kept fixed, FGM rings with positive grading indices are generally expected to be weaker than those with negative grading indices. Consequently, the beneficial shielding effect induced by a positive grading index results from two competing mechanisms: on the one hand, the reduction of the tensile hoop-stress concentration near the hole; on the other hand, the possible decrease in local material strength associated with a lower shear modulus. The optimal grading index should therefore be selected by

balancing these two effects, rather than by minimizing the stress field alone. In this sense, a properly designed FGM ring can improve the strength of the pinned connection by redistributing the hoop stress away from the most critical regions while maintaining sufficient local material resistance.

The normalized distributions of the radial, hoop, and shear stresses, $-\sigma_{rr}$, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$, along the interface between the FGM ring and the surrounding elastic plane are plotted in Fig. 8, for $r_i/r_o = 0.6$ and 0.8. Results are reported for different grading indices m, under the same pin load $P = 20\ \mu_0\delta$.

The radial stress distribution is compressive for values of the angular coordinate θ smaller than the contact half-angle α, whereas it becomes tensile for larger values of θ, as shown in Figs. 8a,b. A clear reduction of the maximum value is observed for negative grading indices. The hoop stress distribution remains tensile along the entire interface, as shown in Figs. 8c,d, with higher maximum values occurring for thinner FGM rings.

These results indicate that large negative grading indices, corresponding to stiffer FGM rings, promote more uniform distributions of both radial and hoop stresses along the interface. Conversely, the shear stress increases for negative grading indices, especially along the upper portion of the interface, namely at large values of θ, as shown in Figs. 8e,f. This behavior denotes a stronger transfer of tangential interaction forces between the FGM ring and the surrounding elastic plane. Overall, the results in Fig. 8 show that the grading index strongly affects the way loads are transferred from the FGM ring to the surrounding elastic medium.

The normalized variations of the hoop stress along the radial directions $\theta = 0$ and $\theta = \alpha$ are plotted in Fig. 9, both for $r_i/r_o = 0.6$ and $r_i/r_o = 0.8$, under the pin load $P = 20\ \mu_0\delta$. Specifically, Figs. 9a,b refer to the radial direction $\theta = 0$, whereas Figs. 9c,d refer to $\theta = \alpha$. These results clearly show that the hoop stress is continuous at the interface between the FGM ring and the surrounding elastic plane, as expected from the continuity of the elastic parameters across the interface. Along the radial direction $\theta = 0$, the hoop-stress distribution within the FGM ring exhibits markedly different trends depending on the sign and magnitude of the grading index. For positive values of m, the hoop stress increases markedly when moving outwards from r_i to r_o, as shown in Figs. 9a,b; it reaches its maximum value at the interface between the FGM ring and the elastic plane and then gradually decreases in the outer elastic plane. By contrast, for $m \leq 0$, the hoop stress attains its maximum at the inner rim of the FGM ring and then decreases along the radial direction $\theta = 0$. The magnitude of this maximum increases with both the thickness of the FGM ring and the magnitude of the negative grading index. This behavior is consistent with previous studies showing that, in FGMs, the stress distribution tends to follow the spatial gradient of the Young's modulus (Fukui and Yamanaka, 1992; Abdalla et al., 2023). In particular, for $m \approx 0.25$, the hoop stress within the FGM ring is found to be nearly uniform along

the radial line $\theta = 0$.

The corresponding variations of the hoop stress along the radial direction $\theta = \alpha$ are shown in Figs. 9c,d for both considered aspect ratios. In this case, the maximum hoop stress occurs at $r = r_i$, for every value of the grading index. However, its magnitude decreases as m increases in agreement with the trends reported in Figs. 6c,d. For $m > 2$, the hoop stress distribution across the FGM ring becomes almost uniform. This result is also consistent with the findings of Mohammadi et al. (2011) and Sburlati (2013), which indicate that the optimal grading index for mitigating the SCF in an FGM ring inserted around a hole in a remotely loaded plane is approximately $m = 1.1$. Overall, the results in Fig. 9 confirm that the grading index plays a key role in controlling not only the magnitude of the hoop stress but also its radial distribution within the FGM ring. Positive grading indices tend to reduce the hoop-stress concentration at the inner rim near the edge of the contact region, where crack initiation is most likely to occur. Conversely, negative grading indices may increase the hoop stress at the inner rim, particularly for thicker rings, although they may be associated with higher local material stiffness and strength.

If the pin is inserted into a hole in a homogeneous material, it is reasonable to expect that reducing the contact pressure leads to an enhancement of the load-bearing capacity of the connection. However, if the material is inhomogeneous, the problem becomes considerably more complex. To evaluate the advantages produced by material grading, it becomes necessary to consider the effects of material grading on both the SCF and the material strength. However, when designing with FGMs, it can be difficult to rely on a fixed allowable stress because of the lack of consistent data on the yield strength of FGMs. Indeed, FGMs have a continuously changing material composition, which complicates the prediction of resulting material properties, with the exception of the Young's modulus. The latter can be accurately estimated by theoretical methods based on the Hashin-Shtrikman bounds in the absence of experimental data. It has been demonstrated that many porous materials, including cellular materials (Gibson, 1989) and bones (Currey, 1988; Martin, 1991), generally exhibit power law variations in both strength and elastic modulus with density. In agreement with the suggestions by Ciavarella (2024), it can thus be concluded that the most effective structural design can be achieved by maintaining closely matched radial variations of effective stress and material strength.

An alternative approach to shape and material optimization is provided by the technique of topological optimization. Silva et al. (2024) employed this technique on the basis of the minimization of the maximum von Mises equivalent stress, instead of trying to keep it under a fixed allowable stress. This technique has the potential to provide valuable insight into the form finding process, as well as facilitating the identification of optimal distributions of multi-materials to minimize stress

concentration in contact problems.

The normalized distributions of the radial and tangential displacements, u_r and u_θ, along the inner rim of the FGM ring are shown in Fig. 10 as functions of the angular coordinate θ, for $r_i/r_o = 0.6$ and $r_i/r_o = 0.8$, under the pin load $P = 20\ \mu_0\delta$. The results reveal that the displacement field is significantly affected by the radial grading of the FGM ring, although the dependence on the grading index m is not monotonic. The radial displacement u_r shown in Figs. 10a,b, reaches its largest magnitude in the loaded region, at $\theta = 0$, and gradually decreases away from the contact zone. As expected, the deformation is more pronounced for positive grading indices because the corresponding elastic modulus decreases near the inner rim in contact with the rigid pin. However, the effect of the grading index is not simply related to the overall stiffness of the ring. Instead, intermediate values of m may lead to larger or smaller radial displacements than those obtained for both negative and large positive grading indices. This non-monotonic behaviour indicates that the radial displacement is governed by a balance between the local compliance near the inner rim, where the pin load is applied, and the stiffness of the outer region, which controls the interaction with the surrounding elastic plane.

A similar non-monotonic dependence is observed for the tangential displacement u_θ reported in Figs. 10c,d. The tangential displacement vanishes at $\theta = 0$ by symmetry and increases in magnitude away from the loading axis, reaching its maximum at intermediate angular positions before decreasing toward the unloaded portion of the hole boundary. The grading index modifies both the amplitude and the angular position of this maximum. In particular, changes in m alter the way the load transferred by the pin is redistributed along the hole rim, thereby affecting the relative contribution of radial compression and tangential sliding. This explains why the curves do not follow a simple monotonic ordering with the grading index. The non-monotonic trend observed in Fig. 10 can be interpreted as the result of two competing mechanisms. On the one hand, increasing the stiffness of the material near the inner rim tends to reduce the local deformation induced by the pin. On the other hand, the radial stiffness gradient modifies the load-transfer path between the FGM ring and the surrounding elastic medium, which may either amplify or reduce the displacement of the hole boundary depending on the angular position and ring thickness. Consequently, the displacement response cannot be predicted solely from the local stiffness at the hole rim, but it depends on the combined stiffness distribution across the whole FGM ring.

Overall, Fig. 10 shows that radial grading provides an additional degree of freedom for controlling the deformation pattern of the pinned connection. While the stress results discussed above indicate that positive grading indices may reduce hoop-stress concentrations near the inner rim, the displacement results show that the associated deformation field may vary in a non-monotonic manner. Therefore, the optimal grading index should be selected by considering both stress mitigation and

displacement control. This is particularly important in applications where excessive hole ovalization or tangential sliding may compromise the dimensional stability, fatigue performance, or long-term integrity of the pinned joint.

The analytical distributions of the von Mises equivalent stress are reported in Figs. 11 and 12 in order to assess the stress level induced by the pin load in both the FGM ring and the surrounding isotropic elastic plane. This representation is particularly useful for evaluating whether the equivalent stress remains below the allowable stress of the graded material and for identifying the regions where stress concentrations are most likely to promote yielding and failure. In this sense, the von Mises stress field provides a direct measure of the effectiveness of the FGM ring in redistributing, reducing, or shifting stress concentrations, which is one of the main objectives of an optimal material gradation design.

To highlight the influence of radial material grading on the equivalent stress distribution, the normalized von Mises stress under plane strain conditions

$$\sigma_{eq} = \sqrt{(1-\nu+\nu^2)(\sigma_{rr}+\sigma_{\theta\theta})^2 - 3\sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2}\ , \tag{5.2}$$

is plotted under the pin load $P = 20\ \mu_0\delta$, for different values of the grading index m. In particular, Fig. 11 refers to the thicker FGM ring, with $r_i/r_o = 0.6$, whereas Fig. 12 refers to the thinner FGM ring, with $r_i/r_o = 0.8$. In both cases, results are shown under plane strain conditions for a positive grading index, $m = 1$, a homogeneous ring, $m = 0$, and a negative grading index, $m = -1$. The maximum equivalent stress consistently occurs at the center of the contact region ($\theta = 0$). This region corresponds to the zone of direct load transfer from the rigid pin to the deformable ring and therefore represents the most critical location from the standpoint of yielding or damage initiation. The stress concentration is also strongly affected by the ring thickness: for the thinner ring the stress field is more localized near the hole boundary, whereas the thicker ring allows a broader redistribution of stresses across the graded layer.

The grading index has a clear influence on the magnitude and spatial distribution of the equivalent stress. For positive grading indices, the shear modulus decreases from the outer boundary toward the inner rim of the FGM ring. As a consequence, the material surrounding the hole is more compliant, and the peak von Mises stress at the hole rim is reduced with respect to the homogeneous case ($m = 0$). This reduction is more evident for the thicker FGM ring shown in Fig. 11, where the larger graded region enhances the stress-shielding effect. For the thinner ring shown in Fig. 12, the same qualitative trend is observed, although the reduction is less pronounced because the smaller thickness limits the ability of the graded layer to redistribute the load. For instance, incorporating an FGM ring with a positive grading index (e.g., $m = 1$) can reduce the peak equivalent stress by approximately 14% compared to the homogeneous case ($m = 0$), provided the ring is sufficiently thick (Fig. 11a). The

reduction is slightly lower for thinner rings (Fig. 12a).

Conversely, for a negative grading index the shear modulus increases toward the inner rim. In this case, the stiffer material close to the hole attracts a larger portion of the load, leading to higher values of the equivalent stress near the contact region. Accordingly, the maximum von Mises stress increases with respect to the homogeneous case. This behaviour is consistent with the general mechanical response of FGMs, in which stress tends to concentrate in regions of higher stiffness. However, these regions are also expected to exhibit higher material strength when the local strength scales with the local elastic modulus. Moreover, regions with higher elastic modulus bear greater loads and typically exhibit improved mechanical integrity because of reduced microdefect/porosity density. Therefore, the increase in equivalent stress observed for negative m should not necessarily be interpreted as detrimental, since it may be accompanied by a corresponding, or even larger, increase in the local material strength. Any compensating increase in allowable stress must be established for the particular constituents, microstructure, and manufacturing process; it is not predicted by the present elastic model.

The comparison between Figs. 11 and 12 further shows that the effect of material grading is more significant for thicker rings. For $r_i/r_o = 0.6$, the larger radial extension of the FGM layer allows the stiffness gradient to modify the stress-transfer mechanism more effectively. As a result, the effect of the grading index is more clearly visible. By contrast, for $r_i/r_o = 0.8$, the FGM ring is thinner, and the stress field is more strongly governed by the local contact action at the hole rim, so that the influence of grading is still present but less pronounced.

Overall, Figs. 11 and 12 confirm that radial material grading can be used as an effective design parameter for controlling the stress field around a pinned hole. A positive grading index reduces the peak equivalent stress at the inner rim and therefore produces a stress-shielding effect, particularly for sufficiently thick FGM rings. Conversely, a negative grading index increases the equivalent stress near the hole, although this effect may be accompanied by an increase in the local material strength, depending on the composition and grading profile. Consequently, the optimal grading index cannot be identified solely by minimizing the von Mises equivalent stress. A meaningful assessment of the structural performance requires the local stress field to be compared with the corresponding spatial distribution of the relevant material strength. From this perspective, the most effective FGM design is the one that maximizes the local safety margin rather than simply minimizing the stress concentration.

A quantitative comparison of the strength or safety margin associated with different values of m would require additional information on the spatial variation of the relevant strength parameter, such as the local yield stress for a ductile FGM or an appropriate fracture or failure strength for a brittle

635 FGM. These properties depend on the specific constituents, microstructure, and manufacturing
636 process and cannot be inferred from the elastic grading law adopted in the present analysis alone.
637 They must therefore be prescribed independently, for example on the basis of experimental
638 characterization or a suitably validated constitutive or micromechanical model for the FGM under
639 consideration.
640 The present study considers a generic FGM within the framework of linear elasticity and does not
641 introduce any specific yielding or failure criterion. Depending on the constituent materials, their
642 spatial distribution, and the resulting microstructure, the graded material may exhibit predominantly
643 ductile or brittle behavior. Accordingly, the von Mises equivalent stress is employed primarily as a
644 convenient scalar measure of the local intensity of the elastic stress state. It should therefore not be
645 interpreted, by itself, as a direct measure of the load-carrying capacity or failure resistance of FGMs
646 characterized by different values of the grading index m.
647 The analytical results reported in Figs. 11 and 12 for $m = 1$ and $m = -1$ are validated through
648 comparison with finite-element simulations performed using COMSOL 6.1 provided in the
649 Supplementary Material. The same comparison also includes the contact half-angle α, for which the
650 maximum deviation between the analytical and numerical predictions remains within approximately
651 5%.
652
653 *5.1 Results under plane stress conditions*
654 This subsection presents some results obtained under plane stress loading conditions, with the aim of
655 highlighting the small differences relative to the corresponding plane strain results. In particular, the
656 normalised variations of the contact half-angle α with the pin load P under plane stress loading
657 conditions are plotted in Fig. 13 for the same geometrical and material parameters considered in Fig.
658 5 under plane strain conditions. These results show that the contact angle exhibits a slight increase
659 under plane stress loading conditions in comparison with the corresponding plane-strain results.
660 Moreover, the contour plots of the normalized von Mises equivalent stress σ_{eq} under plane stress
661 loading conditions, namely

662 $$\sigma_{eq} = \sqrt{\sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2}\ , \tag{5.3}$$

663 are reported in Fig. 14 both in the FGM ring and in the isotropic elastic plane, in order to compare
664 them with the corresponding plots obtained in Figs. 11 and 12 under plane strain loading conditions,
665 under the same pin load $P = 20\mu_0\delta$, for $m = 1$ and $m = -1$. This comparison shows that the von Mises
666 equivalent stress turns out to be a bit higher under plane stress loading conditions, both in the FGM
667 ring and in the outer elastic plane, especially for a positive grading index ($m = 1$).

5. Validation of results by using the *M*-integral

In a homogeneous elastic material, the *M*-integral is an energy-based path-independent integral associated with scaling transformations (dilatation of the coordinate system), which is defined by:

$$M(r) = \oint_{C(r)} (w\mathbf{I} - \nabla \mathbf{u}^{\mathrm{T}} \boldsymbol{\sigma}) \mathbf{n} \cdot \mathbf{r} \, ds \,, \tag{6.1}$$

where $\mathbf{r} = r\,\mathbf{e}_r$ is the position vector and $\mathbf{e}_r$ is the radial unit vector; $w = w(\boldsymbol{\varepsilon})$ is the strain-energy density, with $\boldsymbol{\varepsilon}$ denoting the strain tensor, $C(r)$ is a counterclockwise circular contour enclosing the hole; and $\boldsymbol{n}$ denotes the outward unit normal to the contour $C(r)$. From the physical point of view, the *M*-integral evaluated along a contour enclosing a hole represents the energy release associated with an expansion of the hole (Yi-Feng and Chen, 2011). In a radially inhomogeneous material, the strain energy density depends on both the strain tensor $\boldsymbol{\varepsilon}$ and the radial distance r, namely $w = w(\boldsymbol{\varepsilon}, r)$. In this case, the *M*-integral is no longer path-independent and a correction term due to the material gradient must be added to the classical form (6.1).

In Appendix B it is proved that the following quantity must be invariant for $r_i \le r \le r_o$:

$$M(r) = \oint_{C(r)} (w\mathbf{I} - \nabla \mathbf{u}^{\mathrm{T}} \boldsymbol{\sigma}) \mathbf{n} \cdot \mathbf{r} \, ds + \int_{r}^{r_o} r^2 dr \int_{-\pi}^{\pi} \left. \frac{\partial w}{\partial r} \right|_{\varepsilon} d\theta \,, \tag{6.2}$$

where $\partial w/\partial r|_{\varepsilon}$ denotes the explicit material derivative of the strain energy density at fixed strain. This term vanishes, for $r \ge r_o$, namely in the homogeneous elastic material, so that the area integral in (6.2) also vanishes and $M(r)$ coincides with the classical *M*-integral (6.1) when evaluated along a circle of radius $r \ge r_o$ taken within the homogeneous elastic region. For $r_i \le r \le r_o$, the contour term in Eq. (6.2) evaluated in the FGM ring depends on the radius r, but the combined expression $M(r)$ is invariant (in the absence of body forces and for solutions satisfying equilibrium and compatibility) provided that the area integral is included. In practice, both terms on the right-hand side of Eq. (6.2) must be considered to obtain a contour-independent value in an FGM. This is the key correction noted in the literature for viscoelastic (Hou et al., 2020), thermoelastic (Radi et al., 2016), and damaged/fractured solids (Hou et al., 2022).

One of the most practical and powerful uses of the *M*-integral (especially in FGMs) is to validate analytical or numerical results, because the *M*-integral provides a global, invariant energetic measure of the solution when all correction terms (such as the FGM gradient term) are properly included. Contour independence of the corrected M-integral (6.2) provides a global consistency check on the analytical fields and their numerical evaluation. Agreement with the exact far-field value supplies a further check. Neither property alone proves pointwise satisfaction of all equilibrium, compatibility,

699 constitutive, and interface conditions.

700 Under plane strain or plane stress loading conditions, the strain energy density for a radially
701 inhomogeneous material with a constant Poisson's ratio is defined as

702 $$w = w(\boldsymbol{\varepsilon}, r) = \mu(r)\left[\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2 + 2\varepsilon_{r\theta}^2 + \frac{\overline{\nu}}{1-2\overline{\nu}}(\varepsilon_{rr} + \varepsilon_{\theta\theta})^2\right], \tag{6.3}$$

703 where $\overline{\nu}$ is defined in Eq. (2.4). For $r \geq r_o$, namely in the homogeneous elastic plane, the following
704 quantity is then invariant

705 $$M(r) = \frac{r^2}{2}\int_{-\pi}^{\pi}\left[\sigma_{\theta\theta}^0 \frac{u_{\theta,\theta}^0 + u_r^0}{r} - \sigma_{rr}^0 u_{r,r}^0 - \sigma_{r\theta}^0\left(u_{\theta,r}^0 + \frac{u_\theta^0 - u_{r,\theta}^0}{r}\right)\right] d\theta, \tag{6.4}$$

706 whereas, for $r_i \leq r \leq r_o$, namely in the FGM ring, the invariant quantity (6.2) is given by the sum

707 $$\begin{aligned} M(r) = {} & \frac{r^2}{2}\int_{-\pi}^{\pi}\left[\sigma_{\theta\theta} \frac{u_{\theta,\theta} + u_r}{r} - \sigma_{rr} u_{r,r} - \sigma_{r\theta}\left(u_{\theta,r} + \frac{u_\theta - u_{r,\theta}}{r}\right)\right] d\theta \\ & + \frac{m r_0^m}{4\mu_0}\int_r^{r_o} r^{1-m} dr \int_{-\pi}^{\pi}\left[(1-\overline{\nu})(\sigma_{rr} + \sigma_{\theta\theta})^2 - 2\sigma_{rr}\sigma_{\theta\theta} + 2\sigma_{r\theta}^2\right] d\theta, \end{aligned} \tag{6.5}$$

708 where $\partial w/\partial r|_\varepsilon = m\, w/r$, according to Eqs. (6.3) and (2.1).

709 Note that the *M*-integral in the outer homogeneous elastic region is determined entirely by the leading
710 far-field stress generated by the resultant pin load *P*. At fixed *P*, that leading term depends only on
711 the properties of the surrounding homogeneous material. The grading index and ring thickness affect
712 the faster-decaying terms, whose contributions disappear when the integration contour is taken to
713 infinity. In particular, only the terms containing the coefficient D_1 in the stress and displacement fields
714 (2.5) determine the *M*-integral (6.4) as $r \to \infty$. Their introduction in Eq. (6.4) then gives

715 $$M(r) = -2\pi\mu_0\,\delta^2(1+\rho^2)\,D_1^2. \tag{6.6}$$

716 The coefficient D_1 is related to the applied pin load *P* by the balance condition at infinity along the
717 loading direction, namely from the following relation where only the terms containing the coefficient
718 D_1 in Eqs. (2.5) provide a contribution as $r \to \infty$:

719 $$P = -\lim_{r\to\infty} r\int_{-\pi}^{\pi}\left[\sigma_{rr}^0(r,\theta)\cos\theta - \sigma_{r\theta}^0(r,\theta)\sin\theta\right] d\theta = -4\pi\mu_0\,\delta\,D_1. \tag{6.7}$$

720 Therefore, eliminating D_1 from Eqs (6.6) and (6.7), it follows

721 $$M(r) = -\frac{1+\rho^2}{8\pi\mu_0}P^2. \tag{6.8}$$

722 Although the exterior stress field itself depends on grading and ring thickness, at fixed pin load *P*, the
723 invariant quantity $M(r)$ assumes the constant value (6.8) because it can be evaluated in the far field,

where only the resultant load matters. This means that the M-integral alone cannot rank the reinforcing effectiveness of different FGM rings under the same applied load.

The variations of the quantity $M(r)$, defined in Eqs. (6.4) and (6.5), with the radius r of the closed circular contour enclosing the pin-loaded hole under plane strain conditions, are plotted in Figs.15a and 15b for the aspect ratios $r_i/r_o = 0.6$ and $r_i/r_o = 0.8$, respectively, and for different values of the grading index m, under the pin load $P = 20\mu_0\delta$. As expected, the results plotted in Figs. 15a,b show that the quantity $M(r)$ does not vary with the contour radius r, for $r \geq r_i$. This trend confirms the path-independence of $M(r)$ for the present problem and, together with the predictions of the FE analysis reported in the Supplementary Material, provides an additional check of the analytical results obtained for the stress and displacement fields, both in the FGM ring and in the surrounding homogeneous elastic plane. The near independence of the computed integral from contour radius provides an additional global consistency check on the analytical solution and numerical integration. For all the considered cases, the M-integral is negative. The dashed lines in Fig. 15 denote the exact value given by Eq. (6.8), $M/(\mu_0\,\delta^2) = -450/(7\pi) \approx -20.4628$, for the values of P and ν considered in Fig. 15. The small offsets of the curves from this value are numerical residuals, potentially arising from load-matching tolerance, series truncation, and numerical integration, and not physical grading effects. Both Fig. 15a and Fig.15b display similar values of the M-integral, thus showing that also the thickness of the FGM ring does not affect the value of M.

The sign of the M-integral is governed by the balance between the strain energy stored in the body and the work done by the external contact pressure. Since $M = -\partial\Pi/\partial r_i$, the negative value of M in Eq. (6.8) implies that the total potential energy Π would increase under a virtual expansion of the hole. Therefore, the configuration is energetically resistant to hole growth. Although the M-integral is commonly positive for a circular hole subjected to uniform internal pressure, negative values may occur in the present pin-loaded configuration because the non-uniform contact loading produces a different energetic balance. During a virtual expansion of the hole, the contact pressure performs positive work, which tends to reduce the total potential energy, whereas the surrounding elastic medium and the FGM ring store additional strain energy. When the increase in stored strain energy dominates the work of the external forces, the M-integral becomes negative.

According to Eq. (6.8), the magnitude of the M-integral increases quadratically with the pin load. As the contact half-angle approaches its limiting value, slightly lower than $\pi/2$, the magnitude of the M-integral increases rapidly.

6. Conclusions

An analytical formulation has been developed for frictionless contact between a rigid pin and a

clearance-fit hole reinforced by an FGM ring embedded in an infinite elastic plane. A power-law radial variation of shear modulus and a uniform Poisson's ratio allow a Michell-type representation of the stress and displacement fields. The contact and interface conditions reduce the problem to dual trigonometric series and a linear algebraic system solved by truncation, for both plane stress and plane strain loading conditions.

The main objective of the study was to assess whether a suitable radial grading of the shear modulus can reduce the stress concentrations generated by the load transmitted from the rigid pin to the hole boundary. To this end, series solutions evaluated by truncation have been obtained for the pin load-contact angle relation, contact pressure distribution, displacement field, and stress distributions in both the FGM ring and the surrounding elastic plane, for different grading indices, ring aspect ratios, and loading levels. The analytical results have also been validated against finite-element simulations, showing good agreement.

The contact region grows nonlinearly with the applied load, while radial grading and ring thickness modify the pressure profile and redistribute stresses between the reinforcement and the surrounding elastic medium. For small contact angles, the pressure approaches a semi-elliptical distribution, whereas increasing contact conformity can produce appreciable departures from this profile.

In the cases examined, positive grading makes the inner rim more compliant, enlarges the contact region at a prescribed load, and reduces the peak hoop and equivalent stresses near the hole boundary. This stress-shielding effect is particularly pronounced for thicker rings. Negative grading, by contrast, makes the inner rim stiffer and increases the stress field near the contact region, while reducing it in other parts of the ring and the surrounding elastic plane. The preferred grading profile therefore depends on where stress reduction is required.

These elastic stress trends alone do not establish the relative load-bearing capacity of different grading profiles. In porous materials, reduced porosity commonly increases both stiffness and strength, so the stiffer inner region associated with negative grading may also sustain higher stresses. However, the prescribed elastic grading law does not determine the local strength, and the relationship between these properties must be established independently for the material system considered. Negative grading may therefore be beneficial when the increase in local strength outweighs the accompanying stress increase. Accordingly, the grading index should be selected by assessing the local safety margin using an appropriate failure criterion and the spatial distribution of material strength, rather than by minimizing stress alone.

The accuracy of the derived analytical fields is confirmed by finite-element simulations provided in the Supplementary Material as well as by the invariance of the M-integral calculated along circular contours of arbitrary radius r, with $r > r_i$, enclosing the loaded hole. Therefore, the analytical solution

presented here may also serve as a benchmark for future theoretical and numerical studies on pin-loaded joints reinforced by FGM rings. Possible extensions of the present model include the elastic deformability of the pin, which can be incorporated within the same analytical framework, as well as more general material gradations, variable Poisson's ratio, interfacial imperfections, frictional contact, and finite-geometry effects. Experimental validation is also required to assess the actual load-bearing capacity of FGM-reinforced pinned connections and to quantify the relationship between elastic modulus and strength for specific classes of graded materials.

It should be observed that the present solution is based on an infinite-plane idealization and therefore reflects a specific far-field load-transfer mechanism. In finite structural components, the actual geometry and remote boundary conditions may influence both the circumferential stress distribution along the hole boundary and the contact pressure distribution at the pin-hole interface. Accordingly, for direct design applications, the present analytical results should be complemented by appropriate open-hole or finite-boundary corrections capable of accounting for the actual load path and structural configuration.

In conclusion, the present study provides useful insight for the design of advanced structural joints, including bearing bolt connections and aircraft skin fasteners, by exploiting the tailored mechanical response offered by FGMs.

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References

Abdalla, H.M.A., Casagrande, D., De Bona, F., 2023. Analysis of stress concentration in functionally graded plates with linearly increasing Young's modulus. Mater. 16(21), 6882.

Abdalla, H.M.A., De Bona, F., Casagrande, D., 2024. Optimization of functionally graded materials to make stress concentration vanish in a plate with circular hole. Compos. C: Open Access 15, 100512.

Akbarpour, S., Hallström, S., 2019. Reinforcement around holes in composite materials by use of patched metal inserts. Compos. Struct. 225, 111084.

Alinia, Y., Asiaee, A., Hosseini-nasab, M., 2019. Stress analysis in rolling contact problem of a finite thickness FGM layer. Meccanica, 54(1), 183-203.

Alkunte, S., Fidan, I., Naikwadi, V., Gudavasov, S., Ali, M. A., Mahmudov, M., Hasanov, S., Cheepu, M., 2024. Advancements and challenges in additively manufactured functionally graded materials: A comprehensive review. J. Manuf. Mater. Proc. 8(1), 23.

Andreev, V.I., Cybin, N.Y., 2014. The inhomogeneous plate with a hole: Kirsch's problem. Procedia Eng. 91, 26-31.

Arasan, U., Venkatachalam, S., Murthy, H., 2022. Solution to two-dimensional elastic problems involving functionally graded material in radial co-ordinates. Acta Mech. 233(1), 343-362.

Arslan, E., Haskul, M., 2015. Generalized plane strain solution of a thick-walled cylindrical panel subjected to radial heating. Acta Mech. 226, 1213-1225.

Aswad, M.F., Mohammed, A. J., Faraj, S.R. (2021). Induction surface hardening: A review. J. Phys. Conf. Ser. 1973(1), 012087.

Bahri, A., Akhlaghi, M., Salehi, M., 2017. An exact solution for improved metal-composite joints reinforced by FG inter-layers. Sci. Eng. Compos. Mater. 24(2), 221-235.

Barber, J.R., 2010. Elasticity. 3rd revised ed. Springer International Publishing, New York.

Batra, R.C., Nie, G.J., 2010. Analytical solutions for functionally graded incompressible eccentric and non-axisymmetrically loaded circular cylinders. Compos. Struct. 92(5), 1229-1245.

Bi, G., Sun, C.N., Gasser, A., 2013. Study on influential factors for process monitoring and control in laser aided additive manufacturing. J. Mater. Process. Technol. 213(3), 463-468.

Block, J.M., Keer, L.M., 2007. Partial contact of an elastic coated cylinder pressed by a rigid flat surface. J. Tribol. 129(1), 60-64.

Chau, K.T., Wei, X.X., 2001. Stress concentration reduction at a reinforced hole loaded by a bonded circular inclusion. J. Appl. Mech. 68(3), 405-411.

Chawde, D.P., Bhandakkar, T.K., 2021. Mixed boundary value problems in power-law functionally graded circular annulus. Int. J. Press. Vessel. Pip. 192, 104402.

Ciavarella, M., 2024. Cancelling the effect of sharp notches or cracks with graded elastic modulus materials. J. Mech. Phys. Solids 192, 105809.

Ciavarella, M., Decuzzi, P., 2001a. The state of stress induced by the plane frictionless cylindrical contact. I. The case of elastic similarity. Int. J. Solids Struct. 38(26-27), 4507-4523.

Ciavarella, M., Decuzzi, P., 2001b. The state of stress induced by the plane frictionless cylindrical contact. II. The general case (elastic dissimilarity). Int. J. Solids Struct. 38(26-27), 4525-4533.

Çömez, İ., 2015. Contact problem for a functionally graded layer indented by a moving punch. Int. J. Mech. Sci. 100, 339-344.

Currey, J.D., 1988. The effect of porosity and mineral content on the Young's modulus of elasticity of compact bone. J. Biomech. 21(2), 131-139.

de Jong, T., 1977. Stresses around pin-loaded holes in elastically orthotropic or isotropic plates. J. Compos. Mater. 11(3), 313-331.

Derevyagin, M., Juricic, N., 2020. An asymptotic formula for integrals of products of Jacobi polynomials. J. Stoch. Anal. 1(4), 8.

Duc, N.D., 2025. Pin-loaded hole contact in an anisotropic magnetoelectroelastic plate. Int. J. Mech. Sci. 302, 110551.

Dundurs, J., Tsai, K.C., Keer, L.M., 1973. Contact between elastic bodies with wavy surfaces. J. Elast. 3, 109-115.

Ebhota, W.S., Karun, A.S., Inambao, F.L., 2016. Centrifugal casting technique baseline knowledge, applications, and processing parameters: overview. Int. J. Mater. Res. 107(10), 960-969.

Eroglu, U., 2016. Large deflection analysis of planar curved beams made of functionally graded materials using variational iterational method. Compos. Struct. 136, 204-216.

Fukui, Y., Yamanaka, N., 1992. Elastic analysis for thick-walled tubes of functionally graded material subjected to internal pressure. JSME Int. J. Ser. 1, 35(4), 379-385.

Giannakopoulos, A.E., Suresh, S. 1997a. Indentation of solids with gradients in elastic properties: Part I. Point force. Int. J. Solids Struct. 34(19), 2357-2392.

Giannakopoulos, A.E., Suresh, S. 1997b. Indentation of solids with gradients in elastic properties: Part II. Axisymmetric indentors. Int. J. Solids Struct. 34(19), 2393-2428.

Gibson, L.J., 1989. Modelling the mechanical behavior of cellular materials. Mater. Sci. Eng. A, 110, 1-36.

Goyat, V., Verma, S., Garg, R.K., 2019. Stress concentration reduction using different functionally graded materials layer around the hole in an infinite panel. Strength Fract. Complex. 12(1), 31-45.

Gurtin, M.E., Murdoch, A.I., 1978. Surface stress in solids. Int. J. Solids Struct. 14(6), 431-440.

Guleid, H., Faal, R.T., Fotuhi, A.R., Milani, A.S., 2022. A generalization and application of the Michell solution for functionally graded circular planes. Z. Angew. Math. Phys. 73, 1-31.

Guler, M.A., Erdogan, F., 2006. Contact mechanics of two deformable elastic solids with graded coatings. Mech. Mater. 38(7), 633-647.

Hart, E.L., Terokhin, B.I., 2025a. Finite-element analysis of stress concentration in thin plates and cylindrical shells with a circular hole surrounded by an inclusion of functionally graded material. J. Math. Sci. 291(5), 1-13.

Hart, E.L., Terokhin, B.I., 2025b. Effect of functionally graded inclusion on stress conservation near a circular hole in thin plates for different boundary conditions. J. Optim. Differ. Equ. Appl. 33(1), 110-127.

Hassan, H.A.Z., 1997. Concentrated force acting on an elastic inclusion in a thick-walled tube. J. Eng. Math. 32, 73-86.

Ho, K.C., Chau, K.T., 1997. An infinite plane loaded by a rivet of a different material. Int. J. Solids Struct. 34(19), 2477-2496.

Hou, J.P, Hills, D.A., 2001. Contact between a pin and a plate with a hole under interference-fit and clearance-fit conditions. Proc. Inst. Mech. Eng. Part C: J. Mech. Eng. Sci. 215(6), 629-639.

Hou, J., Lv, J., Ricoeur, A., Hu, Y., Zuo, H., Chen, Y., Li, Q., 2022. The M-integral in fracture and damage mechanics: A review of developments and applications. Eng. Fract. Mech. 273, 108741.

Hou, J., Zhang, C., Li, Q., 2020. The concept and numerical evaluation of M-integral based on domain integral method in cracked viscoelastic materials. Mech. Mater. 145, 103363.

Iyer, K., 2001. Solutions for contact in pinned connections. Int. J. Solids Struct. 38(50-51), 9133-9148.

Ke, L.L., Wang, Y.S., 2006. Two-dimensional contact mechanics of functionally graded materials with arbitrary spatial variations of material properties. Int. J. Solids Struct., 43(18-19), 5779-5798.

Kubair, D.V., Bhanu-Chandar, B., 2008. Stress concentration factor due to a circular hole in functionally graded panels under uniaxial tension. Int. J. Mech. Sci. 50(4), 732-742.

Kumar, P., Sharma, S.K., Singh, R.K.R., 2023. Recent trends and future outlooks in manufacturing methods and applications of FGM: a comprehensive review. Mater. Manuf. Process. 38(9), 1033-1067.

Lanczos, C., 1956. Applied Analysis. Prentice Hall, Englewood Cliffs.

Li, H., Liu, Y., 2014. Functionally graded hollow cylinders with arbitrary varying material properties under nonaxisymmetric loads. Mech. Res. Comm. 55, 1-9.

Li, Y., Huang, R., Zhao, S., Wang, J., 2022. Contact pressure analysis of pin-loaded lug with clearance. Adv. Mech. Eng. 14(6), 16878132221107475.

Liu, C.S., Zhang, K., Yang, L., 2005. The compliance contact model of cylindrical joints with clearances. Acta Mech. Sinica 21, 451-458.

Liu, C. S., Zhang, K., Yang, R., 2007. The FEM analysis and approximate model for cylindrical joints with clearances. Mech. Mach. Theory 42(2), 183-197.

Nie, G.J., Batra, R.C., 2010. Exact solutions and material tailoring for functionally graded hollow circular cylinders. J. Elasticity, 99, 179-201.

Noble, B., Hussain, M.A., 1969. Exact solution of certain dual series for indentation and inclusion problems. Int. J. Eng. Sci. 7(11), 1149-1161.

Martin, R.B. 1991. Determinants of the mechanical properties of bones. J. Biomech. 24, 79-88.

927 Mencik, J., 2013. Mechanics of components with treated or coated surfaces. Vol. 42. Springer Science
928 & Business Media.
929 Michell, J.H., 1899. On the direct determination of stress in an elastic solid, with application to the
930 theory of plates. Proc. London Math. Soc. 31 (1), 100-124.
931 Mizushima, I., Hamada, M., 1983. Stress analysis around circular hole in infinite plate with rigid disk.
932 Bull. JSME 26(218), 1296-1301.
933 Mohammadi, M., Dryden, J.R., Jiang, L., 2011. Stress concentration around a hole in a radially
934 inhomogeneous plane. Int. J. Solids Struct. 48, 483-491.
935 Mohammadi, M., Dryden, J.R. 2008. Thermal stress in a nonhomogeneous curved beam. J. Therm.
936 Stresses 31(7), 587-598.
937 Nguyen, V.L., Nguyen, V.T., Duc, N.D., 2025. Pin-loaded hole contact in an anisotropic
938 magnetoelectroelastic plate. Int. J. Mech. Sci. 302, 110551.
939 Nie, G. J., Zhong, Z., Batra, R.C., 2018. Material tailoring for reducing stress concentration factor at
940 a circular hole in a functionally graded material (FGM) panel. Compos. Struct. 205, 49-57.
941 Nie, G.J., Batra, R.C., 2019. Reducing stress concentration factor by strengthening circular hole with
942 functionally graded incompressible material layer. Thin-Walled Struct. 144, 106223.
943 Noble, B., Whiteman, J.R., 1970. Solution of dual trigonometric series using orthogonality relations.
944 SIAM J. Appl. Math. 18(2), 372-379.
945 Omar, T., Hassan, H.A., 1991. Approximate solution for an indentation problem using dual series
946 equations. Int. J. Eng. Sci. 29(2), 187-194.
947 Osborn, H.B., 1941. Surface hardening by induction. Trans. Electrochem. Soc. 79(1), 215-241.
948 Pei, Y.L., Li, L.X., 2021. A simplified theory of FG curved beams. Eur. J. Mech. A/Solids 85,
949 104126.
950 Persson, A., 1964. On the stress distribution of cylindrical elastic bodies in contact. PhD. Dissertation,
951 Chalmers University, Sweden.
952 Persson, E., Eriksson, I., Hammersberg, P., 1997. Propagation of hole machining defects in pin-
953 loaded composite laminates. J. Compos. Mater. 31(4), 383-408.
954 Pradeep, A.D., Rameshkumar, T. 2021. Review on centrifugal casting of functionally graded
955 materials. Mater. Today: Proc. 45, 729-734.
956 Radi, E., 2024. Dual series solution for the standardized ISRM Brazilian disc test modelled as an
957 advancing contact problem. Int. J. Rock Mech. Min. Sci. 173, 105634.
958 Radi, E., 2027. Surface elasticity effects in pin-hole contact with radial clearance. Eur. J. Mech.
959 A/Solids 121, 106294.

960 Radi, E., Falope, F.O., Lanzoni, L., 2025. Advancing contact between a rigid pin and a FGM circular
961 beam with clearance. Int. J. Solids Struct. 311, 113244.
962 Radi, E., Güler, M.A., 2026a. Effects of microstructure in pin-loaded hole contact with clearance. Int.
963 J. Eng. Sci. 221 (2026) 104454.
964 Radi, E., Güler, M.A., 2026b. Effects of couple tractions in pin-loaded hole contact. Int. J. Solids
965 Struct. 339 (2026) 114121.
966 Radi, E., Morini, L., Sevostianov, I., 2016. Conservation integrals for two circular holes kept at
967 different temperatures in a thermoelastic solid. Int. J. Solids Struct. 85, 1-14.
968 Radi, E., Strozzi, A., 2023. Advancing contact of a 2D elastic curved beam indented by a rigid pin
969 with clearance. Int. J. Nonlin. Mech. 149, 104313.
970 Rao, A.K., Eshwar, V.A., Dattaguru, B., 1977. A comprehensive study of interference and clearance
971 fit pins in large planes. Mech. Res. Commun. 4(5), 325-332.
972 Rao, A.K., 1978. Elastic analysis of pin joints. Compos. Struct. 9, 125-144.
973 Sburlati, R., 2013. Stress concentration factor due to a functionally graded ring around a hole in an
974 isotropic plane. Int. J. Solids Struct. 50(22-23), 3649-3658.
975 Sburlati, R., Atashipour, S.R., Atashipour, S.A., 2014. Reduction of the stress concentration factor in
976 a homogeneous panel with hole by using a functionally graded layer. Compos. B: Eng. 61, 99-
977 109.
978 Silva, R.F., Coelho, P.G., Conde, F.M., Santos, B.R., Oliveira, J.P., 2024. Minimizing the maximum
979 von Mises stress of elastic continuum structures using topology optimization and additively
980 manufactured functionally graded materials. Comput. Struct. 301, 107469.
981 Singh, G., Bhandakkar, T.K., 2019. Simplified approach to solution of mixed boundary value
982 problems on homogeneous circular domain in elasticity. ASME J. Appl. Mech. 86(2), 021007.
983 Sneddon, I.N., 1966. Mixed boundary value problems in potential theory. John Wiley, New York.
984 Sundaram, N., Farris, T.N., 2010a. Mechanics of advancing pin-loaded contacts with friction. J.
985 Mech. Phys. Solids 58(11), 1819-1833.
986 Sundaram, N., Farris, T.N., 2010b. The generalized advancing conformal contact problem with
987 friction, pin loads and remote loading-case of rigid pin. Int. J. Solids Struct. 47(6), 801-815.
988 Suresh, S., 2001. Graded materials for resistance to contact deformation and damage. Sci. 292, 2447-
989 2451.
990 Suresh, S., Olsson, M., Giannakopoulos, A.E., Padture, N.P., Jitcharoen, J., 1999. Engineering the
991 resistance to sliding-contact damage through controlled gradients in elastic properties at contact
992 surfaces. Acta Mater. 47(14), 3915-3926.

993 Tutuncu, N., 2007. Stresses in thick-walled FGM cylinders with exponentially-varying properties.
994 Eng. Struct. 29(9), 2032-2035.
995 Vasu, T.S., Bhandakkar, T.K., 2018. Plane strain cylindrical indentation of functionally graded half-
996 plane with exponentially varying shear modulus in the presence of residual surface tension. Int.
997 J. Mech. Sci. 135, 158-167.
998 Yi-Feng, H., Chen, Y.H., 2011. The area contraction and expansion for a nano-void under four
999 different kinds of loading. Arch. Appl. Mech., 81(9), 1323-1331.
1000 Zhang, B., Jaiswal, P., Rai, R., Nelaturi, S., 2018. Additive manufacturing of functionally graded
1001 material objects: a review. J. Comput. Inf. Sci. Eng., 18(4), 041002.
1002 Zhang, C., Chen, F., Huang, Z., Jia, M., Chen, G., Ye, Y., Lin, Y., Liu, W., Chen, B., Shen, Q., Zhang,
1003 L., Lavernia, E.J., 2019. Additive manufacturing of functionally graded materials: A review.
1004 Mater. Sci. Eng. A 2019; 764, 138209.
1005 Zhou, Y., Lin, Q., Hong, J., Yang, N., 2021. Optimal design of functionally graded material for stress
1006 concentration reduction. Struct. 29, 561-569.
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1010 **Appendix A**
1011 The set of dual equations $(2.16)_1$ and (2.21) can be solved for the coefficients A_n, (n = 0, 1, 2,…) by
1012 using the procedure proposed by Sneddon (1966) and Dundurs et al. (1973), based on the introduction
1013 of an auxiliary stress function φ such that

1014 $$\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos n\theta = H(\alpha-\theta)\cos\frac{\theta}{2}\int_{\theta}^{\alpha}\frac{\varphi(s)\,\mathrm{d}s}{\sqrt{\cos\theta-\cos s}}, \qquad \text{for} \quad 0\le|\,\theta\,|\le\pi. \tag{A.1}$$

1015 where $H(\theta)$ is the Heaviside unit step function. Then, the coefficients A_n, for n = 0, 1, 2,…, of the
1016 Fourier cosine series expansion (A.1) are given by Block and Keer (2007), Sneddon (1966), and
1017 Dundurs et al. (1973):

1018 $$A_0=\frac{2}{\pi}\int_0^{\alpha}\cos\frac{\theta}{2}\,\mathrm{d}\theta\int_{\theta}^{\alpha}\frac{\varphi(s)\,\mathrm{d}s}{\sqrt{\cos\theta-\cos s}}=\frac{2}{\pi}\int_0^{\alpha}\varphi(s)\,\mathrm{d}s\int_0^{s}\frac{\cos(\theta/2)\,\mathrm{d}\theta}{\sqrt{\cos\theta-\cos s}}=\sqrt{2}\int_0^{\alpha}\varphi(s)\,\mathrm{d}s, \tag{A.2}$$

1019 $$\begin{aligned}A_n&=\frac{2}{\pi}\int_0^{\alpha}\cos\frac{\theta}{2}\cos n\theta\,\mathrm{d}\theta\int_{\theta}^{\alpha}\frac{\varphi(s)\,\mathrm{d}s}{\sqrt{\cos\theta-\cos s}}=\frac{2}{\pi}\int_0^{\alpha}\varphi(s)\,\mathrm{d}s\int_0^{s}\frac{\cos(\theta/2)\cos n\theta\,\mathrm{d}\theta}{\sqrt{\cos\theta-\cos s}}=\\&=\frac{1}{\pi}\int_0^{\alpha}\varphi(s)\,\mathrm{d}s\int_0^{s}\frac{\cos(n+1/2)\theta+\cos(n-1/2)\theta}{\sqrt{\cos\theta-\cos s}}\,\mathrm{d}\theta=\\&=\frac{1}{\sqrt{2}}\int_0^{\alpha}[P_n(\cos s)+P_{n-1}(\cos s)]\,\varphi(s)\,\mathrm{d}s, \qquad \text{for} \quad n=1,2,...,\end{aligned} \tag{A.3}$$

1020 where the following Mehler's integral for the Legendre polynomial (Sneddon, 1966, Eq. 2.6.20) has

been used

$$\int_0^s \frac{\cos(n+1/2)\theta}{\sqrt{\cos\theta-\cos s}}\,\mathrm{d}\theta=\frac{\pi}{\sqrt{2}}P_n(\cos s),\qquad \text{for}\quad n=0,1,2,... \tag{A.4}$$

Now, using Eq. (A.3) and the following result (Sneddon, 1966, Eq. 2.6.31)

$$\frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}[P_n(\cos s)+P_{n-1}(\cos s)]\sin n\theta=\frac{H(\theta-s)\cos(\theta/2)}{\sqrt{\cos s-\cos\theta}}, \tag{A.5}$$

from Eq. (2.21) it follows

$$\int_0^{\theta}\frac{\varphi(s)\,\mathrm{d}s}{\sqrt{\cos s-\cos\theta}}=\frac{1}{\cos(\theta/2)}\left[(c\,A_0+d)\,\theta+\sum_{n=1}^{\infty}h_n\,A_n\sin n\theta\right],\qquad \text{for}\quad 0\le|\,\theta\,|\le\alpha, \tag{A.6}$$

which is an integral equation of the Abel type. Then, Eq. (A.6) can be inverted to obtain the auxiliary stress function φ, namely

$$\varphi(s)=\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s\frac{\sin(\theta/2)}{\sqrt{\cos\theta-\cos s}}\left[(c\,A_0+d)\,\theta+\sum_{n=1}^{\infty}h_n\,A_n\sin n\theta\right]\mathrm{d}\theta, \tag{A.7}$$

Using the following integral provided in the appendix of Noble and Hussain (1969)

$$\int_0^s\frac{\theta\sin(\theta/2)}{\sqrt{\cos\theta-\cos s}}\,\mathrm{d}\theta=-\sqrt{2}\,\pi\ln\left(\cos\frac{s}{2}\right), \tag{A.8}$$

and the result

$$\int_0^s\frac{\sin(\theta/2)\sin n\theta}{\sqrt{\cos\theta-\cos s}}\,\mathrm{d}\theta=\frac{\pi}{2\sqrt{2}}[P_{n-1}(\cos s)-P_n(\cos s)],\qquad \text{for}\quad n=1,2,... \tag{A.9}$$

which follows from Eq. (A.4), the integrals in Eq. (A.7) can be calculated in closed form, thus obtaining

$$\varphi(s)=\frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}s}\left\{-4(c\,A_0+d)\ln\left(\cos\frac{s}{2}\right)+\sum_{n=1}^{\infty}h_n\,A_n\,[P_{n-1}(\cos s)-P_n(\cos s)]\right\}, \tag{A.10}$$

namely

$$\varphi(s)=\frac{1}{\sqrt{2}}\left\{2(c\,A_0+d)+\sum_{n=1}^{\infty}n\,h_n\,A_n\,[P_n(\cos s)+P_{n-1}(\cos s)]\right\}\tan\frac{s}{2}. \tag{A.11}$$

Then, using the results

$$\int_0^{\alpha}[P_n(\cos s)+P_{n-1}(\cos s)]\tan\frac{s}{2}\,\mathrm{d}s=\frac{1}{n}[P_{n-1}(\cos\alpha)-P_n(\cos\alpha)]\,, \tag{A.12}$$

$$\begin{aligned}\int_0^{\alpha}[P_n(\cos s)+P_{n-1}(\cos s)][P_j(\cos s)+P_{j-1}(\cos s)]\tan\frac{s}{2}\,\mathrm{d}s&=\frac{(1+\cos\alpha)\sin^2\alpha}{2(n^2-j^2)}\\ &\times[(n+1)P_{j-1}^{(0,1)}(\cos\alpha)P_{n-2}^{(1,2)}(\cos\alpha)-(j+1)P_{n-1}^{(0,1)}(\cos\alpha)P_{j-2}^{(1,2)}(\cos\alpha)],\end{aligned} \tag{A.13}$$

provided in the appendix of Radi and Strozzi (2023), the substitution of Eq. (A.11) into Eqs. (A.2) and (A.3) yields the relations (2.24) and (2.25) for the coefficients A_n, for n = 0, 1, 2,….

Appendix B

Path dependence of the contour M-integral for FGMs follows from the transformation into an area integral evaluated over the region A surrounded by the closed contour C, whose outward normal unit vector is denoted by $\boldsymbol{n}$. Indeed, it is easy to prove that

$$\operatorname{div}(w\boldsymbol{r} - \boldsymbol{\sigma}\nabla\boldsymbol{u}\,\boldsymbol{r}) = 2w + \boldsymbol{r}\cdot\nabla w - \boldsymbol{\sigma}\cdot[\nabla\boldsymbol{u} + (\nabla\nabla\boldsymbol{u})\,\boldsymbol{r}] - (\nabla\boldsymbol{u})\,\boldsymbol{r}\cdot\operatorname{div}\boldsymbol{\sigma}\,. \tag{B.1}$$

Then, by using the balance condition in the absence of body forces, $\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0}$, the compatibility condition $\boldsymbol{\varepsilon} = (\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})/2$, and the definition of strain-energy density, $2w = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$, one obtains

$$\operatorname{div}(w\boldsymbol{r} - \boldsymbol{\sigma}\nabla\boldsymbol{u}\,\boldsymbol{r}) = \boldsymbol{r}\cdot(\nabla w - \boldsymbol{\sigma}\cdot\nabla\nabla\boldsymbol{u}) = \boldsymbol{r}\cdot\left(\nabla w - \frac{\partial w}{\partial\boldsymbol{\varepsilon}}\cdot\nabla\boldsymbol{\varepsilon}\right) = \boldsymbol{r}\cdot\nabla r\left.\frac{\partial w}{\partial r}\right|_{\varepsilon} = r\left.\frac{\partial w}{\partial r}\right|_{\varepsilon}. \tag{B.2}$$

Using Eq. (B.2) and the Gauss-Green theorem, it follows directly that

$$\oint_C (w\boldsymbol{I} - \nabla\boldsymbol{u}^{\mathrm{T}}\boldsymbol{\sigma})\boldsymbol{n}\cdot\boldsymbol{r}\,ds = \int_A \operatorname{div}(w\boldsymbol{r} - \boldsymbol{\sigma}\nabla\boldsymbol{u}\,\boldsymbol{r})\,dA = \int_A \left.\frac{\partial w}{\partial r}\right|_{\varepsilon} r\,dA, \tag{B.3}$$

where C is an arbitrary closed contour surrounding the area A. Eq. (6.2) then follows by applying the divergence theorem to the annular region A between an outer circle of radius r, with $r > r_o$, and an inner circle of arbitrary radius r, with $r_i \le r \le r_o$, by considering that the explicit material derivative of w vanishes outside the graded ring. For a homogeneous elastic material, the strain-energy density w does not depend explicitly on r, and thus the path-independence of the classical M-integral is recovered by Eq. (B.3).